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Disagreement point axioms and the egalitarian bargaining solution

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Abstract We provide new characterizations of the egalitarian bargaining solution on the class of strictly comprehensive n-person bargaining problems. The main axioms used in all of our results are Nash's IIA and disagreement point monotonicity—an axiom which requires a player's payoff to strictly increase in his disagreement payoff. For $n = 2$ these axioms, together with other standard requirements, uniquely characterize the egalitarian solution. For $n > 2$ we provide two extensions of our 2-person result, each of which is obtained by imposing an additional axiom on the solution. Dropping the axiom of anonymity, strengthening disagreement point monotonicity by requiring player i's payoff to be a strictly decreasing function of the disagreement payoff of every other player $j \neq i$, and adding a "weak convexity" axiom regarding changes of the disagreement point, we obtain a characterization of the class of weighted egalitarian solutions. This "weak convexity" axiom requires that a movement of the disagreement point in the direction of the solution point should not change the solution point. We also discuss the so-called "transfer paradox" and relate it to this axiom.

Keywords Bargaining · Egalitarian solution · Disagreement point monotonicity

1 Introduction

In this paper we consider Nash's bargaining problem (Nash 1950) and characterize its egalitarian solution. A bargaining problem is described as follows: n players are facing a set of feasible agreements. Each of these agreements can be achieved if and only if players agree on it unanimously. In case they do not reach a unanimous agreement, the bargaining outcome is some prespecified agreement (the *status quo*), under

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which each player i enjoys d_i (von-Neumann Morgenstern) utility units. We call d, the utility-vector of the status quo, the *disagreement point*, and we call the utility-image of the feasible-agreements-set, which we denote S, the *feasible set*. Informally, a bargaining solution is a rule that chooses a unique point in the feasible set for every such situation.

2 The model

We now turn to a formal description of the bargaining model, for which we will use the following notation and definitions. We denote vector inequalities in $\mathbb{R}^n$ as follows: $x \geq y \Leftrightarrow x_i \geq y_i$ for all i; $x \gneqq y \Leftrightarrow (x \geq y)\&(x \neq y)$; $x > y \Leftrightarrow x_i > y_i$ for all i. Given a vector $x \in \mathbb{R}^n$, x_{-i} denotes the $(n-1)$-dimensional vector which is obtained from x be deleting its i's coordinate. We set $N = \{1, \ldots, n\}$, and denote the set of all permutations on N by Π. The unit vectors of $\mathbb{R}^n$ are denoted e_i for all $i \in N$. Given $x, y \in \mathbb{R}^n$, the line segment with end points x and y is denoted $[x; y]$. Given a set $A \subset \mathbb{R}^n$, the *convex hull* of A is denoted conv$A$, and the interior of A is denoted int$A$. A feasible set S that satisfies $S = \pi S$ for all $\pi \in \Pi$ is called *symmetric*.[1] Given a feasible set S and a point $x \in S$, let $S_x \equiv \{s \in S | s \geq x\}$. The strict and weak *Pareto frontiers* of a feasible set S are:

$$P(S) \equiv \{x \in S | y \gneqq x \Rightarrow y \notin S\}$$

and

$$WP(S) \equiv \{x \in S | y > x \Rightarrow y \notin S\}$$

A feasible set S is *comprehensive* if for all $x, y \in S$ such that $y \leq x$ and for every $z \in \mathbb{R}^n$ that satisfies $y \leq z \leq x, z \in S$. A comprehensive feasible set S for which $P(S) = WP(S)$ is *strictly comprehensive*.

A *bargaining problem* is a pair of a feasible set and a disagreement point, (S, d), which satisfies the following assumptions:

- (A1) $S \subset \mathbb{R}^n$ is closed and convex;
- (A2) $d \in S$, and there exists $x \in S$ such that $x > d$;
- (A3) S_d is bounded; and
- (A4) S is strictly comprehensive.

All of these assumptions have natural economic interpretations. As they are well-known and thoroughly discussed in the literature, we will only discuss them briefly. Convexity means that any lottery over feasible agreements is itself a feasible agreement. Closedness is a technical regularity condition. According to (A2), disagreement is one of the feasible outcomes, and, more importantly, it is strictly Pareto dominated. (A3) is obvious. It is implied by, and is substantially weaker than, a boundedness-from-above assumption on the feasible set. Without it, no satisfactory bargaining solutions

[1] Given $\pi \in \Pi$, $\pi S = \{\pi s | s \in S\}$.

exist. (A4) says that whenever a player i is willing to compromise and give up some utility, there exists some other player j who can gain something.

Let $\mathcal{B}$ be the collection of pairs (S, d) satisfying (A1) through (A4). A *solution* is a map $\mu : \mathcal{B} \longrightarrow \mathbb{R}^n$ which satisfies $\mu(S, d) \in S$ for all $(S, d) \in \mathcal{B}$. We are interested in solutions that satisfy the following axioms, in the statements of which (S, d), (S', d'), and (T, d') are arbitrary elements of $\mathcal{B}$:

Pareto Optimality (PO): $\mu(S, d) \in P(S)$.
Individual Rationality (IR): $\mu_i(S, d) \geq d_i$ for all $i \in N$.
Translation Invariance (TINV): $\mu(S+\{p\}, d+p) = \mu(S, d)+p$ for all $p \in \mathbb{R}^n$.[2]
Symmetry (SY): If S is symmetric and $d = \pi d$ for all $\pi \in \Pi$, then $\mu_i(S, d) = \mu_j(S, d)$ for every $i, j \in N$.
Independence of Irrelevant Alternatives (IIA): If $S \subset T$, $d = d'$ and $\mu(T, d') \in S$, then $\mu(S, d) = \mu(T, d')$.
Disagreement Point Monotonicity (DIM): For all $i \in N$: if $d'_i > d_i$, $d'_j = d_j$ for all $j \in N \backslash \{i\}$, and $S' = S$, then $\mu_i(S', d') > \mu_i(S, d)$.

The first five axioms, introduced by Nash (1950), are well-known. The sixth, DIM, was first presented in Thomson (1987) with a weak inequality.[3] The strict-inequality version appears in Peters and van Damme (1991).[4] We will discuss the significance of the difference in Sect. 8.

One particular solution that satisfies all of these axioms is the *egalitarian solution*, E. Letting $(S, d) \in \mathcal{B}$, $E(S, d) = d + \varepsilon \cdot \mathbf{1}$, where ε is the maximal number such that the expression on the right hand side is in S.[5] Apart from E, the best-known solutions considered in the literature are the *Nash solution*, N (Nash 1950), and the *Kalai–Smorodinsky* solution, K (Kalai and Smorodinsky 1975). Letting (S, d) denote an arbitrary element of $\mathcal{B}$, $N(S, d)$ is the (unique) maximizer of $\Pi_{i=1}^n (x_i - d_i)$ over S_d and $K(S, d)$ is the intersection-point of $WP(S)$ and $[d; a(S, d)]$, where $a(S, d)$ is given by $a_i(S, d) = \max\{x_i | x \in S_d\}$ for all $i \in N$.[6]

The paper is organized as follows. In Sect. 3 we discuss IR. There, we show that it is essentially implied by PO, IIA, and DIM. This implication is important, because in all of our results in the subsequent sections we impose IR on the solution. Section 4 contains our result for 2-person bargaining. There, we show that E is the unique solution that satisfies the axioms listed above. In Sect. 5 we introduce two different ways to extend our 2-person characterization to the multi-person case, by imposing additional axioms on the solution. In Sect. 6 we characterize the class of weighted egalitarian solutions. In Sect. 7 we briefly discuss the phenomenon of “transfer paradoxes”, and in Sect. 8 we conclude.

2 $S + \{p\} = \{s + p | s \in S\}$.

3 A weak inequality version of DIM in the context of 2-person bargaining appears in Livne (1986).

4 In their paper, Peters and van Damme allow the disagreement point to lie in the Pareto frontier. Their axiom requires player i's payoff to strictly increase in d_i only if it is possible. We will refer to their work in more detail later in the paper.

5 $\mathbf{1} = (1, \ldots, 1)$. Similarly, we let $\mathbf{0} = (0, \ldots, 0)$.

6 The point $a(S, d)$ is called the *ideal point* of (S, d).

3 Individual rationality

In this section we prove that, essentially, IR is implied by three other axioms: PO, IIA, and DIM. In the 2-person case the word "essentially" in the last statement can be dropped. In the n-person case, this logical implication holds on a rich and large sub-class of $\mathcal{B}$, which we call $\mathcal{B}^*$. This implication is important, because IR is imposed on the solution in all the characterizations we derive.

3.1 The 2-person case

Lemma 1 *Let $n = 2$. Then, if a solution satisfies PO, IIA, and DIM, then it satisfies IR.*

Proof Let $n = 2$ and let μ be a solution that satisfies PO, IIA, and DIM. Assume by contradiction that there exists an element $(S, d) \in \mathcal{B}$ such that $\mu_i(S, d) < d_i$ for some $i \in \{1, 2\}$. By PO we may assume, without loss of generality, that $\mu_1(S, d) < d_1$ and $\mu_2(S, d) > d_2$. In fact, since S is strictly comprehensive, PO and the fact that $(d_1, a_2(S, d)) \in S$ imply that $\mu_2(S, d) > a_2(S, d)$.

