Int J Game Theory (2008) 37:371–386
DOI 10.1007/s00182-008-0123-8

ORIGINAL PAPER

Cournot–Walras equilibrium as a subgame perfect equilibrium

Francesca Busetto · Giulio Codognato · Sayantan Ghosal

Accepted: 13 February 2008 / Published online: 1 August 2008

Abstract In this paper, we investigate the problem of the strategic foundation of the Cournot–Walras equilibrium approach. To this end, we respecify à la Cournot–Walras the mixed version of a model of simultaneous, noncooperative exchange, originally proposed by Lloyd S. Shapley. We show, through an example, that the set of the Cournot–Walras equilibrium allocations of this respecification does not coincide with the set of the Cournot–Nash equilibrium allocations of the mixed version of the original Shapley's model. As the nonequivalence, in a one-stage setting, can be explained by the intrinsic two-stage nature of the Cournot–Walras equilibrium concept, we are led to consider a further reformulation of the Shapley's model as a two-stage game, where the atoms move in the first stage and the atomless sector moves in the second stage. Our main result shows that the set of the Cournot–Walras equilibrium allocations coincides with a specific set of subgame perfect equilibrium allocations of this two-stage game, which we call the set of the Pseudo–Markov perfect equilibrium allocations.

Keywords Walras equilibrium · Cournot–Nash equilibrium · Cournot–Walras equilibrium · Subgame perfect equilibrium

JEL Classification C72 · D51

We would like to thank Pierpaolo Battigalli, Marcellino Gaudenzi, and an anonymous referee for their comments and suggestions.

F. Busetto (✉) · G. Codognato
Dipartimento di Scienze Economiche, Università degli Studi di Udine,
Via Tomadini 30, 33100 Udine, Italy
e-mail: busetto@uniud.it

S. Ghosal
Department of Economics, University of Warwick, Coventry CV4 7AL, UK

1 Introduction

In this paper, we study the problem of the strategic foundation of the Cournot–Walras equilibrium approach, initiated by Gabszewicz and Vial (1972), by comparing it with the noncooperative market game approach, initiated by Shapley and Shubik (1977).

Gabszewicz and Vial (1972) is one of the first attempts to extend the analysis of oligopolistic interaction proposed by Cournot to a general equilibrium framework. These authors introduced the concept of Cournot–Walras equilibrium for an economy with production, where firms are assumed to be "few" whereas consumers are assumed to be "many." Firms produce consumption goods and distribute them—according to some preassigned shares—to consumers, who are therefore endowed with the bundles of goods which they receive as shareholders of the firms plus some given initial bundles. Consumers are then allowed to exchange their endowments among themselves and the equilibrium prices resulting form these exchanges enable firms to determine the profits associated with their production decisions. A Cournot–Walras equilibrium is a noncooperative equilibrium of a game where the players are the firms, the strategies are their production decisions and the payoffs are their profits.

The denomination of the equilibrium concept introduced by Gabszewicz and Vial (1972) comes from the fact that firms behave "à la Cournot" in making their production decisions while consumers behave "à la Walras" in exchanging goods. The line of research initiated by these authors raised some theoretical problems (see also Roberts and Sonnenschein 1977; Roberts 1980; Mas-Colell 1982; Dierker and Grodal 1986, among others). Gabszewicz and Vial (1972) were already aware that their concept of Cournot–Walras equilibrium depends on the rule chosen to normalize prices and that profit maximization may not be a rational objective of the firms.

Codognato and Gabszewicz (1991) introduced a Cournot–Walras equilibrium concept for exchange economies where "few" traders, called the oligopolists, behave strategically "à la Cournot" in making their supply decisions and share the endowment of a particular commodity while "many small" traders behave "à la Walras" and share the endowments of all the other commodities. The oligopolists are allowed to supply a fraction of their initial endowments. Taking prices as given, each oligopolist is able to determine the income corresponding to his supply decision and to choose a bundle of commodities which gives him the highest utility. All traders, behaving "à la Walras," are then allowed to exchange commodities among themselves until prices clear all the markets. A Cournot–Walras equilibrium is a noncooperative equilibrium of the game where the players are the oligopolists, the strategies are their supply decisions and the payoffs are the utility levels they achieve through the exchange.

