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On ordinal equivalence of the Shapley and Banzhaf values for cooperative games

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Abstract In this paper I consider the *ordinal equivalence* of the Shapley and Banzhaf values for TU cooperative games, i.e., cooperative games for which the preorderings on the set of players induced by these two values coincide. To this end I consider several solution concepts within semivalues and introduce three subclasses of games which are called, respectively, weakly complete, semicoherent and coherent cooperative games. A characterization theorem in terms of the ordinal equivalence of some semivalues is given for each of these three classes of cooperative games. In particular, the Shapley and Banzhaf values as well as the segment of semivalues they limit are ordinally equivalent for weakly complete, semicoherent and coherent cooperative games.

Keywords Cooperative game · Semivalue · Weakly complete game · Semicoherent game · Coherent game

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JEL Classification C71 · D71

1 Introduction

The *ordinal equivalence* of the Shapley and Banzhaf power indices for simple games was first considered by Tomiyama (1987) who proved that, for every weighted majority game, the preorderings induced by the classical Shapley–Shubik and Penrose–

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Banzhaf–Coleman indices coincide. Diffo Lambo and Moulen (2002) extended Tomiyama's result to all complete simple games (also called in the literature: swap robust games, linear games or ordered games). Carreras and Freixas (2008) extended Diffo Lambo and Moulen's result for weakly complete simple games. In this paper I study the ordinal equivalence problem in the general context of cooperative games. To this end I consider three types of cooperative games: weakly complete, semicoherent and coherent. As we shall see these sets of games are related by strict inclusion. Indeed, every weakly complete game is semicoherent and every semicoherent game is coherent.

Complete games have been extensively studied for simple games since Isbell considered the desirability relation in Isbell (1958). Some references on complete simple games are: Carreras and Freixas (1996), Friedman et al. (2006), and Taylor and Zwicker (1999). Weakly complete games have been introduced in the context of simple games in Carreras and Freixas (2008). Although the extension of complete games from simple games to cooperative games is quite obvious, the extension for weakly complete games can be done in several different ways. The one I choose in this paper is a feasible election. As far as I know semicoherent and coherent games are new classes of cooperative games.

For cooperative games, I consider the total preorderings on the set of players (rankings) induced by the application of semivalues. Roughly speaking, any functional from the space of transferable utility cooperative games with a fixed set of players to the space of n-vectors is a *value*. A *semivalue* is a value uniquely defined by a group of axioms due to Dubey et al. (1981). I consider three types of semivalues in this paper: regular, one-point binomial and segment semivalues. Regular and binomial semivalues were introduced and characterized in Carreras and Freixas (1999).

The paper is organized as follows. Three preorderings on the set of players are given in Sect. 2, the completeness of each preordering provides a subclass of cooperative games. The relationship among the considered three types of cooperative games is established. Section 3 recalls semivalues. Several special types of semivalues are introduced as well as the total relation on the set of players (ranking) they induce. The main result of the paper proposes a characterization in terms of the ordinal equivalence of some semivalues for each of the considered three types of cooperative games. Some final comments conclude the paper in Sect. 4.

2 Some subclasses of cooperative games

Let $N = \{1, 2, \ldots, n\}$ be the finite set of *players* and 2^N be the set of *coalitions* (subsets of N). A *cooperative game with transferable utility* (TU) on N is a map $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Games on N form a real vector space $\mathcal{G}_N$ endowed with the usual linear operations given by the sum and the scalar product. As is well known, the dimension of the vector space $\mathcal{G}_N$ is $2^n - 1$. A *value* is a map $f : \mathcal{G}_N \to \mathbb{R}^n$ that assigns to each game $v \in \mathcal{G}_N$ a vector of $\mathbb{R}^n$, whose components represent the payoff that each player receives. A cooperative game is *simple* if $v(S) = 0$ or 1 for all coalition $S \subset N$ and $v(N) = 1$. A cooperative game is *monotonic* if $v(S) \leq v(T)$

whenever $S \subset T$. Hereafter only monotonic cooperative TU games on the set of players N are considered and write v instead of (N, v) or $v \in \mathcal{G}_N$.

Player's strategic positions in v can be compared by means of several relations, three of them are proposed here (Definitions 2.1, 2.3 and 2.5). The basic problem with these relations is that they are not always total. Completeness of these relations suggests the consideration of three types of cooperative games (Definitions 2.2, 2.4 and 2.6). See e.g. Sen (1970) for a reference about some concepts of Linear Algebra applied to Social Sciences and used in the paper, like e.g. relations, orderings, preorderings, sub-preorderings, etc.

Given an arbitrary cooperative game v on a finite set of players N, let us define, for all $i \in N$ and $0 \leq k \leq n-1$, the n^2 numbers:

$$c_i(k) = \sum_{\substack{S \subseteq N \setminus \{i\} \\ |S|=k}} (v(S \cup \{i\}) - v(S)).$$

Definition 2.1 Let v be an arbitrary cooperative game on a finite set of players N and consider

$$i \succsim_d j \quad \text{iff} \quad c_i(k) \geq c_j(k) \quad \text{for all} \quad k = 0, 1, \ldots, n-1.$$

Then $\succsim_d$ is a preordering on N because $\succsim_d$ is reflexive ($i \succsim_d i$ for all i) and transitive ($i \succsim_d j,\ j \succsim_d k \Rightarrow i \succsim_d k$). $\succsim_d$ (resp., $\succ_d$) is called the *weak desirability* (resp., *strict weak desirability*) relation on N and $\approx_d$ the *weak equi-desirability* relation on N.