Define $Q \equiv \text{conv}\{d, \mu(S, d), (a_1(S, d), d_2), (d_1, a_2(S, d))\}$. We argue that $(Q, d) \in \mathcal{B}$. Note that Q is compact and convex, and $x \equiv \frac{1}{2}(a_1(S, d), d_2) + \frac{1}{2}(d_1, a_2(S, d)) \in Q$ is such that $x > d$. Finally, it is easy to see that Q is strictly comprehensive, and therefore assumptions (A1)–(A4) hold, hence $(Q, d) \in \mathcal{B}$. Let $d' \equiv (\mu_1(S, d), d_2)$, and define $Q' \equiv \text{conv}(Q \cup \{d'\})$. Along the lines of the arguments we outlined above, it is easily verified that $(Q', d'), (Q', d) \in \mathcal{B}$. Since $Q \subset S$ and $\mu(S, d) \in Q$, we obtain by IIA that:

$$\mu(Q, d) = \mu(S, d) \tag{1}$$

Note that $P(Q') = P(Q)$, and therefore, by PO we obtain that $\mu(Q', d) \in Q$. Then, since $Q \subset Q'$ we obtain by IIA that:

$$\mu(Q, d) = \mu(Q', d) \tag{2}$$

Combining Eqs. 1 and 2 we conclude that $\mu(Q', d) = \mu(S, d)$. Therefore, applying DIM to (Q', d) we obtain that $\mu_1(Q', d') < \mu_1(S, d)$. However, this implies that $\mu(Q', d') \notin Q'$, a contradiction. □

3.2 The n-person case

Given a feasible set S, let us denote by $\mathcal{C}(S)$ the *comprehensive hull* of S—the set of points each of which is bounded from above by some point of S:

$$\mathcal{C}(S) \equiv \{y \in \mathbb{R}^n | \exists x \in S \text{ such that } y \leq x\}$$

We will call a feasible set S for which $S = \mathcal{C}(S)$ *fully comprehensive*.

Let $\mathcal{B}^* \equiv \mathcal{B}_{\mathcal{C}} \cup \mathcal{B}_+$, where $\mathcal{B}_+ \equiv \{(S,d) \in \mathcal{B} | S = S_x \text{ for some } x \in S\}$, and $\mathcal{B}_{\mathcal{C}} \equiv \{(S,d) \in \mathcal{B} | S = \mathcal{C}(S)\}$.

The class $\mathcal{B}^*$ consists of two important sub-classes of problems. The first, $\mathcal{B}_{\mathcal{C}}$, is the class of problems where players can freely dispose of utility. The second, $\mathcal{B}_+$, contains all the problems (S,d) for which $S = S_d$. This is an important class, because in certain applications, non-individually-rational outcomes are irrelevant.

Therefore, for most economic applications one could restrict attention to $\mathcal{B}^*$. It is easy to see that the inclusion $\mathcal{B}^* \subsetneq \mathcal{B}$ is strict. For example, $(\text{conv}\{(-\frac{1}{2},1),(1,-\frac{1}{2}), \mathbf{0}\}, \mathbf{0})$ is an element of $\mathcal{B} \backslash \mathcal{B}^*$.

Lemma 2 *A solution on $\mathcal{B}^*$ that satisfies PO, IIA, and DIM, satisfies IR.*

Proof Let μ be a solution on $\mathcal{B}^*$ that satisfies PO, IIA, and DIM. Assume by contradiction that there exists an element $(S,d) \in \mathcal{B}^*$ such that $\mu_i(S,d) < d_i$ for some i. Without loss of generality, assume that $\mu_1(S,d) < d_1$. Define $z \equiv \mu(S,d)$ and $t \equiv (z_1, d_{-1})$. Suppose first that $S = \mathcal{C}(S)$. Then, since $d \in S$, $t \leq d$, and S is fully comprehensive, $t \in S$. Suppose on the other hand that $S \neq \mathcal{C}(S)$; that is, $S = S_x$ for some $x \in S$. Since $z, d \in S = S_x$, $t \geq x$. Then, since $x \leq t \leq d$, $x, d \in S$, and S is comprehensive, $t \in S$.

Note that (S_t, t) and (S_t, d) are elements of $\mathcal{B}^*$. Since $S_t \subset S$, by IIA $\mu(S_t, d) = z$. Then, by DIM, $\mu_1(S_t, t) < \mu_1(S_t, d) = z_1 = t_1$. This implies that $\mu(S_t, t) \notin S_t$, a contradiction. □

Even though, as we argued, it is natural to restrict attention to $\mathcal{B}^*$ for most economic applications, our characterizations hold on the entire class $\mathcal{B}$, and therefore—from now on—we will take it to be the domain of our analysis.

4 2-Person bargaining

Before we turn to the main result of this section we prove two lemmas. In the statement of the first lemma we impose the following axiom, in the statement of which (S,d) is an arbitrary element of $\mathcal{B}$:

Strict Individual Rationality (S.IR): $\mu_i(S,d) > d_i \quad$ for all $i \in N$.

Lemma 3 *Let $n = 2$. Then, if a solution satisfies PO, IR, and DIM, then it satisfies S.IR.*

Proof Let $n = 2$. Let $(S,d) \in \mathcal{B}$ and let μ be a solution which satisfies PO, IR, and DIM. Suppose, by way of contradiction, that S.IR is violated. Then, without loss of generality, and by IR, we may assume that $\mu_1(S,d) = d_1$. Let $x \equiv \mu_2(S,d)$. By PO and assumption (A2) $x > d_2$. Let $d' = (d_1, d_2 + \varepsilon)$, where $\varepsilon \in (0, x - d_2)$. Note that (S,d') is a well-defined element of $\mathcal{B}$. Now, by IR, $\mu_1(S,d') \geq d_1$, and by DIM $\mu_2(S,d') > x$. Then, since $\mu(S,d') \in S$, we conclude that $\mu(S,d) \notin P(S)$, a contradiction. □

Lemma 3 is not true for more than two players. For example, consider the following solution in the 3-person case. For every (3-dimensional) feasible set S, define

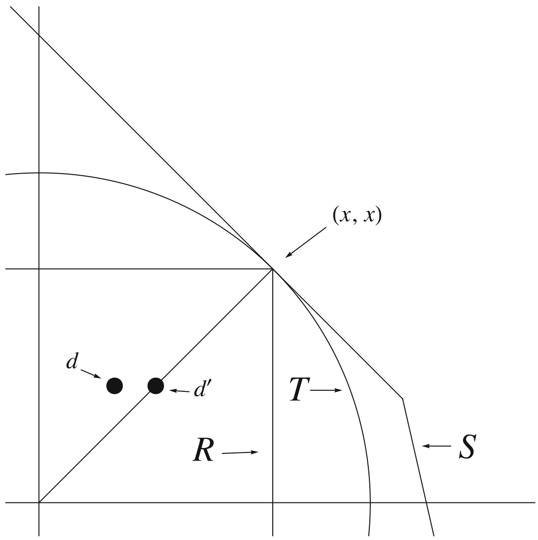

Fig. 1 Lemma 4: Equal solution-payoffs imply equal disagreement-payoffs

$S_{\{1,2\}} \equiv \{(x_1, x_2)|(x_1, x_2, d_3) \in S\}$. Consider the 3-person solution, μ^{12}, which is given by:

$$\mu^{12}(S, d) = (E_1(S_{\{1,2\}}, (d_1, d_2)), E_2(S_{\{1,2\}}, (d_1, d_2)), d_3)$$

for every $(S, d) \in \mathcal{B}$. In words, this solution "slices" every feasible set at the level (or " height") d_3, assigns players 1 and 2 their E-payoffs in the induced 2-person problem, and assigns player 3 his disagreement payoff. It is easy to verify that this solution satisfies PO, IR, DIM, but violates S.IR.

Lemma 4 *Let $n = 2$, and let μ be a solution that satisfies PO, SY, IIA, and DIM. Then, $\mu_1(S, d) = \mu_2(S, d)$ implies $d_1 = d_2$, for all $(S, d) \in \mathcal{B}$.*

Proof Let $(S, d) \in \mathcal{B}$ be such that $\mu_1(S, d) = \mu_2(S, d) \equiv x$. Without loss of generality, assume that $d_1 < d_2$. The following constructions are illustrated in Fig. 1. Define $R \equiv \{r \in \mathbb{R}^2 | r \leq (x, x)\} \cap S$, and $d' \equiv (d_2, d_2)$. By Lemmas 1 and 3, μ satisfies S.IR, and therefore $d_2 < x$. Thus, $d' \in \text{int} R$, and consequently we can find a symmetric feasible set T, such that:

- (i) $R \subset T \subset S$;
- (ii) $(x, x) \in T$; and
- (iii) $(T, d), (T, d') \in \mathcal{B}$.

Now, by PO and SY $\mu(T, d') = (x, x)$. On the other hand, by IIA, $\mu(T, d) = (x, x)$. Finally, $\mu(T, d') = \mu(T, d)$ constitutes a violation of DIM. □

We are now ready to state and prove our main result for the 2-person case.