The line of research initiated by Codognato and Gabszewicz (1991) circumvented the theoretical difficulties of Gabszewicz and Vial's model by defining an equilibrium concept which does not depend on price normalization (see also Codognato and Gabszewicz 1993; d'Aspremont et al. 1997; Gabszewicz and Michel 1997; Shitovitz 1997; Lahmandi-Ayed 2001; Bonnisseau and Florig 2003, among others).

Nevertheless, the whole Cournot–Walras equilibrium approach shares another fundamental problem, stressed, in particular, by Okuno et al. (1980). In fact, all the models mentioned above do not explain why a particular agent behaves strategically rather than competitively.

Taking inspiration from the cooperative analysis of oligopoly introduced by Shitovitz (1973); Okuno et al. (1980) proposed a foundation of agents' behavior that considered the Cournot–Nash equilibria of a model of simultaneous, noncooperative exchange between large traders, represented as atoms, and small traders, represented by an atomless sector. Their model belongs to a line of research initiated by Shapley and Shubik (1977) (see also Dubey and Shubik 1978; Postlewaite and Schmeidler 1978; Mas-Colell 1982; Amir et al. 1990; Peck et al. 1992; Dubey and Shapley 1994, among others). In particular, Okuno et al. (1980) showed that large traders keep their strategic power even when their behavior turns out to be competitive in the cooperative framework considered by Shitovitz (1973).

Codognato (1995) and Codognato and Ghosal (2000b) generalized the analysis of Okuno et al. (1980) by considering the mixed version of a model originally proposed by Lloyd S. Shapley and subsequently analyzed by Sahi and Yao (1989). Within this framework, traders send out bids, i.e., quantity signals, which indicate how much of each commodity they are willing to offer for trade. Every bid of each commodity is tagged by the name of some other commodity for which it has to be exchanged. The pricing rule requires that a single price system, which equates the value of the total amount of bids of any commodity to the value of the total amount available of that commodity, is used to clear the markets.

In particular, Codognato (1995) compared this model with the mixed version of the model in Codognato and Gabszewicz (1991) and provided an example showing that the set of the Cournot–Nash equilibrium allocations may not coincide with the set of the Cournot–Walras equilibrium allocations. There could be two reasons for this result. The first is that the Cournot–Walras equilibrium concept has an intrinsic two-stage nature which cannot be reconciled with the one-stage Cournot–Nash equilibrium of the Shapley's model. The second is that, in the model by Codognato and Gabszewicz (1991), the oligopolists behave à la Cournot in making their supply decisions and à la Walras in exchanging commodities whereas, in the mixed version of the Shapley's model, the large traders always behave à la Cournot. This "twofold behavior" of large traders represents in fact a further problem with the line of research introduced by Codognato and Gabszewicz (1991).

In this paper, we provide a respecification à la Cournot–Walras of the mixed version of the Shapley's model. More precisely, we assume that large traders behave à la Cournot in making bids, as in the Shapley's model, while the atomless sector behaves à la Walras. Given the atoms' bids, prices adjust to equate the aggregate net bids to the aggregate net demands of the atomless sector. Each nonatomic trader then obtains his Walrasian demand whereas each large trader obtains final holdings determined as in the Shapley's model. A Cournot–Walras equilibrium is a noncooperative equilibrium of a game where the players are the large traders, the strategies are their bids and the payoffs are the utility levels they achieve through the exchange process described above. We show that, in the one-stage setting, our respecification of the Shapley's model generates a set of Cournot–Walras equilibrium allocations which does not coincide with the set of the Cournot–Nash equilibrium allocations of the mixed version of the original Shapley's model. This confirms, within a different framework, the result obtained by Codognato (1995). Since large traders always behave à la Cournot in both the respecification à la Cournot–Walras and the original version of the Shapley's model, we could guess that our nonequivalence result is explained by the two-stage implicit nature of the Cournot–Walras equilibrium.

