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# A nonstandard characterization of sequential equilibrium, perfect equilibrium, and proper equilibrium

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**Abstract** New characterizations of sequential equilibrium, perfect equilibrium, and proper equilibrium are provided that use nonstandard probability. It is shown that there exists a belief system $\mu$ such that $(\vec{\sigma}, \mu)$ is a sequential equilibrium in an extensive game with perfect recall iff there exist an infinitesimal $\epsilon$ and a completely mixed behavioral strategy profile $\sigma'$ (so that $\sigma'_i$ assigns positive, although possibly infinitesimal, probability to all actions at every information set) that differs only infinitesimally from $\vec{\sigma}$ such that at each information set $I$ for player $i$, $\sigma_i$ is an $\epsilon$-best response to $\vec{\sigma}'_{-i}$ conditional on having reached $I$. Note that the characterization of sequential equilibrium does not involve belief systems. There is a similar characterization of perfect equilibrium; the only difference is that $\sigma_i$ must be a best response to $\vec{\sigma}'_{-i}$ conditional on having reached $I$. Yet another variant is used to characterize proper equilibrium.

**Keywords** Sequential equilibrium · Perfect equilibrium · Proper equilibrium · Nonstandard probability

## 1 Introduction

*Sequential equilibrium* (Kreps and Wilson 1982) and *perfect equilibrium* (Selten 1975) are refinements of Nash equilibrium in extensive-form games defined in terms of "trembles" or mistakes. The definitions are subtle, involving sequences of strategies with smaller and smaller "trembles". In this paper I provide alternative characterizations of sequential equilibrium and perfect equilibrium using *nonstandard probability*. The definition replaces the sequence of trembles by a single (nonstandard) strategy

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profile and dispenses with belief systems altogether. I would argue that the new definition provides a better conceptual basis for the notion of sequential equilibrium and makes the relationship between sequential equilibrium, perfect equilibrium, and Nash equilibrium more transparent.

The new definition takes as its point of departure the observation that sequential equilibrium and perfect equilibrium have a simple definition for extensive-form games where the equilibrium is a completely mixed strategy profile, so that each strategy assigns positive probability to every action at every information set. A completely mixed strategy profile is a sequential equilibrium iff each player plays a best response *at all information sets*, that is, for each player $i$ and each information set $I$ for player $i$, player $i$ plays a best response to the other players' strategies, conditional on having reached $I$. Moreover, a completely mixed strategy profile is (the strategy component of) a sequential equilibrium iff it is a perfect equilibrium.

All the subtlety in the definition of sequential equilibrium comes in dealing with information sets off the equilibrium path, that is, information sets that are reached with probability 0. To define "best response" at such an information set requires defining what the agent's beliefs are at that information set; this is exactly what is provided by the belief assessment. The consistency requirements imposed on belief assessments ensure that an agent's beliefs are in some sense "reasonable". At an information set that is reached with positive probability, there is an obvious way of defining "reasonable" using conditioning; indeed, an agent's beliefs at such an information set are completely determined by the strategies used. But what should count as a reasonable belief at an information set that the agent believes will not be reached in the first place? This concern is not new. Kreps and Wilson (1982, p. 876) themselves say: "We shall proceed here to develop the properties of sequential equilibrium as defined above; however, we do so with some doubts of our own concerning what 'ought' to be the definition of a consistent assessment that, with sequential rationality, will give the 'proper' definition of a sequential equilibrium." Osborne and Rubinstein (1994, p. 225) say "we do not find the consistency requirement to be natural, since it is stated in terms of limits; it appears to be a rather opaque technical assumption". They go on to quote Kreps (1990, p. 430), who says "[r]ather a lot of bodies are buried in this definition".

The new characterization avoids these difficulties to some extent by considering only completely mixed strategies. Intuitively, actions that are not best responses are given infinitesimal probability rather than 0 probability. As a first step towards the new characterization, consider the following characterization of sequential equilibrium, which is a variant of Kreps and Wilson's (1982) characterization of sequential equilibrium in terms of what they called *weak perfect equilibrium* (see their Proposition 6).

**Proposition 1.1** *If $\Gamma$ is an extensive-form game with perfect recall, then there exists a belief system $\mu$ such that the assessment $(\vec{\sigma}, \mu)$ is a sequential equilibrium in $\Gamma$ iff there exists a sequence $\vec{\sigma}^n$ of completely mixed strategy profiles converging to $\vec{\sigma}$ and a sequence $\epsilon_n$ of nonnegative real numbers converging to 0 such that for each player $i$ and each information set $I$ for player $i$, $\sigma_i^n$ is an $\epsilon_n$-best response to $\vec{\sigma}_{-i}^n$, conditional on having reached $I$.*

Proposition 1.1 defines sequential equilibrium without needing assessments and is arguably quite natural. But it still defines a sequential equilibrium in terms of limits. Using nonstandard reals, we can replace the sequence $\vec{\sigma}^n$ of strategies in Proposition 1.1 by a single strategy and the sequence $\epsilon_n$ by a single infinitesimal to get a characterization that does not involve perturbations of either strategies or utility functions, and does not require assessments.

**Theorem 1.2** *If $\Gamma$ is an extensive-form game with perfect recall, then there exists a belief system $\mu$ such that the assessment $(\vec{\sigma}, \mu)$ is a sequential equilibrium in $\Gamma$ iff there exist an infinitesimal $\epsilon$ and a nonstandard completely mixed strategy profile $\vec{\sigma}'$ that differs infinitesimally from $\vec{\sigma}$ such that, for each player $i$, and each information set $I$ for player $i$, $\sigma_i'$ is an $\epsilon$-best response to $\vec{\sigma}'_{-i}$, conditional on having reached $I$.*

I define all the relevant notions in the statement of Theorem 1.2 in Sect. 2. I continue with a more informal discussion here, in the hope that the reader has enough of an intuitive sense of the relevant notions to follow.