The *strict weak desirability* relation on N between two players, $i \succ_d j$, arises whenever $c_i(k) \geq c_j(k)$ for all $k = 0, 1, \ldots, n-1$ and $c_i(k') > c_j(k')$ for at least one k', whereas the *weak equi-desirability* relation on N, $i \approx_d j$, arises whenever $c_i(k) = c_j(k)$ for all $k = 0, 1, \ldots, n-1$.

Definition 2.2 An arbitrary cooperative game v on a finite set of players N is *weakly complete* whenever $\succsim_d$, the weak desirability relation on N, is total.

Note that if $\succsim_D$ denotes the standard desirability relation on N ($i \succsim_D j$ iff $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$) and $\succ_D$ the strict desirability relation ($i \succ_D j$ iff $i \succsim_D j$ and $v(S \cup \{i\}) > v(S \cup \{j\})$ for some $S \subseteq N \setminus \{i, j\}$) then clearly $\succsim_D$ is a sub-preordering of $\succsim_d$ because $i \succsim_D j$ implies $i \succsim_d j$ and $i \succ_D j$ implies $i \succ_d j$. Hence, any complete cooperative game (for which $\succsim_D$ is a total relation on N) is weakly complete and preorderings $\succsim_D$ and $\succsim_d$ on N coincide.

If each member $i \in N$ is willing to form coalition S with probability a_i and independently of the others then coalition S is formed with probability

$$\prod_{i \in S} a_i \prod_{i \in N \setminus S} (1 - a_i)$$

and the expected value of the game is

$$f(a_1, \ldots, a_n) = \sum_{S \subseteq N} \left[\prod_{i \in S} a_i \prod_{i \in N \setminus S} (1 - a_i) \right] v(S). \tag{1}$$

The function f, here with domain $[0, 1]^n$ is the *multilinear extension* of a cooperative game v on N ($|N| = n$) which was treated in detail by Owen (1975). I will essentially be restricted to the case wherein all components coincide, i.e., $(a_1, \ldots, a_n) = (p, \ldots, p)$ for some $p \in [0, 1]$. I will denote by $f_i(a_1, \ldots, a_n)$ the partial derivative of f with respect to component i in $(a_1, \ldots, a_n) \in (0, 1)^n$.

Definition 2.3 Let v be an arbitrary cooperative game on a finite set of players N and consider the equivalence relation on N (i.e. reflexive, symmetric and transitive)

$$i \approx_C j \quad \text{iff} \quad f_i(p, \ldots, p) = f_j(p, \ldots, p) \quad \text{for all } p \in (0, 1)$$

and the strict preordering on N

$$i \succ_C j \quad \text{iff} \quad f_i(p, \ldots, p) > f_j(p, \ldots, p) \quad \text{for all } p \in (0, 1).$$

The pair $(\approx_C, \succ_C)$ gives rise to the *circumstantial* relation $\succsim_C$ on N.

The term "circumstantial" was coined in Freixas and Pons (2005) for simple games [see also Freixas and Pons (2008a)] to referring $f_i(a_1, \ldots, a_n)$ as a power measure if each player is willing to vote for the issue at hand with independent probability a_i.

Definition 2.4 An arbitrary cooperative game v on a finite set of players N is *semi-coherent* whenever $\succsim_C$, the circumstantial relation on N, is total.

Definition 2.5 Let v be an arbitrary cooperative game on a finite set of players N and consider the equivalence relation on N

$$i \approx_c j \quad \text{iff} \quad \int_0^1 f_i(p, \ldots, p)\, dp = \int_0^1 f_j(p, \ldots, p)\, dp \quad \text{and}$$
$$f_i(1/2, \ldots, 1/2) = f_j(1/2, \ldots, 1/2)$$

and the strict preordering on N

$$i \succ_c j \quad \text{iff} \quad \int_0^1 f_i(p, \ldots, p)\, dp > \int_0^1 f_j(p, \ldots, p)\, dp \quad \text{and}$$
$$f_i(1/2, \ldots, 1/2) > f_j(1/2, \ldots, 1/2).$$

The pair $(\approx_c, \succ_c)$ gives rise to the *ordinally coherent relation* $\succsim_c$ on N.

Definition 2.6 An arbitrary cooperative game v on a finite set of players N is *coherent* whenever $\succsim_c$, the ordinally coherent relation on N, is total.

Owen (1975, 1978) proved the two following identities, which are useful to compute the Banzhaf and Shapley values, respectively, by means of the partial derivative of the multilinear extension:

$$\begin{aligned} \beta_i[v] &= f_i(1/2, \ldots, 1/2), \\ \phi_i[v] &= \int_0^1 f_i(p, \ldots, p)\, dp. \end{aligned} \tag{2}$$

Thus, a game is coherent if the Shapley and Banzhaf values are ordinally equivalent. The term coherent was already considered in Freixas and Pons (2008b) for simple games.

Theorem 2.8 establishes the relationship among weakly complete, semicoherent and coherent cooperative games. The next example shows the existence of semicoherent simple games which are not weakly complete and the existence of coherent simple games which are not semicoherent.