Theorem 1 *Let $n = 2$. Then E is the unique solution that satisfies PO, TINV, SY, IIA, and DIM.*

Proof The fact that E satisfies the five axioms is easy to check. We now prove uniqueness. Let μ be a solution that satisfies the above axioms, and let $(S, d) \in \mathcal{B}$. Define $p \equiv (0, \mu_1(S, d) - \mu_2(S, d))$. By TINV, $\mu(S + \{p\}, d + p) = \mu(S, d) + p = (\mu_1(S, d), \mu_1(S, d))$. Then, by Lemma 4, the two coordinates of $d + p$ are equal. This, together with PO, implies that the solution for $(S + \{p\}, d + p)$ is the egalitarian solution: $\mu(S + \{p\}, d + p) = d + p + (\varepsilon, \varepsilon)$, where ε is the largest number such that $d + p + (\varepsilon, \varepsilon) \in S + \{p\}$. Clearly, ε is the largest number such that $d + (\varepsilon, \varepsilon) \in S$. Thus, we are done, because $\mu(S + \{p\}, d + p) = \mu(S, d) + p = d + p + (\varepsilon, \varepsilon)$, from which we obtain $\mu(S, d) = d + (\varepsilon, \varepsilon)$. □

The five axioms PO, TINV, SY, IIA, and DIM are tight: none of them is implied by the others. PO and TINV are satisfied by virtually all of the solutions considered in the literature, and SY is satisfied by all of the symmetric ones. There is no shortage of solutions that satisfy the first three axioms and exactly one of {IIA, DIM}. For example, N satisfies the first three axioms and IIA (but does not satisfy DIM), and K satisfies the first three axioms and DIM (but not IIA).

5 Multi-person bargaining

In this section our goal is to extend Theorem 1 to the case $n > 2$. We do not know whether this generalization is true without any further assumptions. In the following two subsections we propose two ways to obtain it by imposing additional axioms on the solution. In addition, we use a strengthening of the symmetry axiom. Instead of symmetry, we will impose on the solution the following axiom, in the statement of which (S, d) is an arbitrary element of $\mathcal{B}$:

Anonymity (AN): $\mu(\pi S, \pi d) = \pi \mu(S, d)$ for every $\pi \in \Pi$.

Finally, we state IR explicitly in our results. Recall that if one restricts attention to $\mathcal{B}^*$, then IR is redundant: it is implied by PO, IIA, and DIM.

5.1 Disagreement betweenness

Consider the following axiom, in the statement of which (S, d) and (T, d') are arbitrary elements of $\mathcal{B}$:

Disagreement Betweenness (DB): If $\mu(S, d) = \mu(T, d') \equiv \mathbf{x}$, and if $T = S$, then $\mu(S, \lambda d + (1 - \lambda)d') = \mathbf{x}$ for every $\lambda \in [0, 1]$.[7]

This axiom has been discussed in the literature, for example by Peters and van Damme (1991) and by Thomson (1994). DB has a straightforward interpretation. Suppose that the feasible set is known to be S, but the exact location of the disagreement point is uncertain: it may be either d' or d''.[8] Then, if all agents are indifferent between these two locations, it is natural to assume that they see a coin-flip between these two

[7] It is easy to see that $(S, \lambda d + (1 - \lambda)d')$ is a well-defined element of $\mathcal{B}$ for every $\lambda \in [0, 1]$.

[8] It is straightforward that this argument generalizes to any finite number of disagreement points.

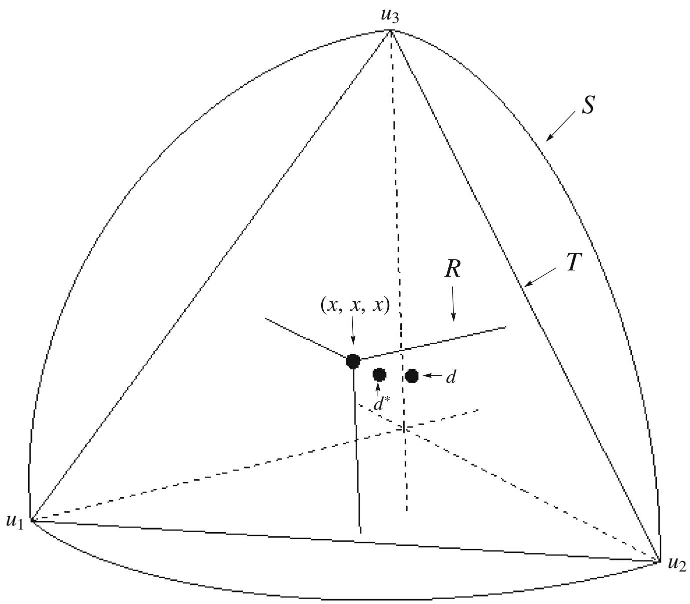

Fig. 2 Illustration of Lemma 5

locations as exactly as good as either of them. Formally, this is captured by representing the coin-flip as the appropriate convex combination of the d's.

The following impossibility result is worth noting. Peters and van Damme (1991) showed that if the disagreement point is allowed to lie on the boundary of the feasible set, and if non-strictly-comprehensive feasible sets are permitted, then DB is inconsistent with PO and IR for $n > 2$.[9]

Lemma 5 *Let μ be a solution which satisfies PO, IR, AN, IIA, DIM, and DB, and let (S, d) be an element of $\mathcal{B}$. Then, if S is fully comprehensive and $\mu(S, d) = x \cdot \mathbf{1}$ for some $x \in \mathbb{R}$, then $d_1 = d_2 = \cdots = d_n$.*

Proof Let $(S, d) \in \mathcal{B}$ be such that S is fully comprehensive. Assume that there exists an $x \in \mathbb{R}$, such that $\mu(S, d) = x \cdot \mathbf{1}$. Assume, by way of contradiction, and without loss of generality, that $d_1 \leq d_2 \leq \cdots \leq d_n$ and that $d_1 < d_n$. The following constructions are illustrated in Fig. 2. Define $R \equiv \{y \in \mathbb{R}^n | y \leq x \cdot \mathbf{1}\}$, and $\Pi_n \equiv \{\pi \in \Pi | \pi(n) = n\}$. By IR, $d \in R$, and $\pi d \in R$ for every $\pi \in \Pi$. Moreover, we can find a symmetric feasible set T such that:

- (i) $R \subset T \subset S$;
- (ii) $x \cdot \mathbf{1} \in T$; and
- (iii) $(T, \pi d) \in \mathcal{B}$ for every $\pi \in \Pi_n$.

[9] Peters and van Damme also noted that, in the 2-person case, the Nash solution N satisfies DB (and obviously PO and IR) also for the case where d may lie on the boundary of S and $P(S) \subsetneq WP(S)$.

By IIA, $\mu(T, d) = x \cdot \mathbf{1}$, and therefore, by AN, $\mu(T, \pi d) = x \cdot \mathbf{1}$ for every $\pi \in \Pi_n$. By DB $\mu(T, d^*) = x \cdot \mathbf{1}$, where $d^* \equiv \frac{1}{(n-1)!} \sum_{\pi \in \Pi_n} \pi d$. However, $\alpha \equiv d_i^* = d_j^* < d_n^*$ for every $1 \leq i, j < n$. Then, on the one hand AN and PO imply that $\mu(T, \alpha \cdot \mathbf{1}) = x \cdot \mathbf{1}$, while on the other hand, by DIM, $\mu_n(T, \alpha \cdot \mathbf{1}) < x$. □

Figure 2 shows the 3-person case, where $d_1 < d_2 = d_3$. There, we obtain d^* from the original d by replacing the first two coordinates by their average. In addition, the lemma's idea can also be seen in Fig. 1. There, d plays the role of d^* and d' plays the rule of $\alpha \cdot \mathbf{1}$.

Before we turn to our first multi-person characterization, we introduce another lemma, in the statement of which we use the following axiom. In the axioms's statement, as usual, (S, d) is an arbitrary element of $\mathcal{B}$:

Independence of Non Individually Rational Outcomes (INIR): $\mu(S, d) = \mu(S_d, d)$.

This axiom was first discussed by Peters (1986). As it is both weak and intuitive, some papers have assumed it implicitly (Kalai and Smorodinsky (1975) is an example).

Lemma 6 *Let μ be a solution that satisfies IR and IIA. Then μ satisfies INIR.*[10]

Proof Let $(S, d), (T, e) \in \mathcal{B}$ be such that $S_d = T_e$. Let $\mathbf{x} \equiv \mu(S, d)$ and $\mathbf{y} \equiv \mu(T, e)$. First, note that $S_d = T_e$ implies that $d = e$. To see this, suppose, without loss of generality, that $d_1 > e_1$. Then $e \in T_e = S_d$ would imply that $e \geq d$, which contradicts our assumption. Now, set $K \equiv S_d = T_d$. By IR $\mathbf{x}, \mathbf{y} \in K$. On the one hand, IIA implies that $\mu(K, d) = \mathbf{x}$; on the other hand, it implies that $\mu(K, d) = \mathbf{y}$.[11] Thus, $\mathbf{x} = \mathbf{y}$. □

Theorem 2 *E is the unique solution that satisfies PO, IR, TINV, AN, IIA, DIM, and DB.*

Proof It is easy to see that E satisfies all the axioms. We now prove uniqueness. Let μ be a solution that satisfies the above axioms, and let $(S, d) \in \mathcal{B}$. By Lemma 6, μ satisfies INIR, and therefore it is sufficient to establish that $\mu(\mathcal{C}(S), d) = E(S, d)$. This follows immediately from Lemma 5 and TINV. □

The axiom DB can be formulated in the following alternative way. It is equivalent to the convexity of the *Status Quo Set* of a solution μ, at the point $x \in S$.[12] Formally, this set is given by:

$$D^{\mu}(S, x) \equiv \{d \in S | (S, d) \in \mathcal{B} \text{ and } \mu(S, d) = x\}$$

We do not know whether DB is implied by the other axioms. However, if IIA is omitted from the axioms list, then DB is not implied by the remaining axioms. For example,

[10] This fact was pointed out in Dagan et al. (2002) in the context of 2-person problems and multi-valued solutions. The proof is essentially the same, and we bring it here for the sake of completeness.

