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Minimum winning coalitions and endogenous status quo

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Abstract I analyze a stochastic bargaining game in which a renewable surplus is divided among $n \geq 5$ committee members in each of an infinite number of periods, and the division implemented in one period becomes the status quo allocation of the surplus in the ensuing period. I establish existence of equilibrium exhibiting minimum winning coalitions, assuming sufficiently mild concavity of stage preferences. The analysis highlights the role of proposal power in committee deliberations and yields a fully strategic version of McKelvey's (J Econ Theory 12:472–482, 1976; Econometrica 47:1086–1112, 1979) dictatorial agenda setting.

Keywords Bargaining · Dictatorial agenda setting · Endogenous status quo · Proposal power

JEL Classification C73 · C78 · D72

1 Introduction

The sequential bargaining approach pioneered by Rubinstein (1982) spawned a non-cooperative literature on collective decision making that has flourished in the last two decades.[1] This literature fills one lacuna in the theory of collective choice by providing a viable solution concept when social choice is plagued by intransitive social preferences. Yet, this literature does not address the dynamic implications of social preference intransitivity as, for the most part, it operates under the assumption that bargaining ceases once the committee has reached a decision. In order to study

[1] E.g., Baron and Ferejohn (1989), Merlo and Wilson (1995), Banks and Duggan (2000, 2006), etc.

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these dynamics, I analyze a multilateral dynamic bargaining game in which agreements require the consent of a majority of the participants and past agreements can be revised *ad infinitum*. Bargaining occurs over the division of a surplus of fixed size. This is a canonical agreement space that is also assumed, for example, in the influential contribution of Baron and Ferejohn (1989). A strong incentive to form minimum winning coalitions (Riker 1962) is present in these agreement spaces when interaction is limited to a proposal and voting stage in each period and collective bargaining ceases once an agreement is reached. In the present study I ask whether such minimum winning coalitions equilibria can be sustained in a dynamic environment in which past agreements constitute future status quos? I also explore the effect of proposal making rights on the equilibrium distribution of the surplus.

I study committees comprising an odd number of five or more players. In each period, one committee member is recognized with some fixed probability and proposes a division of the surplus. If the proposal is approved by a majority, it is implemented in that period; otherwise, the status quo allocation (which is defined as last period's division) is implemented. A non-trivial issue that must be confronted in this environment is that existence of equilibrium in simple Markovian strategies is not guaranteed since the game is stochastic with continuous action and state spaces, and deterministic transitions.[2] I establish existence using a combination of constructive methods coupled with more standard fixed point arguments. Since my focus is on equilibria with minimum winning coalitions, I proceed in two steps. First, I derive equilibrium strategies for status quos such that a bare majority or fewer players receive positive allocations. In the second step, I extend these strategies to the remaining status quo divisions of the surplus and show that the extended minimum winning coalition strategies constitute an equilibrium. This equilibrium has the possibly counter-intuitive property that the proposer extracts the entire surplus in every period except (possibly) the initial two.

In previous work (Kalandrakis 2004) I have derived a similar equilibrium for the special case of a committee with three players. There, I derived the equilibrium in closed form by exploiting the fact that players have identical recognition probabilities and by requiring linear stage payoffs. In the present study, I admit mild concavity in stage payoffs, and I drop the assumption of equal recognition probabilities when these payoffs are linear.[3] Besides covering all odd-sized committees and ensuring robustness of the established equilibrium to heterogeneous recognition probabilities and mild concavity in stage preferences, the additional generality in the present study provides insight on other questions of interest.

First, the equilibrium highlights the importance of proposal rights in securing better (expected) outcomes for committee members since any ex ante allocation of the intertemporal surplus can be achieved in the long run via manipulation of recognition probabilities. This result extends a result of Kalandrakis (2006) cast in an otherwise more general setting but without assuming an endogenous status quo. In addition, as a corollary of the main existence theorem, I obtain a fully strategic version of

[2] For a detailed discussion of the equilibrium existence problem in these settings, see Duggan and Kalandrakis (2007).

[3] Although the present study is otherwise more general, it does not subsume the case of $n = 3$ players due to a difference in equilibrium dynamics for committees of that size.

McKelvey's (McKelvey 1976, 1979) dictatorial agenda setting result. In particular, for every initial status quo and for every discount factor, a committee member who offers proposals with probability one can extract the entire surplus in all periods but the first. To my knowledge, this the first derivation of such dictatorial agenda setting with farsighted voters and the institutions assumed by McKelvey.

I also show that the minimum winning coalitions equilibrium established in this study does not survive when utility from the share of the surplus is sufficiently concave, as concavity generates incentives for sharing the surplus within periods. Consistent with this result, Battaglini and Palfrey (2007) report numerical results in a three-player game in which the set of allocations is restricted on a grid, and find that sufficient concavity in individual allocations can support equilibria in which the surplus is shared among all players within periods. In work complementary to the present paper, Bowen and Zahran (2009) study the same model (requiring concavity in individual allocations) and build on the equilibrium derived in Sect. 3 to derive conditions that may support non-minimum winning allocations for some initial status quo.

Before I move to the detailed presentation and analysis of the model, I further discuss related contributions. Closely related to the present model is that analyzed by Epple and Riordan (1987). They study subgame perfect (as opposed to Markovian) equilibria of a three-player divide-the-dollar game with alternating offers. They establish that at least two radically different sequences of divisions of the surplus can be supported in equilibrium, thus providing evidence that a folk-theorem may hold for these games. The first study of Markov perfect equilibria with the game form I consider in the present study is by Baron (1996), who analyzes the case of a one-dimensional policy space and shows that policies converge to the median in the long run. Diermeier and Fong (2009) study a discretized version of the single proposer, divide-the-dollar game. For any fixed grid on the space of agreements, they show that there exists an equilibrium in which the proposer cannot extract the whole surplus for some initial status quo if the discount factor is high enough. Their equilibrium does not exist for any discount factor when the space of agreements is a continuum, as in the present paper. While the above studies are concerned with applications in special policy spaces, Duggan and Kalandrakis (2007) study a general model with only smoothness conditions imposed on players' preferences and minimal restrictions on the policy space. Among other results, they establish existence of Markovian equilibria in pure strategies. Despite its generality, the existence result of Duggan and Kalandrakis does not apply in the model considered in this study, because they require stochastic shocks on preferences and the status quo.

Related to the setup of Baron (1996), Kalandrakis (2004), and the present paper is the model with a one-dimensional state space of Cho (2005) who studies a multi-party parliamentary democracy with both bargaining and elections. In a model with finite state space and transferable utility, Gomez and Jehiel (2005) study efficiency properties of equilibria. Battaglini and Coate (2007) characterize stationary equilibria in a model of public good provision, private consumption, and taxation in which only the status quo level of the public good is determined by past actions. Bernheim et al. (2006) analyze a dynamic game of sequential proposing and voting such that victorious proposals become the status quo in each voting round (without being implemented) with the proposal surviving the last voting round being the implemented policy, and derive conditions so that this final policy coincides with the ideal policy of the last proposer. In a general setting applying social choice theoretic equilibrium notions, Lagunoff (2009) and Lagunoff (2008) studies the dynamics of institutional stability and reform. Penn (2009) studies dynamic preferences in a model with exogenous proposals and probabilistic voting.

In what follows I present the model, define the equilibrium solution concept, and introduce necessary notation in Sect. 2. I fully characterize an equilibrium when the agreement space is restricted to minimum winning allocations in Sect. 3. In Sect. 4, I establish existence of a minimum winning coalitions equilibrium and discuss properties of this equilibrium. I conclude in Sect. 5. All proofs are relegated to the Appendix.

2 Model and preliminaries

Consider a game among $n = 2m + 1$ players, where m is an integer satisfying $m \geq 2$, contained in the set $N = \{1, \ldots, n\}$. These players convene in committee in each period $t = 1, 2, \ldots$ to reach an agreement x^t drawn from a set X. Period t starts with a status quo policy $q^t \in X$, and then player i is recognized with probability $p_i \geq 0, \sum_{i=1}^{n} p_i = 1$, to make a proposal $y \in X$. Players respond *yes* or *no* and if a majority of $m + 1$ or more players vote *yes*, then the proposed agreement is implemented, i.e., $x^t = y$; otherwise, the status quo policy q^t is implemented, i.e., $x^t = q^t$. The game then moves to period $t+1$, with the status quo now being period t's agreement, $q^{t+1} = x^t$, and a new round of proposal and vote. The agreement space X represents all possible divisions of a fixed budget among the n players, so that $X = \Delta$ where $\Delta = \{x \in \mathbb{R}_+^n : \sum_{i=1}^{n} x_i = 1\}$.[4] Players derive stage utility $u_i : X \rightarrow \mathbb{R}$ from the implemented agreement and I assume players' utility depends only on their share of the surplus. In particular, individual preferences take the form $u_i(x) = u(x_i)$ where the function $u : [0, 1] \rightarrow \mathbb{R}$ is strictly monotonic and concave, and satisfies the normalizations $u(0) = 0$ and $u(1) = 1$. Players discount the future with a common factor $\delta \in (0, 1)$, and their payoff in the game is the discounted sum of stage payoffs.

I focus the analysis on equilibria in which players use Markovian strategies.[5] Existence of such equilibria requires mixing at the proposal stage, so I represent a (mixed) *proposal strategy* for player i as a measurable function $\pi_i : X \rightarrow \mathcal{P}[X]$ that maps status quo q to Borel probability measures over X. I use the somewhat abusive notation $\pi_i[\cdot \mid q] \in \mathcal{P}[X]$ to denote player i's randomization given status quo q. A *voting strategy* is a function $\alpha_i : X \times X \rightarrow \{yes, no\}$, so that, for example, $\alpha_i(q, y) = yes$ indicates player i votes *yes* on proposal y when the status quo is q. In what follows I work with the equivalent representation of voting strategy α_i by a correspondence $A_i : X \rightrightarrows X$ that maps each status quo q to an *acceptance set* $A_i(q) = \{y \in X : \alpha_i(q, y) = yes\}$. Let $\sigma = (\pi_i, A_i)_{i \in N}$ represent a profile of strategies for the n players. Given such a profile, define the *win set* of status quo q as the set of agreements that defeat status quo q by majority rule, namely

[4] In Sect. 3, it will prove convenient to solve an auxiliary game in which the space of possible agreements, X, is restricted to a proper subset of Δ.

[5] There are well developed arguments in the literature (e.g., Maskin and Tirole 2001, and the references therein) that justify this focus on Markov strategies.

$$W(q;\sigma)=\left\{y\in X\mid\sum_{i=1}^{n}I_{A_i(q)}(y)\geq m+1\right\},\tag{1}$$

where $I_A(y)$ is the indicator function. If players use strategies σ and we impose the additional requirement that $support(\pi_i[\cdot\mid q])\subseteq W(q;\sigma)$ for all proposers i and all status quo q, then we can write the expected payoff of player i from an agreement x as

$$U_i(x;\sigma)=(1-\delta)u_i(x)+\delta\sum_{j=1}^{n}p_j\int_X U(y;\sigma)\pi_j[dy\mid x].\tag{2}$$

The supposition in (2) that proposal strategies have support in the win set, $W(q;\sigma)$, does not restrict possible equilibrium outcome distributions in the analysis that follows, in the sense that for every equilibrium in which some player proposes $y\notin W(q;\sigma)$, there exists an otherwise equivalent equilibrium in which this player proposes the status quo q instead.