Example 2.7

(i) There exists a simple game which is semicoherent but not weakly complete.
(ii) There exists a simple game which is coherent but not semicoherent.

(i) Consider the simple game v with $|N| = 10$ in which N decomposes in a tripartition, each of these three subsets is formed by equi-desirable voters. Let $N_1 = \{1, 2, 3\}$, $N_2 = \{4, 5\}$ and $N_3 = \{6, 7, 8, 9, 10\}$ be the tripartition of N. Rather than describing the full set of minimal winning coalitions we give the "types" of minimal winning coalitions, henceforth models. Thus, model (a, b, c) represents *all* coalitions S such that: $|S \cap N_1| = a$, $|S \cap N_2| = b$ and $|S \cap N_3| = c$. The list of models in $\mathcal{W}^m$ appears in Table 1. An exhaustive check leads to the numbers in Table 2. Clearly, v is not weakly complete since, for example, $1 = c_i(2) > c_j(2) = 0$ and $7 = c_i(3) < c_j(3) = 10$ for $i = 1, 2, 3$ and $j = 4, 5$, so that $i \not\succsim_d j$ and $j \not\succsim_d i$. Instead, if now take $i = 1, 2, 3, 4, 5$ and $j = 6, 7, 8, 9, 10$ it yields $i \succ_d j$ and thus $i \succ_C j$. To check that the game is semicoherent it only remains to prove that $f_i(p, \ldots, p) > f_j(p, \ldots, p)$ for each $p \in (0, 1)$, $i = 1, 2, 3$ and $j = 4, 5$. By using the figures in Table 2 and formula (1) it yields:

$$\begin{aligned} f_i(p, \ldots, p) &= p^2(1-p)^3[(1-p)^4 + 7p(1-p)^3 + 81p^2(1-p)^2 \\ &\qquad + 56p^3(1-p) + 22p^4] \\ f_j(p, \ldots, p) &= p^3(1-p)^3[10(1-p)^3 + 70p(1-p)^2 + 35p^2(1-p) + 16p^3], \end{aligned}$$

Table 1 List of the five models representing the 104 minimal winning coalitions of (N, v)

Model	Coalitions	Number of coalitions
(3,0,0)	{1,2,3}	1
(0,2,2)	{4,5,6,7}, …, {4,5,9,10}	10
(2,0,3)	{1,2,6,7,8}, …, {2,3,8,9,10}	30
(2,1,2)	{1,2,4,6,7}, …, {2,3,5,9,10}	60
(1,0,5)	{1,6,7,8,9,10}, …, {3,6,7,8,9,10}	3

Table 2 $c_i(k)$ numbers

i\k	0	1	2	3	4	5	6	7	8	9	Sum
1, 2, 3	0	0	1	7	81	56	22	0	0	0	167
4, 5	0	0	0	10	70	35	16	0	0	0	131
6, 7, 8, 9, 10	0	0	0	6	54	15	6	0	0	0	81

thus

$$\begin{aligned} f_i(p, \ldots, p) - f_j(p, \ldots, p) &= p^2(1-p)^3[(1-p)^4 - 3p(1-p)^3 \\ &\quad + 11p^2(1-p)^2 + 21p^3(1-p) + 6p^4] \\ &= p^2(1-p)^3[1 - 7p + 26p^2 - 14p^3]. \end{aligned}$$

Then, $f_i(p, \ldots, p) - f_j(p, \ldots, p) > 0$ for all $p \in (0, 1)$ iff $1-7p+26p^2-14p^3 > 0$ for all $p \in (0, 1)$. Let $h(p) = 1-7p+26p^2-14p^3$, as $h(0) = 1$, $h(1) = 6$ and $h(p)$ is a continuous function in $[0, 1]$ it is enough to prove that the absolute minimum p^m in $[0, 1]$, which exists by the Weirstrass' theorem, satisfies $h(p^m) > 0$. To this end, I consider equation $h'(p) = 0$, that is,

$$-7 + 52p - 42p^2 = 0$$

which has two real roots and one of them, 0.15369 belongs to $(0, 1)$. By evaluating the second derivative of $h(p)$ in such a value it follows: $h''(0.15369) = 39.09 > 0$ so that 0.15369 is a relative minimum. As $h(0.15369) = 0.48748 < 1 = \min\{h(0), h(1)\}$, it follows that 0.15369 is the absolute minimum p^m for $h(p)$ in $[0, 1]$. Thus $h(p) > 0.48$ for all $p \in (0, 1)$ and, consequently, $f_i(p, \ldots, p) - f_j(p, \ldots, p) > 0$ for all $p \in (0, 1)$ which implies $i \succ_C j$. As a conclusion v is not weakly complete, even though $\succsim_C$ is total and so v is a semicoherent game.

