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Large auctions with risk-averse bidders

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Abstract We study private-value auctions with n risk-averse bidders, where n is large. We first use asymptotic analysis techniques to calculate explicit approximations of the equilibrium bids and of the seller's revenue in any k-price auction ($k = 1, 2, \ldots$). These explicit approximations show that in all large k-price auctions the effect of risk-aversion is $O(1/n^2)$ small. Hence, all large k-price auctions with risk-averse bidders are $O(1/n^2)$ revenue equivalent. The generalization, that all large auctions are $O(1/n^2)$ revenue equivalent, is false. Indeed, we show that there exist auction mechanisms for which the limiting revenue as $n \longrightarrow \infty$ with risk-averse bidders is strictly below the risk-neutral limit. Therefore, these auction mechanisms are not revenue equivalent to large k-price auctions even to leading-order as $n \longrightarrow \infty$.

Keywords Large auctions · Risk aversion · Asymptotic analysis · Revenue equivalence · Equilibrium strategy

1 Introduction

Many auctions, such as those that appear on the Internet, have a large number of bidders. The standard approach to study large auctions has been to consider their limit as n, the number of bidders, approaches infinity. Using this approach, it has been shown for private-value auctions under quite general conditions that as n goes to infinity, the

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equilibrium bid approaches the true value, the seller's expected revenue approaches the maximal possible value, and the auction becomes efficient (Wilson 1977; Pesendorfer and Swinkels 1997; Kremer 2002; Swinkels 1999, 2001; Bali and Jackson 2002). Most of the studies that adopted this approach, however, do not provide the rate of convergence to the limit. Therefore, it is unclear how large n should be (5, 10, 100?) in order for the auction "to be large" i.e., in order for the limiting results for $n = \infty$ to be applicable. Convergence rates were obtained by Satterthwaite and Williams (1989), who showed that the rate of convergence of the bid to the true value in a double auction is $O(1/m)$, where m is the number of traders on each side of the market, by Rustichini et al. (1994), who showed that the rate of convergence of the bid to the true value in a k-double auction is $O(1/m)$ and the corresponding inefficiency is $O(1/m^2)$, and by Hong and Shum (2004), who calculated the convergence rate in common-value multi-unit first-price auctions.

In this paper, we extend the study of large auctions in two ways:

1. We allow bidders to be risk-averse, rather than risk neutral. Our results show that risk-aversion has a small effect in all large k-price auctions. Surprisingly, however, this is not the case for all large auctions. Indeed, we show that there exist other auction mechanisms for which the effect of risk-aversion does not become negligible as $n \longrightarrow \infty$.
2. We use asymptotic analysis techniques in order *to go beyond rate-of-convergence results*, i.e., we explicitly calculate the $O(1/n)$ correction term in the expressions for the equilibrium bids and the seller's revenue. Since our explicit asymptotic approximations include both the limiting value and the $O(1/n)$ correction term, they are $O(1/n^2)$ accurate. Hence, the number of bidders at which they become valid is considerably smaller than the number of bidders at which the limiting values (which are only $O(1/n)$ accurate) become valid. Roughly speaking, if we require 1% accuracy, than our $O(1/n^2)$ asymptotic approximations are already valid for $n = 10$ bidders, whereas the limiting-value approximations become valid only for $n = 100$ bidders.

The paper is organized as follows. In Sect. 2, we introduce the model of symmetric private-value auctions with risk-averse bidders. In Sect. 3, we calculate asymptotic approximations of the equilibrium bids and of the seller's revenue in large first-price auctions with risk-averse bidders. This calculation shows that the differences in the equilibrium bids and in the seller's revenue between risk-neutral and risk-averse bidders are only $O(1/n^2)$.

One measure of a 'good' asymptotic technique is that it can be used, at least in theory, to calculate as many terms in the expansion as desired. To show that this is the case here, we calculate explicitly the next-order, $O(1/n^2)$ terms in the expressions for the equilibrium bids and the revenue. This calculation shows that the $O(1/n^2)$ affect of risk aversion is proportional to the Arrow–Pratt measure of risk aversion at zero. In addition, this calculation provides an analytic estimate for the value of the constant of the $O(1/n^2)$ error term.

In Sect. 3.1, we present numerical examples that suggest that the asymptotic approximations derived in this study are quite accurate even for auctions with as few as $n = 6$ bidders. Although we present only a few numerical examples, we note that the parameters of these examples were chosen "at random", and that we observed the same behavior in numerous other examples that we tested.[1]

In Sect. 4, we calculate asymptotic approximations of the equilibrium bids and of the seller's revenue in large symmetric k-price auctions ($k = 3, 4, \ldots$) with risk-averse bidders. As in the case of first-price auctions, this calculation shows that the differences in the equilibrium bids and in the seller's revenue between risk-neutral and risk-averse bidders are only $O(1/n^2)$. Since in the risk-neutral case all k-price auctions are revenue equivalent, we conclude that all large k-price auctions ($k = 1, 2, \ldots$) with risk-averse or risk-neutral bidders are $O(1/n^2)$ revenue equivalent.[2]

Since the revenue differences among all large k-price auctions with n risk-averse bidders are $O(1/n^2)$, it seems natural to conjecture that this result should extend to all incentive-compatible and individually-rational mechanisms that deliver efficient allocations. This, however, is not the case. Indeed, in Sect. 5 we prove that the limiting revenue as $n \longrightarrow \infty$ in *generalized all-pay auctions*[3] with risk-averse bidders is strictly below the risk-neutral limit, and in Sect. 6 we show that this also true for last-price auctions.[4] Therefore, unlike large k-price auctions where risk-aversion has only an $O(1/n^2)$ effect on the revenue, in the case of large all-pay and last price auctions risk-aversion has an $O(1)$ effect on the revenue. To the best of our knowledge, these are the first examples of private-value auctions whose limiting revenue is not equal to the risk-neutral limit.

The above results raise the question of whether there is a condition that would imply that the limiting revenue with risk-averse bidders is equal to the risk-neutral limit. In Sect. 7 we prove that if the equilibrium payment of the winning bidder approaches his type as $n \longrightarrow \infty$ uniformly for all types, then the limiting revenue with risk-averse bidders is equal to the risk-neutral limit (Proposition 6). We then show that it is sufficient for this condition to hold only at an $O(1/n)$ neighborhood of the highest type.

In Sect. 8 we discuss the advantages and disadvantages of the applied math approach used in Sects. 3 and 4. The Appendix contains proofs omitted from the main body of the paper.

Finally, we note that this paper differs from our previous work, in which we used perturbation analysis techniques to analyze auctions with weakly asymmetric bidders (Fibich and Gavious 2003; Fibich et al. 2004) and with weakly risk-averse bidders (Fibich et al. 2006), in two important ways:

1. In those papers we had to assume that the level of risk-aversion (or asymmetry) is small. The results of this paper are stronger, since no such assumption is made.
2. In those papers we used perturbation techniques that "essentially" amount to Taylor expansions in a small parameter that lead to convergent sums. In contrast,

[1] The fact that an expansion for large n is already valid for $n = 6$ may be surprising to researchers not familiar with asymptotic expansions. However, quite often, this is the case with asymptotic expansions (see e.g., Bender and Orszag 1978).

[2] Recall that revenue equivalence fails under risk aversion (Maskin and Riley 1984; Matthews 1987).

[3] I.e., when the highest bidder wins the object and pays his bid, and the losing bidders pay a fixed fraction of their bids.

[4] I.e., when the highest bidder wins the object and pays the lowest bid.

here we use asymptotic methods (e.g., Laplace method for evaluation of integrals) which typically lead to divergent sums if carried out to all orders (see, e.g., Murray 1984). To the best of our knowledge, these asymptotic methods have not been used in auction theory so far. It is quite likely that these and other asymptotic methods (WKB, method of steepest descent, etc.) will be useful in the asymptotic analysis of other models, e.g., multi-unit auctions with many units (Jackson and Kremer 2004, 2006).

2 The model

Consider a large number ($n \gg 1$) of bidders vying for a single object. Bidder i's valuation v_i is a private information, and bidders are symmetric such that for any $i = 1, \dots, n$, v_i is independently distributed according to a common distribution function $F(v)$ on the interval [0, 1]. Denote by $f = F'$ the corresponding density function. We assume that F is twice continuously differentiable and that $f > 0$ in [0, 1]. We assume that each bidder's utility is given by a function $U(v - b)$, which is twice continuously differentiable, monotonically increasing, concave, and normalized to have a zero utility at zero, i.e.,

$$U(x) \in \mathcal{C}^2, \quad U(0) = 0, \quad U'(x) > 0, \quad U''(x) < 0. \tag{1}$$

3 First-price auctions

Consider a first-price auction with risk-averse bidders, in which the bidder with the highest bid wins the object and pays his bid. In this case, the inverse equilibrium bids satisfy the ordinary-differential equation[5]

$$v'(b) = \frac{1}{n-1} \frac{F(v(b))}{f(v(b))} \frac{U'(v(b) - b)}{U(v(b) - b)}, \quad v(0) = 0. \tag{2}$$

Unlike the risk-neutral case, there are no explicit formulae for the equilibrium bids and for the revenue, except in special cases. Recently, Fibich et al. (2006) obtained explicit approximations of the equilibrium bids for the case of weak risk aversion. Here we relax the assumption that risk aversion is weak.