I now define an equilibrium as a variant of Markov Perfect Nash equilibrium (Maskin and Tirole 2001) with an added standard refinement on voting strategies:

Definition 1 An equilibrium is a profile of proposal and voting strategies $\sigma^*=(\pi_i^*,A_i^*)_{i\in N}$, such that for all players i, and for all status quo q:

$$y\in A_i^*(q)\Leftrightarrow U_i\left(y;\sigma^*\right)\geq U_i\left(q;\sigma^*\right),\text{ and}\tag{3}$$
$$\pi_i^*\left[\arg\max\left\{U_i\left(x;\sigma^*\right)\mid x\in W\left(q;\sigma^*\right)\right\}\mid q\right]=1,\tag{4}$$

Equilibrium condition (3) requires that a player votes *yes* to a proposal if and only if she weakly prefers it to the status quo q. Thus, I eliminate a – rather large – class of uninteresting equilibria that involve arbitrary voting actions when players are not pivotal. Equilibrium condition (4) requires that a committee member chooses a proposal optimally when recognized. Observe that proposers are restricted to choose among the set of alternatives that defeat the status quo, $W(q;\sigma^*)$, a restriction that was assumed in Eq. 2. Note that $q\in W(q;\sigma^*)$ for all $q\in X$ (by equilibrium condition (3)), so that a proposer cannot profitably deviate by proposing $y\notin W(q;\sigma^*)$.

I now proceed to the analysis of the game. To pave the way, I introduce necessary notation. Partition the space of possible divisions of the surplus into subsets $\Delta_\theta\subset\Delta$, where θ, $0\leq\theta\leq n-1$, indicates the number of players receiving zero, i.e., $\Delta_\theta=\left\{x\in\Delta\mid\sum_{i=1}^{n}I_{\{0\}}(x_i)=\theta\right\}$. For any numbers $\beta>\alpha,\alpha,\beta\in\{0,1,\ldots,n-1\}$, define

$$\Delta_\alpha^\beta=\bigcup_{\theta=\alpha}^{\beta}\Delta_\theta.$$

Δ_α^β is the set of all allocations of the surplus with α, or $\alpha+1,\ldots,$ or β players receiving zero. I establish the existence of, and fully characterize, an equilibrium with the property that proposals involve *minimum winning coalitions* (Riker 1962), such that at most $m+1$ players receive a positive fraction of the surplus in each period. Note that if such proposals indeed prevail in equilibrium, then Δ_m^{n-1} is an absorbing set, one that is reached in at most one period from any initial status quo allocation. Capitalizing on the above property of equilibria with minimum winning coalitions, I proceed in two steps. First, I derive equilibrium strategies in closed form for an auxiliary game in which the space of possible agreements is restricted to $X = \Delta_m^{n-1}$ (Sect. 3). Second, I extend the specified equilibrium strategies to the entire space of agreements $X = \Delta$ (Sect. 4).

3 Minimum winning coalition status quo allocations

As an intermediate step to establishing an equilibrium that features minimum winning coalitions, I first establish an equilibrium when the space of possible agreements is restricted to $X = \Delta_m^{n-1}$. This equilibrium is summarized by two properties: First, for every status quo, optimal proposals coincide with the feasible allocations that maximize the proposer's share of the surplus. This is an intuitive property for the agreement space assumed in this study, but nevertheless requires certain restrictions on the concavity of players' stage payoff functions. Second, players with zero status quo allocation are willing to accept proposals $y \in \Delta_{m+1}^{n-1}$ that also allocate them zero.[6] It follows from these two properties that any proposer i obtains the approval of m other players in order to extract the whole surplus for $q \in \Delta_{m+1}^{n-1}$, or for $q \in \Delta_m$, if $q_i > 0$. With these proposal strategies, and using (2), player i's expected payoff from any agreement $x \in \Delta_{m+1}^{n-1}$ is given by

$$U_i(x) = (1-\delta)u(x_i) + \delta p_i. \tag{5}$$

In order to complete the description of proposal strategies for all status quo $q \in \Delta_m^{n-1}$, it remains to specify proposals by players i with zeros status quo allocation and status quo $q \in \Delta_m$. To facilitate the presentation of equilibrium proposals in this case, fix a status quo $q \in \Delta_m$, and, if necessary, relabel players so that $q_{j+1} \geq q_j$, $j = 1, \ldots, n-1$. Any $i \in \{1, \ldots, m\}$ must receive the vote of one among players $j = m+1, \ldots, n$ with status quo allocation $q_j > 0$ in order to pass a proposal. Of course, i wishes to coalesce with the least expensive player who, intuitively, is the player with the lowest positive status quo allocation, i.e., player $m+1$. I will now demonstrate that, depending on the exact value of $q \in \Delta_m$, it is not an equilibrium strategy for proposer i to allocate a positive amount to player $m+1$ with probability one. Indeed, suppose that player $m+1$ is allocated an amount z whenever player $i = 1, \ldots, m$ is the proposer, with i retaining the rest of the surplus. By Eq. 5, the corresponding allocation, say $y \in \Delta_{n-2}$, yields expected utility $U_{m+1}(y) = (1-\delta)u(z) + \delta p_{m+1}$. On the other hand, the expected utility from maintaining the status quo $q \in \Delta_m$ is (given assumed proposal strategies)

6 Indeed, players with zero status quo allocation may even strictly prefer such proposals in equilibrium.

$$
\begin{aligned}
U_{m+1}(q) &= (1-\delta)u(q_{m+1}) \\
&\quad + \delta\left[(1-\delta)\left(\sum_{h=1}^{m} p_h u(z) + p_{m+1}u(1) + \sum_{h=m+2}^{n} p_h u(0)\right) + \delta p_{m+1}\right] \\
&= (1-\delta)u(q_{m+1}) + \delta\left[\sum_{h=1}^{m} p_h(1-\delta)u(z) + p_{m+1}\right].
\end{aligned}
$$

The allocation z required by player $m+1$ can be obtained by solving $U_{m+1}(y) = U_{m+1}(q)$ or

$$u(z) = \frac{u(q_{m+1})}{1-\delta\sum_{h=1}^{m} p_h}.$$

With the above described proposal strategies, the expected payoff of players $j = m+2, \ldots, n$ from the status quo q is $U_j(q) = (1-\delta)u(q_j) + \delta[p_j(1-\delta)u(1) + \delta p_j] = (1-\delta)u(q_j) + \delta p_j$. As a consequence, proposer i can allocate an amount q_j to player $j = m+2, \ldots, n$ in order to obtain j's vote, and retain the rest of the surplus. Thus, the assumed pure proposal strategies are not part of an equilibrium for any $q \in \Delta_m$ such that

$$\frac{u(q_{m+1})}{1-\delta\sum_{h=1}^{m} p_h} > u(q_{m+2}). \tag{6}$$

If (6) holds, player $m+1$ becomes too expensive, because player $m+1$ expects to receive a positive allocation from other players with probability $\sum_{h=1}^{m} p_h$, while players $j = m+2, \ldots, n$ expect zero instead. Thus equilibrium proposals for status quo $q \in \Delta_m$ must generally involve mixed strategies. Specifically, proposer i mixes by allocating an amount I denote by $z_b(q)$ to one among b players $j \in \{m+1, \ldots, m+b\}$. By a similar method to that used above, we conclude that this amount is given by

$$u(z_b(q)) = \frac{\sum_{j=m+1}^{m+b} u(q_j)}{b-\delta\sum_{i=1}^{m} p_i}. \tag{7}$$

Furthermore, the integer $b \in \{1, \ldots, m+1\}$ is determined by two equilibrium conditions:

$$u(z_b(q)) < u(q_{m+b+1}), \quad \text{if } b = 1, \ldots, m, \tag{8}$$

and

$$u(z_b(q)) \geq u(q_j), \quad j = m+1, \ldots, m+b. \tag{9}$$

Condition (8) is a generalization of condition (6) and requires that the allocation received by each of the b players $m+1, \ldots, m+b$ is smaller than that demanded by player $m+b+1$. Thus, (8) ensures that proposers do not have an incentive to coalesce with any of players $m+b+1, \ldots, n$ instead of choosing one among players $m+1, \ldots, m+b$. Condition (9) implies that players $m+1, \ldots, m+b$ receive (and demand) a larger amount than their status quo allocation q_j, $j = m+1, \ldots, m+b$ in order to approve a proposal. On the one hand, these players' utility streams in the event they become the proposer in future periods is identical under the two alternatives, that is, these players can extract the whole surplus in the future whether they accept the equilibrium proposal or retain the status quo. On the other hand, upon accepting an equilibrium proposal, players $j \in \{m+1, \ldots, m+b\}$ receive zero from all proposers $h \neq j$ in future periods, whereas, by maintaining the status quo, these players expect to receive a positive amount as coalition partners with positive probability. Thus, the proposed allocation is larger than the status quo allocation in order for these players to vote against the status quo.

The next lemma states that conditions (8) and (9) jointly determine a unique number of players, b, that are potential recipients of positive allocations in equilibrium when the status quo is $q \in \Delta_m$, thus uniquely pinning down the required mixed strategies in equilibrium.

Lemma 1 *Assume stage payoff function u satisfies (10) of Proposition 1. Fix any $q \in \Delta_m^{n-1}$ and, if necessary, relabel players so that $q_{i+1} \geq q_i$, $i = 1, \ldots, n-1$.*

1. *There exists a unique b, $1 \leq b \leq m+1$, that satisfies (8) and (9).*
2. *If b satisfies (8) and (9), then $u(z_b(q)) \leq \dfrac{(m+1)u(\frac{1}{m+1})}{m+1-\delta\sum_{i=1}^{m} p_i}$.*

Condition (10) is a mild concavity restriction that ensures that the required allocation $z_b(q)$ is feasible. In fact, more stringent concavity restrictions are necessary in order for the strategies described in this section to constitute an equilibrium. In particular,

Proposition 1 *Assume $X = \Delta_m^{n-1}$ and that for all $C \subset N$ such that $|C| = m$*

$$\frac{(m+1)u(\frac{1}{m+1})}{m+1-\delta\sum_{i\in C} p_i} \leq u\left(\tfrac{1}{2}\right), \text{ and} \tag{10}$$

$$\frac{mu(\frac{s}{m})}{m-\delta\sum_{i\in C} p_i} \leq u(s) \tag{11}$$

for all $s \in [0, 1]$ such that $\frac{mu(\frac{s}{m})}{m-\delta\sum_{i\in C} p_i} < u(1-s)$. Then there exists an equilibrium satisfying properties 1 to 3.

1. *Every proposer i extracts the whole surplus if $q \in \Delta_{m+1}^{n-1}$, or if $q \in \Delta_m$ and $q_i > 0$.*
2. *If $q \in \Delta_m$, relabeling players if necessary so that $q_{h+1} \geq q_h$, $h = 1, \ldots, n-1$, then every proposer i with $q_i = 0$ proposes $y^{ij} \in \Delta_{n-2}$ with probability $\mu_i^j = \dfrac{u(z_b(q)) - u(q_j)}{\delta u(z_b(q))\sum_{i=1}^{m} p_i}$, where $j = m+1, \ldots, m+b$, $y_j^{ij} = z_b(q)$ and $y_i^{ij} = 1 - z_b(q)$, $z_b(q)$ satisfies (7), and b satisfies (8) and (9).*
3. *The equilibrium expected utility is continuous and, relabeling players if necessary so that $x_{h+1} \geq x_h$, $h = 1, \ldots, n-1$, takes the form:*

$$U_i(x) = \begin{cases} (1-\delta)u(x_i) + \delta p_i & \text{if } i = m+b+1, \ldots, n, \\ (1-\delta)u(z_b(x)) + \delta p_i & \text{if } i = m+1, \ldots, m+b, \\ \delta(p_i(1-\delta)u(1-z_b(x)) + \delta p_i) & \text{if } i = 1, \ldots, m, \end{cases} \tag{12}$$

where $z_b(x)$ satisfies (7) and b satisfies (8) and (9).

If, in addition, $p_i = \frac{1}{n}$ for all i, then conditions (10) and (11) are necessary for the existence of this equilibrium.