For this reason, we introduce a reformulation of the Shapley's model as a two-stage game, where the atoms move in the first stage and the atomless sector moves in the second stage, and proceed to check whether an equivalence result can be obtained in this setup. In the Cournot–Walras model, different atoms' strategies leading to the same aggregate bids yield the same prices. On the other hand, in the two-stage Shapley's model, there can be subgames associated with atoms' strategies summing to the same aggregate bids that the atomless sector plays in different ways, so generating different prices. In order to avoid this unreasonable behavior, we introduce a subgame perfect equilibrium notion characterized by the fact that, in the second stage, the atomless sector always uses the same strategies when the atoms send out bids which sum to the same total amounts. We call it Pseudo–Markov perfect equilibrium for reasons which will become apparent in Sect. 4, where we discuss the differences between the two notions of Pseudo–Markov and Markov perfect equilibrium. Our main result then follows. The set of the Cournot–Walras equilibrium allocations and the set of the Pseudo–Markov perfect equilibrium allocations of the two-stage game coincide. This theorem reconciles the Cournot–Walras approach with the line of research initiated by Shapley and Shubik (1977) and makes this approach immune from the criticism by Okuno et al. (1980), as it provides an endogenous foundation of strategic and competitive behavior.

The paper is organized as follows. In Sect. 2, we introduce our reformulation of the Cournot–Walras equilibrium concept for mixed exchange economies. In Sect. 3, we compare the Cournot–Walras, Walras, and Cournot–Nash equilibrium concepts for mixed exchange economies in a one-stage framework. In Sect. 4, we show our equivalence theorem in a two-stage framework.

2 The model

We consider an exchange economy with large traders, represented as atoms, and small traders, represented by an atomless sector. The set of traders is denoted by $T = T_0 \cup T_1$, where $T_0 = [0, 1]$ is the set of small traders and $T_1 = \{2, \ldots, m + 1\}$ is the set of large traders. Following Codognato and Ghosal (2000b), it is possible to denote the space of traders by the complete measure space $(T, \mathcal{T}, \mu)$, where $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T and μ is the Lebesgue measure, when restricted to $\mathcal{T}_{T_0} = \{D \cap T_0 : D \in \mathcal{T}\}$, and the counting measure, when restricted to $\mathcal{T}_{T_1} = \{D \cap T_1 : D \in \mathcal{T}\}$. By Propositions 3 and 4 in Codognato and Ghosal (2000b), it is straightforward to show that the measure space $(T_0, \mathcal{T}_{T_0}, \mu)$ is atomless and the measure space $(T_1, \mathcal{T}_{T_1}, \mu)$ is purely atomic; moreover, for each $t \in T_1$, the singleton set $\{t\}$ is an atom of the measure space $(T, \mathcal{T}, \mu)$ (see, for instance, Aliprantis and Border 1999, p. 357). A null set of traders is a set of Lebesgue measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "all" traders, or "each" trader, or "each" trader in a certain set, is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue. Given any function $\mathbf{g}$ defined on T, we denote by ${}^0\mathbf{g}$ and ${}^1\mathbf{g}$ the restrictions of $\mathbf{g}$ to T_0 and T_1, respectively. Analogously, given any correspondence $\mathbf{G}$ defined on T, we denote by ${}^0\mathbf{G}$ and ${}^1\mathbf{G}$ the restriction of $\mathbf{G}$ to T_0 and T_1, respectively.

In the economy, there are l different commodities. A commodity bundle is a point in R^l_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^l_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumptions.

Assumption 1 $\mathbf{w}(t) > 0$, for all $t \in T$, $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$.

An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by an utility function $u_t : R^l_+ \to R$, satisfying the following assumptions.

Assumption 2 $u_t : R^l_+ \to R$ is continuous, strictly monotonic, strictly quasi-concave, for all $t \in T$.

Assumption 3 $u : T \times R^l_+ \to R$, given by $u(t, x) = u_t(x)$, is measurable.

A price vector is a vector $p \in R^l_+$. According to Aumann (1966), we define, for each $p \in R^l_+$, a correspondence $\mathbf{\Delta}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\mathbf{\Delta}_p(t) = \{x \in R^l_+ : px \leq p\mathbf{w}(t)\}$, and a correspondence $\mathbf{\Gamma}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\mathbf{\Gamma}_p(t) = \{x \in R^l_+ : \text{ for all } y \in \mathbf{\Delta}_p(t), u_t(x) \geq u_t(y)\}$. A Walras equilibrium is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector p^* and an allocation $\mathbf{x}^*$, such that, for all $t \in T$, $\mathbf{x}^*(t) \in \mathbf{\Delta}_{p^*}(t) \cap \mathbf{\Gamma}_{p^*}(t)$.