[11] It is easy to see that (K, d) is a well-defined element of $\mathcal{B}$.

[12] The concept of the Status Quo Set is due to Peters (1986).

Peters (1986) showed that $D^K(S, x)$ is, in general, not convex, even for the case $n = 2$. Thomson (1987) proved that K satisfies the following axiom, in the statements of which (S, d) and (S', d') are arbitrary elements of $\mathcal{B}$:

Weak Disagreement Point Monotonicity[13] (W.DIM): For all $i \in N$: if $d'_i > d_i$, $d'_j = d_j$ for all $j \in N \backslash \{i\}$, and $S' = S$, then $\mu_i(S', d') \geq \mu_i(S, d)$.

A slight modification of Thomson's proof shows that on the domain of strictly comprehensive problems K satisfies DIM. Also, it is well known that K satisfies PO, IR, TINV, and AN. The following is an example which is due to Peters (1986), that shows that K may violate DB even in the 2-person case. Let $S = \mathcal{C}(\text{conv}\{(1, \frac{1}{2}), (0, 1)\})$ and $d = (0, 0)$.[14] Then, $K(S, d) = (\frac{2}{3}, \frac{2}{3})$ and $D^K(S, (\frac{2}{3}, \frac{2}{3}))$ is given by the graph of the following function, g:

$$g(t) = \begin{cases} t & \text{if } -\infty < t \leq 0 \\ 2t - (3/2)t^2 & \text{if } 0 \leq t \leq (1/3) \\ (1/2)t + (1/3) & \text{if } (1/3) \leq t < (2/3) \end{cases}$$

As opposed to K, the Nash solution N satisfies DB. We prove this fact in the Appendix.

5.2 DIM*

In the current subsection we provide an alternative generalization of Theorem 1 to the multi-person case, by means of an axiom which we call DIM*. Whereas DIM requires a player's payoff to increase in his own disagreement payoff, everything else being held constant, DIM* requires a player's payoff not to increase as a result of an increase in the disagreement payoff of any of the other players.

Formally, let (S, d) and (S', d') be arbitrary elements of $\mathcal{B}$, and i an arbitrary member of N:

DIM*: If $d'_i > d_i$, $d'_j = d_j$ for all $j \in N \backslash \{i\}$, and $S' = S$, then $\mu_j(S', d') \leq \mu_j(S, d)$ for all $j \in N \backslash \{i\}$.

Once i's disagreement payoff, d_i, is interpreted as an index of bargaining power, DIM* is the natural requirement that a player's payoff is (weakly) negatively affected by an increase in the bargaining power of any of his opponents. As in the previous theorems, we first prove an "equal-disagreement-payoffs lemma" before we turn to the main result.

Lemma 7 *Let μ be a solution that satisfies PO, IR, AN, IIA, DIM, and DIM*, and let (S, d) be an arbitrary element of $\mathcal{B}$. Then, if $\mu(S, d) = x \cdot \mathbf{1}$ for some $x \in \mathbb{R}$, then $d_1 = d_2 = \cdots = d_n$.*

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13 Thomson refers to this version of the axiom simply as "disagreement point monotonicity"; here, we added "weak" in order to distinguish it from our strict-inequality formulation of DIM.

14 This problem, $(\mathcal{C}(\text{conv}\{(1, \frac{1}{2}), (0, 1)\}), (0, 0))$, is not an element of $\mathcal{B}$, because its feasible set is not strictly comprehensive. However, it can be approximated as closely as desired by an element of $\mathcal{B}$, and clearly DB will also fail for the approximating element.

Proof Let $(S, d) \in \mathcal{B}$ be such that $\mu(S, d) = x \cdot \mathbf{1}$ for some $x \in \mathbb{R}$. Assume, by way of contradiction and without loss of generality, that $d_1 \leq d_2 \leq \cdots \leq d_n$ and that $d_1 < d_n$. Let l be the minimal integer in $N \backslash \{n\}$ such that $d_l < d_{l+1}$. We have that $d_i = \alpha$ for all $1 \leq i \leq l$, for some $\alpha \in \mathbb{R}$. Define d^1 by $d_i^1 = d_i$ for all $i \in N \backslash \{l + 1\}$ and $d_i^1 = \alpha$ for $i = l + 1$. Define recursively d^j by $d_i^j = d_i^{j-1}$ for all $i \in N \backslash \{l + j\}$ and $d_i^j = \alpha$ for $i = l + j$, for all $2 \leq j \leq n - 1 - l$.

In words, we apply a procedure of $(n-1-l)$ steps to d: in the j-th step we change the $(l+j)$-th coordinate to α. Note that $d^* \equiv d^{n-1-l}$ is of the form $d^* = (\alpha, ..., \alpha, d_n)$. Note that $(S, d^j) \in \mathcal{B}$ for all $1 \leq j \leq n-1-l$, because S is comprehensive. Defining $R \equiv \{y \in \mathbb{R}^n | y \leq x \cdot \mathbf{1}\} \cap S$, we can find a symmetric feasible set T such that:

- (i) $R \subset T \subset S$;
- (ii) $x \cdot \mathbf{1} \in T$; and
- (iii) $(T, d^j) \in \mathcal{B}$ for every $1 \leq j \leq n - 1 - l$.

BY IIA, $\mu(T, d) = x \cdot \mathbf{1}$, and therefore, by DIM*, we have that $\mu_i(T, d^j) \geq x$ for all $1 \leq i \leq l$ and all $1 \leq j \leq n - 1 - l$. Now, by AN and PO we have that $\mu(T, \alpha \cdot \mathbf{1}) = x \cdot \mathbf{1}$, and therefore, by DIM, we have that $\mu_n(T, d^*) > x$. We conclude that for the feasible set T, when we change d^*'s last coordinate to α, player n's payoff decreases, and since $P(T) = WP(T)$ there exists a player $h \in N \backslash \{n\}$ whose payoff increases. That is, $\mu_h(T, d^*) < x$. Then, it must be that $h > l$. Let $\pi \in \Pi$ be such that $\pi(1) = h, \pi(h) = 1$, and $\pi(k) = k$ for all $k \in N \backslash \{1, h\}$. Note that $(T, d^*) = (\pi T, \pi d^*)$, and therefore, by AN, $\mu(T, d^*) = \pi \mu(T, d^*)$. This contradicts the fact that $\mu_1(T, d^*) \geq x > \mu_h(T, d^*)$. □

The 2-person version of the lemma's idea can be seen in Fig. 1, where d plays the role of d^* and d' plays the rule of $\alpha \cdot \mathbf{1}$.

Theorem 3 *E is the unique solution that satisfies PO, IR, TINV, AN, IIA, DIM, and DIM*.*

Proof It is easy to see that E satisfies all of the axioms. Uniqueness follows immediately from Lemma 7 and TINV. □

6 The weighted egalitarian solution E_p

In this section we characterize the following generalization of E, known as the *weighted egalitarian solution*. Given a vector $p > 0$, this solution, denoted E_p, is given by $E_p(S, d) = d + \varepsilon \cdot p$, where ε is the largest number such that the expression on the right hand side is in S, for all $(S, d) \in \mathcal{B}$. The egalitarian solution corresponds to the special case $p = \mathbf{1}$.[15]

The relevance of E_p emerges when one seeks a way to solve bargaining situations in an egalitarian manner, where players may differ in their bargaining power. Such a

15 Kalai's characterization of E_p is the first axiomatization of the egalitarian solution in the economic literature. Adding SY to Kalai's axioms yields the special case $p = \mathbf{1}$.

motivation may have normative as well as descriptive grounds. For example, in economics, a workers' union and management may differ in their respective bargaining power.

We characterize E_p in two steps. First (Theorem 4), we show that if a solution satisfies certain axioms, then it is necessarily of a *proportional character*. This class of solutions is due to Roth (1979). Informally, these are the weighted egalitarian solutions which are not necessarily Pareto optimal. Formally, a solution μ is of a proportional character if there exists a vector $p > 0$, such that for every $(S, d) \in \mathcal{B}$, $\mu(S, d) = d + \lambda(S, d) \cdot p$ for some $\lambda(S, d) \geq 0$. With Theorem 4 at hand, it is easy to characterize the class of weighted egalitarian solution. First, in Theorem 5, we simply add PO to the axioms of Theorem 4. In Theorems 6 and 7 we add other axioms which together with the ones from Theorem 4 imply PO.