Conditions (10) and (11) are restrictions on the concavity of the stage utility function u. Their role is to ensure that players wish to maximize their own allocation when proposing, a property implicitly assumed in the analysis prior to Proposition 1. Note that conditions (10) and (11) are easier met when the committee is larger (larger m). Both conditions are always satisfied in the case of risk neutrality, $u(x) = x$. I use Proposition 1 in the next section to establish the existence of an equilibrium when the agreement space is unrestricted ($X = \Delta$).

4 Minimum winning coalitions equilibrium

4.1 Existence

The previous section clears the way for the characterization of an equilibrium in which minimum winning coalitions prevail at all status quos. If players' strategies are identical to those characterized in Proposition 1 for $q \in \Delta_m^{n-1}$ then, in order to establish such an equilibrium when the space of agreements encompasses all possible divisions of the surplus, we must extend the proposal strategies of Proposition 1 to $q \in \Delta_0^{m-1}$ with all proposals restricted to the subset Δ_m^{n-1}, and ensure that the resultant strategies are mutual best responses. In particular, it must be shown that proposers using these extended strategies cannot profitably deviate by proposing allocations $y \in \Delta_0^{m-1}$ for all $q \in \Delta$. The last part of Proposition 1 demonstrates that restrictions on the concavity of players' stage utility functions are necessary for the existence of such equilibria, and the arguments in this section require additional such restrictions. Specifically, throughout this section, I assume one of the following conditions holds: First is risk neutrality, namely,

$$u_i(x) = x_i \text{ for all } i. \tag{A1}$$

The second condition allows for mild risk aversion but restricts recognition probabilities as follows:

$$u_i(x) \leq (1+\varepsilon)x_i \text{ for some } \varepsilon > 0, \text{ and } p_i = \frac{1}{n} \text{ for all } i. \tag{A2}$$

As I detail shortly, assumptions (A1) and (A2) enable me to extend players' proposal strategies to $q \in \Delta_0^{m-1}$ with proposals that exhibit minimum winning coalitions.

First, for each $q \in \Delta_0^{m-1}$, I restrict the support of player i's proposals in a set $\Delta(i) \subset \Delta_m^{n-1}$, which is defined as

$$\Delta(i) = \bigcup_{C \subset N \setminus \{i\} : |C| = m} \{x \in \Delta \mid x_i \geq \widehat{z} \text{ and } x_j = 0,\ j \in C\},$$

with

$$\widehat{z} = \begin{cases} 0 & \text{if (A1) holds} \\ u^{-1}\left(\frac{(m+1)u(\frac{1}{m+1})}{m+1-\delta\frac{m}{n}}\right) & \text{if (A2) holds.} \end{cases}$$

The set $\Delta(i)$ is compact as the union of compact sets. When (A1) holds, it contains allocations such that either a bare majority of players *including* player i receive a positive amount, or allocations such that the set of players who receive a positive amount is a minority (possibly excluding i). When (A2) holds, the set $\Delta(i)$ requires that the proposer receive at least $\widehat{z} = u^{-1}\left(\frac{(m+1)u(\frac{1}{m+1})}{m+1-\delta\frac{m}{n}}\right)$. This restriction does not impair the optimality of i's proposals within Δ_m^{n-1} since, if (A1) or (A2) hold and payoffs are given by (12), no allocation in $\Delta_m^{n-1} \setminus \Delta(i)$ can yield a higher expected payoff than the expected payoff i receives from any allocation in $\Delta(i)$. Indeed, this restriction on players' proposals is necessary in order to obtain Lemmas 2 and 3 in the sequel.

Denote i's mixing over proposals at $q \in \Delta_0^{m-1}$ restricted in this manner by $\widehat{\pi}_i \in \mathcal{P}[\Delta(i)]$, and assume (as is required by the equilibrium concept) that $\widehat{\pi}_i$ is such that all proposals in its support are approved and implemented. Denote the vector of such randomizations by all players by $\widehat{\pi} \in \widehat{\Pi}$, where $\widehat{\Pi} = \times_{i\in N}\mathcal{P}[\Delta(i)]$ is the set of all such profiles of mixed proposal strategies at $q \in \Delta_0^{m-1}$. Now, players' expected utility for $q \in \Delta_0^{m-1}$ when players randomize over proposals using $\widehat{\pi}$ can be computed as

$$\widehat{U}_i(q, \widehat{\pi}) = (1-\delta)u(q_i) + \delta \sum_{h=1}^{n} p_h \int U_i(y)\widehat{\pi}_h[dy], \tag{13}$$

where $U_i(y)$ is given by Eq. 12 of Proposition 1. I emphasize that (13) is derived under the assumption that players follow the equilibrium of Proposition 1 for status quos in Δ_m^{n-1}.

Next define $\widehat{A}_i(q, \widehat{\pi}) = \{x \in \Delta_m^{n-1} \mid U_i(x) \geq \widehat{U}_i(q, \widehat{\pi})\}$, i.e., the set of proposals in Δ_m^{n-1} that are accepted by player i when the status quo is $q \in \Delta_0^{m-1}$ and for this status quo players use randomizations $\widehat{\pi} \in \widehat{\Pi}$. For each player i, each $q \in \Delta_0^{m-1}$, and each $\widehat{\pi} \in \widehat{\Pi}$, define the set

$$\widehat{W}_i(q, \widehat{\pi}) = \{y \in \Delta(i) \mid \sum_{h\neq i} I_{\widehat{A}_h(q,\widehat{\pi})}(y) \geq m\}.$$

It follows that $\widehat{W}_i(q, \widehat{\pi})$ contains those among proposals available to player i that are approved by at least m *other* players when $q \in \Delta_0^{m-1}$, players use lotteries over proposals given by $\widehat{\pi}$, and the game is played according to Proposition 1 for status quos in Δ_m^{n-1}. Lemma 2 states that $\widehat{W}_i$ is a non-empty, upper-hemicontinuous correspondence of $\widehat{\pi}$, and that player i has a proposal in $\widehat{W}_i(q, \widehat{\pi})$ such that i's allocation is strictly larger than $\widehat{z}$.

Lemma 2 *Assume either* (A1) *or* (A2) *with* ε *sufficiently small. For all* i*, all* $\widehat{\pi} \in \widehat{\Pi}$*, and all* $q \in \Delta_0^{m-1}$*,*

1. *There exists* $x \in \widehat{W}_i(q, \widehat{\pi})$ *such that* $x_i > \widehat{z}$*.*
2. $\widehat{W}_i(q, \widehat{\pi})$ *is upper-hemicontinuous at* $\widehat{\pi}$*.*

For each player i, define the correspondence of best response proposals

$$M_i(q, \widehat{\pi}) = \arg\max\{U_i(x) \mid x \in \widehat{W}_i(q, \widehat{\pi})\}.$$

Suppose that for any initial or provisional randomizations $\widehat{\pi}$, we obtain new best response mixed strategies $\widehat{\pi}'$ by restricting players to choose optimal proposals (i.e., those in $M_i(q, \widehat{\pi})$). Thus, define the correspondence $B_i(q, \widehat{\pi}) = \mathcal{P}[M_i(q, \widehat{\pi})]$, and require $\widehat{\pi}_i' \in B_i(q, \widehat{\pi})$. Lastly, define the correspondence $B : \widehat{\Pi} \rightrightarrows \widehat{\Pi}$ as $B(q, \widehat{\pi}) = \times_{h=1}^n B_h(q, \widehat{\pi})$. Lemma 3 establishes a decisive step in proving existence of equilibrium:

Lemma 3 *Assume either* (A1) *or* (A2) *with* ε *sufficiently small. For all* $q \in \Delta_0^{m-1}$

1. *There exists a fixed point* $\widehat{\pi}^* \in B(q, \widehat{\pi}^*)$*.*
2. *If* $\widehat{U}_h(q, \widehat{\pi}^*) \leq \delta p_h$ *for some* h*,* $\widehat{\pi}^* \in B(q, \widehat{\pi}^*)$*, then* $\widehat{\pi}_i^*(\{x \in \Delta(i) \mid x_h = 0\}) = 1$ *for all* $i \neq h$*.*

Lemma 3 relies on the fact that there exist feasible proposals for i in $\widehat{W}_i(q, \widehat{\pi})$ that yield an allocation $x_i > \widehat{z}$. This property, established in part 1 of Lemma 2, ensures the necessary continuity property of the best proposals correspondence, M_i, thus allowing the application of a standard fixed point theorem. Furthermore, Lemma 3 also provides a partial characterization of optimal proposals as it establishes a minimum expected payoff necessary for a player $h \neq i$ to receive a positive allocation from proposer i at a fixed point $\widehat{\pi}^*$.

The reader may have noticed that the fixed point mapping B only requires that proposers optimize over the restricted set of available alternatives that receive the vote of m *other* players. This specification is necessary in order to obtain Lemma 2 but does not guarantee that the proposer prefers such optimal proposals over the status quo. Perhaps more threatening to this construction is the fact that we have a priori restricted proposers not to consider proposals in Δ_0^{m-1}. The next lemma addresses both of these issues by ensuring that such alternative proposals cannot constitute profitable deviations for any proposer at any status quo.

Lemma 4 *Assume either* (A1) *or* (A2) *with* ε *sufficiently small. For all* $q \in \Delta_0^{m-1}$*, all fixed points* $\widehat{\pi}^* \in B(q, \widehat{\pi}^*)$*, and all coalitions* $C \subset N$ *such that* $|C| = m + 1$*, there exists* $x \in \Delta_m^{n-1}$ *such that* $U_i(x) \geq \widehat{U}_i(q, \widehat{\pi}^*)$ *for all* $i \in C$*.*

Lemma 4 precludes profitable deviations since it ensures that if at $q' \in \Delta$ proposer i can pass a proposal $q \in \Delta_0^{m-1}$ (with q possibly equal to q') with the votes of coalition C (with $i \in C$), and at q players use proposal strategies $\widehat{\pi}^* \in B(q, \widehat{\pi}^*)$, then i can also pass $x \in \Delta_m^{n-1}$ at q' that yields at least as high of payoff as q to all members of C. Thus, proposal q at status quo q' cannot be strictly better than what i can obtain by optimizing over acceptable proposals in Δ_m^{n-1}. The order of the quantifiers is important in the statement of Lemmas 2 to 4 because, given requisite $\varepsilon > 0$ when assumption (A2) holds, the conclusions of the Lemmas hold for all $i, q, \hat{\pi}$, etc. Thus, by combining these Lemmas we obtain the chief result of this section:

Theorem 1 *Assume $X = \Delta$, and either (A1) or (A2) with ε sufficiently small. There exists an equilibrium σ^* that satisfies properties 1 to 3 of Proposition 1 and proposal strategies that satisfy $\pi_i^*[\Delta_m^{n-1} \mid q] = 1$ for all i and all $q \in \Delta_0^{m-1}$.*

In combination with Proposition 1, Theorem 1 provides a sharp description of the equilibrium, which has the property that within a maximum of three periods, all proposers extract the entire surplus. This property is not shared with the version of this game with $n = 3$ players and equal recognition probabilities analyzed by Kalandrakis (2004). In that case, it is possible that decisions are drawn outside the absorbing set Δ_{n-1} with positive probability for any finite period t. This is because a proposer cannot extract the whole surplus when that player is the only one with zero status quo allocation. On the contrary, when $n \geq 5$, there always exists a bare minority of m players *other* than the proposer who have zero status quo allocation in period $t = 3$.