(ii) Consider the simple game v on N, with $|N| = 8$, defined by

$$\begin{aligned} \mathcal{W}^m = \{&\{1, 4, 5, 6, 7\}, \{1, 4, 5, 6, 8\}, \{1, 4, 5, 7, 8\}, \{1, 4, 6, 7, 8\}, \\ &\{1, 5, 6, 7, 8\}, \{2, 3, 4, 5, 6, 7\}, \{2, 3, 4, 5, 6, 8\}, \{2, 3, 4, 5, 7, 8\}, \\ &\{2, 3, 4, 6, 7, 8\}, \{2, 3, 5, 6, 7, 8\}\}. \end{aligned}$$

Table 3 $c_i(k)$ and $f_i(p, \ldots, p)$ numbers for $p = 0.25, 0.5$ and 0.75

i\k	$c_i(k)$						Sum	$f_i(p, \ldots, p)$		
	...	3	4	5	6	7		$p = 0.25$	$p = 0.5$	$p = 0.75$
1		0	5	11	2	0	18	0.0146	0.1406	0.2769
2, 3		0	0	5	1	0	6	0.0117	0.0469	0.1187
4, 5, 6, 7, 8		0	4	12	4	0	20	0.0139	0.1562	0.3757

The rankings induced for ϕ and β coincide since

$$\phi[v] = \frac{1}{210}(25, 10, 10, 33, 33, 33, 33, 33) \text{ and } \beta[v] = \frac{1}{64}(9, 3, 3, 10, 10, 10, 10, 10).$$

Hence, the game is coherent. But it is not semicoherent, since, e.g.,

$$f_1(0.25, \ldots, 0.25) > f_4(0.25, \ldots, 0.25) \quad \text{and} \quad f_1(0.5, \ldots, 0.5) < f_4(0.5, \ldots, 0.5).$$

This is illustrated in Table 3 below which is obtained from the expressions:

$$f_1(p, \ldots, p) = p^4(1-p)(5 + p - 4p^2)$$
$$f_2(p, \ldots, p) = p^5(1-p)(5 - 4p)$$
$$f_4(p, \ldots, p) = p^4(1-p)(1 + p - p^2)$$

2.1 The relationship among the three classes of cooperative games

Theorem 2.8 *Let v be an arbitrary cooperative game on a finite set of players N.*

(i) *If v is weakly complete then v is semicoherent.*
(ii) *If v is semicoherent then v is coherent.*

However, the converses are not true even in the subclass of simple games.

Proof (i) $i \succ_d j$ iff $c_i(k) \geq c_j(k)$ for all $k = 0, 1, \ldots, n-1$ and $c_i(k') > c_j(k')$ for at least one k', which *implies*

$$\sum_{k=0}^{n-1} p^k (1-p)^{n-k-1} c_i(k) > \sum_{k=0}^{n-1} p^k (1-p)^{n-k-1} c_j(k) \quad \text{for all } p \in (0,1)^n. \tag{3}$$

Furthermore,

$$f_i(p,\ldots,p) = \sum_{k=0}^{n-1} p^k (1-p)^{n-k-1} c_i(k) \quad \text{for all } p \in (0,1)^n \tag{4}$$

because by taking the partial derivative in (1), for all $(a_1,\ldots,a_n) \in (0,1)^n$, it follows:

$$f_i(a_1,\ldots,a_n) = \sum_{S \subseteq N\setminus\{i\}} \left[\prod_{j\in S} a_j \prod_{j \notin N\setminus(S\cup\{i\})} (1-a_j) \right] (v(S\cup\{i\}) - v(S)) \tag{5}$$

and by taking $(a_1,\ldots,a_n) = (p,\ldots,p)$ in formula (5) it yields

$$\begin{aligned} f_i(p,\ldots,p) &= \sum_{s=0}^{n-1} p^s (1-p)^{n-s-1} \left[\sum_{\substack{S\subseteq N\setminus\{i\} \\ |S|=s}} (v(S\cup\{i\}) - v(S)) \right] \\ &= \sum_{s=0}^{n-1} p^s (1-p)^{n-s-1} c_i(s) \end{aligned} \tag{6}$$

where $s = |S|$. Thus, inequality (3) and Eq. 4 yield $f_i(p,\ldots,p) > f_j(p,\ldots,p)$ for all $p \in (0,1)^n$ and hence $i \succ_C j$.

On the other hand, $i \approx_d j$ iff $c_i(k) = c_j(k)$ for all $k = 0, 1, \ldots, n-1$, which *implies*

$$\sum_{k=0}^{n-1} p^k (1-p)^{n-k-1} c_i(k) = \sum_{k=0}^{n-1} p^k (1-p)^{n-k-1} c_j(k) \quad \text{for all } p \in (0,1)^n.$$

Thus, $f_i(p,\ldots,p) = f_j(p,\ldots,p)$ for all $p \in (0,1)^n$ and whence $i \approx_C j$.

Therefore $i \succsim_d j$ implies $i \succsim_C j$. Then preordering $\succsim_d$ on N is a *sub-preordering* of $\succsim_C$ on N. As $\succsim_d$ is total (because v is weakly complete) then so is $\succsim_C$ and both preorderings on N coincide giving rise to the same rankings.

The simple game considered in Example 2.7(i) is semicoherent but not weakly complete. Thus the converse is not true for simple games.

(ii) $i \succ_C j$ iff $f_i(p,\ldots,p) > f_j(p,\ldots,p)$ for all $p \in (0,1)$ where f is the multilinear extension of game v, then by (2) $\phi_i[v] > \phi_j[v]$ and $\beta_i[v] > \beta_j[v]$ which implies $i \succ_c j$. Analogously, $i \approx_C j$ iff $f_i(p,\ldots,p) = f_j(p,\ldots,p)$ for all $p \in (0,1)$ implies $\phi_i[v] = \phi_j[v]$ and $\beta_i[v] = \beta_j[v]$ and thus $i \approx_c j$. Moreover, the completeness of $\succsim_C$ for semicoherent games implies the completeness of $\succsim_c$ and both preorderings on N coincide giving rise to the same rankings.