One advantage of Theorem 1.2 is that it makes the connection between sequential equilibrium, Nash equilibrium, and perfect equilibrium more transparent. For Nash equilibrium, the following theorem, whose straightforward proof I leave to the reader, makes it clear that sequential equilibrium is a refinement of Nash equilibrium in the most obvious sense: we get a characterization of Nash equilibrium by dropping the requirement that $\sigma_i$ be an $\epsilon$-best response to $\vec{\sigma}_{-i}$ *at every information set*.

**Theorem 1.3** *If $\Gamma$ is an extensive-form game with perfect recall, then $\vec{\sigma}$ is a Nash equilibrium in $\Gamma$ iff there exist an infinitesimal $\epsilon$ and a nonstandard completely mixed strategy profile $\vec{\sigma}'$ that differs infinitesimally from $\vec{\sigma}$ such that for each player $i$, $\sigma_i'$ is an $\epsilon$-best response to $\vec{\sigma}'_{-i}$.*

The connection to perfect equilibrium is equally straightforward. Note that in Theorem 1.2 I could have equally well required that $\sigma_i$ be an $\epsilon$-best response to $\vec{\sigma}'_{-i}$, since $\sigma_i$ and $\sigma_i'$ differ only infinitesimally. But if we focus on $\sigma_i$, we can ask for even more. We can require $\sigma_i$ to be a *best response* to $\vec{\sigma}'_{-i}$, not just an $\epsilon$-best response. As the following theorem shows, this strengthening of sequential equilibrium is perfect equilibrium.

**Theorem 1.4** *If $\Gamma$ is an extensive-form game with perfect recall, then $\vec{\sigma}$ is a perfect equilibrium in $\Gamma$ iff there exists a nonstandard completely mixed strategy profile $\vec{\sigma}'$ that differs infinitesimally from $\vec{\sigma}$ such that, for each player $i$ and each information set $I$ for player $i$, $\sigma_i$ is a best response to $\vec{\sigma}'_{-i}$, conditional on having reached $I$.*

This characterization of perfect equilibrium is not surprising; it just replaces the sequence of strategies in the standard definition of perfect equilibrium by a single strategy, just as in the transition from Proposition 1.1 and Theorem 1.2. However, it makes it clear that the only difference between sequential equilibrium and perfect equilibrium is whether $\sigma_i$ is required to be a best response or an $\epsilon$-best response to $\sigma'_{-i}$.

*Proper equilibrium* (Myerson 1978) further refines perfect equilibrium by, roughly speaking, requiring that bigger mistakes get smaller probability. This intuition can also be captured easily using nonstandard probability. Since proper equilibrium is typically defined for normal-form games, I give the characterization for normal-form games.

**Theorem 1.5** *If* $\Gamma$ *is a normal-form game, then a mixed strategy profile* $\vec{\sigma}$ *is a proper equilibrium in* $\Gamma$ *iff there exists a completely mixed nonstandard strategy profile* $\vec{\sigma}'$ *that differs infinitesimally from* $\vec{\sigma}$ *such that, for all players* $i$*, if* $s_1$ *and* $s_2$ *are two pure strategies for player* $i$ *such that* $\mathrm{EU}_i(s_1, \vec{\sigma}'_{-i}) < \mathrm{EU}_i(s_2, \vec{\sigma}'_{-i})$ *and* $\mathrm{EU}_i(s_2, \vec{\sigma}'_{-i}) - \mathrm{EU}_i(s_2, \vec{\sigma}'_{-i})$ *is not infinitesimal, then* $\sigma'_i(s_1)/\sigma'_1(s_2)$ *is infinitesimal.*

It is easy to see that if $\vec{\sigma}$ is a proper equilibrium and $\sigma_i(s) > 0$, then $s$ must be a best response to $\vec{\sigma}'_{-i}$. (Otherwise $\sigma'_i(s)$ would be infinitesimal, so $\sigma_i(s)$ would have to be 0, since $\sigma_i(s)$ and $\sigma'_i(s)$ differ only infinitesimally.) It follows that a proper equilibrium must be a perfect equilibrium.

Theorems 1.2, 1.4, and 1.5 serve to illustrate a well-known general phenomenon: sequences of real objects (strategies, numbers) can often by replaced by a single nonstandard object, leading to simpler statements of results. Of course, it may seem that, if the object is nonstandard, then there is no real gain in transparency or understanding. In practice, the sequence of "trembles" required to generate the off-equilibrium beliefs in a sequential equilibrium or the strategies converging to $\vec{\sigma}$ in a perfect or proper equilibrium is usually described in terms of a polynomial or rational function of $\epsilon$, where $\epsilon$ goes to 0. Lemma 3.1 shows that this way of describing the trembles is almost without loss of generality; they can always be described using probabilities that are power series of $\epsilon$ (thus, we may need to go beyond rational functions to functions that involve sines, cosines, and exponentials). Moreover, it follows (Theorem 3.2) that the nonstandard probabilities can always be taken to be a power series in a single nonstandard $\epsilon$. Thus, the nonstandard probabilities involved are not so mysterious.[1]

Theorem 1.2 also suggests an alternative interpretation of sequential equilibrium. It shows that (the strategy component of) a sequential equilibrium is the standard part of an $\epsilon$-sequential equilibrium for an infinitesimal $\epsilon$. (For every nonstandard number $r$, there is a unique closest standard real number $r'$, which differs from $r$ by an infinitesimal; $r'$ is called the *standard part* of $r$. We can then define the standard part of a nonstandard behavioral strategy profile in the obvious way.) Under this interpretation, the focus is on the $\epsilon$-sequential equilibrium $\vec{\sigma}'$, not $\vec{\sigma}$; $\vec{\sigma}$ is just the closest standard approximation to the "true" equilibrium.