4.2 Proposal power and McKelvey's dictatorial agenda setter

Assuming linear payoffs as in assumption (A1), the conclusion of Theorem 1 holds for all possible values of recognition probabilities, p_i. Thus, players' long-run equilibrium expected payoff can be any fraction of the available surplus, depending on recognition probabilities. If we take the perspective that a player's expected payoff represents her *power* in this setting, then Proposition 1 yields a partial extension[7] of the result of Kalandrakis (2006) on the relation between recognition probabilities and political power:

Corollary 1 (Proposal Power) *Assume $X = \Delta$ and (A1). For all $w \in \Delta$ there exist recognition probabilities $p_1, \ldots, p_n$ and an equilibrium σ^* for the associated game such that*

$$E_{x^t}[U_i(x^t; \sigma^*)] = w_i,$$

for all i and all $t > 3$.

Of note, the case of a single proposer i with $p_i = 1$ yields an equilibrium derivation of dictatorial agenda setting under the institution assumed by McKelvey (1976, 1979). McKelvey's dictator manages to implement her ideal point via a sequence of binary votes between the status quo and appropriate proposals where, as in the present analysis, each proposal that passes becomes the status quo. In McKelvey's analysis, voters approve these proposals to their eventual detriment, because they are assumed to be myopic ($\delta = 0$). In the present setup, this type of dictatorial agenda setting is obtained as part of a Markovian Nash equilibrium, under the assumption that voters

[7] This is a partial extension because Kalandrakis (2006) considers all monotonic voting rules and general (possibly heterogeneous) discount factors.

are farsighted, and for every value of the discount factor $\delta < 1$. Remarkably, it only takes two periods for player i to extract the whole surplus. In particular, the proposer i with $p_i = 1$ implements some $x \in \Delta(i)$ in period $t = 1$. Thus, $q^t = x$ in period $t = 2$ and at least m players other than i have zero status quo allocation, so that the proposer i can extract the whole surplus with probability one in period $t = 2$. Hence, we obtain the following result:

Corollary 2 (Smooth Dictator) *Assume $X = \Delta$, (A1), and $p_i = 1$. There exists an equilibrium such that, for all initial status quo $q \in X$, i extracts the whole surplus in every period $t \geq 2$.*

Recently, Diermeier and Fong (2009) have studied a discretized version of the single proposer model considered in Corollary 2 and have established an equilibrium in which McKelvey's dictatorial agenda setting is not achieved.[8] In particular, in that equilibrium, players with zero proposal probability can obtain a positive share of the surplus in the long-run, depending on the initial status quo allocation. Unlike the equilibrium of Corollary 2 that requires mixed proposal strategies by the agenda setter, the equilibrium of Diermeier and Fong is in pure strategies, but requires high enough discount factors for any fixed grid on the space of divisions of the surplus. In particular, it is straightforward to show that for any discount factor $\delta < 1$, their equilibrium does not survive as the grid on the space of allocations becomes finer. As a result, the equilibrium of Diermeier and Fong (2009) does not exist in the model assumed in Corollary 2 for any $\delta < 1$. Indeed, it is an open question whether any pure strategy equilibria exist in the continuum model studied in the present paper when $X = \Delta$.

4.3 Concavity and equilibria without minimum winning coalitions

Equilibria with the properties stated in Theorem 1 need not be unique, although all such equilibria are essentially identical in that they involve the same expected payoffs for allocations in the absorbing set Δ_m^{n-1}. When the utility function u is concave as is allowed by condition (A2), the equilibrium of Theorem 1 is inefficient because players could collectively benefit by sharing the surplus in each period. Thus concavity generates incentives countervailing those that support the equilibrium of Theorem 1. These countervailing incentives are already evident in Proposition 1, since the existence of that equilibrium relies on the concavity restrictions (10) and (11). Furthermore, as is evident by the arguments in Lemmas 2 to 4, the extension of the equilibrium to status quo such that an oversized coalition of players receive a positive allocation requires more stringent concavity restrictions in the form of assumption (A2) and small ε. This is intuitive as a move to an allocation with minimum winning coalitions is most costly (collectively) at status quo in which all players receive a positive allocation.

In view of the above discussion, both Proposition 1 and Theorem 1 leave open the possibility that equilibria without minimum winning coalitions may prevail when stage payoffs are sufficiently concave. Indeed, two related papers have recently explored

8 In their setup, the discount factor is equivalently interpreted as the probability that the proposer will have the opportunity to offer another proposal.

such equilibria. Using numerical methods, Battaglini and Palfrey (2007) compute equilibria in a model in which the possible divisions of the surplus are restricted on a grid and stochastic shocks on players' payoff from each action are allowed. When stage preferences exhibit considerable concavity, they find equilibria in which players share the surplus in all periods. More related and complementary to the present study is the work of Bowen and Zahran (2009). Relying on the results of Proposition 1, they explore conditions for the existence of equilibria in which players share the surplus when the initial status quo belong in a subset of Δ_0^{m-1}. Their conditions are stated in terms of the discount factor, and require that players are neither too patient nor too impatient. These conditions can be restated in terms of the concavity of stage preferences on individual allocations, and this alternative equilibrium explored by Bowen and Zahran (2009) does not survive when players are risk neutral. It is an open question whether alternative equilibria that do not exhibit minimum winning coalitions and the one established in the present study exist for the same parameter values.

5 Conclusion

The canonical divide-the-dollar bargaining environment is theoretically significant not because it emulates an actual political space of agreements, but because it lays bare certain of the incentives present in more realistic policy environments. In the one-shot version of that model with one round of proposing and voting, equilibrium incentives lead to allocations that exhibit minimum winning coalitions. In the present study I established that these incentives can also be sustained in equilibrium with dynamic interactions in which present decisions serve as future status quo. In fact, minimum winning coalitions in this dynamic environment lead to an extreme manifestation of the incentives in the one-shot environment in that proposers are eventually able to extract the entire surplus. I established that this equilibrium is robust to small concavity in players' individual allocations and to variation on proposal probabilities beyond the benchmark of random recognition. But the analysis also reveals that strong incentives for sharing the surplus may prevail in equilibrium with sufficient concavity in stage preferences because the equilibrium collapses in the absence of restrictions on the concavity of individual payoffs. Thus it appears plausible to conjecture that incentives for compromise outcomes are stronger in more typical policy environments, in which individual payoffs cannot be independently manipulated without imposing positive (or negative) externalities on coalitions of players.

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Appendix

This appendix contains the proofs of all results.

Proof of Lemma 1 We start by showing the following equivalence:

$$[u(z_b(q)) < u(q_{m+b+1}) \Leftrightarrow u(z_{b+1}(q)) < u(q_{m+b+1})], b = 1, \ldots, m. \quad (14)$$

Indeed, using (7) we write

$$\begin{aligned}
& u(z_b(q)) < u(q_{m+b+1}) \Leftrightarrow \\
& \sum_{j=m+1}^{m+b} u(q_j) < (b - \delta \sum_{i=1}^{m} p_i) u(q_{m+b+1}) \Leftrightarrow \\
& \sum_{j=m+1}^{m+b+1} u(q_j) < (b + 1 - \delta \sum_{i=1}^{m} p_i) u(q_{m+b+1}) \Leftrightarrow \\
& u(z_{b+1}(q)) < u(q_{m+b+1}).
\end{aligned}$$

Now, to show existence of b satisfying (8) and (9) consider the algorithm:

1. Start with $b = 1$; if $u(z_1(q)) < u(q_{m+2})$ then $b = 1$. Otherwise proceed to step 2.
2. If $u(z_b(q)) \geq u(q_{m+b+1})$, consider $b' = b + 1$. By the contra-positive of (14), b' satisfies (9). If b' also satisfies (8), then existence of the required integer is established. Otherwise, if $u(z_{b'}(q)) \geq u(q_{m+b'+1})$, proceed as in 2 until $u(z_b(s)) < u(q_{m+b+1})$ for some $b \leq m$.

3. If condition (8), $u(z_b(q)) < u(q_{m+b+1})$, fails for all $b \leq m$, then $u(z_m(q)) \geq u(q_n)$, and $b = m + 1$.

Note that the concavity of u and condition (10) ensures that $z_{m+1}(q) \leq 1$ for all q, ensuring existence of some b that satisfies both (8) and (9).

To show uniqueness, suppose there exist distinct b, b' with $b < b'$ that satisfy (8) and (9) to get a contradiction. Then, we have $u(z_b(q)) < u(q_{m+b+1})$ from (8) and certainly $u(z_{b'}(q)) \geq u(q_{m+b+\lambda})$, $\lambda = 1, \ldots, (b' - b)$ from (9). From the last $(b' - b)$ inequalities we deduce

$$\begin{aligned}
(b' - b) u(z_{b'}(q)) &\geq \sum_{\lambda=1}^{(b'-b)} u(q_{m+b+\lambda}) \Leftrightarrow \\
(b' - b) \frac{\sum_{h=m+1}^{m+b} u(q_h) + \sum_{\lambda=1}^{(b'-b)} u(q_{m+b+\lambda})}{b + (b' - b) - \delta \sum_{i=1}^{m} p_i} &\geq \sum_{\lambda=1}^{(b'-b)} u(q_{m+b+\lambda}) \Leftrightarrow \\
\frac{\sum_{h=m+1}^{m+b} u(q_h)}{b - \delta \sum_{i=1}^{m} p_i} &\geq \frac{\sum_{\lambda=1}^{(b'-b)} u(q_{m+b+\lambda})}{b' - b} \Leftrightarrow \\
u(z_b(s)) \geq \frac{\sum_{\lambda=1}^{(b'-b)} u(q_{m+b+\lambda})}{b' - b} &\geq u(q_{m+b+1}),
\end{aligned}$$

which contradicts condition (8) for b. This concludes the proof of part 1.

To show part 2, assume b satisfies (8) and (9). We first show that $u(z_b(q)) \leq u(z_{m+1}(q))$. This is trivial if $b = m + 1$, so consider the case $b \leq m$. Then, by condition (8) and the fact that $q_{i+1} \geq q_i$, $i = 1, \ldots, n-1$, we have $u(z_b(s)) < u(q_{m+b+1})$, which implies that

$$\frac{\sum_{i=m+1}^{m+b} u(q_i)}{b-\delta\sum_{h=1}^{m} p_h} < \frac{\sum_{i=m+b+1}^{n} u(q_i)}{m+1-b} \Leftrightarrow$$

$$(m+1-b)\sum_{i=m+1}^{m+b} u(q_i) < (b-\delta\sum_{h=1}^{m} p_h)\sum_{i=m+b+1}^{n} u(q_i) \Leftrightarrow$$

$$(m+1)\sum_{i=m+1}^{m+b} u(q_i) < b\sum_{i=m+1}^{n} u(q_i) - \delta\sum_{h=1}^{m} p_h \sum_{i=m+b+1}^{n} u(q_i) \Leftrightarrow$$

$$(m+1-\delta\sum_{h=1}^{m} p_h)\sum_{i=m+1}^{m+b} u(q_i) < (b-\delta\sum_{h=1}^{m} p_h)\sum_{i=m+1}^{n} u(q_i) \Leftrightarrow$$

$$u(z_b(q)) = \frac{\sum_{i=m+1}^{m+b} u(q_i)}{b-\delta\sum_{h=1}^{m} p_h} < \frac{\sum_{i=m+1}^{n} u(q_i)}{m+1-\delta\sum_{h=1}^{m} p_h} = u(z_{m+1}(q)).$$