Proposition 1 *Consider a symmetric first-price auction with n bidders with utility function $U(x)$ that satisfies (1). Then, the equilibrium bid for sufficiently large n is given by*

$$b(v) = v - \frac{1}{n-1} \frac{F(v)}{f(v)} + O\left(\frac{1}{n^2}\right), \tag{3}$$

[5] Under the conditions of Sect. 2, existence of a symmetric equilibrium follows from Maskin and Riley (1984).

and the seller's expected revenue is given by

$$R_n^{ra} = 1 - \frac{2}{n}\frac{1}{f(1)} + O\left(\frac{1}{n^2}\right). \tag{4}$$

Proof Since $\lim_{n\to\infty} v(b) = b$, we can look for a solution of (2) of the form

$$v(b) = b + \frac{1}{n-1}v_1(b) + O\left(\frac{1}{n^2}\right).$$

Substitution in (2) gives

$$1 + O\left(\frac{1}{n}\right) = \frac{1}{n-1}\frac{F(b) + (v_1/(n-1))f(b) + O(n^{-2})}{f(b) + (v_1/(n-1))f'(b) + O(n^{-2})} \times \frac{U'(0) + (v_1/(n-1))U''(0) + O(n^{-2})}{U(0) + (v_1/(n-1))U'(0) + O(n^{-2})}.$$

Since $U(0) = 0$ and $U'(0) > 0$, the balance of the leading order terms gives

$$1 = \frac{F(b)}{f(b)} \cdot \frac{U'(0)}{v_1 U'(0)}.$$

Therefore, $v_1(b) = F(b)/f(b)$ and the inverse equilibrium bids are given by

$$v(b) = b + \frac{1}{n-1}\frac{F(b)}{f(b)} + O\left(\frac{1}{n^2}\right).$$

Inverting this relation (see Lemma 1) shows that the equilibrium bids are given by (3).

To calculate the expected revenue, we use (3) to obtain

$$R_n^{ra} = \int_0^1 b(v)\,dF^n(v) = b(1) - \int_0^1 b'(v)F^n(v)\,dv$$
$$= 1 - \frac{1}{n}\frac{1}{f(1)} + O\left(\frac{1}{n^2}\right) - \int_0^1 [1 + O(1/n)]F^n(v)\,dv.$$

Therefore, by Lemma 2 (see Appendix A), the result follows. □

It is worth noting here the power of this asymptotic analysis approach. While it is not possible to determine the exact expression for the equilibrium bids, it only required a few lines of calculation to obtain an expression with an $O(1/n^2)$ accuracy. We note that Caserta and de Vries (2002) used extreme value theory to derive an asymptotic expression for the revenue which is equivalent to (4). However, the result of Caserta and de Vries (2002) holds only in the risk-neutral case, where an explicit expression for the revenue is available. In addition, the calculation using extreme value theory requires considerably more work.

Since the utility function $U(x)$ does not appear in the $O(1/n)$ term, Proposition 1 shows that the differences in equilibrium bids and in the seller's revenue between first-price auctions with risk-neutral and with risk-averse bidders are at most $O(1/n^2)$. In other words, risk aversion has (at most) an $O(1/n^2)$ effect on the equilibrium bids and on the revenues in symmetric first-price auctions.

Proposition 1 raises two questions:

1. Is the effect of risk-aversion truly $O(1/n^2)$, or is it even smaller?
2. Can we estimate the constants in the $O(1/n^2)$ error terms?

We answer these questions by calculating explicitly the $O(1/n^2)$ terms:

Proposition 2 *Consider a symmetric first-price auction with n bidders whose utility function $U(x)$ satisfy (1). Then, the equilibrium bid for sufficiently large n is given by*

$$b(v) = v - \frac{1}{n-1}\frac{F(v)}{f(v)} + \frac{1}{(n-1)^2}\left[\frac{F(v)}{f(v)} - \frac{F^2(v)f'(v)}{f^3(v)} - \frac{F^2(v)}{2f^2(v)}\frac{U''(0)}{U'(0)}\right] + O\left(\frac{1}{n^3}\right), \tag{5}$$

and the seller's expected revenue is given by

$$R_n^{ra} = 1 - \frac{1}{n}\frac{2}{f(1)} + \frac{1}{n^2}\left[\frac{2}{f(1)} - \frac{3f'(1)}{f^3(1)} - \frac{1}{2f^2(1)}\frac{U''(0)}{U'(0)}\right] + O\left(\frac{1}{n^3}\right). \tag{6}$$

Proof The proof is the same as for Proposition 1, except that one has to keep also the $O(1/(n-1)^2)$ terms, see Appendix B. □

Thus, risk-aversion has an $O(1/n^2)$ effect on the bid when $U''(0) \neq 0$, but a smaller effect if $U''(0) = 0$. As expected, the bids and revenue increase (decrease) for risk-averse (risk-loving) bidders. Note that the magnitude of risk-aversion effect is determined, to leading-order, by $-U''(0)/U'(0)$, i.e., by the value of the Arrow–Pratt absolute risk-aversion at zero.[6]

The observation that risk-aversion has a small effect on the revenue in large first-price auctions has the following intuitive explanation. Since in large first-price auctions the bids are close to the values, one can approximate $U(v-b) \approx (v-b)U'(0)$, which is the risk-neutral case. Similarly, adding the next term in the Taylor expansion gives

$$U(v-b) \approx U'(0)\left[(v-b) + \frac{(v-b)^2}{2}\frac{U''(0)}{U'(0)}\right]. \tag{7}$$

Hence, for large n, the leading-order effect of risk-aversion is proportional to $-U''(0)/U'(0)$.

[6] Although we assume in (1) that bidders are risk averse, the results of this section hold for risk loving bidders as well.

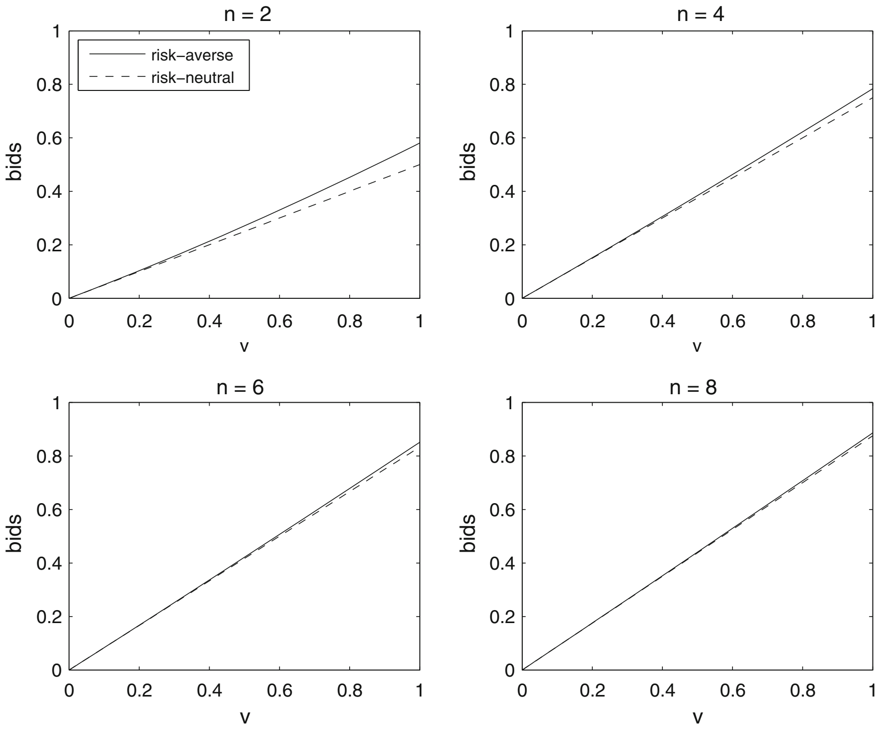

Fig. 1 Equilibrium bids in a first-price auction with risk-averse (*solid*) and risk-neutral (*dashes*) bidders

3.1 Examples

Consider a first-price auction where bidders' valuations are uniformly distributed on [0, 1], i.e., $F(v) = v$. Assume first that each bidder has a CARA utility function $U(x) = 1 - e^{-\lambda x}$ where $\lambda > 0$. In Fig. 1 we compare the (exact) equilibrium bid[7] for $\lambda = 2$, with the equilibrium bid in the risk-neutral case, for $n = 2, 4, 6$, and 8 bidders.[8] Already for $n = 6$ bidders, the equilibrium bids in the risk-neutral and risk averse cases are almost identical. This observation is consistent with Dyer et al. (1989), who found in experiments that in a first-price auction the actual highest bid was much higher than the theoretical risk-neutral equilibrium bid with three players, but very close to the risk-neutral one with six players.

Next, we consider the revenue in a first-price auction with $n = 6$ risk-averse players, denoted by R_n^{ra} (where $n = 6$). Recall that when $F(v) = v$, the revenue in the risk-neutral case with n players is equal to $R_n^{rn} = (n-1)/(n+1)$. Therefore, in the case of six players, $R_6^{rn} = 5/7 \approx 0.7143$. In Table 1 we give the value of R_6^{ra} and the relative change in the revenue due to risk-aversion for four different utility

[7] Namely, the numerical solution of Eq. 2.

[8] Observe that as $\lambda \to 0$, $U(x) \sim \lambda x$, i.e., the utility of risk-neutral bidders. Therefore, $\lambda = 2$ corresponds to a significant deviation from risk-neutrality.