An interpretation of Theorems 1.2 and 1.4 that is somewhat closer to the standard interpretation is that they show that in a sequential equilibrium (resp., perfect equilibrium) $\vec{\sigma}$, each player $i$ can be viewed as making an $\epsilon$-best response (resp., best response) at each information set, under the belief that the other players are playing $\vec{\sigma}'_{-i}$ (so that player $i$'s beliefs off the equilibrium path are determined by $\vec{\sigma}'_{-i}$). This belief is reasonable: even repeated observations of the game will not contradict it (the distribution induced by $\vec{\sigma}$ is essentially indistinguishable from that induced by $\vec{\sigma}'$). However, this interpretation has an obvious difficulty: why should we be interested in the particular beliefs induced by $\vec{\sigma}'$? Although they are reasonable, why not consider the beliefs induced by other strategies $\vec{\sigma}''$ that are also infinitesimally close to $\vec{\sigma}$? Put

[1] The suggestion to prove Lemma 3.1 and Theorem 3.2, as well as sketches of their proof, were provided by Hari Govindan.

another way, why is it good enough to know that $\vec{\sigma}$ is an ($\epsilon$-)best response to *some* beliefs? In the more standard definition of sequential equilibrium, this question can be reformulated as asking why we should care that $\sigma_i$ is a best response to the particular beliefs given by the belief assessment. The advantage of the first interpretation is that it avoids this issue altogether.

*Related work.* As I said, Proposition 1.1 is much in the spirit of Kreps and Wilson's characterization of sequential equilibrium in terms of weak perfect equilibrium. Rather than comparing Proposition 1.1 to Kreps and Wilson's results, I compare it to a reformulation due to Blume and Zame (1994). In their Proposition A, Blume and Zame show that $(\vec{\sigma}, \mu)$ is a sequential equilibrium in a game $\Gamma$ with utility functions $\vec{u}$ iff there exists a sequence $\vec{\sigma}^n$ of completely mixed strategies converging to $\vec{\sigma}$ and a sequence $\vec{u}^n$ of utility functions converging to $\vec{u}$ such that for each player $i$ and each information set $I$ for player $i$, $\sigma_i^n$ is a best response to $\vec{\sigma}_{-i}^n$, conditional on having reached $I$, with respect to the utility function $\vec{u}^n$. Clearly if, for all $n$, $\sigma_i^n$ is a best response to $\vec{\sigma}_{-i}^n$, conditional on having reached $I$, with respect to the utility function $\vec{u}^n$, then we can find a sequence $\epsilon^n$ converging to 0 such that $\sigma_i^n$ is an $\epsilon^n$-best response to $\vec{\sigma}_{-i}^n$ conditional on having reached $I$ with respect to $\vec{u}$. Conversely, if the condition stated in Proposition 1.1 holds, it can be shown that Blume and Zame's condition holds as well, although it is actually easier to show directly that there is a sequential equilibrium. Since the characterization of Theorem 1.2 does not need to use sequences of trembles and does not require modifying the game, it is arguably more natural, although "naturalness" is clearly in the eye of the beholder here.

The results of this paper are far from the first use of nonstandard analysis in game theory. Perhaps closest to the results of this paper, Blume et al. (1991) characterize perfect and proper equilibrium using *lexicographic probability systems* (LPS's). There are deep connection between LPS's and nonstandard probability. Hammond (1994) proves that they are equivalent in finite spaces; Halpern (2001) generalizes Hammond's results. Using these connections, it is not hard to prove Theorems 1.4 and 1.5 from the characterizations given by Blume, Brandenburger, and Dekel, and vice versa. More generally, Hammond (1994) argued for thinking of notions such as perfect and proper equilibrium in terms of nonstandard probabilities. Rajan (1998) characterizes a number of solution concepts, including perfect equilibrium and proper equilibrium (but not sequential equilibrium), in terms of agents whose beliefs are given by nonstandard probability distributions. Nonstandard utilities and lexicographic utilities have also been considered by a number of authors (see, e.g., Fishburn 1972; Richter 1971; Skala 1974 and the references therein).

*Outline.* The rest of this paper is organized as follows: In Sect. 2 I give all the relevant formal definitions (and include a review of the definition of sequential equilibrium). Theorems 1.2 and 1.4 are proved in Sect. 3. These proofs depend on a deep theorem of first-order logic, the *compactness theorem* (Enderton 1972), which is explained in Sect. 2. The proof of Theorem 1.5 is so similar to the other two that it is omitted; the proof of Theorem 1.1 is similar in spirit to, but simpler than, that of Theorem 1.2, so is also omitted.

## 2 Definitions

To make the paper self-contained, I briefly review the definitions of sequential equilibrium, perfect equilibrium, and proper equilibrium, and then discuss relevant concepts from logic. I assume that the reader is familiar with the standard definition of games in extensive form and normal form. All of the notions I consider are standard in the game theory and logic literatures.

### 2.1 Sequential equilibrium

Sequential equilibrium is defined with respect to an *assessment*, a pair $(\vec{\sigma}, \mu)$ where $\vec{\sigma}$ is a strategy profile and $\mu$ is a *belief system*, that is, a function that determines for every information set $I$ a probability $\mu_I$ over the histories in $I$. Intuitively, if $I$ is an information set for player $i$, then $\mu_I$ is $i$'s subjective assessment of the relative likelihood of the histories in $I$. Roughly speaking, an assessment is a sequential equilibrium if (a) at every information set where a player moves he chooses a best response given the beliefs he has about the histories in that information set and the strategies of other players, and (b) his beliefs are consistent with the strategy profile being played. Consistency at an information set $I$ is easy to define if $I$ is reached with positive probability by $\vec{\sigma}$; in that case, it is just the result of conditioning on $I$. The definition is a little more subtle if $I$ is reached with probability 0.

Given a strategy profile $\vec{\sigma}$, let $\Pr_{\vec{\sigma}}$ be the probability distribution on complete histories of the game induced by $\vec{\sigma}$. That is, if $h$ is a complete history, then $\Pr_{\vec{\sigma}}(h)$ is the product of the probability of each of the moves made in $h$. If we identify a partial history with the set of complete histories that extend it, this statement holds for partial histories as well. If $I$ is an information set, we take $\Pr_{\vec{\sigma}}(I)$ to be the probability of the set of complete histories extending some partial history in $I$. The conditional probability $\Pr_{\vec{\sigma}}(h \mid I)$ is defined in the standard way.