Thus, $u(z_b(q)) \leq u(z_{m+1}(q))$ when $b = 1, \ldots, m+1$ satisfies (8) and (9), as we wished to show. Now, concavity of u implies that

$$u(z_{m+1}(q)) = \frac{\sum_{i=m+1}^{n} u(q_i)}{m+1-\delta\sum_{i=1}^{m} p_i} \leq \frac{(m+1)u(\frac{1}{m+1})}{m+1-\delta\sum_{i=1}^{m} p_i},$$

which completes the proof of part 2 and the lemma. □

Proof of Proposition 1 To ensure that the associated proposals are well defined, we show that mixing probabilities lie between zero and one and sum up to one. In particular it suffices to show that $\sum_{j=m+1}^{m+b} \mu_i^j = 1$ and that $\mu_i^j \geq 0$ for all j. The latter is equivalent to $u(z_b(q)) \geq u(q_j)$, $j = m+1, \ldots, m+b$, which is true by (9). We also have $\sum_{j=m+1}^{m+b} \mu_i^j = \frac{bu(z_b(q)) - \sum_{j=m+1}^{m+b} u(q_j)}{\delta u(z_b(q)) \sum_{i=1}^{m} p_i} = 1$, after substitution from (7). Straightforward algebra yields the expected utilities in (12), which are clearly continuous. To establish the equilibrium, we must show that proposals are optimal. Note that extracting the entire surplus is a global maximizer of the expected utilities in (12), yielding $(1-\delta)u(1) + \delta p_i = \max\{U_i(x) \mid x \in \Delta_m^{n-1}\}$. Thus, we only need consider cases when the proposer does not extract the entire surplus. This occurs for status quo $q \in \Delta_m$ and proposer i with $q_i = 0$. Assume that for such status quo and proposer, the integer b uniquely satisfies (8) and (9). Prescribed equilibrium proposals $y^{ih} \in \Delta_{n-2}$, $h = m+1, \ldots, m+b$ for proposer i are optima among feasible alternatives in Δ_{m+1}^{n-1}, because by simple inspection of (12), i's maximization over alternatives in Δ_{m+1}^{n-1} amounts to maximization of i's allocation. Note that part 2 of Lemma 1 and (10) ensure that

$$z_b(q) = y_h^{ih} \leq u^{-1}\left(\frac{(m+1)u\left(\frac{1}{m+1}\right)}{m+1-\delta\sum_{k=1}^m p_k}\right) < \frac{1}{2},$$

and as a result $y_i^{ih} > \frac{1}{2}$. We need to show that there exists no $y' \in \arg\max\{U_i(x) \mid x \in W(q;\sigma^*) \bigcap \Delta_m\}$ such that $U_i(y') > U_i(y^{ih})$. So further assume, in order to get a contradiction, that there exists such a $y' \in W(q;\sigma^*) \cap \Delta_m$ such that $U_i(y') > U_i(y^{ih})$. By (12) and part 2 of Lemma 1, we have $y_i' > y_i^{ih}$. Without loss of generality relabel players so that $y_{j+1}' \geq y_j'$, $j = 1, \ldots, n-1$, still indexing the proposer by i, and let integer b' uniquely satisfy (8) and (9) for y'. After relabeling, we have $i = n$ and $b' \leq m$, by (8), (10), and part 2 of Lemma 1 that jointly imply $z_{b'}(y') \leq \frac{1}{2} < y_i'$. We also have $U_j\left(y'\right) \geq U_j(q) > \delta p_j$ for at least one player j with $q_j > 0$, else $y' \notin W(q;\sigma^*)$. Since $q_j > 0$, (8) and (12) now imply that

$$(1-\delta)^{-1}(U_j(y') - \delta p_j) \geq u(z_b(q)) > 0, \tag{15}$$

so that $y_j' > 0$ and, due to the fact that we have relabeled players, $j \in \{m+1, \ldots, n-1\}$. We now have three cases:

Case 1, $j > m+b'$: Then from (12) we have $U_j\left(y'\right) = (1-\delta)u(y_j') + \delta p_j \geq U_j(q)$ and $y_j' \geq z_b(q) = y_h^{ih}$. But then $1 - y_h^{ih} = y_i^{ih} < y_i' < 1 - y_j'$, which yields $y_h^{ih} > y_j'$, a contradiction.

Case 2, $j \leq m + b'$, $b' = m$: Then we have $U_j\left(y'\right) = (1-\delta)u(z_{b'}(y')) + \delta p_j$. From (15) we have $u(z_{b'}(y')) = \frac{\sum_{k=m+1}^{n-1} u(y_k')}{m-\delta\sum_{k=1}^m p_k} \geq u\left(z_b\left(q\right)\right)$. By (11) and the concavity of u we deduce that

$$u\left(z_b\left(q\right)\right) \leq \frac{\sum_{k=m+1}^{n-1} u(y_k')}{m-\delta\sum_{k=1}^m p_k} \leq \frac{mu(\overline{y}')}{m-\delta\sum_{k=1}^m p_k} \leq u\left(m\overline{y}'\right) = u\left(\sum_{k=m+1}^{n-1} y_k'\right),$$

where $\overline{y}' = \frac{\sum_{k=m+1}^{n-1} y_k'}{m}$. But note that $y_i' > y_i^{ih} \Leftrightarrow 1 - y_i' < 1 - y_i^{ih} \Leftrightarrow y_h^{ih} > \sum_{k=m+1}^{n-1} y_k'$, so that we conclude that $u(z_b(q)) = u(y_h^{ih}) > u(\sum_{k=m+1}^{n-1} y_k') \geq u(z_b(q))$, a contradiction.

Case 3, $j \leq m + b'$, $b' < m$: Then $U_j\left(y'\right) = (1-\delta)u(z_{b'}(y')) + \delta p_j$ and $U_{m+b'+1}\left(y'\right) = (1-\delta)u(y_{m+b'+1}') + \delta p_{m+b'+1}$. From (8) and (15) we have $u\left(y_{m+b'+1}\right) > u(z_{b'}(y')) \geq u\left(z_b\left(q\right)\right) = u(y_h^{ih})$, which implies $y_i^{ih} = 1 - y_h^{ih} > 1 - y_{m+b'+1}' > y_i' \Rightarrow U_i(y') < U_i(y^{ih})$, a contradiction.

In all three cases we obtained a contradiction, due to the absurd hypothesis that $U_i(y') > U_i(y^{ih})$. Thus proposals y^{ih} are optimal, as we wished to show.

In order to complete the proof, we need to show that conditions (10) and (11) are necessary for the existence of the equilibrium when $p_i = \frac{1}{n}$ for all i, which follows from Examples 4 and 5 (pp. 21–22) in Kalandrakis (2007). □

Proof of Lemma 2 Fix $q \in \Delta_0^{m-1}$ and $\hat{\pi} \in \hat{\Pi}$. The proof consists of four steps. First, we establish a lower bound on $\widehat{U}_i\left(q, \widehat{\pi}\right)$. Then, we show an upper bound on the sum of the demands of an appropriate set of players. In steps 3 and 4 we use these results to prove the lemma.

Step 2.1: *For all i,*

$$\widehat{U}_i(q,\widehat{\pi}) \geq \begin{cases} -\frac{\delta^2 p_i(1-p_i)(1-\delta)}{m+1-\delta} + \delta^2 p_i & \text{if (A1) holds} \\ \frac{\delta(1-\delta)\widehat{z}}{n}\left(1-\frac{\delta(n-1)}{n}\right) + \frac{\delta^2}{n} & \text{if (A2) holds.} \end{cases} \tag{16}$$

From (12) we have that i cannot receive a payoff less than

$$\min\{U_i(x) : x \in \Delta(i)\} = (1-\delta)u(\hat{z}) + \delta p_i$$

when proposing. Similarly, the minimum possible payoff that can be received by player i when other players propose satisfies

$$\min\left\{U_i(x) : x \in \Delta_m^{n-1}\right\} \geq \begin{cases} \delta(p_i(1-\delta)u\left(1-\frac{1}{m+1-\delta}\right)+\delta p_i) & \text{if (A1) holds} \\ \delta(\frac{(1-\delta)}{n}u(1-\widehat{z})+\frac{\delta}{n}) & \text{if (A2) holds} \end{cases}$$
$$\geq \begin{cases} -\frac{\delta p_i(1-\delta)}{m+1-\delta}+\delta p_i & \text{if A1 holds} \\ -\frac{\delta(1-\delta)\widehat{z}}{n}+\frac{\delta}{n} & \text{if (A2) holds.} \end{cases}$$

The first inequality follows from the fact that

$$\max\{z_b(x) : b \text{ satisfies (8) and (9)}, x\in\Delta_m^{n-1}\} \leq \begin{cases} \frac{1}{m+1-\delta} & \text{if (A1) holds} \\ u^{-1}\left(\frac{(m+1)u(\frac{1}{m+1})}{m+1-\delta\frac{m}{n}}\right) & \text{if (A2) holds,} \end{cases}$$

by part (ii) of Lemma 1, while the second inequality follows from the fact that $u(1-\widehat{z}) \geq 1-\widehat{z}$ and simple algebra. By combining the above inequalities and noting that i is the proposer with probability p_i, we obtain

$$\widehat{U}_i(q,\widehat{\pi})$$
$$\geq \begin{cases} (1-\delta)q_i + \delta\left(p_i(\delta p_i) + (1-p_i)\left(-\frac{\delta p_i(1-\delta)}{m+1-\delta}+\delta p_i\right)\right) & \text{if (A1) holds} \\ (1-\delta)u(q_i) + \delta\left(\frac{1}{n}((1-\delta)u(\widehat{z})+\frac{\delta}{n}), +\frac{n-1}{n}\left(-\frac{\delta(1-\delta)\widehat{z}}{n}+, \frac{\delta}{n}\right)\right) & \text{if (A2) holds} \end{cases}$$
$$\geq \begin{cases} -\frac{\delta^2 p_i(1-p_i)(1-\delta)}{m+1-\delta} + \delta^2 p_i & \text{if (A1) holds} \\ \frac{\delta(1-\delta)\widehat{z}}{n}\left(1-\frac{\delta(n-1)}{n}\right) + \frac{\delta^2}{n} & \text{if (A2) holds,} \end{cases}$$

proving (16).

For the next two steps we define $\widehat{d}_j(q,\widehat{\pi}) = (1-\delta)^{-1}\max\left\{0, \widehat{U}_i(q,\widehat{\pi}) - \delta p_i\right\}$ and assume without loss of generality that $\widehat{d}_{h+1}(q,\widehat{\pi}) \geq \widehat{d}_h(q,\widehat{\pi})$, $h = 1,\ldots,n-1$.

Step 2.2: $\sum_{h=1}^{m+1}\widehat{d}_h(q,\widehat{\pi}) < 1-\widehat{z}$. Let $\ell = \min\left\{i \in N : \widehat{d}_i(q,\widehat{\pi}) > 0\right\}$. Obviously, if $\ell > m+1$ then $\sum_{h=2}^{m+1}\widehat{d}_h(q,\widehat{\pi}) = 0$, so we only need consider cases with $\ell \leq m+1$.

