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ORIGINAL PAPER

Coalitional strategy-proofness and resource monotonicity for house allocation problems

Onur Kesten

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Abstract We consider the problem of allocating houses to agents when monetary compensations are not allowed. We present a simple and independent proof of a result due to Ehlers and Klaus (Int J Game Theory 32:545–560, 2004) that characterizes the class of rules satisfying *efficiency, strategy-proofness, resource monotonicity,* and *nonbossiness*.

Keywords Indivisible goods · Strategy-proofness · Resource monotonicity

1 Introduction

We consider the problem of allocating a set of distinct indivisible objects to a set of agents (for example, houses to applicants, tasks to workers, etc.) when agents have preferences over objects, and monetary compensations are not allowed. We assume that agents do not have property rights over the objects, but instead, that all the objects are collectively owned. This problem is known as the *house allocation problem*. The recent literature on indivisible good allocation considers various axiomatic approaches to this problem. In a related paper, Ehlers et al. (2002) show that a special subclass of the inheritance rules introduced by Pápai (2000), which they call *restricted inheritance rules,* are the only rules satisfying *efficiency, strategy-proofness,* and *population monotonicity*.

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O. Kesten (✉)
Tepper School of Business, Carnegie Mellon University, 5000 Forbes Avenue,
Pittsburgh, PA 15213, USA
e-mail: okesten@andrew.cmu.edu

We show that *a rule satisfies efficiency, coalitional strategy-proofness, and resource monotonicity if and only if it is the agent-optimal rule associated with an acyclic priority structure.* Interestingly, this class of rules coincides with the class of restricted inheritance rules (for example, see Kesten 2006). Upon completing this work, we were made aware of the existence of a paper by Ehlers and Klaus (2004) which previously discovers a more general result via a significantly longer proof.[1] Nevertheless, we believe that the contribution of this paper would be its simple and transparent proof which proceeds along completeley different lines.

2 The house allocation problem and properties

Let $N \equiv \{1, 2, \ldots, n\}$ denote the (finite) set of agents. Let $\mathcal{X} \equiv \{a, b, \ldots\}$ denote the (finite) set of potential objects. Let 0 denote the **null object** (which represents not being allotted any object in $\mathcal{X}$). Each agent $i \in N$ is equipped with a complete, transitive, and strict preference relation R_i over $\mathcal{X} \cup \{0\}$. Let $\mathcal{R}$ denote the class of all such preferences. Let P_i denote the strict preference relation associated with R_i. A **problem** is a pair $(R;\ X)$ such that $R \in \mathcal{R}^N$ and $X \subseteq \mathcal{X}$.

Given a problem $(R;\ X)$ an **allocation** is a list of allotments $(\alpha_i)_{i \in N}$ such that for all $i \in N, \alpha_i \in X \cup \{0\}$, and no object in X is allotted to more than one agent. The null object can be allotted to any number of agents. Given a problem $(R;\ X)$ let $\mathcal{A}(X)$ be the set of all allocations. A **rule** is a function φ that associates to each problem $(R;\ X)$ an allocation $\varphi(R;\ X) = (\varphi_i(R;\ X))_{i \in N}$. Next are the basic properties.

Efficiency: For all $(R;\ X)$, there is no $\alpha \in \mathcal{A}(X)$ such that for all $i \in N$, $\alpha_i\ R_i\ \varphi_i(R;\ X)$, and for some $j \in N$, $\alpha_j\ P_j\ \varphi_j(R;\ X)$.

We next require that no group of agents ever gain by jointly misrepresenting their preferences.

Coalitional strategy-proofness: For all $(R;\ X)$, and all $M \subseteq N$, there is no $R'_M \in \mathcal{R}^M$ such that for all $i \in M$, $\varphi_i(R'_M,\ R_{-M};\ X))\ R_i\ \varphi_i(R;\ X)$, and for some $j \in M$, $\varphi_j(R'_M,\ R_{-M};\ X) P_j\ \varphi_j(R;\ X)$.

When $|M| = 1$, this property is called *strategy-proofness*. Next is the requirement that if an agent's allotment does not change when his preferences change, nobody else's does either.

Nonbossiness: For all $(R;\ X)$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi_i(R) = \varphi_i(R'_i,\ R_{-i};\ X) \Rightarrow \varphi(R) = \varphi(R'_i,\ R_{-i};\ X)$.

A rule is coalitional strategy-proof if and only if it is strategy-proof and nonbossy (Pápai 2000). Last we require that an increase or a decrease in resources affects all agents in the same direction, i.e., they all gain or, all lose together.