Formally, an assessment $(\vec{\sigma}, \mu)$ is a *sequential equilibrium* of an extensive-form game $\Gamma$ if it satisfies the following properties:

- *Sequential rationality.* For every information set $I$ of player $i$ and every behavioral strategy $\tau$ for player $i$, $\sigma_i$ is a best response to $\vec{\sigma}_{-i}$ at $I$ given belief system $\mu$; that is,

$$\mathrm{EU}_i((\vec{\sigma}, \mu) \mid I) \geq \mathrm{EU}_i(((\tau, \vec{\sigma}_{-i}), \mu) \mid I),$$

  where $\mathrm{EU}_i((\vec{\sigma}, \mu) \mid I) = \sum_{h \in I} \sum_{z \in Z} \mu_I(h) \Pr_{\vec{\sigma}}(z \mid h) u_i(z)$ and $Z$ is the set of terminal histories of the game $\Gamma$.
- *Consistency between belief system and strategy profile.* There exists a sequence of assessments $(\vec{\sigma}^n, \mu^n)$, $n = 1, 2, 3, \ldots$, such that, for all $n$, $\vec{\sigma}^n$ is a profile of completely mixed strategies, $\vec{\sigma}^n \to \vec{\sigma}$, and $\mu_I(h) = \lim_{n \to \infty} \Pr_{\vec{\sigma}^n}(h|I)$. (Since each strategy in $\vec{\sigma}^n$ is completely mixed, $\Pr_{\vec{\sigma}^n}(I) > 0$, so the conditional probability is well defined.)

## 2.2 Perfect equilibrium and proper equilibrium

To define perfect equilibrium, I first need to define the notion of a best response at an information set.

**Definition 2.1** If $\epsilon \geq 0$ and $I$ is an information set for player $i$, $\sigma_i$ is an *$\epsilon$-best response to $\vec{\sigma}_{-i}$ for i conditional on having reached I* if for every strategy $\tau$ for player $i$ that agrees with $\sigma$ except possibly at $I$ and information sets preceded by $I$ we have $\mathrm{EU}_i(\vec{\sigma}) \geq \mathrm{EU}_i(\tau, \vec{\sigma}_{-i}) - \epsilon$. A *best response* is a 0-best response.

**Definition 2.2** $\vec{\sigma}$ is a *perfect equilibrium* in an extensive-form game $\Gamma$ iff there exists a sequence $\vec{\sigma}^n$ of completely mixed behavior strategies such that $\vec{\sigma}^n \rightarrow \vec{\sigma}$ and, for all $n$ and each information set $I$ of player $i$, $\sigma_i$ is a best response to $\vec{\sigma}^n_{-i}$ conditional on having reached $I$.[2]

Proper equilibrium is typically defined in normal-form games. As I said earlier, proper equilibrium tries to capture the intuition that worse strategies get smaller probability.

**Definition 2.3** $\vec{\sigma}$ is a *proper equilibrium* in a normal-form game if there exists a sequence of completely mixed strategy $\vec{\sigma}^n$ and a sequence of positive real numbers $\epsilon_1, \epsilon_2, \ldots$ such that $\vec{\sigma}^n \rightarrow \vec{\sigma}$, $\epsilon_n \rightarrow 0$, and, for all players $i$ and all pure strategies $s_1$ and $s_2$ for player $i$, if $\mathrm{EU}_i(s_1, \vec{\sigma}^k_{-i}) < \mathrm{EU}_i(s_2, \vec{\sigma}^k_{-i})$, then $\sigma_i(s_1) < \epsilon_k \sigma_i(s_2)$.

## 2.3 Nonstandard probabilities

An *elementary extension $F$ of the reals* is an ordered field that includes the real numbers, at least one *infinitesimal* (i.e., a number $\epsilon$ that is less than every positive real number, but still greater than 0), and is *elementarily equivalent to the real numbers*. The fact that $F$ and $\mathbb{R}$ are elementarily equivalent means that every formula that can be expressed in first-order logic and uses the function symbols $+$ and $\times$ and a constant symbol $\mathbf{r}$ for each real number $r$ is true in $F$ iff it is true in $\mathbb{R}$ (where, in $\mathbb{R}$, $+$ is interpreted as addition, $\times$ is interpreted as multiplication, and the constant symbol $\mathbf{r}$ is interpreted as the real number $r$).

The existence of such elementary extensions of the reals follows from the *compactness theorem* of first-order logic. The compactness theorem says that if every finite subset of an infinite collection of formulas has a model, then the whole infinite set has a model. To see how the existence of elementary extensions of the reals follows from the compactness theorem, add an extra constant symbol $\mathbf{c}$ to the language of the reals, and consider the (uncountable) set $\Phi$ of formulas consisting of

[2] It is more standard to define perfect equilibrium as above in normal-form games, and then define a perfect equilibrium in an extensive-form game $\Gamma$ to be a perfect equilibrium of the normal-form game $\Gamma'$ corresponding to the *agent-normal* form of $\Gamma$, where the agent-normal form of $\Gamma$ has one agent for each information set in $\Gamma$; the agents corresponding to information sets of player $i$ in $\Gamma$ make decisions independently but have the same utility as player $i$. It is not hard to show [using arguments similar to those of Selten (1975, Lemma 6)] that, in games with perfect recall, the standard definition is equivalent to the one given here.

(a) all first-order formulas in the language true of the reals (this includes, e.g., a formula such as $\forall x \forall y (x + y = y + x)$, which says that addition is commutative, as well as formulas such as $\mathbf{2} + \mathbf{3} = \mathbf{5}$ and $\sqrt{\mathbf{2}} \times \sqrt{\mathbf{3}} = \sqrt{\mathbf{6}}$);
(b) the countable collection of formulas $\mathbf{c} < 1/\mathbf{n}$, for each positive natural $n$;
(c) the formula $\mathbf{c} > 0$.

Clearly each finite subset $\Phi'$ of $\Phi$ has a model; namely, the reals; there is always an interpretation for $\mathbf{c}$ as a real number if only finitely many statements in (b) are considered. Compactness says that $\Phi$ must have a model, say $F$. In $F$, $\mathbf{c}$ must be interpreted as an infinitesimal, so $F$ must be an elementary extension of $\mathbb{R}$.