By the definition of $\widehat{d}_i(q,\widehat{\pi})$ we have

$$\sum_{h=\ell}^{n}\widehat{d}_h(q,\widehat{\pi})=(1-\delta)^{-1}\sum_{h=1}^{n}\left(\widehat{U}_h(q,\widehat{\pi})-\delta p_i\right)-(1-\delta)^{-1}\sum_{h=1}^{\ell-1}\left(\widehat{U}_h(q,\widehat{\pi})-\delta p_h\right).$$

We now invoke (16) to deduce

$$\sum_{h=\ell}^{n}\widehat{d}_h(q,\widehat{\pi}) \le \frac{\sum_{h=1}^{n}\left(\widehat{U}_h(q,\widehat{\pi})-\delta p_i\right)}{(1-\delta)} + \begin{cases} \sum_{h=1}^{\ell-1}\frac{\delta^2 p_i(1-p_i)}{m+1-\delta}+\sum_{h=1}^{\ell-1}\delta p_i & \text{if (A1) holds} \\ \frac{\delta(\ell-1)}{n}\left(1-\widehat{z}\left(1-\frac{\delta(n-1)}{n}\right)\right) & \text{if (A2) holds.} \end{cases} \tag{17}$$

Also, since $\widehat{d}_{h+1}(q,\widehat{\pi})\ge\widehat{d}_h(q,\widehat{\pi})$, we deduce that

$$\frac{\sum_{h=\ell}^{m+1}\widehat{d}_h(q,\widehat{\pi})}{m+2-\ell}\le\frac{\sum_{h=\ell}^{n}\widehat{d}_h}{2m+2-\ell}. \tag{18}$$

Finally, note that for any $\widehat{\pi}\in\widehat{\Pi}$ and for any status quo $q\in\Delta_0^{m-1}$,

$$\frac{\sum_{h=1}^{n}\left(\widehat{U}_h(q,\widehat{\pi})-\delta p_i\right)}{1-\delta}\le\begin{cases} 1 & \text{if (A1) holds} \\ 1+\varepsilon\left(1+\delta+\delta^2\right) & \text{if (A2) holds,} \end{cases} \tag{19}$$

where the second line is obtained because $u(y)\le y(1+\varepsilon)$, and $\sum_{h=1}^{n}u_h(x)=1$ for all allocations $x\in\Delta_{n-1}$ which prevail two periods after any proposal in Δ_m^{n-1} is approved (by Proposition 1). We now distinguish two cases:

- (A1) holds, $\widehat{z}=0$: Then $\sum_{h=1}^{n}\widehat{U}_h(q,\widehat{\pi})=1$, and we can combine (17)–(19) to obtain

$$\sum_{h=\ell}^{m+1}\widehat{d}_h(q,\widehat{\pi})\le\frac{m+2-\ell}{2m+2-\ell}\left(1+\delta^2\sum_{h=1}^{\ell-1}\frac{p_h(1-p_h)}{m+1-\delta}+\delta\sum_{h=1}^{\ell-1}p_h\right).$$

  The above immediately yields $\sum_{h=\ell}^{m+1}\widehat{d}_h(q,\widehat{\pi})\le\frac{m+1}{2m+1}<1$, when $\ell=1$. Note that if $\ell>1$, $\sum_{h=1}^{\ell-1}p_h(1-p_h)\le(\ell-1)\left(\frac{1}{\ell-1}\right)\left(1-\frac{1}{\ell-1}\right)=\frac{\ell-2}{\ell-1}$. Thus, since $\delta<1$ and $\widehat{d}_h(q,\widehat{\pi})=0$, $h<\ell$, we deduce

$$\sum_{h=1}^{m+1}\widehat{d}_h(q,\widehat{\pi})=\sum_{h=\ell}^{m+1}\widehat{d}_h(q,\widehat{\pi})<\frac{m+2-\ell}{2m+2-\ell}\left(2+\frac{1}{m}\left(\frac{\ell-2}{\ell-1}\right)\right),$$
$$\ell=2,\ldots,m+1.$$

The right-hand side of the last inequality is equal to 1 when $\ell = 2$, and decreases with ℓ, proving the step in this case.

- (A2) holds, $\widehat{z} = u^{-1}\left(\frac{(m+1)u(\frac{1}{m+1})}{m+1-\delta\frac{m}{n}}\right)$: In this case we combine (17)–(19) to obtain

$$\sum_{h=\ell}^{m+1} \widehat{d}_h\,(q,\widehat{\pi}) \leq \frac{m+2-\ell}{2m+2-\ell}$$
$$\left(1+\varepsilon\left(1+\delta+\delta^2\right)+\frac{\delta(\ell-1)}{n}\left(1-\widehat{z}\left(1-\frac{\delta(n-1)}{n}\right)\right)\right)$$
$$\leq \frac{m+1}{2m+1}\left(1+\varepsilon\left(1+\delta+\delta^2\right)\right),$$

since the right-hand side of the first inequality decreases with ℓ, $m+1 \geq \ell \geq 1$. Since u^{-1} is convex with $u^{-1}(1) = 1$, $n = 2m+1$, and $\delta < 1$, we obtain

$$\sum_{h=1}^{m+1} \widehat{d}_h\,(q,\widehat{\pi}) + \widehat{z} \leq \frac{m+1}{2m+1}\left(1+\varepsilon\left(1+\delta+\delta^2\right)\right) + u^{-1}\left(\frac{(m+1)u(\frac{1}{m+1})}{m+1-\delta\frac{m}{n}}\right)$$
$$< \frac{m+1}{2m+1}(1+3\varepsilon) + \frac{1+\varepsilon}{m+1-\frac{m}{2m+1}}$$
$$< \frac{64}{65} + \frac{142\varepsilon}{65}$$
$$\leq 1.$$

The last inequality holds for $\varepsilon \leq \frac{1}{142}$. The expression in the third line is obtained by setting $m = 2$ which is the value of $m \geq 2$ that maximizes the right-hand side of the second inequality.

Step 2.3: *There exists $x \in \widehat{W}_i\,(q,\widehat{\pi})$ such that $x_i > \widehat{z}$.* Let $C = \{1,\ldots,m+1\} \setminus \{i\}$ if $i \leq m+1$ or $C = \{1,\ldots,m\}$, otherwise. By Step 2, $\sum_{h=1}^{m+1} \widehat{d}_h\,(q,\widehat{\pi}) < 1-\widehat{z}$, so that we can construct proposal $x \in \Delta\,(i)$ with $x_h = \widehat{d}_h\,(q,\widehat{\pi})$ for all $h \in C$, and $x_i = 1 - \sum_{h\in C} \widehat{d}_h\,(q,\widehat{\pi}) > \widehat{z}$. By (12) and the definition of $\widehat{d}_h$ and the fact that u is concave, we easily infer that $U_h(x) \geq \widehat{U}_h\,(q,\widehat{\pi})$ for all $h \in C$, so that $x \in \widehat{W}_i\,(q,\widehat{\pi})$.

Note that Step 3 also establishes non-emptiness of the correspondence $\widehat{W}_i$. Thus, it remains to establish upper-hemicontinuity which we prove with the last step.

Step 2.4: *$\widehat{W}_i\,(q,\widehat{\pi})$ is upper-hemicontinuous as a correspondence of $\widehat{\pi}$.* To establish upper-hemicontinuity, notice that $U_i\,(x)$, $\widehat{U}_i\,(q,\widehat{\pi})$ are continuous in x, $\widehat{\pi}$ respectively; thus, $\widehat{A}_i\,(q,\widehat{\pi})$ has closed graph for all i. As a consequence, $\widehat{W}_i\,(q,\widehat{\pi})$ has closed graph, since finite unions and intersections of closed sets are closed. Thus, since it also has a compact Hausdorff range, $\widehat{W}_i\,(q,\widehat{\pi})$ is upper-hemicontinuous, by the Closed Graph Theorem (Aliprantis and Border 1999, 16.12, p. 529). □

Proof of Lemma 3 Throughout assume that in the case of assumption (A2), $\varepsilon > 0$ is small enough so that the conclusions of Lemma 2 hold. The proof consists of a number of steps:

Step 3.1 *$M_i(q,\widehat{\pi})$ is non-empty.* This follows from the fact that $\widehat{W}_i(q,\widehat{\pi})$ is non-empty and compact by Lemma 2.

The following step proves a stronger version of the second part of the lemma.

Step 3.2 *If $\widehat{U}_h(q,\widehat{\pi}) \leq \delta p_h$ for some h, then $M_i(q,\widehat{\pi}) \cap \{x \in \Delta(i) \mid x_h > 0\} = \emptyset$ for all $i \neq h$.* Fix $i \neq h$, $y \in \arg\max\{U_i(x) \mid x \in \widehat{W}_i(q,\widehat{\pi})\}$ and assume $y_h > 0$ for some h such that $\widehat{U}_h(q,\widehat{\pi}) \leq \delta p_h$. Let $C(y) = \{j \in N \mid y_j > 0\}$. By Lemma 2 we have $i \in C(y)$. Without loss of generality, relabel players so that $y_{j+1} \geq y_j$, $j = 1,\ldots,n-1$, and let i still index the proposer and h index the player satisfying $\widehat{U}_h(q,\widehat{\pi}) \leq \delta p_h$. Let b satisfy (8) and (9) for y, and distinguish two cases:

1. $h > m+b$: Consider allocation y' with $y'_i = y_i + \eta$, $y'_h = y_h - \eta$, $y'_j = y_j$, $j \neq i,h$ for small enough $\eta > 0$. Now (12) ensures that $U_j(y') \geq U_j(y)$ for all $j \neq i,h$, $U_i(y') > U_i(y)$ and $U_h(y') = \delta p_h \geq \widehat{U}_h(q,\widehat{\pi})$, so that $y' \in \widehat{W}_i(q,\widehat{\pi})$, contradicting the optimality of y.
2. $h \in \{m+1,\ldots,m+b\}$: Consider allocation y' with $y'_h = 0$, $y'_j = u^{-1}\left(\frac{\sum_{k=m+1}^{m+b} u(y_k)}{b-\delta\sum_{k=1}^{m} p_k}\right)$ for $j \in \{m+1,\ldots,m+b\}\backslash\{i,h\}$, $y'_j = y_j$, $j \notin \{m+1,\ldots,m+b\} \cup \{i\}$, and $y'_i = 1 - \sum_{k\neq i} y'_k$. We now conclude from (12) that $U_j(y') \geq U_j(y)$ for all $j \neq i,h$ and $U_h(y') = \delta p_h \geq \widehat{U}_h(q,\widehat{\pi})$ so that $y' \in \widehat{W}_i(q,\widehat{\pi})$. It remains to show that $U_i(y') > U_i(y)$ to obtain the desired contradiction. This is true if $b = 1$, since then $y'_i = y_i + y_h$. For $b > 1$ it suffices to show that

$$(b-1)u^{-1}\left(\frac{\sum_{k=m+1}^{m+b} u(y_k)}{b-\delta\sum_{k=1}^{m} p_k}\right) < \sum_{k=m+1}^{m+b} y_k,$$

which also ensures that $y'_i > y_i$. By the monotonicity of u, the above inequality is equivalent to

$$\frac{\sum_{k=m+1}^{m+b} u(y_k)}{b-\delta\sum_{k=1}^{m} p_k} < u\left(\frac{\sum_{k=m+1}^{m+b} y_k}{(b-1)}\right),$$

which is true if (A1) holds. In the case of assumption (A2), for $\varepsilon \leq \frac{1-\delta\frac{m}{n}}{m-1}$ we conclude that

$$\frac{\sum_{k=m+1}^{m+b} u(y_k)}{b-\delta\sum_{k=1}^{m} p_k} \leq \frac{(1+\varepsilon)\sum_{k=m+1}^{m+b} y_k}{b-\delta\sum_{k=1}^{m} p_k} < \frac{\sum_{k=m+1}^{m+b} y_k}{(b-1)} \leq u\left(\frac{\sum_{k=m+1}^{m+b} y_k}{(b-1)}\right),$$

thus proving the step.

The next step involves a direct proof that, unlike the typical line of argument that uses the Theorem of the Maximum, relies on part 1 of Lemma 2.