Table 1 Expected revenue in a symmetric first-price auction with six players with a utility function $U(x)$

$U(x)$	R_6^{ra}	$\frac{R_6^{ra}-R_6^{rn}}{R_6^{ra}}$
$x - x^2/2$	0.7220	1.08%
$\ln(1+x)$	0.7209	0.92%
CARA ($\lambda = 1$)	0.7214	0.99%
CARA ($\lambda = 2$)	0.7278	1.89%

Table 2 Expected revenue in a symmetric first-price auction with two players with a utility function $U(x)$

$U(x)$	R_2^{ra}	$\frac{R_2^{ra}-R_2^{rn}}{R_2^{rn}}$
$x - x^2/2$	0.3592	7.7%
$\ln(1+x)$	0.3508	5.25%
CARA ($\lambda = 1$)	0.3541	6.2%
CARA ($\lambda = 2$)	0.3741	12.2%

functions. The first thing to observe is that in all four cases the effect of risk-aversion is small (less that 2%), even though the number of players is not really large, and the utility functions are not close to risk-neutrality. The second thing to observe is that in the first three cases the effect of risk-aversion on the revenue is nearly the same (≈1%), even though the three utility functions are quite different. The reason for this is that the difference between the revenue in the risk-averse and risk-neutral case is given by, see Proposition 2,

$$R_n^{ra} - R_n^{rn} \approx -\frac{1}{n^2}\frac{1}{2f^2(1)}\frac{U''(0)}{U'(0)}.$$

Hence, the effect of risk-aversion is proportional to the value of the absolute risk aversion $-U''(0)/U'(0)$. In the first three cases $-U''(0)/U'(0)$ is identical (=1), explaining why they have "the same" effect on the revenue. In the fourth case $-U''(0)/U'(0) = 2$, and indeed, the change in the revenue nearly doubles.

In Table 2, we repeat the simulations of Table 1, but with $n = 2$ players. In this case, the revenue in the risk-neutral case is equal to $R_2^{rn} = 1/3 \approx 0.3333$. As expected, the relative effect of risk-aversion is much larger than for $n = 6$ players, showing that risk-aversion cannot be neglected in small first-price auctions. In addition, as before, the effect of risk-aversion depends predominantly on $-U''(0)/U'(0)$, which is why the additional revenues due to risk aversion are roughly the same in the first three cases, but roughly double in the fourth case.

4 *k*-price auctions

Consider k-price auctions in which the bidder with the highest bid wins the auction and pays the k-th highest bid. The results of Proposition 1 can be generalized to any k-price auction as follows:

Proposition 3 *Consider a symmetric k-price auction ($k = 1, 2, 3, \ldots$) with n bidders, each with a utility function $U(x)$ that satisfies (1). Then, the equilibrium bid for sufficiently large n is*

$$b(v) = v + \frac{k-2}{n-k}\frac{F(v)}{f(v)} + O\left(\frac{1}{n^2}\right), \tag{8}$$

and the seller's expected revenue is given by (4).

Proof The case $k = 1$ was proved in Proposition 1. When $k = 2$ the result follows immediately, since $b(v) = v$. Therefore, we only need to consider $k \geq 3$. In that case, the equilibrium strategies in k-price auctions are the solutions of (see Monderer and Tennenholtz 2000)

$$\int_0^v U(v - b(t))F^{n-k}(t)(F(v) - F(t))^{k-3} f(t)\, dt = 0. \tag{9}$$

Defining $m = n - k$ and $t = v - s$, we can rewrite Eq. 9 as

$$\begin{aligned} 0 &= \int_0^v U(v - b(t))F^m(t)(F(v) - F(t))^{k-3} f(t)\, dt \\ &= \int_0^v e^{m \ln(F(t))} U(v - b(t))(F(v) - F(t))^{k-3} f(t)\, dt \\ &= e^{m \ln F(v)} \int_0^v e^{m \cdot h(s,v)}\, g(s, v)\, ds, \end{aligned} \tag{10}$$

where

$$h = \ln F(v - s) - \ln F(v), \quad g = U(v - b(v - s))(F(v) - F(v - s))^{k-3} f(v - s).$$

Since h and g are independent of m, we can calculate an asymptotic approximation of the last integral using the Laplace method for integrals (see, e.g., Murray 1984). This method is based on two key observations:

1. As $m \longrightarrow \infty$, essentially all the contribution to the integral comes from the neighborhood of the point $s_{\max}$ where $h(s)$ attains its global maximum in $[0, v]$.
2. Therefore, one can compute the integral, with exponential accuracy, by replacing h and g with their Taylor expansions around $s_{\max}$.

Since the maximum of h for $0 \leq s \leq v$ is attained at $s_{\max} = 0$, we make the change of variables $x(s) = [\ln F(v) - \ln F(v - s)]$ and expand all the terms in the last integral in a Taylor series in s near $s=0$. Expansion of $x(s)$ near $s=0$ gives $x = sf(v)/F(v) + O(s^2)$. Therefore,

$$\frac{dx}{ds} = f(v)/F(v) + O(s), \quad s = x\frac{F(v)}{f(v)} + O(s^2), \quad ds = \frac{dx}{f(v)/F(v)}\left[1 + O(x)\right].$$

Let us expand the solution $b(v)$ in a power series in m, i.e.,

$$b(v) = b_0(v) + \frac{1}{m}b_1(v) + O\left(\frac{1}{m^2}\right).$$

Therefore, near $s=0$,

$$b(v-s) = b_0(v) - sb_0'(v) + \frac{1}{m}b_1(v) - \frac{1}{m}sb_1'(v) + O(s^2) + O\left(\frac{1}{m^2}\right).$$

In addition,

$$(F(v) - F(v-s))^{k-3} = (sf(v) + O(s^2))^{k-3} = s^{k-3}f^{k-3}(v)[1 + O(s)],$$

and

$$f(v-s) = f(v) + O(s).$$

Substitution all the above in (10) gives

$$\begin{aligned}
0 &= \int_0^v \left\{e^{-mx}\, U\left[v - \left(b_0(v) - sb_0'(v) + \frac{b_1(v)}{m} - \frac{sb_1'(v)}{m} + O(s^2) + O\left(\frac{1}{m^2}\right)\right)\right]\right.\\
&\quad \left. \times\, s^{k-3}f^{k-3}(v)\left[1 + O(s)\right]\left[f(v) + O(s)\right]\right\} ds\\
&\sim \int_0^\infty \left\{e^{-mx}\, U\left[v - \left(b_0(v) - x\frac{F(v)}{f(v)}b_0'(v) + \frac{1}{m}b_1(v) - \frac{x}{m}\frac{F(v)}{f(v)}b_1'(v) + O(x^2)\right.\right.\right.\\
&\quad \left.\left. + O\left(\frac{1}{m^2}\right)\right)\right] \times x^{k-3}F^{k-3}(v)\left[1 + O(x)\right]\left[f(v) + O(x)\right]\\
&\quad \left. \times \frac{dx}{f(v)/F(v)}\left[1 + O(x)\right]\right\}\\
&= F^{k-2}(v)\int_0^\infty \left\{e^{-mx}\left[U(v - b_0(v)) + U'(v - b_0(v))\left(x\frac{F(v)}{f(v)}b_0'(v) - \frac{b_1(v)}{m}\right.\right.\right.\\
&\quad \left.\left.\left. + \frac{x}{m}\frac{F(v)}{f(v)}b_1'(v)\right) + O(x^2) + O\left(\frac{1}{m^2}\right)\right] x^{k-3}\left[1 + O(x)\right]\right\} dx. \qquad (11)
\end{aligned}$$

We recall that for p integer, $\int_0^\infty e^{-mx} x^p \, dx = p!/m^{p+1} \neq 0$. Therefore, balancing the leading $O(m^{-(k-2)})$ terms gives

$$U(v - b_0(v)) F^{k-2}(v) \int_0^\infty e^{-mx} \, x^{k-3} \, dx = 0.$$

Since $U(z) = 0$ only at $z = 0$, this implies that $b_0(v) \equiv v$. Using this and $U'(0) \neq 0$, Eq. 11 reduces to

$$0 = \int_0^\infty \left\{ e^{-mx} \left(x \frac{F(v)}{f(v)} - \frac{1}{m} b_1(v) + \frac{x}{m} \frac{F(v)}{f(v)} b_1'(v) \right) \left[x^{k-3} + O(x^{k-2}) \right] \right\} dx.$$

Therefore, balance of the next-order $O(m^{-(k-1)})$ terms gives

$$\frac{F(v)}{f(v)} \int_0^\infty e^{-mx} x^{k-2} \, dx - \frac{1}{m} b_1(v) \int_0^\infty e^{-mx} x^{k-3} \, dx = 0,$$

or

$$\frac{F(v)}{f(v)} \frac{(k-2)!}{m^{k-1}} - \frac{(k-3)!}{m^{k-1}} b_1(v) = 0.$$

Therefore,

$$b_1(v) = (k-2) \frac{F(v)}{f(v)}.$$

Hence, we proved (8).

The seller's expected revenue in a k-price auction is given by $R_n^k = \int_0^1 b(v) dF_k(v)$, where $b(v)$ is the equilibrium bid in the k price auction and $F_k(v)$ is the distribution of the k-th valuation in order (i.e., kth-order statistic of the bidders private valuations). Substituting the asymptotic expansion for the equilibrium bids gives

$$R_n^k = \int_0^1 \left[v + \frac{k-2}{n-k} \frac{F(v)}{f(v)} \right] dF_k(v) + O\left(\frac{1}{n^2} \right).$$

Since the asymptotic expansion for the equilibrium bid is independent of the utility function U until order $O(\frac{1}{n^2})$, the revenue in the risk-averse case is the same as in the risk-neutral case, up to $O(\frac{1}{n^2})$ accuracy. By the revenue equivalence theorem, the latter is given by (4). □

In the risk-neutral case $U(x) = x$, the equilibrium bids in k-price auctions ($k = 2, 3, \ldots$) are given by (Wolfstetter 1995)

$$b(v) = v + \frac{k-2}{n-k+1}\frac{F(v)}{f(v)}. \quad (12)$$

Comparison with Eq. 8 shows that in large symmetric k-price auctions, risk aversion only has an $O(1/n^2)$ effect on the equilibrium bids. Proposition 3 also shows that risk aversion only has an $O(1/n^2)$ effect on the revenue in large symmetric k-price auctions.[9] Since all k-price auctions are revenue equivalent in the risk-neutral case, this implies, in particular, that all large symmetric k-price auctions with risk-averse bidders are $O(1/n^2)$ revenue equivalent.