Resource monotonicity:[2] For all $(R;\ X)$, and all $X' \subseteq X$, either for all $i \in N$, $\varphi_i(R;\ X)\ R_i \varphi_i(R;\ X')$, or for all $i \in N$, $\varphi_i(R;\ X')\ R_i\ \varphi_i(R;\ X)$.

[1] Their work differs from ours in two aspects: first, their result applies also to a smaller domain of preferences (where all objects are desirable) and secondly, they use the weaker property of "independence of irrelevant alternatives" instead of coalitional strategy-proofness.

[2] See Thomson (2008) for a number of applications of this property.

3 The agent-optimal rule and a characterization

A closely related problem is *school choice* where additionally to the present model each object is also assigned a *strict* order of agents. These orders represent the "priorities" of agents for objects. Formally, let $\mathcal{S}$ denote the set of all bijections from N to N. Given $x \in X$, **priority order** $f_x \in \mathcal{S}$ assigns priorities to agents for object x. We interpret $f_x(i) < f_x(j)$ as *agent i has higher priority for object x than agent j*. The collection $f \equiv (f_x)_{x \in X}$ is called the **priority structure.**

A rule is **fair** with respect to a priority structure f if no agent ever prefers somebody else's allotment to his own when he has higher priority for the allotment, i.e., for all $i, j \in N$ and all $R \in \mathcal{R}^N$, $\varphi_j(R; X)\ P_i\ \varphi_i(R; X) \Rightarrow f_{\varphi_j(R)}(j) < f_{\varphi_j(R)}(i)$. Given f, the associated **agent-optimal rule** ψ^f always finds a fair allocation via the following **deferred acceptance algorithm** (Gale and Shapley 1962):

Step 1 Each agent applies to his favorite object. If more than one agent applies to the same object, then all but the highest priority agent are rejected. Such an object is temporarily allotted to the highest priority agent.

Step k, $k \geq 2$ Each rejected agent applies to his next favorite object. If more than one agent applies to the same object, then all but the highest priority agent are rejected. Such an object is temporarily allotted to the highest priority agent.

The algorithm terminates when no agent is rejected any more.[3] Rule ψ^f Pareto dominates any other rule that is *fair with respect to f* (Balinski and Sönmez 1999). However, it may violate *efficiency* and *coalitional strategy-proofness*. These properties are recovered if certain "acyclicity" restrictions are imposed on the priority structure (Ergin 2002). Formally, a **cycle** of a given priority structure f is three agents i, j, and k such that there are two objects x and y with $f_x(i) < f_x(j) < f_x(k)$ and $f_y(k) < f_y(i)$. Priority structure f is **acyclic** if it has no cycles. Rule ψ^f is *efficient* or alternatively, *coalitional strategy-proof* if and only if f is *acyclic* (Ergin 2002). Combining this result with that of Balinski and Sönmez (1999) leads to the following proposition.

Proposition 1 *A rule is efficient and fair with respect to a given priority structure if and only if it is the associated agent-optimal rule, and the given priority structure is acyclic.*

If f is acyclic, ψ^f is also equivalent to a *restricted inheritance rule* (Ehlers et al. 2002), that interprets priority structure f as an "inheritance table" (see, for example, Pápai 2000; Kesten 2006). We are now ready to present our main result.

Theorem 1 *A rule satisfies strategy-proofness, efficiency, resource monotonicity, and nonbossiness if and only if it is the agent-optimal rule associated with an acyclic priority structure.*

[3] Given $X \subseteq \mathcal{X}$, let $f|_X = (f_x)_{x \in X}$ be the priority structure that is induced by f for X. Given $R \in \mathcal{R}^N$ and $X \subseteq \mathcal{X}$, let $R|_X$ be the preference profile induced by R for X. Then given a problem $(R; X)$, let $\psi^f(R; X) \equiv \psi^{f|_X}(R|_X)$.

Proof Recall that for an acyclic f, ψ^f satisfies *coalitional strategy-proofness* and, *efficiency.* It is also easy to check that ψ^f is *resource monotonic* (regardless of the acyclicity of f). Conversely, let φ be a rule satisfying *coalitional strategy-proofness, efficiency,* and *resource monotonicity.*

Step 1 Identification of f. Given $x \in \mathcal{X}$, let $R^x \in \mathcal{R}^N$ be such that for all $i \in N$ and all $y \in \mathcal{X} \backslash \{x\}$, $x\ P_i^x\ 0\ P_i^x y$. Also let $R^0 \in \mathcal{R}^N$ be such that for all $i \in N$ and all $y \in \mathcal{X}$, $0\ P_i^0\ y$. By *efficiency*, there is $i \in N$ such that $\varphi_i(R^x; \{x\}) = x$. Let $f_x(i) \equiv 1$ and $f_x{}^{-1}(1) \equiv i$. By *efficiency*, there is $j \in N \backslash \{i\}$ such that $\varphi_j(R_i^0, R_{-i}^x; \{x\}) = x$. Let $f_x(j) \equiv 2$ and $f_x{}^{-1}(2) \equiv j$. Continuing in a similar way, identification of f_x is completed.