Step 3.3 *$M_i(q,\widehat{\pi})$ is upper-hemicontinuous.* Since M_i has compact Hausdorff range, it suffices to show that $M_i(q,\widehat{\pi})$ has closed graph (by the Closed Graph Theorem, Aliprantis and Border 1999, 16.12, p. 529). Suppose $M_i(q,\widehat{\pi})$ does not have closed graph to get a contradiction. Then there exists a sequence

$$(\widehat{\pi}^k, y^k) \in GrM_i = \{(\pi, x) \in \widehat{\Pi} \times \Delta(i) \mid x \in M_i(q, \pi)\}$$

such that $(\widehat{\pi}^k, y^k) \to (\widehat{\pi}, y) \notin GrM_i$. By Lemma 2, $y \in \widehat{W}_i(q, \widehat{\pi})$, i.e., y is feasible. Thus, since $(\widehat{\pi}, y) \notin GrM_i$, there exists $z \in \arg\max\{U_i(x) \mid x \in \widehat{W}_i(q, \widehat{\pi})\}$ such that $U_i(z) > U_i(y)$. Note that by Lemma 2 we must have $z_i > \widehat{z}$. Otherwise, Lemma 2 guarantees the existence of $w \in \widehat{W}_i\,(q, \widehat{\pi})$ such that $w_i > \widehat{z}$, hence $U_i(w) > (1-\delta)u(\widehat{z}) + \delta p_i \geq U_i(z)$ if $z_i = \hat{z}$, a contradiction. Thus, $z_i > \widehat{z}$. Without loss of generality, relabel players so that $z_{j+1} \geq z_j$, $j = 1, \ldots, n-1$, maintaining i as the index for the proposer. By Step 3.2 we can assume that $U_j(z) \geq \widehat{U}_j(q, \widehat{\pi})$ for all $j \in \{m+1, \ldots, n\} \setminus \{i\}$. Let b satisfy (8) and (9) for z. We distinguish two cases:

Case 1, $i > m + b$: Construct z' such that $z'_i = z_i - \eta$, $z'_j = z_j + \frac{\eta}{m}$ for $j \in \{m+1, \ldots, n\} \setminus \{i\}$. For sufficiently small $\eta > 0$ we have $U_i(y) < U_i(z') < U_i(z)$ and $U_j(z') > U_j(z) \geq \widehat{U}_j(q, \widehat{\pi})$ for all $j \in \{m+1, \ldots, n\} \setminus \{i\}$. Since $\widehat{U}_h, U_h$ are continuous and $(\widehat{\pi}^k, y^k) \to (\widehat{\pi}, y)$, there exists large enough k such that $U_j(z') > \widehat{U}_j(q, \widehat{\pi}^k) \to \widehat{U}_j(q, \widehat{\pi})$ and $U_i(z') > U_i(y^k) \to U_i(y)$. Then $z' \in \widehat{W}_i(q, \widehat{\pi}^k)$, contradicting $(\widehat{\pi}^k, y^k) \in GrM_i$.

Case 2, $i \leq m+b$: By the definition of $\widehat{z}$, this case pertains only to assumption A1. Construct z' such that $z'_j = z_j + \frac{\eta}{m}$ for $j \in \{m+b+1, \ldots, n\}$ and $z'_j = \frac{\sum_{h=m+1}^{m+b} z_h}{b - \delta \sum_{h=1}^{m} p_h} + \frac{\eta}{m}$ for $j \in \{m+1, \ldots, m+b\} \setminus \{i\}$, and $z'_i = \sum_{h=m+1}^{m+b} z_h - \frac{(b-1)\sum_{h=m+1}^{m+b} z_h}{b - \delta \sum_{h=1}^{m} p_h} - \eta = \frac{(1-\delta \sum_{h=1}^{m} p_h)\sum_{h=m+1}^{m+b} z_h}{b - \delta \sum_{h=1}^{m} p_h} - \eta$. Note that $U_i(z) - U_i(z') = -\frac{(1-\delta)\eta}{(1-\delta \sum_{h=1}^{m} p_h)}$. Thus, for small enough $\eta > 0$ it follows that $U_i(y) < U_i(z') < U_i(z)$ and $U_j(z') > U_j(z) \geq \widehat{U}_j(q, \widehat{\pi})$ for all $j \in \{m+1, \ldots, n\} \setminus \{i\}$, leading to a contradiction as in the previous step.

In both cases we arrived at a contradiction due to the absurd hypothesis that GrM_i is not closed. Hence, M_i is upper-hemicontinuous and the proof of the step is complete.

The second part of the lemma is proved by Step 3.2, and we are now ready to prove the first part. Since $B_i(q, \widehat{\pi}) = \mathcal{P}[M_i(q, \widehat{\pi})]$, B_i and B are non-empty, upper-hemicontinuous, and convex valued by Theorem 16.14 of Aliprantis and Border (1999), page 530. Thus, a fixed point exists by the theorem of Glicksberg (1952). □

Proof of Lemma 4 Throughout assume that in the case of assumption (A2), $\varepsilon > 0$ is small enough so that the conclusions of Lemmas 2 and 3 hold. Fix $q \in \Delta_0^{m-1}$ and a fixed point $\widehat{\pi}^* \in B(q, \widehat{\pi}^*)$. Define

$$d_h(q, \widehat{\pi}^*) = u^{-1}((1-\delta)^{-1} \max\{\widehat{U}_h(q, \widehat{\pi}^*) - \delta p_h, 0\}).$$

Assume, without loss of generality, that $d_{h+1}(q, \widehat{\pi}^*) \geq d_h(q, \widehat{\pi}^*)$, $h = 1, \ldots, n-1$. We will show a number of steps.

Step 4.1 *Assume* (A1) *and* $d_h(q, \widehat{\pi}^*) = 0$ *for some* h. *Then* $\widehat{\pi}_i^*(\{x \in \Delta_m^{n-1} \mid x_h = 0\}) = 1$ *for all* $i \neq h$. This follows from the second part of Lemma 3.

Next we show a similar result for the case in which assumption (A2) holds.

Step 4.2 *Assume* (A2) *and* $d_h(q, \widehat{\pi}^*) > d_{m+1}(q, \widehat{\pi}^*)$ *for some* h. *Then* $\widehat{\pi}_i^*(\{x \in \Delta_m^{n-1} \mid x_h = 0\}) = 1$ *for all* $i \neq h$. Assume a proposer $i \neq h$, $y \in \arg\max\{U_i(x) \mid x \in \widehat{W}_i(q, \widehat{\pi}^*)\}$ and assume $h \in \arg\max\{d_k(q, \widehat{\pi}^*) \mid k \neq i, y_k > 0\}$ is such that

$d_h(q, \widehat{\pi}^*) > d_{m+1}(q, \widehat{\pi}^*)$ to get a contradiction. Relabel players if necessary so that $y_{j+1} \geq y_j$, $j = 1, \ldots, n-1$, maintaining i as an index for the proposer and h as an index for the player with $d_h(q, \widehat{\pi}^*) > d_{m+1}(q, \widehat{\pi}^*)$ and $y_h > 0$. Let b satisfy (8) and (9) for y. By Lemma 2 we conclude that $y_i > \widehat{z}$ so that $i > m + b$. Define $K = \{k \in N \setminus \{i\} \mid U_k(y) \geq \widehat{U}_k(q, \widehat{\pi}^*)\}$. We now distinguish four cases:

Case 1, $U_h(y) < \widehat{U}_h(q, \widehat{\pi}^*)$ or $|K| > m$, and $h > m + b$: Construct y' such that $y'_h = z_b(y) < y_h$, $y'_i = y_i + (y_h - z_b(y))$, $y'_j = y_j$ for all $j \neq i, h$. We now conclude from (12) that $U_j(y') \geq U_j(y)$ for all $j \neq i, h$ so that $y' \in \widehat{W}_i(q, \widehat{\pi}^*)$, and $U_i(y') > U_i(y)$, contradicting the optimality of y.

Case 2, $U_h(y) < \widehat{U}_h(q, \widehat{\pi}^*)$ or $|K| > m$, and $h \leq m + b$: Let $Y = \sum_{k=m+1}^{m+b} y_k$. If $b > 1$, since $b \leq m$ we deduce from A2 when $\varepsilon \leq \frac{1-\delta\frac{m}{n}}{b-\delta\frac{m}{n}}$ that

$$u\left(\frac{Y}{b-1}\right) \geq \frac{Y}{b-1} > \frac{Y(1+\varepsilon)}{b-\delta\frac{m}{n}} \geq \frac{bu(\frac{Y}{b})}{b-\delta\frac{m}{n}} \geq u(z_b(y)).$$

Thus, we can construct y' such that $y'_h = 0$, $y'_j = z_b(y)$ for any $j \in \{m+1, \ldots, m+b\} \setminus \{h\}$, $y'_i = y_i + (Y - (b-1)z_b(y)) > y_i$, and $y'_j = y_j$ for all remaining players. We now conclude from (12) that $U_j(y') > U_j(y)$ for all $j \neq i, h$ so that $y' \in \widehat{W}_i(q, \widehat{\pi}^*)$, and $U_i(y') > U_i(y)$, contradicting the optimality of y.

Case 3, $U_h(y) \geq \widehat{U}_h(q, \widehat{\pi}^*)$, $|K| = m$, and $h > m + b$: Then, since $|\{k \mid d_k(q, \widehat{\pi}^*) < d_h(q, \widehat{\pi}^*)\}| \geq m + 1$, there exists $j \notin K$ with $U_j(y) < \widehat{U}_j(q, \widehat{\pi}^*)$ and $d_j(q, \widehat{\pi}^*) < d_h(q, \widehat{\pi}^*)$. For small enough $\eta > 0$, we can construct y' such that $y'_h = 0$, $y'_i = y_i + \eta$, $y'_j = y_h - \eta$, and $y'_k = y_k$ for all $k \neq i, j, h$. We now conclude from (12) that $U_k(y') = U_k(y)$ for all $k \neq i, j, h$ and $U_j(y) \geq \widehat{U}_j(q, \widehat{\pi}^*)$, so that $y' \in \widehat{W}_i(q, \widehat{\pi}^*)$, and $U_i(y') > U_i(y)$, contradicting the optimality of y.

Case 4, $U_h(y) \geq \widehat{U}_h(q, \widehat{\pi}^*)$, $|K| = m$, and $h \leq m+b$: Then, there exists $S \subseteq \{k \neq i \mid d_k(q, \widehat{\pi}^*) < d_h(q, \widehat{\pi}^*)\}$ with $|S| = m$. Construct y' such that $y'_i = y_i + \eta$, $y'_j = \frac{(1-y_i-\eta)}{m}$ for all $j \in S$, and $y'_j = 0$ otherwise. Since $|S| = m$ and $0 \leq U_h(y) - \widehat{U}_h(q, \widehat{\pi}^*) < U_j(y') - \widehat{U}_j(q, \widehat{\pi}^*)$ for all $j \in S$ when $\eta > 0$ is small enough, we conclude from (12) that $y' \in \widehat{W}_i(q, \widehat{\pi}^*)$ and $U_i(y') > U_i(y)$, once more contradicting the optimality of y.

For the next step, define $M = \{m-a+1, \ldots, m+1, \ldots, m+b\} = \{j \mid d_j(q, \widehat{\pi}^*) = d_{m+1}(q, \widehat{\pi}^*)\}$, and note that $|M| = a + b$.