The simple game considered in Example 2.7(ii) is coherent but not semicoherent. Thus the converse is not true for simple games. □

While the concept of coherent game is associated with the pair Shapley–Banzhaf, the concept of semicoherent game is quite general because it is relative to the choice of parameters all equal to p in the multilinear extension function. The most general concept among the three considered is the one of weakly complete game which is associated with sums of marginal contributions for each allowable size. Theorem 2.8 establishes that sufficient conditions for a game to be semicoherent and coherent are, respectively, to be weakly complete and semicoherent, but these conditions are not necessary.

3 Some types of semivalues

Semivalues were first introduced in Weber (1979) only for simple games and extended to all cooperative games in Dubey et al. (1981). The axiomatic characterization given in Weber (1988) for this class of games has been adapted to simple games in Carreras and Freixas (2003). Following Dubey et al. (1981) a *semivalue* is a solution concept $\psi : \mathcal{G}_N \to \mathbb{R}^N$ that satisfies *linearity*, *positivity* (monotonic games receive nonnegative payoff vectors), the *dummy player* (a dummy player i in game v, if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N \setminus \{i\}$, gets just $\psi_i[v] = v(\{i\})$) axioms (hence, probabilistic values) and *symmetry* (which, in particular, implies the equal treatment property, i.e. if $i \approx_D j$ then $\psi_i[v] = \psi_j[v]$). As is well known, the *Shapley value* ϕ (Shapley 1953) becomes the only *efficient* semivalue, in the sense that $\sum_{i \in N} \phi_i[v] = v(N)$ for every v. There is an interesting characterization of semivalues in terms of *weighting coefficients*, provided by Dubey et al. (1981) from the group of axioms for TU cooperative games. For every weighting vector

$$p = (p_0, p_1, \ldots, p_{n-1}) \text{ such that } p_k \geq 0 \text{ for all } k \text{ and } \sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1, \tag{7}$$

let ψ be defined by

$$\psi_i[v] = \sum_{S \subseteq N \setminus \{i\}} p_s[v(S \cup \{i\}) - v(S)] \tag{8}$$

for all $i \in N$ and all $v \in \mathcal{G}_N$, where $s = |S|$ hereafter. Then ψ is a semivalue. Conversely, any semivalue can be defined in this way, and the correspondence $p \mapsto \psi$ is one-to-one. For example, the Shapley value ϕ is defined by weighting coefficients $p_k = 1/\left(n\binom{n-1}{k}\right)$, and the *Banzhaf value* β (Owen 1975) is the only semivalue whose weighting coefficients are constant, i.e., $p_k = 1/2^{n-1}$ for all k.

Some particular subclasses of semivalues for TU cooperative games belonging to the space associated with the set of games with the fixed set of players N deserve consideration. A *semivalue* ψ is *regular* (Carreras and Freixas 1999) if $p_k > 0$ for all $k = 0, 1, \ldots, n-1$. In Carreras and Freixas (1999) there are characterizations for regular semivalues and for semivalues with binomial coefficients, i.e. a semivalue ψ^p is *p-binomial* whenever $p_k = p^k(1-p)^{n-k-1}$ for all $k = 0, 1, \ldots, n-1$ and $p \in [0, 1]$. The Banzhaf value β is the 1/2-binomial semivalue (so that $\beta = \psi^{1/2}$) and the Shapley value is a regular semivalue. The only binomial non-regular semivalues are the marginal and dictatorial indices introduced by Owen (1978) for $p = 0$ and $p = 1$, respectively. Coalitional semivalues are studied in Albizuri and Zarzuelo (2004) and general properties of semivalues in Dragan (1999).

The following result provides a simple way to calculate regular binomial semivalues by using the multilinear extension of the game.

Lemma 3.1 *Let v be a cooperative game in N, $(p, \ldots, p) \in (0, 1)^n$, and ψ^p be a regular p-binomial semivalue ψ^p. Then, $f_i(p, \ldots, p) = \psi_i^p[v]$.*

Proof By using equation (5) which follows by taking the partial derivative in (1) for all $(a_1, \ldots, a_n) \in (0, 1)^n$, it yields (6), thus $f_i(p, \ldots, p) = \sum_{s=0}^{n-1} p^s(1-p)^{n-s-1}c_i(s)$. On the other hand, by replacing the coefficients for p-binomial semivalues in the semivalue formula (8) it yields $\psi_i^p[v] = \sum_{s=0}^{n-1} p^s(1-p)^{n-s-1}c_i(s)$ which concludes the proof. □

The family formed by all binomial semivalues geometrically depict a closed curve in the n-dimensional simplex given by equation $\sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1$ with $p_k \geq 0$ for all k. The two extreme points of this closed curve are the dictatorial ($p_0 = 1$ and $p_k = 0$ if $k \neq 0$) and the marginal ($p_{n-1} = 1$ and $p_k = 0$ if $k \neq n-1$) semivalues. If only regular binomial semivalues are considered then the curve, denoted hereafter $\mathcal{B}$, becomes an open curve. The Shapley semivalue, which belongs to the n-dimensional simplex, corresponds to an external point of $\mathcal{B}$ which can be interpreted as the point at infinity. Following the topological notion of one-point compactification of an open curve, see e.g. Munkres (1975), it can be establish the following.