6.1 ADC

Formally, our goal in this section is to generalize Theorems 2 and 3 when AN is dropped from the axioms list. We first note that even for $n = 2$, the remaining axioms are not enough to characterize E_p. To see this, consider a strictly increasing, non-linear, function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_{++}$. The *monotone path solution* induced by f is:[16]

$$M_f(S, d) \equiv \{(x, f(x)) | (x, 0) \in S - d\} \cap P(S - d) + d$$

for all $(S, d) \in \mathcal{B}$.
It is easy to verify that this solution satisfies PO, IR, TINV, IIA, DIM, DB, and DIM*. This example shows that the property that needs to be recovered is—loosely speaking—some linearity of the status quo set. The following axiom presents itself as a natural candidate for this aim; in its statement (S, d) is an arbitrary element of $\mathcal{B}$:

Agreement-disagreement convexity (ADC): $\mu(S, d) = \mu(S, \lambda d + (1 - \lambda)\mu(S, d))$, for all $\lambda \in (0, 1]$.[17]

This axiom was introduced by Peters and van Damme (1991). Thomson (1994), who calls this property *star-shaped inverse*, succinctly summarized its idea by saying that a movement of the disagreement point in the direction of the compromise should not change the compromise. This idea arises as a natural requirement from the solution, when the above "direction" is taken to represent the players' relative bargaining powers.

The following is an additional motivation for ADC, which is due to Dagan et al. (2002). Suppose that the players play an infinite-horizon stationary extensive form bargaining game. To fix ideas, think of $n = 2$ in Rubinstein (1982) alternating offers model, with a common discount factor δ. Think of the (unique) subgame perfect

[16] This solution is due to Thomson and Myerson (1980).

[17] It is easy to see that $(S, \lambda d + (1 - \lambda)\mu(S, d))$ is a well-defined element of $\mathcal{B}$ whenever (S, d) is, for all $\lambda \in (0, 1]$.

equilibrium (SPE) of the game: an immediate agreement in the first period. Call this SPE agreement a^*. Naturally, let d represent the utilities from perpetual disagreement. Discounting an agreement reached in period t by δ^{t-1}, $\delta \in (0, 1)$, we have $d = \mathbf{0} \in \mathbb{R}^2$. Now, fix some $t > 0$ and construct the following game from the original game: if the players reach the subgame beginning at t, the game ends and the agreement chosen is a^*. The new game is a finite-horizon extensive form game, in which constant disagreement leads to a^* in period t. The unique SPE of this game is the same as that of the original game: choosing a^* in the first round of negotiations. The disagreement point of the new game is then $d' = \lambda d + (1 - \lambda)a^*$, where $\lambda = 1 - \delta^{t-1}$. With this illustration, we have that ADC may appear as an appropriate requirement when one has in mind stationary extensive form games, the axiomatic model being the "reduced form" of which.

To show that the solutions of a proportional character are the only solutions satisfying a certain set of axioms, we modify Theorem 3 in the following way: we drop PO and AN, add ADC, and, finally, we strengthen a little bit the disagreement monotonicity axiom. Instead of DIM*, we will impose the following axiom, Strong DIM*.

6.2 Strong DIM*

Consider the following axiom, in the statement of which (S, d) and (S', d') are arbitrary elements of $\mathcal{B}$, and i is an arbitrary member of N:

Strong DIM* (S.DIM*): If $d'_i > d_i, d'_j = d_j \quad \text{for all } j \in N\backslash\{i\}$, and $S' = S$, then $\mu_j(S', d') < \mu_j(S, d) \quad \text{for all } j \in N\backslash\{i\}$.

That is, S.DIM* is obtained by replacing the weak inequalities in DIM* by strict inequalities. In words, it requires each player's payoff to be a strictly decreasing function of each of his opponents' outside option.

Obviously, S.DIM* implies DIM*. Also, as is easy to see, if a solution satisfies PO and S.DIM* then it satisfies DIM. On the other hand, S.DIM* does *not* follow from DIM, DIM*, and PO together. To see this, recall the solution μ^{12} from the beginning of Sect. 4.[18]

6.3 The characterization

Lemma 8 *If a solution satisfies IR and S.DIM*, then it satisfies S.IR.*

Proof Let μ be a solution that satisfies IR and S.DIM*. Assume by contradiction that there exists an element $(S, d) \in \mathcal{B}$ such that $\mu_i(S, d) = d_i$ for some $i \in N$. Without loss of generality, assume that $i = 1$. By assumption there exists $x \in S$ such that $x > d$. Let $\varepsilon > 0$ be small enough such that $d' \equiv d + \varepsilon e_2 < x$. Since S is comprehensive, $(S, d') \in \mathcal{B}$. By IR, $\mu_1(S, d') \geq d'_1 = d_1$. On the other hand, by S.DIM*, $\mu_1(S, d') < d_1$, a contradiction. □

[18] Using the notation of this section, $\mu^{12} = E_{(1,1,0)}$.

Lemma 9 *Let μ be a solution that satisfies S.IR, IIA, and TINV, and let $D \equiv \{x \in \mathbb{R}^n | \sum_{i=1}^n x_i \leq n\}$. Then, there exists a vector q such that $(D, q) \in \mathcal{B}$ and $\mu(D, q) = \mathbf{1}$.*

Proof Let $c \in D$ be such that $\sum_i c_i = n - \varepsilon$ for some $\varepsilon > 0$. The problem (D, c) is a well-defined element of $\mathcal{B}$. Let $y \equiv \mathbf{1} - \mu(D, c)$, $\delta \equiv \sum_i y_i$. By S.IR, $\delta < \varepsilon$. By TINV, $\mathbf{1} = \mu(D, c) + y = \mu(D + \{y\}, c + y)$.

Claim 1: $D \subset D + \{y\}$.

Let $a \in D$. We need to find $b \in D$ such that $a = b + y$. Let $b \equiv a - y$. Note that $\sum_i b_i = \sum_i a_i - \sum_i y_i \leq n - \delta \leq n$, so $b \in D$.

Claim 2: $c + y \in \text{int}D$.

$$\sum_i c_i + \sum_i y_i = n - \varepsilon + \delta < n.$$

By *Claim 2*, $(D, c + y)$ is a well-defined element of $\mathcal{B}$. By *Claim 1*, IIA, the fact that $\mathbf{1} \in D$, and the fact that $\mathbf{1} = \mu(D + \{y\}, c + y)$, $\mu(D, c + y) = \mathbf{1}$. Set $q \equiv c + y$. □

Theorem 4 *If a solution satisfies IR, TINV, IIA, ADC, and S.DIM*, then it is of a proportional character.*

Proof Let μ be a solution that satisfies the axioms. By TINV it is enough to show that there exists a $p > \mathbf{0}$ such that for all $(S, d) \in \mathcal{B}$ with $\mu(S, d) = \mathbf{1}$ there exists a number $\lambda(S, d) \geq 0$ such that $\mu(S, d) = d + \lambda(S, d) \cdot p$.[19] Let then $(S, d) \in \mathcal{B}$ be such that $\mu(S, d) = \mathbf{1}$. Let D and q be as defined in Lemma 9. We will prove that $\mu(S, d) = \mu(S, d) = d + \lambda(S, d) \cdot p$ where $\lambda(S, d)$ is some non-negative number, for $p \equiv \mathbf{1} - q$. Note that $p > 0$, because $\mu(D, q) = \mathbf{1}$ and by Lemma 8 μ satisfies S.IR, hence $q < \mathbf{1}$. The fact that $q < \mathbf{1}$ allows us to assume without loss of generality, in view of ADC, that $q < d$.

Also, note that since μ satisfies IR and $\mu(S, d) = \mathbf{1}$, we can assume that $\sum_i d_i < n$. Because, IR implies that $d_i \leq 1$ for all $i \in N$, and therefore $\sum_i d_i \leq n$. If this inequality holds as equality, then $d = \mathbf{1}$ and hence what we want to prove holds trivially. That is, $\mu(S, d) = d + \lambda(S, d) \cdot p$ with $\lambda(S, d) = 0$. Therefore, we assume $\sum_i d_i < n$.

Finally, since μ satisfies IR and IIA, by Lemma 6 it satisfies INIR. Hence we can assume $S = \mathcal{C}(S)$. In particular, this assumption implies that $q \in S$, because $d \in S$ and $q < d$. The importance of $q \in S$ and $\sum_i d_i < n$ will be clear shortly (in *Case 2* below).

Case 1: $d \in [q; \mathbf{1}]$. Then $d = \lambda q + (1 - \lambda) \cdot \mathbf{1}$ for some $\lambda \in (0, 1)$ (recall that μ satisfies S.IR, hence $\lambda > 0$, and $q < d$ hence $\lambda < 1$). Rearranging yields $\mathbf{1} - d = \lambda(\mathbf{1} - q)$. We obtain:

$$\mathbf{1} = d + (\mathbf{1} - d) = d + \lambda(\mathbf{1} - q) = d + \lambda p$$

[19] To see this, suppose that indeed there exists a $p > 0$ such that $\mu(S, d) = d + \lambda(S, d) \cdot p$ for all $(S, d) \in \mathcal{B}$ with $\mu(S, d) = \mathbf{1}$. Let $(S, d) \in \mathcal{B}$ with $\mu(S, d) = x$. Let $y \equiv \mathbf{1} - x$. By TINV, $\mathbf{1} = \mu(S, d) + y = \mu(S + \{y\}, d + y) = d + y + \lambda(S + \{y\}, d + y) \cdot p$, where the last equality is by assumption. Canceling y gives $\mu(S, d) = d + \lambda(S + \{y\}, d + y) \cdot p$.