5 All-pay auctions

In Proposition 3 we saw that all large k-price auctions with risk-averse bidders are $O(1/n^2)$ revenue equivalent. A natural conjecture is that this $O(1/n^2)$ asymptotic revenue equivalence holds for "all" auction mechanisms. To see that this is not the case, consider an all-pay auction with risk-averse bidders in which the highest bidder wins the object and all bidders pay their bid. In this case, the limiting value of the revenue is strictly below the risk-neutral limit:

Proposition 4 *Consider a symmetric all-pay auction with n bidders that have a utility function U that satisfies (1), and let R_n^{ra} be the seller's expected revenue in equilibrium. Then,*

$$\lim_{n\to\infty} R_n^{ra} < \lim_{n\to\infty} R_n^{rn}.$$

Proof This is a special case of Proposition 5. □

Therefore, even the limiting revenue of all-pay auctions is not revenue equivalent to that of k-price auctions with risk-averse bidders. Surprisingly, the effect of risk-aversion does not disappear as $n \longrightarrow \infty$.

In (Fibich et al. 2006) it was shown that in the case of all-pay auctions, risk-aversion lowers the equilibrium bids of the low types but increases the bids of the high types. As a result, the seller's revenue may either increase or decrease due to risk-aversion. In the case of large all-pay auctions, however, Proposition 5 shows that risk aversion always lowers the expected revenue.

Example 1 In Fig. 2 we graph the expected revenue as a function of n for an all-pay action with $F(v) = v$ and $U(x) = x - 0.5x^2$. In this case, risk-aversion increases the expected revenue when the number of bidders is small. As n increases, however,

[9] As in the case of first-price auctions (see Sect. 3), we can calculate explicitly the $O(1/n^2)$ terms in order to see that the leading-order effect of risk-aversion is truly $O(1/n^2)$ and is proportional to $-U''(0)/U'(0)$. Indeed, since $\lim_{n\to\infty} b(v) = v$, see Eq. 8, this conclusion follows from Eq. 7.

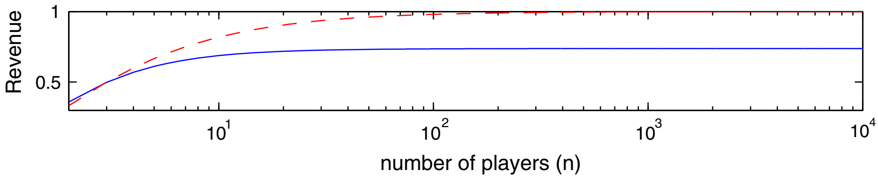

Fig. 2 Expected revenue in all-pay auction with risk-averse (*solid*) and risk-neutral (*dashes*) players, as a function of the number of players. Data plotted on a semi-logarithmic scale

this trend reverses and risk-aversion decreases the expected revenue. In particular, as $n \longrightarrow \infty$, the expected revenue in the risk-averse case approaches ≈ 0.74, which is well below the risk-neutral limit of $1 = \lim_{n\to\infty} R_n^{rn}$.[10]

Proposition 5 holds also for generalized all-pay auctions, where losing bidders pay α times their bid, where $0 \leq \alpha \leq 1$[11]:

Proposition 5 *Consider a generalized all-pay auction where bidders have a utility function U that satisfies (1). Then,*

$$\lim_{n\to\infty} R_n^{ra} < \lim_{n\to\infty} R_n^{rn}, \quad for\ 0 < \alpha \leq 1.$$

Proof See Appendix C. □

6 Last-price auctions

So far, the only case where risk-aversion reduced the limiting revenue was of generalized all-pay auctions, in which the losing bidders pay a fixed portion of their bid. We therefore ask whether risk-aversion can reduce the limiting revenue even when only the winner pays. To see that this is possible, we consider an auction in which the highest-bidder wins the object and pays the lowest bid, i.e., a last-price auction.

Example 2 Consider a last-price auction with n bidders that are risk averse with the CARA utility function $U(x) = 1 - e^{-\lambda x}$, where $\lambda > 0$. Assume that bidders values are distributed uniformly in $[0, 1]$. Then,

$$\lim_{n\to\infty} R_n^{last\text{-}price} < \lim_{n\to\infty} R_n^{rn}.$$

Proof See Appendix D. □

Although a last-price auction is a k-price auction with $k = n$, the results of Sect. 4 do not apply here. In a last-price auction $k \longrightarrow \infty$ as $n \longrightarrow \infty$, hence the kth value approaches 0 as $n \longrightarrow \infty$. In contrast, in the k-price auctions of Sect. 4, k is held fixed as $n \longrightarrow \infty$. Hence, the kth value approaches 1 as $n \longrightarrow \infty$.

[10] In the case of risk-loving bidders the limiting revenue is above the risk-neutral limit. For example, we find numerically for $U(x) = x + 0.5x^2$ that $lim_{n\to\infty} R_n^{ra} \approx 1.26$.

[11] Thus, $\alpha = 1$ corresponds to the standard all-pay auction, and $\alpha = 0$ to the first-price auction.

7 An asymptotic revenue equivalence theorem

We saw that in the case of risk-averse bidders, all large k-price auctions are $O(1/n^2)$ revenue equivalent to each other, but not to large all-pay auctions or last-price auctions. In particular, the limiting revenue approaches the risk-neutral limit for all k-price auctions, but not for all-pay auctions or last-price auctions. Here we give a sufficient condition for the limiting revenue to approach the risk-neutral limit.

Proposition 6 *Consider any symmetric auction where bidders have a utility function U that satisfies (1). Let $\beta^{win}(v_i, v_{-i})$ denote the equilibrium payment of bidder i when he wins with type v_i, and the other bidders have types $v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$. Assume that $\beta^{win}(v_i, v_{-i}) \longrightarrow v_i$ uniformly as $n \longrightarrow \infty$, i.e., that there exists a sequence $\epsilon_n \longrightarrow 0$ such that*

$$|v_i - \beta^{win}(v_i, \mathbf{v}_{-i})| \le \epsilon_n, \quad 0 \le v_i \le 1, \; 0 \le \mathbf{v}_{-i} \le v_i. \tag{13}$$

Then, the limiting revenue approaches the risk-neutral limit, i.e.,

$$\lim_{n\to\infty} R_n^{ra} = \lim_{n\to\infty} R_n^{rn}. \tag{14}$$

Proof See Appendix E. □

Remark The opposite direction is not necessarily true, see Example 3 below.

Condition (13) says that the equilibrium payment of a player who wins with value v_i approaches v_i uniformly as the number of bidders goes to infinity. The motivation for this condition is as follows. When the bidder wins and Condition (13) is satisfied, then his utility is $U(v - \beta^{win}) \sim (v - \beta^{win})U'(0)$. Therefore, $U(x)$ can be approximated by $U'(0)x$, the utility of a risk-neutral bidder.

In principle, there should be a second condition in Proposition 6 that would imply that when the bidder loses, his utility is $U(-\beta^{lose}) \sim -U'(0)\beta^{lose}$, i.e., the utility of a risk-neutral bidder, where β^{lose} is the equilibrium payment of a losing bidder. This second condition is not needed, however, for the following reason. The seller's revenue can be written as

$$R_n^{ra} = R_n^{win} + R_n^{lose}, \tag{15}$$

where

$$\begin{aligned} R_n^{win} &= \sum_{i=1}^{n} \int_0^1 E_{\mathbf{v}_{-i}}[\beta^{win}(v_i, \mathbf{v}_{-i})] F^{n-1}(v) f(v)\, dv, \\ R_n^{lose} &= \sum_{i=1}^{n} \int_0^1 E_{\mathbf{v}_{-i}}[\beta^{lose}(v_i, \mathbf{v}_{-i})](1 - F^{n-1}(v)) f(v)\, dv, \end{aligned} \tag{16}$$

are the revenues due to payments of the winning and losing bidders, respectively. When Condition (13) is satisfied, then $\lim_{n\to\infty} R^{lose} = 0$, see Eq. 35, or equivalently,

$$\lim_{n\to\infty} R_n^{ra} = \lim_{n\to\infty} R_n^{win}. \tag{17}$$

Therefore, even if the payments of the losing bidders are affected by risk-aversion, this has no effect on the limiting revenue.

From the proof of Proposition 6 it follows immediately that the pointwise Condition (13) can be replaced with the weaker condition that $E_{\mathbf{v}_{-i}}[v_i - \beta^{win}(v_i, v_{-i})] \le \epsilon_n$ for $0 \le v_i \le 1$. An even weaker condition can be derived as follows. As noted, the limiting revenue is only due to the contribution of the payments of the winning bidders. Because R^{win} has the multiplicative term $F^{n-1}(v)$ which is exponentially small except in an $O(1/n)$ region near the maximal value, Condition (13) can be relaxed to hold only in this shrinking region:

Corollary 1 *Proposition 6 remains valid if we replace Condition (13) with the weaker condition that there exists a sequence $\epsilon_n \longrightarrow 0$, such that for any $C > 0$,*

$$|v_i - \beta^{win}(v_i, \mathbf{v}_{-i})| \le \epsilon_n, \quad 1 - C/n \le v_i \le 1, \;\; 0 \le \mathbf{v}_{-i} \le v_i. \tag{18}$$

Proof See Appendix F. □

Example 3 Consider a generalized all-pay auction with $F(v) = v$, $U(x) = x - 0.5x^2$, and $\alpha = 1/n$. As $\alpha \longrightarrow 0$, the equilibrium bids are highly influenced by risk-aversion. Indeed, the bids are everywhere exponentially small (see Fig. 3, left panel), except in an $O(1/n)$ region near $v = 1$ where they approach the first-price bids (Fig. 3, right panel).[12] Therefore, Condition (13) is not satisfied. Nevertheless, the $O(1/n)$ small region near the maximal value where Condition (18) holds is sufficient to have the limiting revenue go to 1, the risk-neutral limit (Fig. 4).[13]

In the case of a generalized all-pay auction with a fixed α, Condition (13) is satisfied at $v = 1$, the maximal value, i.e., $\lim_{n\to\infty} b(1) = 1$, see Lemma 4. However, it is not satisfied in an $O(1/n)$ neighborhood of 1. Indeed, the heart of the proof of Proposition 5 is the key observation that

$$\lim_{n\to\infty} b(1 - 1/n) \ne 1,$$

see Eq. 30.