Definition 3.2 Let $\mathcal{B}$ be the open curve of regular binomial semivalues inside the n-dimensional simplex. Take semivalue ϕ (the Shapley value) outside $\mathcal{B}$, and adjoin it to $\mathcal{B}$ forming the set $\mathcal{P} = \mathcal{B} \cup \{\phi\}$. The set $\mathcal{P}$ is called the *one-point compactification* of $\mathcal{B}$ and its elements are the *one-point binomial semivalues*.

In the same way, that the one-point compactification of the real line $\mathbb{R}$ is homeomorphic with the circle or the one-point compactification of the $\mathbb{R}^2$ is homeomorphic to the sphere, the one-point compactification of all regular binomial semivalues is homeomorphic with the "circle" formed by all regular binomial semivalues with the addition of the Shapley semivalue. It is also meaningful to introduce here the notion of segment semivalues.

Definition 3.3 Let v be an arbitrary cooperative game on a finite set of players N and ψ and ψ' be two semivalues. It is obvious from (7) that $(1-a)\psi + a\psi'$ also defines a semivalue for all $a \in (0, 1)$. The closed segment limited by ψ and ψ' contains the semivalue $(1-a)\psi + a\psi'$ for all $a \in [0, 1]$ and its elements are the ψ-ψ' *segment semivalues*.

Finally consider the separability relation by semivalues or by subclasses of them.

Definition 3.4 Let v be a cooperative game on N and ψ be a semivalue. Consider the equivalent relation on N

$$i \approx_{\psi} j \quad \text{iff} \quad \psi_i[v] = \psi_j[v]$$

and the strict preordering on N

$$i \succ_{\psi} j \quad \text{iff} \quad \psi_i[v] > \psi_j[v].$$

The pair $(\approx_{\psi}, \succ_{\psi})$ gives rise to the *separability* relation on N (with respect to semivalue ψ).

Then $\succsim_{\psi}$ is a total preordering on N called the *separability* relation $\succsim_{\psi}$ on N (with respect to the semivalue ψ). In particular, the notations $\succsim_{\phi}$ and $\succsim_{\beta}$ represent the separability relations with respect to the Shapley and Banzhaf values; whereas $\succsim_{\psi^p}$ stands for the separability relation with respect to the p-binomial semivalue on N.

3.1 The main result

I provide a characterization for: weakly complete, semicoherent and coherent cooperative games in terms of the ordinal equivalence of some special types of semivalues: regular, binomial and segment semivalues.

Theorem 3.5 *Let v be an arbitrary cooperative game on a finite set of players N. Then:*

(i) *v is weakly complete iff all separability relations on N given by regular semivalues coincide.*
(ii) *v is semicoherent iff all separability relations on N given by regular binomial semivalues coincide.*
(iii) *v is coherent iff all separability relations on N given by ϕ-β segment semivalues coincide.*

Proof

(i) ($\Rightarrow$) Assume that v is weakly complete and ψ is a regular semivalue. I need to check that $i \approx_d j$ implies $i \approx_{\psi} j$ and $i \succ_d j$ implies $i \succ_{\psi} j$. I start by writing formula (8) in an equivalent way

$$\psi_i[v] = \sum_{s=0}^{n-1} p_s \left[\sum_{\substack{S \subseteq N \setminus \{i\} \\ |S|=s}} (v(S \cup \{i\}) - v(S)) \right] = \sum_{s=0}^{n-1} p_s c_i(s). \tag{9}$$

Thus

$$\psi_i[v] - \psi_j[v] = \sum_{s=0}^{n-1} p_s [c_i(s) - c_j(s)]. \tag{10}$$

$i \approx_d j$ implies $c_i(s) = c_j(s)$ for all $s = 0, 1, \ldots, n-1$ and by (10) $\psi_i[v] = \psi_j[v]$ for all semivalue (not necessarily regular) and whence $i \approx_\psi j$.
$i \succ_d j$ implies $c_i(s) \geq c_j(s)$ for all $s = 0, 1, \ldots, n-1$ and $c_i(s') > c_j(s')$ for at least one s'. As $p_s > 0$ for all s, it follows from (10) that $\psi_i[v] - \psi_j[v] > 0$ and whence $i \succ_\psi j$.
($\Leftarrow$) Let us assume that v is not weakly complete. Then there are $i, j \in N$ such that $i \not\succsim_d j$ and $j \not\succsim_d i$. Thus, there exist h and ℓ such that $c_i(h) < c_j(h)$ and $c_i(\ell) > c_j(\ell)$. For any $\epsilon \in (0, 1)$ let ψ' be the regular semivalue defined by means of a weighting vector $p' = (p'_0, p'_1, \ldots, p'_{n-1})$ such that $p'_h = (1-\epsilon)/\binom{n-1}{h}$; $p'_k = \epsilon/\left((n-1)\binom{n-1}{k}\right)$ if $k \neq h$. Then, setting $\epsilon \approx 0$ and by applying formula (10), it follows

$$\psi'_i[v] \approx \frac{c_i(h)}{\binom{n-1}{h-1}} \quad \text{and} \quad \psi'_j[v] \approx \frac{c_j(h)}{\binom{n-1}{h-1}},$$

so that $\psi'_i[v] < \psi'_j[v]$ and hence $i \not\succsim_{\psi'} j$ (or, equivalently by completeness, $j \succ_{\psi'} i$). In a similar way, by interchanging roles, it can be defined a regular semivalue ψ'' such that $j \not\succsim_{\psi''} i$ (equivalently, $i \succ_{\psi''} j$). Therefore, the separability relations on N $\succsim_{\psi'}$ and $\succsim_{\psi''}$ differ.