Case 2: $d \notin [q; \mathbf{1}]$. Let $Q \equiv S \cap D$.[20] By IIA, $\mu(Q, q) = \mu(Q, d) = \mathbf{1}$.

Note that for each $i \in N$, there exists a unique $\lambda_i \in (0, 1)$ such that $d_i = \lambda_i q_i + (1 - \lambda_i)$. Without loss of generality, assume that $\lambda_1 \leq \cdots \leq \lambda_n$. Then, since $d \notin [q; \mathbf{1}]$, $\lambda_1 < \lambda_n$.

Let $\lambda \equiv \lambda_1$. Then, there exists an l, $1 \leq l < n$, such that $\lambda_i = \lambda$ if and only if $1 \leq i \leq l$. Set $d^l \equiv d$, and for each $l + 1 \leq k \leq n$ define d^k by:

$$d_i^k = \begin{cases} \lambda q_i + (1 - \lambda) & \text{if } i = k \\ d_i^{k-1} & \text{otherwise} \end{cases}$$

In words, we apply a procedure of $n - l$ steps to d, where in each step we increase exactly one of the coordinates indexed by $\{l+1, \ldots, n\}$. Note that $d^n = \lambda q + (1 - \lambda) \cdot \mathbf{1}$, and recall that $\mu(Q, q) = \mathbf{1}$. Therefore, by ADC, $\mu(Q, d^n) = \mathbf{1}$. On the other hand, by S.DIM* we have that each one of these steps decreases player 1's payoff, in contradiction to $\mu_1(Q, d^n) = \mu_1(Q, d) = 1$. □

We are now ready to state and prove the main result of this section.

Theorem 5 *A solution μ satisfies PO, IR, TINV, IIA, ADC, and S.DIM*, if and only if there exists a $p > 0$ such that $\mu = E_p$.*

Proof It is easy to check that every weighted egalitarian solution satisfies the axioms. Conversely, let μ be a solution that satisfies the axioms. By Theorem 4, there exists a $p > 0$ such that $\mu(S, d) = d + \lambda(S, d) \cdot p$ for some $\lambda(S, d) \geq 0$, for all $(S, d) \in \mathcal{B}$. By PO, $\lambda(S, d)$ is the largest number such that the expression on the right hand side is in S. That is, $\mu = E_p$. □

An alternative way to characterize the class of weighted egalitarian solutions is to combine Theorem 4 with an additional axiom which together with (some of) the axioms of Theorem 4 would imply PO. The following axiom has this property.

Disagreement Point Continuity (D.CONT): If d^n is such that $d^n \to d$, $(S, d^n) \in \mathcal{B}$ for all n and $(S, d) \in \mathcal{B}$, then $\mu(S, d^n) \to \mu(S, d)$.

In words, D.CONT requires small changes in the disagreement point not to translate to large jumps in solution payoffs.

Lemma 10 *If a solution satisfies S.IR, ADC, and D.CONT, then it satisfies PO.*

Proof Let μ be a solution that satisfies S.IR, ADC, and D.CONT. Assume by contradiction that there exists an element $(S, d) \in \mathcal{B}$ such that $\mu(S, d) \notin P(S)$. Then $(S, \mu(S, d)) \in \mathcal{B}$. Taking a sequence $\{d_k\}_{k=1}^{\infty}$ of points in $[d; \mu(S, d)]$ which converges to $\mu(S, d)$, and applying ADC and D.CONT, we obtain $\mu(S, \mu(S, d)) = \mu(S, d)$, in contradiction to S.IR. □

20 It is easy to verify that both (Q, d) and (Q, q) are elements of $\mathcal{B}$. The inequality $\sum_i d_i < n$ assures that $(Q, d) \in \mathcal{B}$. Similarly, $q \in S$ assures that $(Q, q) \in \mathcal{B}$.

Theorem 6 *A solution μ satisfies IR, TINV, IIA, ADC, S.DIM*, and D.CONT if and only if there exists a $p > 0$ such that $\mu = E_p$.*

Proof Combine Lemma 8, Lemma 10, and Theorem 5. □

Another alternative to Theorem 5 is to impose on the solution the following axiom, in the statement of which (S, d) is an arbitrary element of $\mathcal{B}$:

Homogeneity (HOM): For all $\alpha \in \mathbb{R}_{++}$, $\mu(\alpha S, \alpha d) = \alpha\mu(S, d)$.[21]

One interpretation of HOM is that if all utilities are measured on the same scale, then the solution should be invariant to joint rescalings. For additional interpretations, see Kalai (1977).

Lemma 11 *If a solution satisfies TINV, S.IR, IIA, and HOM, then it satisfies PO.*

Proof Let μ be a solution that satisfies TINV, S.IR, IIA, and HOM. Assume by contradiction that there exists an element $(S, d) \in \mathcal{B}$ such that $x \equiv \mu(S, d) \notin P(S)$. By TINV we may assume, without loss of generality, that $d = \mathbf{0}$. By Lemma 6 μ satisfies INIR, and hence we can assume that $S = S_{\mathbf{0}}$. By S.IR, $x > \mathbf{0}$. Since $x \notin P(S)$, there exists a $\lambda \in (0, 1)$, close to 1, such that $x \in \lambda S$. Since $\lambda S \subset S$, by IIA, $\mu(\lambda S, \mathbf{0}) = x$. On the other hand, by HOM, $\mu(\lambda S, \mathbf{0}) = \mu(\lambda S, \lambda\mathbf{0}) = \lambda x$, a contradiction. □

Theorem 7 *A solution μ satisfies IR, TINV, IIA, ADC, S.DIM*, and HOM if and only if there exists a $p > 0$ such that $\mu = E_p$.*

Proof Combine Lemma 8, Lemma 11, and Theorem 5. □

6.4 Tightness of the axioms

In this subsection we discuss the tightness of the axioms of Theorem 5. Recall that once we restrict attention to $\mathcal{B}^*$, IR is redundant[22] (we do not know whether this redundancy holds on $\mathcal{B}$). As for PO, we do not know whether it is redundant. However, from Theorems 6 and 7 we know that a solution that violates PO and satisfies the rest of the axioms of Theorem 5 violates both D.CONT and HOM. Hence, if such a solution exists, it is extremely ill-behaved. As for TINV, we do not know whether it is redundant in general. However, as the following example shows, it is not redundant in the case $n = 2$.

Consider the following solution for the 2-person case. Let $p_1 = (\frac{2}{3}, \frac{1}{3})$, $p_2 = (\frac{1}{3}, \frac{2}{3})$, and let $\mu^{\neg\text{TINV}}$ be the following solution:

$$\mu^{\neg\text{TINV}}(S, d) \equiv \begin{cases} E_{p_1}(S, d) & \text{if } d_1 \geq d_2 \\ E_{p_2}(S, d) & \text{otherwise} \end{cases}$$

[21] $\alpha S = \{\alpha s | s \in S\}$.

[22] PO and S.DIM* imply DIM, and by Lemma 2, PO , IIA, and DIM imply IR.

It is easy to see that this solution satisfies all the axioms of Theorem 5 besides TINV. The following is a similar construction of a solution (for the n-person case) that satisfies all the axioms besides IIA. Let $p, q > \mathbf{0}$ be such that $p \neq q$. Fix a feasible set S^* and consider the following solution, $\mu^{\neg \text{IIA}}$:

$$\mu^{\neg \text{IIA}}(S, d) \equiv \begin{cases} E_p(S, d) & \text{if } S = S^* + \{t\} \text{ for some } t \in \mathbb{R}^n \\ E_q(S, d) & \text{otherwise} \end{cases}$$

Finally, the monotone path solution satisfies all the axioms besides ADC, and the Nash solution N satisfies all the axioms besides S.DIM*.[23]

6.5 ADC and DB

ADC shares a very similar flavor with DB. Naturally, the question arises whether ADC, perhaps together with other mild conditions, implies DB. In the 2-person case the answer is affirmative: DB is implied by PO, IR, and ADC.

Proposition 1 *Let $n = 2$, and let μ be a solution that satisfies PO, IR, and ADC. Then μ satisfies DB.*

Proof Let $(S, d), (T, e) \in \mathcal{B}$ be such that $T = S$ and $\mu(S, d) = \mu(T, e)$. Without loss of generality, suppose that $\mu(S, d) = \mu(T, e) = \mathbf{1}$. Let $\lambda \in [0, 1]$ and let $f = \lambda d + (1 - \lambda)e$. Assume, by way of contradiction, that $\mu(S, f) = (1 + \varepsilon, 1 + \delta) \neq (1, 1)$. Let $l_k \equiv [k; \mathbf{1}]$, for $k \in \{d, e\}$. By PO and IR, l_k is a line in the plane with a non-negative slope. If there exists $w \in l_d \cap l_e$ such that $w \neq \mathbf{1}$, then one of the line segments l_k is a subset of the other line segment $l_{k'}$, where $\{k, k'\} = \{d, e\}$, and then ADC contradicts $\mu(S, f) \neq 1$. Suppose then that $l_d \cap l_e = \{\mathbf{1}\}$. Also by ADC, we may assume, without loss of generality, that $d_1 > e_1$ and $d_2 < e_2$. Let $l_f \equiv [f; (1 + \varepsilon, 1 + \delta)]$. By PO, there are exactly two possible cases:

Case 1: $\varepsilon < 0$ and $\delta > 0$. There is exactly one point in the intersection $l_f \cap l_e$. Call this point z. Then, applying ADC to e and $\mathbf{1}$, we conclude that $\mu(S, z) = \mathbf{1}$. On the other hand, applying ADC to f and $(1 + \varepsilon, 1 + \delta)$, we conclude that $\mu(S, z) = (1 + \varepsilon, 1 + \delta)$.