Step 2 Given $i \in N$, $a \in X \subseteq \mathcal{X}$, *and* $R \in R^N$, *if for all* $j \in N \backslash \{i\}$ *with* $f_a(j) < f_a(i)$ *we have* $\varphi_j(R; X) \neq a$, *then* $\varphi_i(R; X)\ R_i a$. Given $i \in N$, $a \in X \subseteq \mathcal{X}$, and $R \in \mathcal{R}^N$, suppose that for all $j \in N$ with $f_a(j) < f_a(i)$, we have $\varphi_j(R; X) \neq a$. Let $J \equiv \{j \in N |\ f_a(j) < f_a(i)\}$. If $0\ R_i\ a$, then by *efficiency*, $\varphi_i(R; X)\ R_i\ a$. So, suppose $a\ P_i\ 0$. By *strategy-proofness*, for all $j \in J$, $\varphi_j(R_j^{\varphi_j(R;X)}, R_{-j}; X) = \varphi_j(R; X)$, and by *nonbossiness*, for all $j \in J$, $\varphi(R_j^{\varphi_j(R;X)}, R_{-j}; X) = \varphi(R; X)$. Continuing to change the preferences of each agent in J in a similar way, we have

$$\varphi\left((R_j^{\varphi_j(R;X)})_{j\in J},\ R_{-J};\ X\right) = \varphi(R;\ X). \tag{1}$$

By Step 1, $\varphi_i((R_J^0,\ R_{-J}^a;\ \{a\}) = a$. By *efficiency*, for all $j \in J$, we have $\varphi_j((R_j^{\varphi_j(R;X)})_{j\in J}, R_i^a, R_{N\backslash(J\cup\{i\})}; \{a\}) = 0$. By *coalitional strategy-proofness*, for all $k \in N \backslash (J \cup \{i\})$, we have $\varphi_k((R_j^{\varphi_j(R;X)})_{j\in J}, R_i^a, R_{N\backslash(J\cup\{i\})}; \{a\}) = 0$. Hence, by *efficiency*, $\varphi(R_J^0,\ R_{-J}^a;\ \{a\}) = \varphi((R_j^{\varphi_j(R;X)})_{j\in J},\ R_i^a,\ R_{N\backslash(J\cup\{i\})}; \{a\})$. By *strategy-proofness*, $\varphi_i((R_j^{\varphi_j(R;X)})_{j\in J}, R_i^a, R_{N\backslash(J\cup\{i\})}; \{a\}) = \varphi_i((R_j^{\varphi_j(R;X)})_{j\in J}, R_{-J}; \{a\})$. Then, by *resource monotonicity*, $\varphi_i((R_j^{\varphi_j(R;X)})_{j\in J}, R_{-J}; X)\ R_i\ a$. Using (1), $\varphi_i(R; X)\ R_i\ a$.

Step 3 Given $X \subseteq \mathcal{X}$, φ is *fair with respect to* $f|_X$. Suppose by contradiction that there are $i, j \in N$, $R \in \mathcal{R}^N$, and $a \in X \subseteq \mathcal{X}$ such that $a = \varphi_j(R; X)\ P_i\ \varphi_i(R; X)$ and $f_a(j) > f_a(i)$. Then, for all $k \in N$ with $f_a(k) < f_a(i)$, we have $\varphi_k(R; X) \neq a$, but $a P_i\ \varphi_i(R; X)$, a contradiction to Step 2. Finally, if $\varphi(R; X)$ is *efficient* for $(R; X)$, then it is also *efficient* for $(R|_X; X)$. By Step 2 for all $X \subseteq \mathcal{X}$, φ is *fair with respect to* $f|_X$. Thus, for all $R \in \mathcal{R}^N$ and all $X \subseteq \mathcal{X}$, by Proposition 1, $\varphi(R; X) = \psi^{f|_X}(R|_X)$. Hence, $\varphi = \psi^f$ where f is acyclic. □

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