If $F$ is an elementary extension of $\mathbb{R}$, I refer to the elements of $F$ that are in $\mathbb{R}$ as standard reals. It is not hard to show that for all $b \in F$ such that $-r < b < r$ for some standard real $r > 0$, there is a unique closest real number $a$ such that $|a - b|$ is an infinitesimal. (In fact, $a$ is the inf of the set of real numbers that are at least as large as $b$.) Let $st(b)$ denote the closest standard real to $b$; $st(b)$ is sometimes read "the standard part of $b$".

For the purposes of this paper, I take a nonstandard probability to be one whose range lies in an elementary extension of the reals. Two nonstandard probabilities Pr and Pr$'$ are *infinitesimally close* if, for each event $E$, $|\Pr(E) - \Pr'(E)|$ is an infinitesimal. Given a nonstandard probability measure Pr, there is a unique standard probability measure Pr$'$ that is infinitesimally close to it, obtained by taking $\Pr'(E) = st(\Pr(E))$. A behavior strategy for player $i$ just defines a probability distribution over actions at each information set of player $I$. A *nonstandard* behavior strategy is one that uses a nonstandard probability distribution; a *standard behavior strategy* is one that uses a standard probability distribution. Two behavior strategies $\sigma$ and $\sigma'$ for player $i$ are infinitesimally close if, for every information set $I$ of player $i$, the probability distributions defined by $\sigma$ and $\sigma'$ are infinitesimally close.

In Sect. 3.4, I show that in all the main theorems we can use one particular, quite natural, elementary extension of the reals, denoted $\mathbb{R}^*(\epsilon)$. To describe $\mathbb{R}^*(\epsilon)$, first consider $\mathbb{R}(\epsilon)$, the smallest non-Archimedean field strictly containing the reals. $\mathbb{R}(\epsilon)$ consists of all rational expressions $f(\epsilon)/g(\epsilon)$ in $\epsilon$, where $f$ and $g$ are polynomials and $g \neq 0$. [The construction of $\mathbb{R}(\epsilon)$ apparently goes back to Robinson (1973); $\mathbb{R}(\epsilon)$ is used extensively by Hammond (1994).] While $\mathbb{R}(\epsilon)$ is an ordered non-Archimedean field, it is not an elementary extension of $\mathbb{R}$. For example, $\epsilon$ does not have a cube root in $\mathbb{R}(\epsilon)$, although it is a theorem of $\mathbb{R}$ that every number has a cube root. As a first step to obtaining an elementary extension of $\mathbb{R}$, let $\mathbb{R}^+(\epsilon)$ consist of all formal power series in $\epsilon$ of the form $\sum_{i \geq n} r_i \epsilon^i$, where $n$ is an integer and $r_i \in \mathbb{R}$. $\mathbb{R}^+(\epsilon)$ is also easily seen to be an ordered non-Archimedean field, where inverses are computed by formal division of power series. (Note that we need to allow the summation to start at a negative number since, e.g., the inverse of $\epsilon^3$ is $\epsilon^{-3}$.) Although $\mathbb{R}^+(\epsilon)$ includes $\mathbb{R}(\epsilon)$ (every rational function can be expressed as a power series by formal division), it is still not an elementary extension of $\mathbb{R}$; again $\epsilon$ has no cube root. In order to extend $\mathbb{R}^+(\epsilon)$ to an elementary extension of $\mathbb{R}$, we need a field that includes, for example $\epsilon^{1/3}$, so that $\epsilon$ has a cube root. This suggests that, in order to construct an elementary extension of $\mathbb{R}(\epsilon)$, it is necessary to allow rational exponents in power series. Let $\mathbb{R}^*(\epsilon)$ consist of all series of the form $\sum_{i \geq n} r_i \epsilon^{i/k}$, where $n$ is an integer and and $k$ is a positive natural number. $\mathbb{R}^*(\epsilon)$ is known as the field of *Puiseux series in* $\epsilon$, and is an elementary extension of $\mathbb{R}$ (Basu et al. 2003, Chapter 2).

## 3 Proofs of Theorems

### 3.1 Proof of Proposition 1.1

Suppose that the assessment $(\vec{\sigma}, \mu)$ is a sequential equilibrium. Then there exists a sequence of assessments $(\vec{\sigma}^n, \mu^n)$ converging to $(\vec{\sigma}, \mu)$. It is easy to see that there must be a sequence $\epsilon_n$ of nonnegative real numbers converging to 0 such that $\vec{\sigma}^n$ is an $\epsilon_n$-sequential equilibrium.

For the converse, if $\vec{\sigma}^n$ is a sequence of completely mixed strategy profiles converging to $\vec{\sigma}$, then let $\mu^n$ be the belief system determined by $\vec{\sigma}^n$. By compactness, there must be a convergent subsequence of $\mu^n$. Let $\mu$ be the belief system to which this convergent subsequence converges. It easily follows from the fact that $\vec{\sigma}^n$ is an $\epsilon_n$-sequential equilibrium for some sequence of nonnegative reals $\epsilon_n$ converging to 0 that $(\vec{\sigma}, \mu)$ is a sequential equilibrium.

### 3.2 Proof of Theorem 1.2

First suppose that $\vec{\sigma}'$ satisfies the conditions of the theorem. Define a belief system $\mu$ by taking $\mu_I(h) = st(\Pr_{\vec{\sigma}}(h \mid I))$; that is, the standard part of the conditional probability of $h$ given $I$, according to $\Pr_{\vec{\sigma}'}$. Since $\vec{\sigma}'$ consists of completely mixed strategies, this conditional probability is well defined. We want to show that $(\vec{\sigma}, \mu)$ is a sequential equilibrium. Because there exists an infinitesimal $\epsilon$ such that $\sigma_i$ is an $\epsilon$-best response to $\vec{\sigma}'_{-i}$ conditional on having reached $I$, and $\vec{\sigma}'$ differs infinitesimally from $\vec{\sigma}$, it easily follows that, for all standard strategies $\tau$ for player $i$, we have $\mathrm{EU}_i((\vec{\sigma}, \mu) \mid I) \geq \mathrm{EU}_i(((\tau, \vec{\sigma}_{-i}), \mu) \mid I)$; thus, sequential rationality holds. To show that we have consistency between the belief system and strategy profile, we use the fact that $F$, the field which is the range of the nonstandard probabilities used in $\vec{\sigma}'$, is an elementary extension of $\mathbb{R}$. The idea is, for each $n$, to write a first-order formula $\varphi_n$ that says that a strategy profile $\vec{\sigma}^n$ with the right properties exists. Since $\varphi_n$ will be easily seen to be true in $F$, it must also be true in $\mathbb{R}$. In more detail, we proceed as follows.