In order to formulate the concept of Cournot–Walras equilibrium, we first focus on the atomless sector's behavior. By Assumption 2, for each $p \in R^l_{++}$, it is possible to define the small traders' Walrasian demands as a function ${}^0\mathbf{x}(\cdot, p) : T_0 \to R^l_+$ such that, for each $t \in T_0$, ${}^0\mathbf{x}(t, p) = {}^0\mathbf{\Delta}_p(t) \cap {}^0\mathbf{\Gamma}_p(t)$. We are now able to show the following proposition.

Proposition *Under Assumptions 1–3, the function* ${}^0\mathbf{x}(\cdot, p)$ *is integrable, for each* $p \in R^l_{++}$.

Proof Let $p \in R^l_{++}$. From Aumann (1966), we know that the function ${}^0\mathbf{x}(\cdot, p)$ is a Borel measurable function since the correspondences ${}^0\mathbf{\Delta}_p$ and ${}^0\mathbf{\Gamma}_p$ are Borel measurable and ${}^0\mathbf{x}(t, p) = {}^0\mathbf{\Delta}_p(t) \cap {}^0\mathbf{\Gamma}_p(t)$, for each $t \in T_0$. Moreover, ${}^0\mathbf{x}(\cdot, p)$ is integrably bounded, since ${}^0\mathbf{x}^i(t, p) \leq \frac{\sum_{j=1}^l p^j \mathbf{w}^j(t)}{p^i}$, $i = 1, \ldots, l$, for all $t \in T_0$. But then, by Theorem 2 in Aumann (1965), the function ${}^0\mathbf{x}(\cdot, p)$ is integrable. □

Consider now the atoms' strategies. Let $e \in R^{l^2}$ be a vector such that $e = (e_{11}, e_{12}, \ldots, e_{ll-1}, e_{ll})$. A strategy correspondence is a correspondence $\mathbf{E} : T_1 \to \mathcal{P}(R^{l^2})$ such that, for each $t \in T_1$, $\mathbf{E}(t) = \{e \in R^{l^2} : e_{ij} \geq 0,\ i, j = 1, \ldots, l;\ \sum_{j=1}^l e_{ij} \leq \mathbf{w}^i(t),\ i = 1, \ldots, l\}$. A strategy selection is an integrable function $\mathbf{e} : T_1 \to R^{l^2}$ such that, for all $t \in T_1$, $\mathbf{e}(t) \in \mathbf{E}(t)$. For each $t \in T_1$, $\mathbf{e}_{ij}(t),\ i, j = 1, \ldots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. Let E be the set of all strategy selections. Moreover, let $\mathbf{e} \setminus e(t)$ be a strategy selection obtained by replacing $\mathbf{e}(t)$ in $\mathbf{e}$ with $e(t) \in \mathbf{E}(t)$. Finally, let $\pi(\mathbf{e})$ denote the correspondence which associates, with each $\mathbf{e} \in E$, the set of the price vectors such that

$$\int_{T_0} {}^{0}\mathbf{x}^{j}(t,p)\,d\mu + \sum_{i=1}^{l}\int_{T_1} \mathbf{e}_{ij}(t)\,d\mu \frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^{j}(t)\,d\mu + \sum_{i=1}^{l}\int_{T_1} \mathbf{e}_{ji}(t)\,d\mu, \quad j = 1,\ldots,l. \tag{1}$$

Assumption 4 For each $\mathbf{e} \in E$, $\pi(\mathbf{e}) \neq \emptyset$ and $\pi(\mathbf{e}) \subset R^l_{++}$.

A price selection $p(\mathbf{e})$ is a function which associates, with each $\mathbf{e} \in E$, a price vector $p \in \pi(\mathbf{e})$ and is such that $p(\mathbf{e}') = p(\mathbf{e}'')$ if $\int_{T_1} \mathbf{e}'(t)\,d\mu = \int_{T_1} \mathbf{e}''(t)\,d\mu$.[1] For each strategy selection $\mathbf{e} \in E$, we define atoms' final holdings as a function ${}^{1}\mathbf{x}(\cdot, \mathbf{e}(\cdot), p(\mathbf{e})) : T_1 \to R^l_+$ such that

$${}^{1}\mathbf{x}^{j}(t,\mathbf{e}(t),p(\mathbf{e})) = \mathbf{w}^{j}(t) - \sum_{i=1}^{l} \mathbf{e}_{ji}(t) + \sum_{i=1}^{l} \mathbf{e}_{ij}(t)\frac{p^i(\mathbf{e})}{p^j(\mathbf{e})}, \tag{2}$$