Case 2: $\varepsilon > 0$ and $\delta < 0$. There is exactly one point in the intersection $l_f \cap l_d$. Call this point z. Then, applying ADC to d and $\mathbf{1}$, we conclude that $\mu(S, z) = \mathbf{1}$. On the other hand, applying ADC to f and $(1 + \varepsilon, 1 + \delta)$, we conclude that $\mu(S, z) = (1 + \varepsilon, 1 + \delta)$.

Since either case leads to a contradiction, we conclude that DB holds. □

All the axioms used in Proposition 1 are necessary: dropping one of them renders the conclusion of the proposition false. The solution K, for example, satisfies PO, IR, but neither ADC nor DB. The following is an example of a 2-person solution that satisfies IR, ADC, but violates DB. Let S' be the part of the plane's unit

[23] The fact that N satisfies ADC was proved by Peters and van Damme (1991).

disc that lies in the non-negative orthant. That is, $S' \equiv \{x \in \mathbb{R}^2_+ | \|x\| \leq 1\}$. Let $l_1 \equiv \left[\left(0, \sqrt{\frac{1}{2}}\right); \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)\right]$ and $l_2 \equiv \left[\left(\sqrt{\frac{1}{2}}, 0\right); \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)\right]$. Consider the following solution, F. For all $(S, d) \in \mathcal{B}$ such that $S \neq S'$ set $F(S, d) = E(S, d)$. For $(S, d) \in \mathcal{B}$ such that $S = S'$ set $F(S, d) = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$ if $d \in l_1 \cup l_2$, $F(S, d) = d$ if $\mathbf{0} \leq d < \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$, and $F(S, d) = E(S, d)$ otherwise. It is easy to see that F satisfies IR and ADC. However, F violates DB. To see this, consider $\left(S', \left(0, \sqrt{\frac{1}{2}}\right)\right)$ and $\left(S', \left(\sqrt{\frac{1}{2}}, 0\right)\right)$. We have that $F\left(S', \left(0, \sqrt{\frac{1}{2}}\right)\right) = F\left(S', \left(\sqrt{\frac{1}{2}}, 0\right)\right) = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$, but $F\left(S', \left(\frac{1}{2}\sqrt{\frac{1}{2}}, \frac{1}{2}\sqrt{\frac{1}{2}}\right)\right) = \left(\frac{1}{2}\sqrt{\frac{1}{2}}, \frac{1}{2}\sqrt{\frac{1}{2}}\right)$.

Proposition 1 is very close to a theorem proved by Peters and van Damme (1991).[24] They consider the 2-person case on the domain of problems that are not necessarily comprehensive and where the disagreement point may lie on the boundary. Call this domain $\tilde{\mathcal{B}}$. On $\tilde{\mathcal{B}}$, naturally, some modifications are required in formulating the axioms. In particular, in the definition of ADC convex combinations of the disagreement point and the solution point where all the weight is on the latter ($\lambda = 0$ in our definition of ADC) are allowed, and in the definition of S.IR the requirement is that a player $i \in N$ receives a payoff strictly above his d_i only if there exists a point x in the feasible set with $x_i > d_i$. Call these modified versions ADC* and S.IR*, respectively. Similarly, let D.CONT* be the disagreement point continuity axiom for $\tilde{\mathcal{B}}$ (replace $\mathcal{B}$ by $\tilde{\mathcal{B}}$ in the definition of D.CONT). They prove that on $\tilde{\mathcal{B}}$, DB is implied by ADC*, S.IR*, and D.CONT*. A crucial difference between the two models is the fact that the disagreement point may lie in the Pareto frontier of the feasible set. In particular, this means that S.IR* and ADC* imply PO.[25] On $\mathcal{B}$, by contrast, S.IR and ADC do not imply PO. To see this, consider the following solution, $\tilde{F}$, for the 2-person case. Let S', l_1, and l_2 be defined as in the previous example above. For all $(S, d) \in \mathcal{B}$ such that $S \neq S'$ set $\tilde{F}(S, d) = E(S, d)$. For $(S, d) \in \mathcal{B}$ such that $S = S'$ set $\tilde{F}(S, d) = E(S, d)$ if $d_i \geq \sqrt{\frac{1}{2}}$ for some i; otherwise, set $\tilde{F}(S, d) = \{d + \varepsilon \cdot \mathbf{1} | \varepsilon > 0\} \cap (l_1 \cup l_2)$. It is easy to see that $\tilde{F}$ satisfies S.IR and ADC, but violates PO. However, by Lemma 10, when D.CONT is added PO is recovered. Therefore, it is a corollary of Proposition 1 that S.IR, ADC, and D.CONT imply DB, which is Peters and van Dammes' theorem for our model.

In the case $n > 2$, the conclusion of Proposition 1 is not true: there are solutions that satisfy PO, IR, and ADC, but not DB. We now turn to a description of such a solution, for which we will use the following notation and definition. Denote by

[24] Peters and van Damme (1991), Theorem 4.1.

[25] To see this, let μ be a solution that satisfies S.IR* and ADC*, and let (S, d) be an arbitrary element of $\tilde{\mathcal{B}}$. Suppose that there exists a $z \in S$ with $z \gneqq \mu(S, d)$. In particular, $z_i > \mu_i(S, d)$ for some $i \in N$. Now, by ADC* we have that $\mu(S, \mu(S, d)) = \mu(S, d)$, and therefore S.IR* is violated for player i.

D^i the i-th *dictatorial bargaining solution*- the solution that assigns player i his maximal payoff in S_d. That is, given $i \in N$ and $(S, d) \in \mathcal{B}$:

$$D^i(S, d) \equiv \text{argmax}_{x \in S_d}\{x_i\}$$

It is easy to see that this solution is well-defined.[26] Let $n = 3$ and let $S^* = \text{conv}\{\mathbf{0}, e_1, e_2, e_3\}$. That is, S^* is the set of points that lie between the origin and the unit simplex of $\mathbb{R}^3$. Let $l = [(\frac{1}{2}, 0, 0); e_1]$ and let $l' = [(0, \frac{1}{2}, 0,); e_1]$. Consider the following solution, G. For $(S, d) \in \mathcal{B}$ such that $S \neq S^*$, set $G(S, d) = E(S, d)$. For $(S, d) \in \mathcal{B}$ such that $S = S^*$, let $G(S, d) = e_1$ if $d \in l \cup l'$, and $G(S, d) = D^3(S, d)$ if $d \notin l \cup l'$. G is well-defined, and it is easy to see that it satisfies PO, IR, and ADC. However, it violates DB. To see this, note for example, that $G(S^*, (\frac{1}{2}, 0, 0)) = G(S^*, (0, \frac{1}{2}, 0)) = e_1$, but $G(S^*, (\frac{1}{4}, \frac{1}{4}, 0)) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

We conclude this section by observing that both DB and ADC are implied, together with other weak conditions, by the following axiom, in the statement of which (S, d) and (T, d') are arbitrary element of $\mathcal{B}$:

Disagreement Point Quasi-Concavity (DQC): For all $i \in N$ and every $\lambda \in [0, 1]$, if $T = S$ then $\mu_i(S, \lambda d + (1 - \lambda)d') \geq \min\{\mu_i(S, d), \mu_i(T, d')\}$.[27]

The fact that PO and DQC imply DB is easy to check. A proof of the fact that ADC is implied by PO, IR, and DQC, can be found, for example, in Dagan et al. (2002). This axiom has a natural interpretation in the context of an uncertain disagreement point. Suppose that the feasible set is known to be S, but the disagreement point d is uncertain: it may be either d' or d'' with known probabilities. Suppose that the players can solve the problem before the uncertainty is resolved, by taking the appropriate convex combination of d' and d'' to represent the expected disagreement point, or they can wait until the exact location of d becomes known and solve the problem then. In such a case, DQC is a necessary condition for an early agreement. A significant strengthening of it is the following axiom, in the statement of which (S, d) and (T, d') are arbitrary elements of $\mathcal{B}$:

Disagreement Point Concavity (D.CAV): For all $\lambda \in [0, 1]$, if $T = S$ then $\mu(S, \lambda d + (1 - \lambda)d') \geq \lambda\mu(S, d) + (1 - \lambda)\mu(S, d')$.

In the context of an uncertain disagreement point, D.CAV says that all agents always (weakly) prefer early compromises. Chun and Thomson (1990a) proved that, together with a few additional mild conditions, D.CAV suffices for a characterization of E_p.

7 Transfer paradoxes

We conclude the paper with a discussion of the so-called "transfer paradox". Suppose that for some bargaining problem (S, d) the following change is applied. The disagreement payoff of some player $i \in N$ increases, the disagreement payoff of some

[26] Recall that S is strictly comprehensive.