(ii) ($\Rightarrow$) Assume that v is semicoherent. If $i \approx_C j$, then $f_i(p, \ldots, p) = f_j(p, \ldots, p)$ for all $p \in (0, 1)$. Formula (6) and Lemma 3.1 lead to $\psi_i^p[v] = \psi_j^p[v]$ for all $p \in (0, 1)$, whence $i \approx_{\psi^p} j$ for all regular binomial semivalue ψ^p.
If $i \succ_C j$, then $f_i(p, \ldots, p) > f_j(p, \ldots, p)$ for all $p \in (0, 1)$. Applying again formula (6) and Lemma 3.1 it follows $\psi_i^p[v] > \psi_j^p[v]$ for all $p \in (0, 1)$, and whence $i \succ_{\psi^p} j$ for all regular binomial semivalue ψ^p.
Then preordering $\succsim_C$ on N is a *sub-preordering* of $\succsim_{\psi^p}$ on N whenever $p \in (0, 1)$. As $\succsim_C$ is total then so is $\succsim_{\psi^p}$ and both preorderings on N coincide.
($\Leftarrow$) Obviously, if $\succsim_C$ coincides with the total preordering $\succsim_{\psi^p}$ on N for some regular binomial semivalue ψ^p, $\succsim_C$ is also total and whence v is semicoherent.

(iii) ($\Rightarrow$) Assume that v is coherent and $\psi = (1-a)\phi + a\beta$ for some $a \in (0, 1)$ is a ϕ-β segment semivalue. I need to check that $i \approx_c j$ implies $i \approx_\psi j$ and $i \succ_c j$ implies $i \succ_\psi j$. These implications are obvious if $n < 3$ because $\phi = \beta$. Assume below $n \geq 3$.
$i \approx_c j$ iff $\phi_i[v] = \phi_j[v]$ and $\beta_i[v] = \beta_j[v]$ iff $\psi_i[v] = ((1-a)\phi + a\beta)_i[v] = (1-a)\phi_i[v] + a\beta_i[v] = (1-a)\phi_j[v] + a\beta_j[v] = ((1-a)\phi + a\beta)_j[v] = \psi_j[v]$; whence $i \approx_\psi j$.
$i \succ_c j$ iff $\phi_i[v] > \phi_j[v]$ and $\beta_i[v] > \beta_j[v]$ iff $\psi_i[v] = ((1-a)\phi + a\beta)_i[v] = (1-a)\phi_i[v] + a\beta_i[v] > (1-a)\phi_j[v] + a\beta_j[v] = ((1-a)\phi + a\beta)_j[v] = \psi_j[v]$; whence $i \succ_\psi j$. (Note that a and $1-a$ are both positive).
($\Leftarrow$) This part is obvious by taking $a = 0$ and $a = 1$. □

4 Some final comments

Note that the proof of the main characterization (Theorem 3.5) is fulfilled for arbitrary pair of players i, j. Thus, the following claims are proved:

(i) A player i is strictly more (equally) weakly desirable than the player j in a TU game iff its payoff prescribed him/her by every regular semivalue is greater than (equal to) the payoff of player j.

(ii) A player i is strictly more (equally) circumstantially desirable than the player j in a TU game iff its payoff prescribed him/her by every binomial semivalue is greater than (equal to) the payoff of player j.

(iii) A player i is strictly more (equally) coherently desirable than the player j in a TU game iff its payoff prescribed him/her by every ϕ-β segment semivalue is greater than (equal to) the payoff of player j.

It is important to note that if all separability relations on N given by binomial semivalues coincide (left part of Theorem 3.5(ii)) then all separability relations on N given by one-point binomial semivalues coincide, which may be easily derived from the second identity in (2) and, moreover, all these separability relations coincide with the separability relations on N given by ϕ-β segment semivalues. Thus, the right implication in Theorem 3.5(ii) could be stated in this more general setting, which is a useful version to fit it well with Theorem 2.8(i). However, the given version is preferable because binomial semivalues are enough to check whether the game is semicoherent (left implication).

The subspace spanned by all semivalues within the vector space of valuations on $\mathcal{G}_N$ has dimension n, i.e., semivalues form an n-dimensional simplex within this subspace, and every semivalue is a convex combination of any basis of this subspace. Thus, Theorem 3.5 can be stated as follows:

(i) A *necessary and sufficient* condition for a game v to be weakly complete is that the members of a basis (of that subspace) formed by *regular* semivalues are ordinally equivalent, i.e., their separability relations on N coincide.

(ii) A *necessary and sufficient* condition for a game v to be semicoherent is that the members of a basis (of that subspace) formed by *one-point binomial* semivalues are ordinally equivalent, i.e., their separability relations on N coincide.

(iii) A *necessary* condition for a game to be coherent is the ordinal equivalence of *two* arbitrary semivalues within the ϕ-β segment semivalue. Equivalently, if we are able to find two arbitrary ϕ-β segment semivalues which are not ordinally equivalent then the game is not coherent. In other words, ϕ-β segment semivalues are *inescapably* ordinally equivalent for any coherent game. However, this condition is *not sufficient* if only the ϕ-β *open* segment semivalue was considered, as is illustrated in the next example.