An obvious weakness of Proposition 6 is that Condition (13) involves the unknown bidding strategies. In the case of the first-price auction, one can prove that Condition (13) holds, by utilizing the known result that risk-aversion increases the equilibrium bids (Maskin and Riley 1984). We also note that for all the auction mechanisms

[12] The transition from exponentially-small bids to the first-price bids has nothing to do with risk-aversion, as it exists also in the risk-neutral case, see Eq. 38.

[13] In this case, however risk aversion does affect the $O(1/n)$ correction to the revenue.

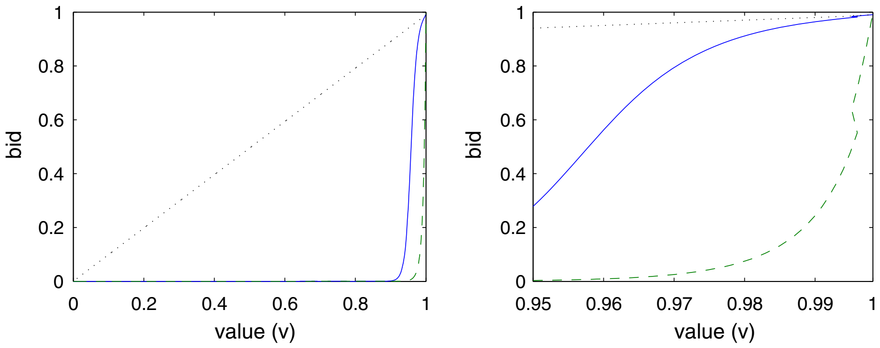

Fig. 3 Equilibrium bids in generalized all-pay auction with $\alpha = 1/n$ with risk-averse players (*solid line*) for $n = 100$. Also plotted are the equilibrium bids in the first-price (*dots*) and all-pay (*dashes*) auctions. *Right panel* is a magnification of the $O(1/n)$ region near the maximal value

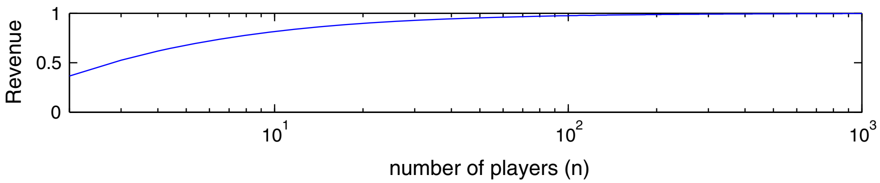

Fig. 4 Expected revenue in generalized all-pay auction with $\alpha = 1/n$ with risk-averse players as a function of the number of players. Data plotted on a semi-logarithmic scale

considered here, Condition (13) is satisfied in the risk-averse case if and only if it is satisfied in the risk-neutral case (see Appendix G). Indeed, generically, if for a given auction mechanism Condition (13) is not satisfied in the risk-neutral case, there is “no reason” for Condition (13) to be satisfied in the risk-averse case, hence it is “likely” that this auction mechanism will not be asymptotically revenue equivalent to large k-price auctions in the risk-averse case. Since in the risk-neutral case the bidding strategies are usually known explicitly, it is easy to check whether they satisfy Condition (13). For example, from Eq. 12 it immediately follows that any k-price auction with $k = n - 1, k = n - 2, k = n - 3, \ldots$, or with $k = n/2, k = n/3, \ldots$, would not satisfy Condition (13) in the risk-neutral case. This suggests that these auction mechanisms are not asymptotically revenue equivalent to large k-price auctions in the risk-averse case, as can be confirmed by a minor modification of the proof for last-price auctions in Appendix D.

Remark At a first sight, it seems that the results for k-price auctions in Sect. 4 are a special case of Proposition 6. This is not the case, since in order to apply Proposition 6 to k-price auctions, one needs to prove that the equilibrium payment of the winning bidder in k-price auctions approaches his type as $n \longrightarrow \infty$ uniformly. In addition, the result of Proposition 6 is weaker, since it only shows that the limiting revenue is unaffected by risk-aversion, whereas in Sect. 4 we show that the $O(1/n)$ correction is also unaffected by risk-aversion.

8 Final remarks

In this study we used two different mathematical approaches for studying the effect of risk-aversion in large private-value auctions. The results on all-pay auctions, last-price auctions, and the asymptotic revenue equivalence theorem (Sects. 5–7) were proved using rigorous techniques which are standard in auction theory. Hence, these results are of interest mostly for their economic implications. In contrast, the results on first-price and k-price auctions (Sects. 3, 4) were proved using asymptotic analysis techniques such as the Laplace method for integrals which, to the best of our knowledge, have not been used before in auction theory. Since these techniques can be applied to numerous other problems in auction theory, these results are also of interest for their "applied math approach".

There are some obvious disadvantages for using applied math techniques such as asymptotic analysis, as compared with the standard rigorous approach used in auction theory. Thus, the results are not "exact", but "only" approximate.[14] Moreover, one can sometimes construct pathological "counter-examples" for which the results "do not hold".[15] In other words, while the results hold "generically", it is not always easy or even possible to formulate the exact conditions under which they hold.

Addressing this "criticism" would probably seem out of place in a physical or an engineering journal, where such applied math techniques have been in use for over 200 years. Indeed, these (and similar) techniques and approximations have been routinely used in the design of airplanes, nuclear plants, bridges, medical instruments, etc. However, as these techniques have not been in use in the auction (and, more generally, the economic) literature, in what follows we will "defend" the legitimacy of this "applied math approach".

One obvious advantage of these applied math techniques is that they can be used to solve ("to leading-order") hard problems, which cannot be solved exactly. For example, the explicit expressions for the bids and revenue in large first-price auctions with risk-averse bidders cannot be calculated exactly, but can be easily calculated asymptotically (Proposition 1). Essentially, this is because by solving "to leading-order" we replace the original "hard" nonlinear problem, with a linear problem for the leading-order correction.

Even when results can be calculated using the standard techniques, in many cases the applied math techniques require substantially less work. For example, in the special case of risk-neutrality, the result of Proposition 1 is equivalent to the one obtained by Caserta and de Vries (2002) using extreme value theory. Their calculation, however, required considerably more work.

The assumptions made on F and U do not cover all possible cases. For example, our asymptotic results do not cover the case when $f(1) = 0$. We stress, however, that this case can be analyzed using the same techniques, with only minor modifications. When $f(1)$ is positive but very small, our asymptotic results are valid as $n \longrightarrow \infty$. In that case, however, the value of n should be significantly larger for them to be

[14] E.g., they are $O(1/n^2)$ accurate in this study.

[15] E.g., one can find distribution functions F for which $f(1)$ is positive but extremely small, for which our asymptotic expansions provide a poor approximation at values of n which are already "large".

accurate. It is possible to estimate the value of n for which they become accurate (e.g., by calculating the next-order term).

As was discussed in the Introduction, the asymptotic expansions allowed us to obtain results which are stronger than the limiting results and also from the rate-of-convergence results. In particular, the asymptotic expansions become applicable at much smaller values of n. This is important when one wants to analyze a specific auction, since then the number of bidders is given, and does not go to infinity.

Finally, we acknowledge that k-price auctions rarely appear in real-life auctions. Nevertheless, they are of interest from a theoretical point of view. Moreover, the result that the effect of risk aversion decays as $1/n^2$ for all k-price auctions, makes our discovery that there exist auction mechanisms for which the effect of risk aversion does not disappear as $n \longrightarrow \infty$, all the more surprising.

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A Auxiliary Lemmas

Lemma 1 *Let* $n \gg 1$, *let* $b(v) = v + (1/n)B_1(v) + (1/n^2)B_2(v) + O(1/n^3)$, *and let* $v(b) = b + \frac{1}{n}v_1(b) + \frac{1}{n^2}v_2(b) + O(1/n^3)$ *be the inverse function of* $b(v)$. *Then,*

$$B_1(v) = -v_1(v), \quad B_2(v) = -B_1(v)v_1'(v) - v_2(v). \tag{19}$$

Proof We substitute the two expansions into the identity $v \equiv v(b(v))$ and expand in $1/n$:

$$\begin{aligned} v = v(b(v)) &= b(v) + \frac{1}{n}v_1(b(v)) + \frac{1}{n^2}v_2(b(v)) + O(1/n^3) \\ &= v + \frac{1}{n}B_1(v) + \frac{1}{n^2}B_2(v) + \frac{1}{n}v_1\left(v + \frac{1}{n}B_1(v)\right) + \frac{1}{n^2}v_2(v) + O(1/n^3) \\ &= v + \frac{1}{n}\left[B_1(v) + v_1(v)\right] + \frac{1}{n^2}\left[B_2(v) + B_1(v)v_1'(v) + v_2(v)\right] + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Balancing the $O(1/n)$ and $O(1/n^2)$ terms proves (19). □

In the following we calculate an asymptotic expansion of the integral $\int_0^v F^n(x)\,dx$ using integration by parts (for an introduction to asymptotic evaluation of integrals using integration by parts, see, e.g., Murray 1984):

Lemma 2 *Let* $F(v)$ *be a twice-continuously differentiable, function and let* $f = F' > 0$. *Then, for a sufficiently large* n,

$$\int_0^v F^n(x)\,dx = \frac{1}{n}\frac{F^{n+1}(v)}{f(v)}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{20}$$