For each information set $I$ of player $i$, if $h$ is in $I$, let $q_h = \Pr_{\vec{\sigma}'}(h|I)$; let $r_h = \mu_I(h) = st(q_h)$. In addition, for each action $a$ that $i$ can perform in information set $I$, let $q_{I,a} = \sigma'_i(I)(a)$; that is, $q_{I,a}$ is the probability that $i$ performs action $a$ in information set $I$ according to $\sigma'_i$. Let $r_{I,a} = \sigma_i(I)(a) = st\left(q_{I,a}\right)$. Suppose that there are $M$ histories in the game $\Gamma$, $K$ information sets in $\Gamma$, and for each information set $I_j$ in $\Gamma$, there are $N_j$ actions that can be performed at $I_j$. Now, for each $n$, let $\varphi_n$ be a formula $\exists x_{h_1} \ldots \exists x_{h_M} \exists x_{I_1,a_{11}} \ldots \exists x_{I_1,a_{1N_1}} \ldots \exists x_{I_K,a_{K1}} \ldots \exists x_{I_K,a_{KN_K}} \varphi'_n$, where $\varphi'_n$ is described below. Intuitively, $\varphi'_n$ says that $x_h$, the conditional probability of $h$ given $I$ (the information set containing $h$) is within $1/n$ of $q_h$, and that $x_{I_j,a_{jk}}$, the probability of performing action $a_{jk}$ in information set $I_j$, is within $1/n$ of $q_{I_j,a_{jk}}$. Finally, for each history $h$ in information set $I$, $\varphi'_n$ relates $x_h$ to the probability of reaching $h$ conditional on reaching $I$. Thus, $\varphi'_n$ is the conjunction of the following formulas:

(a) formulas of the form $|x_h - \mathbf{r}_h| < 1/n$ for all histories $h$;
(b) formulas of the form $(|x_{I,a} - \mathbf{r}_{I,a}| < 1/n) \wedge (x_{I,a} > 0)$ for each information set $I$ and action $a$ that can be performed at $I$;
(c) for each information set $I_j$, a formula $\sum_{k=1}^{N_j} x_{I_j,a_{jk}} = 1$;
(d) for each history $h$ in an information set $I$, a formula saying that $x_h$ is the product of the probabilities $x_{I',a}$ over the actions $a$ taken to reach $h$, divided by the sum of the corresponding products taken over all histories $h' \in I$.

The formula $\varphi_n$ is clearly satisfiable in $F$ by taking $x_h$ to be $q_h$ and $x_{I,a}$ to be $q_{I,a}$. Thus, $\varphi_n$ is also satisfiable in $\mathbb{R}$. The values of $x_{I,a}$ that satisfy the formula in $\mathbb{R}$ determine the strategy profile $\vec{\sigma}^n$; if $I$ is an information set of player $I$, then $\sigma_i^n(I)(a) = x_{I,a}$. The formulas in (b) and (c) guarantee that $\sigma_i^n$ is a completely mixed strategy that is within $1/n$ of $\sigma_i$. Thus $\vec{\sigma}^n \rightarrow \vec{\sigma}$. Taking $\mu_I^n(h) = x_h$, the formulas in clause (d) guarantee that $\mu_I^n$ is the (unique) belief assessment corresponding to the strategy profile $\vec{\sigma}^n$ (it is unique because $\vec{\sigma}^n$ consists of completely mixed strategies). The formulas in clause (a) guarantee that $\mu^n$ is within $1/n$ of $\mu$, so $\mu^n \rightarrow \mu$.

For the converse, suppose that $(\vec{\sigma}, \mu)$ is a sequential equilibrium. Thus, there exists a sequence of assessments $(\vec{\sigma}^n, \mu^n) \rightarrow (\vec{\sigma}, \mu)$ such that, for each $n$, $\vec{\sigma}^n$ is a profile of completely mixed strategies. Since, at each information set $I$ for player $i$, $\sigma_i$ is a best response to $\vec{\sigma}_{-i}$ conditional on having reached $I$, given belief system $\mu$, it must be the case that, for all $\epsilon$, there exists $n$ such that $\sigma_i^n$ is an $\epsilon$-best response to $\vec{\sigma}_{-i}^n$ at $I$, given belief system $\mu^n$. But since $\vec{\sigma}^n$ consists of completely mixed strategies, so that $\mu^n$ is determined by $\vec{\sigma}^n$, it is easy to see that $\sigma_i^n$ is in fact an $\epsilon$-best response to $\sigma_{-i}^n$ conditional on having reached $I$, in the sense of Definition 2.1.

With this observation, we can now use the compactness theorem to show that an appropriate $\vec{\sigma}'$ exists. The proof is similar in spirit to that of the first half. We consider the language of the reals with additional constants $\mathbf{c}_h$ for each history $h$, a constant $\mathbf{c}_{I,a}$ for each (information set, action) pair, and a constant $\mathbf{d}$. Consider the following collection $\Phi$ of formulas:

(a) all first-order formulas in the language true of the reals;
(b) for each information $I$ and action $a$ that can be performed at $I$, the formula $\mathbf{c}_{I,a} > 0$ and, for each natural number $n > 0$, the formula $|\mathbf{c}_{I,a} - \mathbf{r}_{I,a}| < 1/\mathbf{n}$;
(c) for each information set $I_j$, a formula $\sum_{k=1}^{N_j} \mathbf{c}_{I_j,a_{jk}} = 1$;
(d) the formula $\mathbf{d} > 0$ and, for each natural number the formula $\mathbf{d} < 1/\mathbf{n}$;
(e) for each player $i$, a formula saying that the strategy $\sigma_i$ is a $\mathbf{d}$-best response to the strategy profile $\vec{\sigma}_{-i}$ defined by the probabilities $\mathbf{c}_{I,a}$. The statement that $\sigma_i$ is a $\mathbf{d}$-best response is easily definable using a universally quantified formula with linear inequalities.