Example 4.1 Let v be a game defined on N where $|N| = 5$ and with three different output values: $0 < \epsilon < 1$. The list of nonempty coalitions with value 0 are (brackets and commas omitted): 1, 2, 3, 5 and 23. The list of coalitions with value ϵ are: 4, 24, 34, 45, 234, 245 and 345. The remaining coalitions have value 1. From Table 4 it follows: $1 \succ_d 2 \approx_d 3 \succ_d 5$.

Table 4 $c_i(k)$ numbers

i\k	0	1	2	3	4
1	0	$3-\epsilon$	$6-3\epsilon$	$3-3\epsilon$	0
2, 3	0	1	2	$1-\epsilon$	0
4	ϵ	1	$1+3\epsilon$	0	0
5	0	0	1	$1-\epsilon$	0

Moreover,

$$\phi[v] = \frac{1}{12}(6, 2, 2, 1, 1) + \frac{\epsilon}{20}(-6, -1, -1, 6, -1),$$
$$\beta[v] = \frac{1}{16}(12 - 7\epsilon, 4 - \epsilon, 4 - \epsilon, 2 + 4\epsilon, 2 - \epsilon).$$

If $\epsilon < 5/21$ it is straightforward to check that v is coherent because ϕ and β show the same rankings. However, $\phi_i[v] = \phi_4[v]$ and $\beta_i[v] > \beta_4[v]$ for $i = 2, 3$ if $\epsilon = 5/21$, whereas all ϕ-β open segment semivalues are ordinally equivalent.

Some new classes of TU cooperative games have been introduced in this paper: weakly complete, semicoherent and coherent games, as well as some new subclasses of semivalues. The paper provides characterizations for weakly complete, semicoherent and coherent TU cooperative games in terms of the ordinal equivalence of some types of semivalues. All these characterizations share the ordinal equivalence of the Shapley and Banzhaf values. As a future research it would be valuable getting new characterizations of these classes of games as well as finding applications for them. Concerning one-point binomial or ϕ-β segment semivalues (or ψ-ψ' segment semivalues for two arbitrary semivalues ψ and ψ'), two subclasses of semivalues considered in the paper, it would be nice obtaining axiomatic characterizations for them. The study of the ordinal equivalence for other values, like e.g. the nucleolus with the Shapley or Banzhaf values, is another interesting point to investigate. I would like to encourage future research in some of these lines.

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References

Albizuri MJ, Zarzuelo JM (2004) On coalitional semivalues. Games Econ Behav 49:221–243

Carreras F, Freixas J (1996) Complete simple games. Math Soc Sci 32:139–155

Carreras F, Freixas J (1999) Some theoretical reasons for using regular semivalues. In: De Swart H (ed) Logic, game theory and social choice. Proceedings of the international conference, LGS, Tilburg, the Netherlands, pp 140–154

Carreras F, Freixas J (2008) On ordinal equivalence of power measures given by regular semivalues. Math Soc Sci 55:221–234

Carreras F, Freixas J, Puente MA (2003) Semivalues as power indices. Eur J Oper Res 149:676–687

Diffo Lambo L, Moulen J (2002) Ordinal equivalence of power notions in voting games. Theory Decis 53:313–325

Dubey P, Neyman P, Weber RJ (1981) Value theory without efficiency. Math Oper Res 6:122–128

Dragan I (1999) Potential and consistency for semivalues of finite cooperative TU games. Int J Math, Game Theory Algebra 9:85–97

Freixas J, Pons M (2005) Two measures of circumstantial power: influences and bribes. Homo Oecon 22:569–588

Freixas J, Pons M (2008a) Circumstantial power: optimal persuadable voters. Eur J Oper Res 186: 1114–1126

Freixas J, Pons M (2008b) Hierarchies achievable in simple games. Theory Decis. doi:10.1007/s11238-008-9108-0

Friedman J, McGrath L, Parker C (2006) Achievable hierarchies in voting games. Theory Decis 61:305–318

Isbell JR (1958) A class of simple games. Duke Math J 25:423–439

Munkres JR (1975) Topology. Prentice-Hall Inc., Englewood Cliffs, New Jersey

Owen G (1975) Multilinear extensions and the Banzhaf value. Nav Res Logist Quart 22:741–750

Owen G (1978) Characterization of the Banzhaf–Coleman index. SIAM J Appl Math 35:315–327

Sen AK (1970) Collective choice and social welfare. Holden-Day, San Francisco, USA

Shapley LS (1953) A value for n-person games. In: Tucker AW, Kuhn HW (eds) Contributions to the theory of games II. Princeton University Press, Princeton, USA, pp 307–317

Taylor AD, Zwicker WS (1999) Simple games: desirability relations, trading, and pseudoweightings. Princeton University Press, New Jersey, USA

Tomiyama Y (1987) Simple game, voting representation and ordinal power equivalence. Int J Policy Inf 11: 67–75

Weber RJ (1979) Subjectivity in the valuation of games. In: Moeschlin O, Pallaschke D (eds) Game theory and related topics. North Holland, Amsterdam, pp 129–136

Weber RJ (1988) Probabilistic values for games. In: Tucker AW, Kuhn HW (eds) Contributions to the theory of games II. Princeton University Press, Princeton, pp 101–119