for all $t \in T_1$, $j = 1,\ldots,l$. Given a strategy selection $\mathbf{e} \in E$, taking into account the structure of the traders' measure space, the Proposition, and Eq. (1), it is straightforward to show that the function $\mathbf{x}(t)$ such that $\mathbf{x}(t) = {}^{0}\mathbf{x}(t, p(\mathbf{e}))$, for all $t \in T_0$, and $\mathbf{x}(t) = {}^{1}\mathbf{x}(t, \mathbf{e}(t), p(\mathbf{e}))$, for all $t \in T_1$, is an allocation.

At this stage, we are able to define the concept of Cournot–Walras equilibrium.

Definition 1 A pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, consisting of a strategy selection $\tilde{\mathbf{e}}$ and an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(t) = {}^{0}\mathbf{x}(t, p(\tilde{\mathbf{e}}))$, for all $t \in T_0$, and $\tilde{\mathbf{x}}(t) = {}^{1}\mathbf{x}(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))$, for all $t \in T_1$, is a Cournot–Walras equilibrium, with respect to a price selection $p(\mathbf{e})$, if $u_t({}^{1}\mathbf{x}(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))) \geq u_t({}^{1}\mathbf{x}(t, e(t), p(\tilde{\mathbf{e}} \setminus e(t))))$, for all $t \in T_1$ and for all $e(t) \in \mathbf{E}(t)$.

3 Cournot–Walras, Walras, and Cournot–Nash equilibrium

In this section, we begin with investigating the relationship between the concepts of Cournot–Walras and Walras equilibrium for the mixed exchange economy defined above. Next, we compare the Cournot–Walras equilibrium concept introduced in this paper with the Cournot–Walras equilibrium concept proposed by Codognato and Gabszewicz (1991). Finally, we introduce the mixed version of the original Shapley's model and the related notion of Cournot–Nash equilibrium and we analyze the relationship between the concepts of Cournot–Walras and Cournot–Nash equilibrium in a one-shot structure.

[1] Assumption 4 is quite strong. In our framework, it guarantees that the price-correspondence is non-empty and that the atomless sector's demand is well-defined. Analogous strong assumptions on the price-correspondence or the price selection are used in all the previous models belonging to the Cournot–Walras approach.

Within a cooperative context, Shitovitz (1973) showed that, counterintuitively, the core allocations of a mixed exchange economy are Walrasian when the atoms have the same endowments and preferences (but not necessarily the same size). The following example shows that this unsatisfying result can be avoided within a noncooperative setting. It analyzes an exchange economy with two identical atoms facing an atomless continuum of traders and proves that, in this economy, there is a Cournot–Walras equilibrium allocation which is not Walrasian.

Example 1 Consider the following specification of an exchange economy satisfying Assumptions 1–4, where $l = 2$, $T_1 = \{2, 3\}$, $T_0 = [0, 1]$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = lnx^1 + lnx^2$, for all $t \in T_1$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = lnx^1 + lnx^2$, for all $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = lnx^1 + lnx^2$, for all $t \in [\frac{1}{2}, 1]$. For this economy, there is a Cournot–Walras equilibrium allocation which does not correspond to any Walras equilibrium.

Proof The only symmetric Cournot–Walras equilibrium is the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{e}}_{12}(2) = \tilde{\mathbf{e}}_{12}(3) = \frac{1+\sqrt{13}}{12}$, $\tilde{\mathbf{x}}^1(2) = \tilde{\mathbf{x}}^1(3) = \frac{11+\sqrt{13}}{12}$, $\tilde{\mathbf{x}}^2(2) = \tilde{\mathbf{x}}^2(3) = \frac{1+\sqrt{13}}{20+8\sqrt{13}}$, $\tilde{\mathbf{x}}^1(t) = \frac{1}{2}$, $\tilde{\mathbf{x}}^2(t) = \frac{3}{10+4\sqrt{13}}$, for all $t \in [0, \frac{1}{2}]$, $\tilde{\mathbf{x}}^1(t) = \frac{5+2\sqrt{13}}{6}$, $\tilde{\mathbf{x}}^2(t) = \frac{1}{2}$, for all $t \in [\frac{1}{2}, 1]$. On the other hand, the only Walras equilibrium of the economy considered is the pair $(\mathbf{x}^*, p^*)$, where $\mathbf{x}^{*1}(2) = \mathbf{x}^{*1}(3) = \frac{1}{2}$, $\mathbf{x}^{*2}(2) = \mathbf{x}^{*2}(3) = \frac{1}{10}$, $\mathbf{x}^{*1}(t) = \frac{1}{2}$, $\mathbf{x}^{*2}(t) = \frac{1}{10}$, for all $t \in [0, \frac{1}{2}]$, $\mathbf{x}^{*1}(t) = \frac{5}{2}$, $\mathbf{x}^{*2}(t) = \frac{1}{2}$, for all $t \in [\frac{1}{2}, 1]$, $p^* = \frac{1}{5}$. $\square$