[27] This axiom was introduced by Chun and Thomson (1990b); it is also discussed in Thomson (2008).

player $j \in N \backslash \{i\}$ decreases, and everything else is unchanged.[28] We say that a *transfer paradox* occurs if i's payoff decreases, or if j's payoff increases, or both. Not allowing this paradox to arise is closely related to the idea of disagreement point monotonicity. Below we show that in the 2-person case, ADC, when combined with PO and IR, prevents such a paradox. To state this formally, we formulate the following axiom, in the statement of which (S, d) is an arbitrary element of $\mathcal{B}$:

No Transfer Paradox (NTP): For all $\varepsilon, \delta > 0$ and for all $i, j \in N$ such that $i \neq j$, if $(S, d + \varepsilon e_i - \delta e_j) \in \mathcal{B}$, then $\mu_i(S, d + \varepsilon e_i - \delta e_j) \geq \mu_i(S, d)$ and $\mu_j(S, d + \varepsilon e_i - \delta e_j) \leq \mu_j(S, d)$.

As was noted by Thomson (1987), NTP is implied by PO and DIM*.

Proposition 2 *Let $n = 2$, and let μ be a solution that satisfies PO, IR, and ADC. Then μ satisfies NTP.*

Proof Suppose that there exists $(S, d) \in \mathcal{B}$, and two numbers $\varepsilon, \delta > 0$, such that $(S, d') \in \mathcal{B}$, but either $\mu_1(S, d') < \mu_1(S, d)$ or $\mu_2(S, d') > \mu_2(S, d)$, where $d' \equiv (d_1 + \varepsilon, d_2 - \delta)$. Let $l_k \equiv [k; \mu(S, k)]$, for $k \in \{d, d'\}$. By PO and IR we have that these lines intersect in an interior point of S. Call this point e. Now, by applying ADC to k and $\mu(S, k)$, we conclude that $\mu(S, e) = \mu(S, k)$ for both $k \in \{d, d'\}$, a contradiction. □

The conclusion of Proposition 2 is not true if there are more than two players. For example, consider the solution G and the feasible set S^* defined in the previous section for the case $n = 3$. Let $d = (0, \frac{1}{2}, 0)$ and $d' = (\varepsilon, \frac{1}{2} - \varepsilon, 0)$ for some $\varepsilon > 0$ arbitrarily small. We have that $(G_1(S^*, d), G_2(S^*, d)) = (1, 0)$ and $(G_1(S^*, d'), G_2(S^*, d')) = (\varepsilon, \frac{1}{2} - \varepsilon)$.

The axiom NTP can be strengthened in various ways. One particular such strengthening, which is interesting and natural, is to require, in addition to NTP, that every player who is not involved in the disagreement-point-change (every $k \in N \backslash \{i, j\}$ in the formulation of NTP above) should not benefit more than the "winner" (i) or lose more than the "loser" (j). Bossert (1994) showed that if there are at least 3 players, then E is the unique solution on $\mathcal{B}$ that satisfies PO, IR, D.CONT, and this stronger no-transfer-paradox axiom.[29]

8 Conclusion

The theorems presented here provide alternative characterizations of the egalitarian and weighted egalitarian solutions. In all of our proofs, we rely on strict inequalities when we consider changes in the disagreement point. This is crucial. If instead we considered weak inequalities, then Theorems 1 and 2 would not be true, as the Nash solution satisfies all of the axioms of these theorems when DIM is replaced

[28] Calling the new disagreement point d', we assume that $(S, d') \in \mathcal{B}$.

[29] In 2-person case, both dictatorial solutions D^i, $i \in \{1, 2\}$, satisfy PO, IR, D.CONT, and the above stronger version of NTP.

by W.DIM.[30] Imposing weak-inequality-based monotonicity axioms on bargaining solutions is very common in the literature. Notable examples are Kalai's *monotonicity* and *individual monotonicity* axioms (Kalai 1977), Thomson and Myerson's *twisting* (Thomson and Myerson 1980), and W.DIM. Conditions involving strict inequalities are much more rare. However, in the class of strictly comprehensive problems, where utility can always be transfered among players, strict inequalities seem reasonable. As we see, they may also be fruitful in terms of the characterizations one can obtain.[31] This suggests a line of research that may result in new characterizations of known solutions. Rachmilevitch (2008) is an example, where a new characterization of K is derived for the 2-person case, by replacing Kalai and Smorodinsky's individual monotonicity by DIM, a weak version of IIA, and an additional axiom, midpoint domination. The main question left open in this paper is whether the axioms in Theorems 2 and 3 are tight.

Appendix

In this Appendix we prove that the Nash solution satisfies DB. We will use the following notation. Given $(S, d) \in \mathcal{B}$ and two distinct players $i, j \in N$, let $S_d(i, j) \equiv \{(x_i, x_j)|x \in S \text{ and } x_k = N_k(S, d) \text{ for all } k \notin \{i, j\}\}$. That is, $S_d(i, j)$ is the two-dimensional feasible set that is obtained from S by fixing the payoff of all players different from i and j at their Nash-payoffs and looking at the payoff pairs which are now feasible for i and j.

Lemma 12 *$N(S_d(i, j), (d_i, d_j)) = (N_i(S, d), N_j(S, d))$, for all $(S, d) \in \mathcal{B}$ and $i, j \in N$ such that $i \neq j$.*

Proof To see that this is true, consider an arbitrary element $(S, d) \in \mathcal{B}$ and let i and j be two arbitrary players. Note that $(N_i(S, d), N_j(S, d))$ is a legitimate choice in the maximization of the Nash product of the problem $(S_d(i, j), (d_i, d_j))$. Assume, by way of contradiction, that $N(S_d(i, j), (d_i, d_j)) \neq (N_i(S, d), N_j(S, d))$. Then, since the maximizer of the Nash product is unique, we have that there exists a point $(z_i, z_j) \in S_d(i, j)$, such that:

$$(z_i - d_i)(z_j - d_j) > (N_i(S, d) - d_i)(N_j(S, d) - d_j)$$

from which we get:

$$(z_i - d_i)(z_j - d_j)\psi > (N_i(S, d) - d_i)(N_j(S, d) - d_j)\psi$$

where $\psi \equiv \Pi_{k \notin \{i,j\}}(N_k(S_d) - d_k)$. This contradicts the fact that N maximizes the Nash product of (S, d). □

Also, we will use the fact that N satisfies the axiom of *Scale Invariance* given below, in the statement of which (S, d) is an arbitrary element of $\mathcal{B}$ and $(f_1, ..., f_n)$ is an

[30] The fact that N satisfies W.DIM was proved by Thomson (1987).

[31] Of course, such conditions will be equally as fruitful where the strict comprehensiveness requirement is dropped, and an appropriate continuity axiom is imposed on the feasible sets.

arbitrary vector of *positive linear transformations*. That is, for all $i \in N f_i(x) = \alpha_i x$ where $\alpha_i > 0$, for all $x \in \mathbb{R}$.

Scale Invariance (SINV): $f(S, d) = (fS, fd)$.[32]

Proposition 3 *The Nash solution satisfies DB.*

Proof Let (S, d) and (T, d') be two elements of $\mathcal{B}$ such that $S = T$ and $\mu(S, d) = \mu(T, d') \equiv \mathbf{x}$, and let λ be an arbitrary number in [0, 1]. Since N satisfies SINV and TINV we may assume, without loss of generality, that $\mathbf{x} = \mathbf{1}$, and that $d' = \mathbf{0}$. Let i be the minimal integer in $\{1, ..., n\}$ such that $(d_i, d_{i+1}) \neq (0, 0)$.[33] By Lemma 12, $N(S_d(i, i+1), (d_i, d_{i+1})) = (1, 1)$. Then, simple geometry tells us that there exists some $\alpha \in \mathbb{R}$ such that $(d_i, d_{i+1}) = (\alpha, \alpha)$. By applying this argument iteratively to the pairs $(i+1, i+2)$,...,$(i-1, i)$, we conclude that $d = \alpha \cdot \mathbf{1}$. Thus, we need to show that $N(S, d'') = \mathbf{1}$, where $d'' \equiv \lambda\alpha \cdot \mathbf{1}$. We will do this by a simple application of a theorem of Shapley (1969). Shapley showed that there exist positive numbers b_i such that:

$$b_i(z_i - d_i) = b_i(z_j - d_j)$$

for all $i, j \in N$, and:

$$\sum_{k \in N} b_k(z_k - d_k) \geq \sum_{k \in N} b_k(w_k - d_k)$$

for all $w \in S$, if and only if $z = N(S, d)$.

Pick a number $\theta > 0$. Since $N(S, \mathbf{0}) = \mathbf{1}$, we obtain by Shapley's theorem that $\theta n \geq \sum_{k \in N} w_k$ for all $w \in S$. Therefore, $\sum_{k \in N} \theta(1 - \lambda\alpha) \geq \sum_{k \in N} \theta(w_k - \lambda\alpha)$ for all $w \in S$. Setting $b_i = \theta$ for all $i \in N$, we conclude by Shapley that $N(S, d'') = \mathbf{1}$, as desired. □

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[32] $fS = \{fs | s \in S\}$.

[33] If this minimal i equals n, take $i + 1 = 1$.

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