Proof Using integration by parts,

$$\int_0^v F^n(x)\,dx = \int_0^v [F^n(x)f(x)]\frac{1}{f(x)}\,dx = \frac{1}{n+1}\frac{F^{n+1}(v)}{f(v)}$$
$$+\frac{1}{n+1}\int_0^v [F^{n+1}(x)f(x)]\frac{f'(x)}{f^3(x)}\,dx = \frac{1}{n+1}\frac{F^{n+1}(v)}{f(v)}$$
$$+\frac{1}{n+1}\frac{1}{n+2}F^{n+2}(v)\frac{f'(v)}{f^3(v)} - \frac{1}{n+1}\frac{1}{n+2}\int_0^v F^{n+2}(x)\left(\frac{f'(x)}{f^3(x)}\right)'\,dx.$$

Therefore, the result follows. □

B Proof of Proposition 2

Since $\lim_{n\to\infty} v(b) = b$, we can look for a solution of the form

$$v(b) = b + \frac{1}{n-1}v_1(b) + \frac{1}{(n-1)^2}v_2(b) + O\left(\frac{1}{n^3}\right). \tag{21}$$

Substituting (21) in (2) and using $U(0) = 0$ and $0 < U'(0) < \infty$ gives

$$1 + \frac{1}{n-1}v_1'(b) + O\left(\frac{1}{n^2}\right) = \frac{1}{n-1}\frac{F(b) + \frac{v_1}{n-1}f(b) + O(n^{-2})}{f(b) + \frac{v_1}{n-1}f'(b) + O(n^{-2})}$$
$$\times \frac{U'(0) + \frac{v_1}{n-1}U''(0) + O(n^{-2})}{U(0) + \left(\frac{v_1}{n-1} + \frac{v_2}{(n-1)^2}\right)U'(0) + \frac{v_1^2}{2(n-1)^2}U''(0) + O(n^{-3})}$$
$$= \frac{F(b) + \frac{v_1}{n-1}f(b) + O(n^{-2})}{f(b) + \frac{v_1}{n-1}f'(b) + O(n^{-2})} \cdot \frac{U'(0) + \frac{v_1}{n-1}U''(0) + O(n^{-2})}{\left(v_1 + \frac{v_2}{(n-1)}\right)U'(0) + \frac{v_1^2}{2(n-1)}U''(0) + O(n^{-2})}$$
$$= \left(\frac{F(b)}{f(b)} + \frac{v_1}{n-1} + O(n^{-2})\right)\left(1 - \frac{v_1}{n-1}\frac{f'(b)}{f(b)} + O(n^{-2})\right)$$
$$\times \left(\frac{1}{v_1} + \frac{1}{n-1}\frac{U''(0)}{U'(0)} + O(n^{-2})\right)$$
$$\times \left(1 - \frac{1}{(n-1)}\frac{v_2}{v_1} - \frac{v_1}{2(n-1)}\frac{U''(0)}{U'(0)} + O(n^{-2})\right)$$
$$= \frac{F(b)}{f(b)}\frac{1}{v_1} + \frac{1}{n-1}\left[1 - \frac{f'(b)F(b)}{f^2(b)} + \frac{F(b)}{2f(b)}\frac{U''(0)}{U'(0)} - \frac{F(b)}{f(b)}\frac{v_2}{v_1^2}\right] + O(n^{-2}).$$

Balancing the $O(1)$ terms gives, as before,

$$v_1(b) = \frac{F(b)}{f(b)}. \tag{22}$$

Balancing the $O\left(\frac{1}{n-1}\right)$ terms gives

$$v_1'(b) = 1 - \frac{f'(b)F(b)}{f^2(b)} + \frac{F(b)}{2f(b)}\frac{U''(0)}{U'(0)} - \frac{F(b)}{f(b)}\frac{v_2}{v_1^2}.$$

Substituting $v_1(b) = F(b)/f(b)$ and $v_1'(b) = 1 - \frac{F(b)f'(b)}{f^2(b)}$ gives

$$v_2(b) = \frac{F^2(b)}{2f^2(b)}\frac{U''(0)}{U'(0)}. \tag{23}$$

Using Lemma 1 and (22,23) to invert the expansion (21) gives

$$b(v) = v + \frac{1}{n-1}B_1(v) + \frac{1}{(n-1)^2}B_2(v) + O\left(\frac{1}{n^3}\right),$$

where

$$B_1(v) = -\frac{F(v)}{f(v)}, \quad B_2(v) = \frac{F(v)}{f(v)} - \frac{F^2(v)f'(v)}{f^3(v)} - \frac{F^2(v)}{2f^2(v)}\frac{U''(0)}{U'(0)}.$$

This completes the proof of (5).

To calculate the expected revenue, we first use (5) to obtain

$$\begin{aligned} R_n^{ra} &= \int_0^1 b(v)\,dF^n(v) = b(1) - \int_0^1 b'(v)F^n(v)\,dv \\ &= 1 - \frac{1}{n-1}\frac{1}{f(1)} + \frac{1}{(n-1)^2}\left[\frac{1}{f(1)} - \frac{f'(1)}{f^3(1)} - \frac{1}{2f^2(1)}\frac{U''(0)}{U'(0)}\right] \\ &\quad - \int_0^1 \left[1 - \frac{1}{n-1}\left(\frac{F(v)}{f(v)}\right)'\right]F^n(v)\,dv + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Integration by integration by parts (as in Lemma 2) gives,

$$\int_0^1 F^n(v)\,dv = \frac{1}{n+1}\frac{1}{f(1)} + \frac{1}{n+1}\frac{1}{n+2}\frac{f'(1)}{f^3(1)} + O\left(\frac{1}{n^3}\right),$$

and

$$\int_0^1 \left(\frac{F(v)}{f(v)}\right)' F^n(v)\,dv = \frac{1}{n+1}\frac{1}{f(1)}\left(\frac{F(v)}{f(v)}\right)'_{v=1} + O\left(\frac{1}{n^2}\right).$$

Therefore,

$$\int_0^1 \left[1 - \frac{1}{n-1}\left(\frac{F(v)}{f(v)}\right)'\right] F^n(v)\,dv$$
$$= \frac{1}{n}\frac{1}{1+\frac{1}{n}}\frac{1}{f(1)} + \frac{1}{n^2}\frac{f'(1)}{f^3(1)} - \frac{1}{n^2}\frac{1}{f(1)}\left(1 - \frac{f'(1)}{f^2(1)}\right) + O\left(\frac{1}{n^3}\right)$$
$$= \frac{1}{n}\frac{1}{f(1)} - \frac{1}{n^2}\frac{2}{f(1)} + \frac{2}{n^2}\frac{f'(1)}{f^3(1)} + O\left(\frac{1}{n^3}\right).$$

Substitution in the expression for R_n^{ra} proves (6). □

C Proof of Proposition 5

We first show that the maximal bid $b(1)$ is monotonically increasing in α:

Lemma 3 *Consider a generalized all-pay auction where bidders valuations are distributed according to $F(v)$ in $[0, 1]$, and bidders have a utility function U that satisfies (1). Then,*

$$\frac{\partial b(1)}{\partial \alpha} > 0, \quad 0 \le \alpha \le 1.$$

Proof Let

$$V(v) = F^{n-1}(v)U(v - b(v)) + (1 - F^{n-1}(v)U(-\alpha b(v)) \tag{24}$$

be the expected utility of a bidder with value v. By the envelope theorem,

$$V'(v) = F^{n-1}(v)U'(v - b(v)). \tag{25}$$

In addition, differentiating (24) with respect to α gives

$$\frac{\partial V(v)}{\partial \alpha} = -\frac{\partial b}{\partial \alpha}\left(F^{n-1}(v)U'(v - b(v)) + (1 - F^{n-1}(v))U'(-\alpha b(v))\right)$$
$$-b(v)(1 - F^{n-1}(v))U'(-\alpha b(v)). \tag{26}$$

We now prove that $\frac{\partial V(v)}{\partial \alpha} < 0$ for all $0 < \alpha, v \le 1$. By negation, assume that $\frac{\partial V(v)}{\partial \alpha} \ge 0$ for some $0 < v_1, \alpha_1 \le 1$. Then, from Eq. 26 it follows that $\frac{\partial b}{\partial \alpha}|_{v_1,\alpha_1} < 0$.

Hence, by risk aversion and (25),

$$\frac{\partial}{\partial\alpha}V'(v)\bigg|_{v_1,\alpha_1} = -\frac{\partial b}{\partial\alpha}F^{n-1}(v)U''(v-b(v))\bigg|_{v_1,\alpha_1} < 0. \tag{27}$$

Denote $y(v) = V_{\alpha_1+\Delta\alpha}(v) - V_{\alpha_1}(v)$, where $0 < \Delta\alpha$. By the negation assumption, if $\Delta\alpha$ is sufficiently small, then $y(v_1) \geq 0$. Hence, by (27), $y'(v_1) = V'_{\alpha+\Delta}(v_1) - V'_{\alpha}(v_1) < 0$. Thus, $y(t) = V_{\alpha+\Delta}(t) - V_{\alpha}(t) > 0$ for t slightly below v_1, and therefore by a continuation argument for every $0 \leq t < v_1$. This contradicts the fact that $y(0) = V_{\alpha+\Delta}(0) - V_{\alpha}(0) = 0$, since $V(0) = 0$ for every α.