Any finite subset $\Phi'$ of $\Phi$ is satisfiable by the reals. Since $F$ contains only finitely many formulas of the form $\mathbf{d} < 1/n$, there is some real number $\epsilon$ that satisfies these formulas. We can then choose $\mathbf{c}_{I,a}$ to be the probabilities determined by $\vec{\sigma}^n$ for $n$ sufficiently large so that, for each information set $I$ for player $i$, $\sigma_i^n$ is an $\epsilon$-best response to $\vec{\sigma}_{-i}^n$. That is, for the appropriate choice of $n$, if $I$ is an information set for player $i$, then $(c)_{I,a}$ is interpreted as $\sigma_i^n(I)(c)$.

Since every finite subset of $\Phi$ is satisfiable, by compactness, there is a model $F$ for $\Phi$. The formulas in (a) force $F$ to be an elementary extension of $\mathbb{R}$; the formulas in (c) ensure that the interpretation of the constants $\mathbf{c}_{I,a}$ determines a strategy profile $\vec{\sigma}'$; the formulas in (b) guarantee that $\vec{\sigma}$ and $\vec{\sigma}'$ differ only infinitesimally; the formulas in (d) guarantee that the interpretation $\epsilon$ of $\mathbf{d}$ is an infinitesimal; and the formulas in (e) guarantee that $\sigma_i$ is an $\epsilon$-best response to $\vec{\sigma}'$. This proves the desired result.

### 3.3 Proof of Theorem 1.4

The structure of the proof of Theorem 1.4 is almost identical to that of Theorem 1.2, so here I just highlight the differences.

Suppose that there exists a completely mixed nonstandard strategy profile $\vec{\sigma}'$ that differs infinitesimally from $\vec{\sigma}$ such that at each information set $I$ for player $i$, $\sigma_i$ is a best response to $\vec{\sigma}'_{-i}$, conditional on having reached $I$. For each $n$, we can define a formula $\psi_n$ that can be used to define the strategy profile $\vec{\sigma}^n$; $\psi_n$ has same form as the formula $\varphi_n$ in the proof of Theorem 1.2. We just need to add another finite collection of conjunctions to the formula $\varphi_n'$ saying that, at each information set $I$ of player $i$, $\sigma_i$ is a best response to the strategy profile defined by the values $x_{I,a}$. Again, the formula $\varphi_n$ is satisfied in $F$, the field which is the range of the nonstandard probabilities used in $\vec{\sigma}'$ by interpreting $x_{I,a}$ as $q_{I,a}$ and $x_h$ as $q_h$. Thus, $\varphi_n$ is satisfied in $\mathbb{R}$. The interpretation of the variables $x_{I,a}$ in $\mathbb{R}$ determines the strategy $\vec{\sigma}^n$.

For the converse, we again take an infinite collection of formulas and apply compactness. The formulas are identical to those used in the proof of Theorem 1.2, except that we do not need the constant $\mathbf{d}$ and the constraints on $\mathbf{d}$ described by the formulas in (d), and we modify the formulas in (e) to say that $\sigma_i$ is a best response to the strategy profile defined by the probabilities $\mathbf{c}_{I,a}$, rather than just a $\mathbf{d}$-best response.

### 3.4 A power series representation of nonstandard strategy profiles

As I said in the introduction, Theorem 1.2, 1.4, and 1.5 do not give any hint as to how to think of the nonstandard strategy profiles $\vec{\sigma}'$ that differ infinitesimally from $\vec{\sigma}$. In this section, I show that they can be thought of in a way that is close in spirit to the way trembles are typically presented in the literature; we can take all the probabilities to lie in $\mathbb{R}^*(\epsilon)$.

As a first step, we need the following lemma. Note that a standard strategy in a (finite) extensive-form game can be identified with a vector in $\mathbb{R}^N$ for an appropriate choice of $N$, since the strategy is just characterized by a tuple of numbers describing the probabilities of moves at each information set. Of course, the same is true for a normal-form game as well. In the following lemma, I assume that $N$ is chosen appropriately. Recall that an analytic function is one that is locally given by a convergent power series.

**Lemma 3.1** *Suppose that $\vec{\sigma}$ is a (standard) strategy profile.*

(a) *$\vec{\sigma}$ is a sequential equilibrium of an extensive-form $\Gamma$ with perfect recall iff there exists analytic functions $\bar{\sigma} : [0, 1] \to \mathbb{R}^N$ and $f : [0, 1] \to [0, 1]$ such that $\bar{\sigma}(0) = \vec{\sigma}$, $f(0) = 0$, and, for all $\epsilon > 0$, $\bar{\sigma}(\epsilon)$ is a completely mixed strategy profile such that, for each player $i$ and each information set $I$ for player $i$, $\bar{\sigma}_i(\epsilon)$ is an $f(\epsilon)$-best response to $\bar{\sigma}_{-i}(\epsilon)$, conditional on having reached $I$.*

(b) *$\vec{\sigma}$ is a perfect equilibrium of an extensive-form game $\Gamma$ with perfect recall iff there exists an analytic functions $\bar{\sigma} : [0, 1] \to \mathbb{R}^N$ such that $\bar{\sigma}(0) = \vec{\sigma}$ and, for all $\epsilon > 0$, $\bar{\sigma}(\epsilon)$ is a completely mixed strategy profile such that, for each player $i$ and each information set $I$ of player $i$, $\sigma_i$ is a best response to $\bar{\sigma}_{-i}(\epsilon)$ conditional on having reached $I$.*