Step 4.3 *If (A2) holds and* $\varepsilon \leq \frac{1-\frac{\delta(n-1)}{n}}{m+1-\frac{\delta m}{n}}$, *then*

$$u(d_{m+1}(q, \widehat{\pi}^*)) \leq \frac{bu\left(\frac{\sum_{j\in M} q_j}{b}\right)}{b-\delta\frac{m}{n}}.$$

Obvious if $d_{m+1}(q, \widehat{\pi}^*) = 0$. If $d_{m+1}(q, \widehat{\pi}^*) > 0$, then note that by Step 4.2, any proposer $i \in \{1, \ldots, m-a\}$ must obtain the votes of $a+1$ players from M, while the remaining proposers need the votes of a players from $M \setminus \{i\}$. By Lemma 2, proposer i optimizes offering y such that $y_i > \widehat{z}$, so that (12) implies that i's optimization amounts to maximizing the allocation y_i and that any player $j \in M$ such that $y_j > 0$ must receive expected utility exactly equal to $\widehat{U}_j(q, \widehat{\pi}^*) = (1-\delta)u(d_{m+1}(q, \widehat{\pi}^*)) + \frac{\delta}{n}$. As a result, we conclude that

$$\sum_{j\in M} \widehat{U}_j(q, \widehat{\pi}^*)$$
$$\leq (1-\delta)\sum_{j\in M} u(q_j) + \delta\left(\tfrac{a+b}{n}\widehat{U}_p + \tfrac{(n-1)a+m}{n}\widehat{U}_j(q, \widehat{\pi}^*) + \tfrac{b(n-1)-m}{n}\widehat{U}_e\right)$$
$$= (1-\delta)\sum_{j\in M} u(q_j) + \delta\tfrac{(n-1)a+m}{n}u(d_{m+1}(q, \widehat{\pi}^*)) + \tfrac{(a+b)\delta}{n},$$

where $\widehat{U}_p = (1-\delta) + \frac{\delta}{n}$ is the maximum that any $j \in M$ can extract as a proposer and $\widehat{U}_e = \frac{\delta}{n}$ is the maximum that any $j \in M$ can expect when an allocation y prevails in which $y_j = 0$. Since $\sum_{j\in M} \widehat{U}_j(q, \widehat{\pi}^*) = (1-\delta)(a+b)u(d_{m+1}(q, \widehat{\pi}^*)) + \frac{(a+b)\delta}{n}$, we obtain that

$$u(d_{m+1}(q, \widehat{\pi}^*)) \leq \frac{\sum_{j\in M} u(q_j)}{a+b-\delta\frac{(n-1)a+m}{n}} \leq \frac{(a+b)u\left(\frac{\sum_{j\in M} q_j}{a+b}\right)}{a+b-\delta\frac{(n-1)a+m}{n}}.$$

This proves the step if $a = 0$. If $a \geq 1$, then note that by assumption (A2), since $\varepsilon \leq \frac{1-\frac{\delta(n-1)}{n}}{m+1-\frac{\delta m}{n}}$, we conclude

$$\frac{(a+b)u\left(\frac{\sum_{j\in M} q_j}{a+b}\right)}{a+b-\delta\frac{(n-1)a+m}{n}} \leq \frac{(1+\varepsilon)\sum_{j\in M} q_j}{a+b-\delta\frac{(n-1)a+m}{n}} \leq \frac{\sum_{j\in M} q_j}{b-\delta\frac{m}{n}} \leq \frac{bu\left(\frac{\sum_{j\in M} q_j}{b}\right)}{b-\delta\frac{m}{n}}$$

for all a, b, as desired.

The next step essentially establishes a bound on the cost of coalition building when (A1) holds.

Step 4.4 *If (A1) holds, then for all coalitions $C \subset N$ with $|C| = m+1$,*

$$\sum_{i\in C} d_i(q, \widehat{\pi}^*) - \delta \sum_{i\notin C} p_i d_\ell(q, \widehat{\pi}^*) \leq 1,$$

where $\ell = \min\{i \mid i \in C\}$. Choose any majority coalition $C \subset N$ with $|C| = m+1$. Define l as $l = \min\{i \in N \mid d_i(q, \widehat{\pi}^*) > 0\}$. We have $\sum_{i=1}^n (U_i(q, \widehat{\pi}^*) - \delta p_i) = 1-\delta$, thus, if $l = 1$ we must have $\sum_{i\in C} d_i(q, \widehat{\pi}^*) \leq 1$. Hence, to show $\sum_{i\in C} d_i(q, \widehat{\pi}^*) - \delta\sum_{i\notin C} p_i d_\ell(q, \widehat{\pi}^*) < 1$, it remains to consider the case $l > 1$. In this case, we define $D(q, \widehat{\pi}^*) = \sum_{i=1}^{m+1} d_i(q, \widehat{\pi}^*)$, and we deduce that

$$\widehat{U}_i(q, \widehat{\pi}^*) \geq (1-\delta)q_i + \delta(p_i(1-\delta)(1-D(q, \widehat{\pi}^*)) + \delta p_i), i = 1, \ldots, l-1 \quad (20)$$

To see why (20) holds, note that all players other than i propose alternatives in Δ_{m+1}^{n-1} and allocate zero to i by Lemma 3, so that i obtains utility δp_i with probability $(1-p_i)$. Also, i can secure utility of at least $(1-\delta)(1-D(q,\widehat{\pi}^*))+\delta p_i$ when proposing with probability p_i, simply by allocating $d_j(q,\widehat{\pi}^*)$ to all $j\in\{1,\ldots,m+1\}\backslash\{i\}$, since $D(q,\widehat{\pi}^*)<1$ by Step 2.2 of Lemma 2. By summing both sides of (20) for $i=1,\ldots,l-1$ and rearranging terms, we get

$$\sum_{i=1}^{l-1}(\widehat{U}_i(q,\widehat{\pi}^*)-\delta p_i)\geq(1-\delta)\sum_{i=1}^{l-1}x_i-\delta(1-\delta)D(q,\widehat{\pi}^*)\sum_{i=1}^{l-1}p_i. \qquad (21)$$

Since $j>i\Rightarrow d_j(q,\widehat{\pi}^*)\geq d_i(q,\widehat{\pi}^*)$, we also have

$$d_h(q,\widehat{\pi}^*)+\sum_{i\notin C}d_i(q,\widehat{\pi}^*)\geq D(q,\widehat{\pi}^*)=\sum_{i=1}^{m+1}d_i(q,\widehat{\pi}^*). \qquad (22)$$

If $\ell\geq l$, we have $\sum_{i=1}^{l-1}p_i\leq\sum_{i\notin C}p_i$, while if $\ell<l$, $d_\ell(q,\widehat{\pi}^*)=0$. In either case we deduce from (22) that

$$\sum_{i\notin C}p_i d_\ell(q,\widehat{\pi}^*)+\sum_{i\notin C}d_i(q,\widehat{\pi}^*)\geq\sum_{i=1}^{l-1}p_i D(q,\widehat{\pi}^*).$$

Since $0<\delta<1$ and $\sum_{i=1}^{l-1}q_i\geq 0$, the above implies

$$\sum_{i=1}^{l-1}q_i+\delta\sum_{i\notin C}p_i d_\ell(q,\widehat{\pi}^*)+\sum_{i\notin C}d_i(q,\widehat{\pi}^*)\geq\delta D(q,\widehat{\pi}^*)\sum_{i=1}^{l-1}p_i.$$

Adding $\sum_{i\in C}d_i(q,\widehat{\pi}^*)$ on both sides and rearranging terms, this is equivalent to

$$\sum_{i=1}^{l-1}q_i-\delta D(q,\widehat{\pi}^*)\sum_{i=1}^{l-1}p_i+\sum_{i=1}^{n}d_i(q,\widehat{\pi}^*)\geq\sum_{i\in C}d_i(q,\widehat{\pi}^*)-\delta\sum_{i\notin C}p_i d_\ell(q,\widehat{\pi}^*).$$

Since we have that $\sum_{i=1}^{l-1}d_i(q,\widehat{\pi}^*)=0$, and from (21), we deduce

$$(1-\delta)^{-1}\sum_{i=1}^{l-1}\left(\widehat{U}_i(q,\widehat{\pi}^*)-\delta p_i\right)+\sum_{i=l}^{n}d_i(q,\widehat{\pi}^*)$$
$$\geq\sum_{i\in C}d_i(q,\widehat{\pi}^*)-\delta\sum_{i\notin C}p_i d_\ell(q,\widehat{\pi}^*).$$

But the left-hand side is equal to 1, by the fact that $d_i(q,\widehat{\pi}^*)=(1-\delta)^{-1}(\widehat{U}_i(q,\widehat{\pi}^*)-\delta p_i)$ for $i=l,\ldots,n$, and since $(1-\delta)^{-1}\sum_{i=1}^{n}(\widehat{U}_i(q,\widehat{\pi}^*)-\delta p_i)=1$. Thus, $\sum_{i\in C}d_i(q,\widehat{\pi}^*)-\delta(\sum_{i\notin C}p_i)d_\ell(q,\widehat{\pi}^*)\leq 1$, proving the step.

It is now possible to prove the lemma by constructing the required allocation x in the two cases:

Case (A1): We construct x with

$$x_j = \begin{cases} d_j(q, \widehat{\pi}^*) & \text{if } j \in C \setminus \{\ell\} \\ 1 - \sum_{i \in C \setminus \{\ell\}} d_i(q, \widehat{\pi}^*) & \text{if } \quad j = \ell \\ 0 & \text{otherwise.} \end{cases}$$

By Step 4.4, $x_\ell = 1 - \sum_{i \in C \setminus \{\ell\}} d_i(q, \widehat{\pi}^*) \geq (1 - \delta \sum_{i \notin C} p_i) d_\ell(q, \widehat{\pi}^*)$ and it is immediate from (12), (7), and the definition of d_j that $U_j(x) \geq \widehat{U}_j(q, \widehat{\pi}^*)$ for all $j \in C$.

Case (A2): In this case, due to the anonymity of (12) from the fact that $p_i = \frac{1}{n}$, it suffices to show the existence of the required allocation for the most expensive coalition $C = \{m+1, \ldots, n\}$. We construct $x \in \Delta_m$ with

$$x_j = \begin{cases} q_j & \text{if } j \in \{m+b+1, \ldots, n\} \\ \frac{\sum_{i=1}^{m+b} q_i}{b} & \text{if } j \in \{m+1, \ldots, m+b\} \\ 0 & \text{otherwise.} \end{cases}$$

By Step 4.3 we conclude that $U_j(x) \geq \widehat{U}_j(q, \widehat{\pi}^*)$ for all $j \in \{m+1, \ldots, m+b\}$ since $\frac{\sum_{i=1}^{m+b} u(x_i)}{b - \frac{\delta m}{n}} \geq \frac{bu\left(\frac{\sum_{j \in M} q_j}{b}\right)}{b - \delta \frac{m}{n}} \geq d_j(q, \widehat{\pi}^*)$. Similarly, by Step 4.2 we conclude that $U_j(x) \geq \widehat{U}_j(q, \widehat{\pi}^*)$ for all $j \in \{m+b+1, \ldots, n\}$ since in that case $q_j \geq d_j(q, \widehat{\pi}^*)$. □

Proof of Theorem 1 In the case of assumption (A2), assume ε so that the conclusions of Lemmas 2, 3, and 4 hold. Note that both (10) and (11) are satisfied when (A1) holds, or for small enough ε when (A2) holds. Define the correspondence $B^* : \Delta_0^{m-1} \rightrightarrows \widehat{\Pi}$ that maps status quo $q \in \Delta_0^{m-1}$ to the fixed points $\widehat{\pi}^* \in \widehat{\Pi}$ of $B(q, \widehat{\pi})$. Thus, combining a selector (which can be assumed measurable by an application of the Kuratowski-Ryll-Nardzewski selection theorem) from B^* with the proposal strategies of Proposition 1, we obtain proposal strategies $\pi_i^* : \Delta \to \mathcal{P}[\Delta]$ for each player i. From these proposal strategies we calculate expected payoffs $U_i(x; \sigma^*)$, $x \in \Delta$ for each i, in accordance with Eq. 2. In particular, $U_i(x; \sigma^*)$ coincides with $U_i(x)$ defined in (12) for all $x \in \Delta_m^{n-1}$, so that we can trivially compute $U_i(x; \sigma^*)$ for $x \in \Delta_0^{m-1}$, in which case $U_i(x; \sigma^*) = \hat{U}_i(x, \pi^*)$ for the selected fixed point $\pi^* \in B^*(x)$. Using these expected payoffs $U_i(x; \sigma^*)$, we obtain voting strategies A_i^* that satisfy condition (3) for all i. Thus, it remains to show that proposal strategies π_i^* satisfy condition (4) for all i. But this follows from Lemma 4 for status quo $q \in \Delta_0^{m-1}$ and from Proposition 1 and Lemma 4 for status quo $q \in \Delta_m^{n-1}$. □

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