The model introduced in Sect. 2 can be viewed as a respecification à la Cournot–Walras of a noncooperative market game first proposed by Lloyd S. Shapley and next analyzed by Sahi and Yao (1989) and Codognato and Ghosal (2000a). Here, we introduce a mixed version of the original Shapley's model, where the space of traders is as in Sect. 2.

We first consider traders' strategy decisions. Let $b \in R^{l^2}$ be a vector such that $b = (b_{11}, b_{12}, \ldots, b_{ll-1}, b_{ll})$. A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^{l^2})$ such that, for each $t \in T$, $\mathbf{B}(t) = \{b \in R^{l^2} : b_{ij} \geq 0, i, j = 1, \ldots, l; \sum_{j=1}^{l} b_{ij} \leq \mathbf{w}^i(t), i = 1, \ldots, l\}$. A strategy selection is an integrable function $\mathbf{b} : T \to R^{l^2}$, such that, for all $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. For each $t \in T$, $\mathbf{b}_{ij}(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. Given a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}}$ as $\bar{\mathbf{B}} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ a strategy selection obtained by replacing $\mathbf{b}(t)$ in $\mathbf{b}$ with $b(t) \in \mathbf{B}(t)$. Then, we are able to introduce the following definition (see Sahi and Yao 1989).

Definition 2 Given a strategy selection $\mathbf{b}$, a price vector p is market clearing if

$$p \in R^l_{++}, \sum_{i=1}^{l} p^i \bar{\mathbf{b}}_{ij} = p^j \left(\sum_{i=1}^{l} \bar{\mathbf{b}}_{ji} \right), \quad j = 1, \ldots, l. \tag{3}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (3) if and only if $\bar{\mathbf{B}}$ is irreducible. Denote by $p(\mathbf{b})$ the function which associates, with each strategy selection $\mathbf{b}$ such that $\bar{\mathbf{B}}$ is irreducible, the unique, up to a scalar multiple, market clearing price vector p. Given a strategy selection $\mathbf{b}$ such that p is market clearing and unique, up to a scalar multiple, consider the assignment determined as follows:

$$\mathbf{x}^j(t, \mathbf{b}(t), p(\mathbf{b})) = \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{b}_{ji}(t) + \sum_{i=1}^{l} \mathbf{b}_{ij}(t) \frac{p^i(\mathbf{b})}{p^j(\mathbf{b})},$$

for all $t \in T$, $j = 1, \ldots, l$. It is easy to verify that this assignment is an allocation. Given a strategy selection $\mathbf{b}$, the traders' final holdings are

$$\mathbf{x}^j(t) = \mathbf{x}^j(t, \mathbf{b}(t), p(\mathbf{b})) \quad \text{if } p \text{ is market clearing and unique,}$$
$$\mathbf{x}^j(t) = \mathbf{w}^j(t) \quad \text{otherwise,}$$

for all $t \in T$, $j = 1, \ldots, l$.

This reformulation of the Shapley's model for mixed exchange economies allows us to define the following concept of Cournot–Nash equilibrium (see Codognato and Ghosal 2000b).

Definition 3 A strategy selection $\hat{\mathbf{b}}$ such that $\bar{\hat{\mathbf{B}}}$ is irreducible is a Cournot–Nash equilibrium if

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for all $t \in T$ and for all $b(t) \in \mathbf{B}(t)$.