(c) *$\vec{\sigma}$ is a proper equilibrium of a normal-form game $\Gamma$ iff there exist analytic functions $\bar{\sigma} : [0, 1] \to \mathbb{R}^N$ and $f : [0, 1] \to [0, 1]$ such that $\bar{\sigma}(0) = \vec{\sigma}$, $f(0) = 0$, and, for all $\epsilon > 0$, $\bar{\sigma}(\epsilon)$ is a completely mixed strategy profile such that each player $i$ and all pure strategies $s_1$ and $s_2$ of player $i$, if $\mathrm{EU}_i(s_1, \bar{\sigma}_{-i}(\epsilon) < \mathrm{EU}_i(s_2, \bar{\sigma}_{-i}(\epsilon)$, then $\bar{\sigma}_i(s_1) < f(\epsilon)\bar{\sigma}_i(s_2)$.*

*Proof* Sufficiency in all three cases is clear (for part (a), this depends on Proposition 1.1). For necessity, I first prove part (b). Let $A$ be the set of completely mixed strategy profiles $\vec{\tau}$ such that, for each player $i$ and each information set $I$ of player $i$, $\sigma_i$ is a best response to $\vec{\tau}_{-i}$ conditional on having reached $I$. $A$ is easily seen to be a *semialgebraic* set (i.e., a set defined by a finite sequence of polynomial equations and inequalities, or the union of such sets); moreover, it is immediate from the definition of perfect equilibrium that $\vec{\sigma}$ is in the closure of $A$. The result now follows immediately from the Nash Curve Selection Lemma (Bochnak et al. 1998, Proposition 8.1.13).

The modifications required to prove parts (a) and (c) are straightforward. For (a), let $A$ consist of pairs $(\vec{\tau}, \epsilon)$ such that, for each player $i$ and each information set $I$ for player $i$, $\tau_i$ is an $\epsilon$-best response to $\vec{\tau}_{-i}$, conditional on having reached $I$; for (c), let $A$ consist of pairs $(\vec{\tau}, \epsilon)$ such that if $\mathrm{EU}_i(s_1, \vec{\tau}_{-i}(\epsilon) < \mathrm{EU}_i(s_2, \vec{\tau}_{-i}(\epsilon)$, then $\tau_i(s_1) < \epsilon\tau_i(s_2)$. In both cases, $(\vec{\sigma}, 0)$ is in the closure of $A$ (for part (a), this follows from Proposition 1.1). Again, the result follows from the Nash Curve Selection Lemma. □

**Theorem 3.2** *In Theorems 1.2, 1.4, and 1.5, the extension field $F$ can be taken to be $\mathbb{R}^*(\epsilon)$, and all the probabilities can be taken to be analytic functions of $\epsilon$ (i.e., power series of the form $\sum_{i \geq 0} r_i \epsilon^i$).*

*Proof* The fact that the probabilities can be taken to be analytic functions of $\epsilon$ in Theorems 1.2, 1.4, and 1.5 is immediate from Lemma 3.1. We can simply take $\vec{\sigma}'$ in each of these theorems to be $\bar{\sigma}(\epsilon)$. Since $\bar{\sigma}(0) = \vec{\sigma}$, it follows that each component of $\vec{\sigma}' - \vec{\sigma}$ is a power series whose leading coefficient is 0, and hence is infinitesimal. For (a) and (c), since $f(0) = 0$, $f(\epsilon)$ is an infinitesimal.

The fact that $\vec{\sigma}$ is a sequential (resp., perfect; proper) equilibrium if the conditions in Theorem 1.2 (resp., Theorem 1.4; Theorem 1.5) hold for a nonstandard mixed strategy $\vec{\sigma}$ whose probabilities are all analytic functions of $\epsilon$ is immediate from the proofs of these theorems (where it is shown that these results hold as long as the probabilities come from any elemenatry extension of $\mathbb{R}$). □

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## References

Basu S, Pollack R, Roy M-F (2003) Algorithms in real algebraic geometry. Algorithms and computation in mathematics, vol 10. Springer, Berlin
Blume L, Brandenburger A, Dekel E (1991) Lexicographic probabilities and equilibrium refinements. Econometrica 59(1):81–98
Blume LE, Zame WR (1994) The algebraic geometry of perfect and sequential equilibrium. Econometrica 62:783–794
Bochnak J, Coste M, Roy M-F (1998) Real algebraic geometry. Springer, Berlin
Enderton HB (1972) A mathematical introduction to logic. Academic Press, New York
Fishburn PC (1972) On the foundations of game theory: the case of non-Archimedean utilities. Int J Game Theory 1:65–71
Halpern JY (2001) Lexicographic probability, conditional probability, and nonstandard probability. In: Theoretical aspects of rationality and knowledge: proceedings of eighth conference (TARK 2001). Morgan Kaufmann, San Francisco, pp 17–30. A full version of the paper can be found at http://arxiv.org/abs/cs/0306106
Hammond PJ (1994) Elementary non-Archimedean representations of probability for decision theory and games. In: Humphreys P (ed) Patrick Suppes: scientific philosopher, vol 1. Kluwer, Dordrecht, pp 25–49
Kreps DM (1990) Game theory and economic modeling. Oxford University Press, Oxford
Kreps DM, Wilson RB (1982) Sequential equilibria. Econometrica 50:863–894
Myerson R (1978) Refinements of the Nash equilibrium concept. Int J Game Theory 7:73–80
Osborne MJ, Rubinstein A (1994) A course in game theory. MIT Press, Cambridge
Rajan U (1998) Trembles in the Bayesian foundation of solution concepts. J Econ Theory 82:248–266
Richter MK (1971) Rational choice. In: Chipman J, Hurwicz L, Richter MK, Sonnenschein HF (eds) Prefrence, utility, and demand. Harcourt, Brace, and Jovanovich, New York, pp 29–58
Robinson A (1973) Function theory on some nonarchimedean fields. Am Math Mon: Papers Found Math 80:S87–S109
Selten R (1975) Reexamination of the perfectness concept for equilibrium points in extensive games. Int J Game Theory 4:25–55
Skala HJ (1974) Nonstandard utilities and the foundations of game theory. Int J Game Theory 3(2):67–81