We have thus proved that

$$0 > \frac{\partial V(1)}{\partial\alpha} = -\frac{\partial b(1)}{\partial\alpha}U'(1-b(1)).$$

Therefore, the result follows. □

Therefore, the maximal bid approaches the maximal value:

Lemma 4 *Under the conditions of Lemma 3,*

$$\lim_{n\to\infty} b(1) = 1, \quad 0 \leq \alpha \leq 1.$$

Proof From Lemma 3 we have that $b(1)$ is monotonically increasing in α. Therefore,

$$b(1; \alpha = 0) < b(1; \alpha) \leq 1.$$

Since for $\alpha = 0$ we have a first price auction, from Eq. 3 it follows that $\lim_{n\to\infty} b(1; \alpha = 0) = 1$. Therefore, the result follows. □

We now prove Proposition 5. Let $V(v)$, defined by (24), be the expected utility of a bidder with value v in equilibrium. Then,

$$0 \leq n\int_0^1 V(v)f(v)\,dv = nU'(0)\int_0^1 [vF^{n-1}(v) - \alpha b - (1-\alpha)F^{n-1}(v)b]f(v)\,dv$$
$$-C_n = U'(0)A_n - U'(0)R_n^{ra} - C_n,$$

where

$$C_n = nU'(0)\int_0^1 [vF^{n-1}(v) - \alpha b - (1-\alpha)F^{n-1}(v)b]f(v)\,dv - n\int_0^1 V(v)f(v)\,dv,$$

$$A_n = n\int_0^1 vF^{n-1}(v)f(v)\,dv,$$

$$R_n^{ra} = n\int_0^1 [bF^{n-1} + \alpha b(1-F^{n-1}]f\,dv.$$

Therefore,

$$R_n^{ra} \le A_n - \frac{C_n}{U'(0)}.$$

Since

$$A_n = \int_0^1 v(F^n)' = 1 - \int_0^1 F^n = 1 + O(1/n),$$

see Eq. 20, then $\lim_{n\to\infty} A_n = 1$. Therefore, to complete the proof, we only need to show that $\lim_{n\to\infty} C_n > 0$.

Now,

$$\begin{aligned} C_n &= -n\int_0^1 \Big[F^{n-1}(v)\big(U(v-b) - (v-b)U'(0)\big) + (1-F^{n-1}(v))\,(U(-\alpha b) \\ &\quad +\alpha bU'(0))\Big]\,f(v)\,dv \\ &= -n\int_0^1 \left[F^{n-1}(v)\frac{(v-b)^2}{2}U''(\theta_1(v)) + (1-F^{n-1}(v))\frac{\alpha^2 b^2}{2}U''(\theta_2(v))\right]f(v)\,dv, \end{aligned}$$

where $0 < \theta_1(v) < v - b(v)$ and $-b(v) < \theta_2(v) < 0$. Since $-U'' \ge M > 0$, we have that

$$\begin{aligned} C_n &\ge Mn\int_0^1 \left[F^{n-1}(v)\frac{(v-b)^2}{2} + (1-F^{n-1}(v))\frac{\alpha^2 b^2}{2}\right]f(v)\,dv \\ &\ge Mn\int_0^1 F^{n-1}(v)\frac{(v-b)^2}{2}\,f(v)\,dv. \end{aligned} \tag{28}$$

We now show that the limit of (28) is strictly positive. Indeed,

$$\int_0^1 nF^{n-1}(v)f(v)(v-b)^2\,dv = \int_0^1 \left(F^n(v)\right)'(v-b)^2\,dv$$
$$= F^n(v)(v-b)^2\Big|_0^1 - 2\int_0^1 F^n(v)(v-b)(1-b')\,dv$$
$$= (1-b(1))^2 - 2\int_0^1 F^n(v)(v-b)\,dv + 2\int_0^1 F^n(v)(v-b)b'\,dv. \quad (29)$$

We claim that the first two terms go to zero, but the third term goes to a positive constant. Since $\lim_{n\to\infty} b(1) = 1$, by Lemma 4, the first term in (29) approaches zero. Since $(v-b)$ is bounded, by Lemma 2 the second term also goes to zero. As for the third term,

$$\int_0^1 F^n(v)(v-b)b'dv \geq \int_{1-1/n}^1 F^n(v)(v-b)b'\,dv.$$

Now, $F^n(1-1/n) \geq C_1 > 0$. Indeed,

$$F\left(1-\frac{1}{n}\right) = 1 - \frac{1}{n}f(\theta), \quad 1-\frac{1}{n} < \theta < 1.$$

Therefore,

$$F^n\left(1-\frac{1}{n}\right) \geq \left(1-\frac{\max f}{n}\right)^n \longrightarrow e^{-\max f}.$$

Therefore,

$$\int_{1-1/n}^1 F^n(v)(v-b)b'dv \geq C_1 \int_{1-1/n}^1 (v-b)b'dv.$$

In addition,

$$\int_{1-1/n}^{1} (v-b)b'\,dv = vb\Big|_{1-1/n}^{1} - \int_{1-1/n}^{1} b\,dv - \frac{b^2}{2}\Bigg|_{1-1/n}^{1}$$
$$= b(1) - (1-1/n)b(1-1/n) - \int_{1-1/n}^{1} b\,dv - \frac{b^2(1)}{2} + \frac{b^2(1-1/n)}{2}.$$

As n goes to infinity, $b(1) \to 1$ and $\int_{1-1/n}^{1} b\,dv \to 0$. Hence,

$$\lim_{n\to\infty} \int_{1-1/n}^{1} (v-b)b'\,dv = \frac{1}{2}(1-X_\infty)^2,$$

where

$$X_\infty = \lim_{n\to\infty} X_n, \quad X_n = b(1-1/n).$$

We now show that

$$X_\infty < 1, \tag{30}$$

and this will complete the proof. By Taylor expansion,

$$1 - X_n = 1 - b(1) + \frac{1}{n}b'(\theta), \quad 1 - 1/n < \theta < 1. \tag{31}$$

Recall that

$$b'(v) = (n-1)F^{n-2}(v)f(v)\frac{U(v-b(v)) - U(-\alpha b(v))}{F^{n-1}(v)U'(v-b(v)) + \alpha(1-F^{n-1}(v))U'(-\alpha b(v))}.$$

Now, for $v \in (1-1/n, 1)$, as $n \longrightarrow \infty$,

$$F^{n-2}(v) \ge F^n(v) \ge C_1, \quad f(v) \ge \min f(v),$$

$$U(v-b(v)) - U(-\alpha b(v)) = (v-(1-\alpha)b(v))U'(\theta_2)$$
$$\ge (v-(1-\alpha)v)U'(\theta_2) \ge \alpha(1-1/n)U'(1),$$

and

$$F^{n-1}(v)U'(v-b(v)) + \alpha(1-F^{n-1}(v))U'(-\alpha b(v)) \le U'(-1).$$

Therefore, there exists $C_2 > 0$ such that

$$b'(v) \geq (n-1)C_2, \quad 1 - 1/n < v < 1.$$

Thus, since $1 - b(1) \to 0$, Eq. 31 implies that $\lim_{n\to\infty}(1 - X_n) \geq C_2 > 0$. □

D Last-price auctions

The equilibrium bid function in a last-price auctions with $F(x) = x$ and a CARA utility $U(x) = 1 - e^{-\lambda x}$ is the solution of, see Eq. 9,

$$\int_0^v [1 - e^{-\lambda(v - b(t))}](v-t)^{n-3}\, dt = 0.$$

Therefore,

$$\int_0^v e^{\lambda b(t)}(v-t)^{n-3}\, dt = e^{\lambda v}\frac{v^{n-2}}{n-2}.$$

Differentiating $n - 3$ times with respect to v gives

$$(n-3)! \int_0^v e^{\lambda b(t)}\, dt = \frac{d^{n-3}}{dv^{n-3}}\left(e^{\lambda v}\frac{v^{n-2}}{n-2}\right).$$

One more differentiation gives

$$(n-2)! e^{\lambda b(v)} = \frac{d^{n-2}}{dv^{n-2}}\left(e^{\lambda v} v^{n-2}\right).$$

Therefore,

$$\begin{aligned} b(v) &= \frac{1}{\lambda}\ln\left[\frac{1}{(n-2)!}\frac{d^{n-2}}{dv^{n-2}}\left(e^{\lambda v}v^{n-2}\right)\right] = \frac{1}{\lambda}\ln\left[e^{\lambda v}\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{1}{k!}\lambda^k v^k\right] \\ &= v + \frac{1}{\lambda}\ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{1}{k!}\lambda^k v^k\right] \leq v + \frac{1}{\lambda}\ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\lambda^k v^k\right] \\ &= v + \frac{1}{\lambda}\ln\left[(1+\lambda v)^{n-2}\right] \leq v + \frac{1}{\lambda}\ln\left[(e^{\lambda v})^{n-2}\right] = (n-1)v = b_{rn}(v), \end{aligned}$$

where $b_{rn}(v)$ is the equilibrium strategy in the risk-neutral case, see Eq. 12. Hence,

$$\begin{aligned}
\lambda(b_{rn} - b) &\geq \ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\lambda^k v^k\right] - \ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{1}{k!}\lambda^k v^k\right] \\
&\geq \ln\left[\sum_{k=0}^{n-2}\binom{n-2}{k}\lambda^k v^k\right] \\
&\quad - \ln\left[-\frac{1}{2}\binom{n-2}{k}\lambda^2 v^2 + \sum_{k=0}^{n-2}\binom{n-2}{k}\lambda^k v^k\right] \\
&= \ln\left[(1+\lambda v)^{n-2}\right] - \ln\left[-\frac{1}{2}\binom{n-2}{2}\lambda^2 v^2 + (1+\lambda v)^{n-2}\right].
\end{aligned}$$

Since $\ln b - \ln a \geq (b-a)/b$ for $0 < a < b$, we get that

$$\lambda(b_{rn} - b) \geq \frac{1}{2}\binom{n-2}{2}\lambda^2 v^2 \frac{1}{(1+\lambda v)^{n-2}}.$$

The distribution function of the lowest value is $F_{(n)} = 1 - (1-v)^n$, hence the expected revenue is given by $R_n^{last\text{-}price} = \int_0^1 b(v)\, dF_{(n)} = n\int_0^1 b(v)(1-v)^{n-1}\, dv$. Therefore,

$$\begin{aligned}
R_n^{rn} - R_n^{last\text{-}price} &= n\int_0^1 [b_{rn}(v) - b(v)](1-v)^{n-1}\, dv \\
&\geq \frac{\lambda}{4} n(n-2)(n-3)\int_0^{1/n} v^2 \frac{1}{(1+\lambda v)^{n-2}}(1-v)^{n-1}\, dv \\
&\geq \frac{\lambda}{4} n(n-2)(n-3)\int_0^{1/n} v^2 \frac{1}{(1+\lambda/n)^{n-2}}(1-1/n)^{n-1}\, dv \\
&= \frac{\lambda}{4} n(n-2)(n-3)\frac{1}{3n^3}\frac{1}{(1+\lambda/n)^{n-2}}(1-1/n)^{n-1}.
\end{aligned}$$

Taking the limit, we have that

$$\lim_{n\to\infty}(R_n^{rn} - R_n^{last\text{-}price}) \geq \frac{\lambda}{12}e^{-\lambda-1} > 0.$$

□

E Proof of Proposition 6

Let $P(v) = F^{n-1}(v)$ be the probability of winning of a bidder with value v. Since $n\int_0^1 P(v)f(v)\,dv = 1$, from Condition (13) it follows that

$$n\int_0^1 E_{\mathbf{v}_{-i}}[\beta^{win}(v_i, \mathbf{v}_{-i}) - v_i]P(v_i)f(v_i)\,dv_i = O(\epsilon_n). \tag{32}$$

Let

$$S_i(v_i) = E_{\mathbf{v}_{-i}}\left[U(v_i - \beta(v_i, \mathbf{v}_{-i}))|i\; wins\right]P(v_i) \\ +E_{\mathbf{v}_{-i}}\left[U(-\beta(v_i, \mathbf{v}_{-i}))|i\; loses\right](1 - P(v_i)),$$

be the expected surplus of a risk-averse bidder i when his type is v_i. From now on, we suppress the subindex i and the dependence on v_{-i}, and introduce the notations β^{win} and β^{lose} for the equilibrium payment when bidder i wins or loses, respectively. Therefore, the last relation can be rewritten as

$$S(v) = E\left[U(v - \beta^{win}(v))\right]P(v) + E\left[U(-\beta^{lose}(v))\right](1 - P(v)). \tag{33}$$

Similarly, the expected revenue can be written as $R = R^{win} + R^{lose}$, where

$$R_n^{win} = n\int_0^1 E[\beta^{win}(v)]P(v)f(v)\,dv,$$

$$R_n^{lose} = n\int_0^1 E[\beta^{lose}(v))](1 - P(v))f(v)\,dv.$$

From Eq. 32 it follows that

$$R_n^{win} = n\int_0^1 vP(v)f(v)\,dv + O(\epsilon_n) = 1 + O(1/n) + O(\epsilon_n) \\ = R_n^{rn} + O(1/n) + O(\epsilon_n). \tag{34}$$

We now show that relation (32) implies that

$$R_n^{lose} = O(\epsilon_n). \tag{35}$$

Indeed, from Eq. 33 and the fact that $S_i \geq 0$, we have that

$$-E\left[U(-\beta^{lose}(v))\right](1 - P(v)) \leq E\left[U(v - \beta^{win}(v))\right]P(v). \tag{36}$$

Since $U(-x) = -xU'(0) + x^2/2U''(\theta(-x)) < -xU'(0)$, it follows that $xU'(0) \leq -U(-x)$. Therefore, by (36) and the fact that the payments are positive,

$$0 \leq U'(0)E\left[\beta^{lose}(v)\right](1 - P(v)) \leq E\left[U(v - \beta^{win}(v))\right]P(v).$$

Hence,

$$\begin{aligned} 0 &\leq U'(0)n \int_0^1 E\left[\beta^{lose}(v))\right](1 - P(v))f(v)\,dv \\ &\leq n \int_0^1 E\left[U(v - \beta^{win}(v))\right]P(v)f(v)\,dv = O(\epsilon_n), \end{aligned}$$

where is the last stage we used (32). Therefore, we proved (35).

Combining (34,35) we get that $R_n^{ra} = R_n^{rn} + O(1/n) + O(\epsilon_n)$. Therefore, we proved Eq. 14. □

F Proof of Corollary 1

In the proof of Proposition 6 we used Condition (13) to conclude that

$$\lim_{n\to\infty} n \int_0^1 E[\beta^{win}(v) - v]P(v)f(v)\,dv = 0.$$

Therefore, we need to show that this limit does not change even if (13) holds "only" for $1 - C/n \leq v \leq 1$. To see that, we note that

$$n \int_0^1 E[\beta^{win}(v) - v]P(v)f(v)\,dv = I_1 + I_2,$$

where

$$I_1 = n \int_0^{1-C/n} E[\beta^{win}(v) - v]P(v)f(v)\,dv,$$

$$I_2 = n \int_{1-C/n}^1 E[\beta^{win}(v) - v]P(v)f(v)\,dv.$$

Since $0 \le E[\beta^{win}(v)] \le v \le 1$,

$$I_1 \le n \int_0^{1-C/n} P(v) f(v)\, dv = F^n(1 - C/n).$$

Now,

$$F(1 - C/n) = 1 - C/n f(\theta_n), \quad 1 - C/n < \theta_n < 1.$$

Therefore, as $n \longrightarrow \infty$,

$$F^n(1 - C/n) = \left(1 - \frac{C f(\theta_n)}{n}\right)^n \longrightarrow e^{-Cf(1)}. \tag{37}$$

Therefore, we can choose C sufficiently large so that $|I_1| \le \epsilon/2$. In addition,

$$|I_2| \le \epsilon_n n \int_{1-C/n}^{1} P(v) f(v)\, dv \le \epsilon_n n \int_0^1 P(v) f(v)\, dv = \epsilon_n.$$

Therefore, we can choose n sufficiently large so that $|I_2| \le \epsilon/2$. Therefore, the result follows. □

G Condition (13) in the risk-neutral case

- The risk-neutral equilibrium bids, hence payments, in the first-price and all-pay auctions are given by

$$\beta_{rn}^{1st}(v) = b_{rn}^{1st}(v) = v - \frac{1}{F^{n-1}(v)} \int_0^v F^{n-1}(s)\, ds,$$
$$\beta_{rn}^{all}(v) = b_{rn}^{all}(v) = F^{n-1}(v) \beta_{rn}^{1st}(v).$$

  Hence, by Lemma 2, as $n \longrightarrow \infty$,

$$\beta_{rn}^{1st}(v) \sim v - \frac{1}{n} \frac{F(v)}{f(v)}, \quad \beta_{rn}^{all}(v) \sim F^{n-1}(v) \left[v - \frac{1}{n} \frac{F(v)}{f(v)}\right].$$

  Therefore, Condition (13) is satisfied for (risk-neutral) first-price auction, but not for the all-pay auction.

- In the case of generalized all-pay auctions, the equilibrium bid function is the solution of

$$b'(v) = (n-1)F^{n-2}(v)f(v) \times \frac{U(v-b(v)) - U(-\alpha b(v))}{F^{n-1}(v)U'(v-b(v)) + \alpha(1-F^{n-1}(v))U'(-\alpha b(v))},$$
$$b(0) = 0.$$

This equation can be explicitly solved in the risk-neutral case, yielding

$$\left(\beta_{rn}^{gen\text{-}all}\right)^{win}(v) = b_{rn}^{gen\text{-}all}(v) = \frac{vF^{n-1}(v) - \int_0^v F^{n-1}(s)\,ds}{\alpha + (1-\alpha)F^{n-1}(v)}$$
$$= \frac{F^{n-1}(v)}{\alpha + (1-\alpha)F^{n-1}(v)} b_{rn}^{1st}(v). \tag{38}$$

Hence, $(\beta_{rn}^{gen\text{-}all})^{win}(v) \longrightarrow v$ provided that $\frac{F^{n-1}(v)}{\alpha+(1-\alpha)F^{n-1}(v)} \longrightarrow 1$. If α is held constant, then $\frac{F^{n-1}(v)}{\alpha+(1-\alpha)F^{n-1}(v)} \longrightarrow 1$ when $F^{n-1}(v) \longrightarrow 1$, i.e., for $1 - v \ll 1/n$ but not for $1 - v = O(1/n)$. Therefore, Condition (13) is not satisfied. If, however, $\alpha = \alpha(n)$ and $\lim_{n\to\infty} \alpha = 0$, then by (37) $\frac{F^{n-1}(v)}{\alpha+(1-\alpha)F^{n-1}(v)} \longrightarrow 1$ for $1 - v = O(1/n)$, but not for all $0 \le v \le 1$. Therefore, Condition (13) is not satisfied, but its weaker form (see Corollary 1) is satisfied.
- In the case of k-price auctions, $\lim_{n\to\infty} b_{rn}^{k-price}(v) = v$, see Eq. 12. In addition, as $n \longrightarrow \infty$, the kth value approach the value of the winning bidder. Therefore, $\lim_{n\to\infty} \beta_{rn}^{k-price}(v) = v$, so that Condition (13) is satisfied.
- Finally, in the case of last-price auctions,

$$(\beta_{rn}^{last})^{win}(v_i, \mathbf{v}_{-i}) = b_{rn}^{last}(v_{\min}) = v_{\min} + (n-2)\frac{F(v_{\min})}{f(v_{\min})}, \quad v_{\min} = \min_{j\neq i} v_j,$$

see Eq. 12. In addition, since $b_{rn}^{last}(v_{\min})$ is independent of v, then it is not converging to v for all v as $n \longrightarrow \infty$. Therefore, Condition (13) is not satisfied. □

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