Codognato and Ghosal (2000a) showed that, in limit exchange economies, the set of the Cournot–Nash equilibrium allocations and the set of the Walras equilibrium allocations of the Shapley's model coincide. On the other hand, Okuno et al. (1980) showed that the Cournot–Nash equilibrium allocations of a mixed exchange economy with two commodities are not Walrasian even under those circumstances where the core turns out to be competitive (see Shitovitz 1973). It can be shown that an analogous result holds for the Shapley's model introduced in this section by simply verifying that the allocation corresponding to a Cournot–Walras equilibrium in Example 1 also corresponds to a Cournot–Nash equilibrium as in Definition 3.

In effect, if we consider the mixed version of the original Shapley's model, all traders behave strategically but those belonging to the atomless sector have a negligible influence on prices. The strategic behavior of the atomless sector could consequently be interpreted as competitive behavior. On the other hand, in our version à la Cournot–Walras of the Shapley's model, the atomless sector is supposed to behave competitively while the atoms have strategic power. Therefore, it seems to be reasonable to conjecture that the set of the Cournot–Walras equilibrium allocations of our variant of the Shapley's model coincides with the set of the Cournot–Nash equilibrium allocations of the mixed version of the original Shapley's model. This conjecture turns out to be false, as is shown by the following example.

Example 2 Consider the following specification of an exchange economy satisfying Assumptions 1–4, where $l = 2$, $T_1 = \{2, 3\}$, $T_0 = [0, 1]$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = lnx^1 + lnx^2$, for all $t \in T_1$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = lnx^1 + lnx^2$, for all $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = x^1 + lnx^2$, for all $t \in [\frac{1}{2}, 1]$. For this economy, there is a Cournot–Walras equilibrium allocation which does not correspond to any Cournot–Nash equilibrium.

Proof The only symmetric Cournot–Walras equilibrium of the economy considered is the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{e}}_{12}(2) = \tilde{\mathbf{e}}_{12}(3) = \frac{-1+\sqrt{37}}{12}$, $\tilde{\mathbf{x}}^1(2) = \tilde{\mathbf{x}}^1(3) = \frac{11-\sqrt{37}}{12}$, $\tilde{\mathbf{x}}^2(2) = \tilde{\mathbf{x}}^2(3) = \frac{-1+\sqrt{37}}{14+4\sqrt{37}}$, $\tilde{\mathbf{x}}^1(t) = \frac{1}{2}$, $\tilde{\mathbf{x}}^2(t) = \frac{3}{7+2\sqrt{37}}$, for all $t \in [0, \frac{1}{2}]$, $\tilde{\mathbf{x}}^1(t) = \frac{1+2\sqrt{37}}{6}$, $\tilde{\mathbf{x}}^2(t) = \frac{6}{7+2\sqrt{37}}$, for all $t \in [\frac{1}{2}, 1]$. On the other hand, the only symmetric Cournot–Nash equilibrium is the strategy selection $\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}_{12}(2) = \hat{\mathbf{b}}_{12}(3) = \frac{1+\sqrt{13}}{12}$, $\hat{\mathbf{b}}_{12}(t) = \frac{1}{2}$, for all $t \in [0, \frac{1}{2}]$, $\hat{\mathbf{b}}_{21}(t) = \frac{5+2\sqrt{13}}{11+2\sqrt{13}}$ for all $t \in [\frac{1}{2}, 1]$, which generates the allocation $\hat{\mathbf{x}}^1(2) = \hat{\mathbf{x}}^1(3) = \frac{11+\sqrt{13}}{12}$, $\hat{\mathbf{x}}^2(2) = \hat{\mathbf{x}}^2(3) = \frac{1+\sqrt{13}}{22+4\sqrt{13}}$, $\hat{\mathbf{x}}^1(t) = \frac{1}{2}$, $\hat{\mathbf{x}}^2(t) = \frac{3}{11+2\sqrt{13}}$, for all $t \in [0, \frac{1}{2}]$, $\hat{\mathbf{x}}^1(t) = \frac{5+2\sqrt{13}}{6}$, $\hat{\mathbf{x}}^2(t) = \frac{6}{11+2\sqrt{13}}$, for all $t \in [\frac{1}{2}, 1]$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}, p(\hat{\mathbf{b}}))$, for all $t \in T$. □

4 Cournot–Walras equilibrium as a subgame perfect equilibrium

Example 2 shows the nonequivalence between Cournot–Walras and Cournot–Nash equilibrium allocations in mixed exchange economies. As this nonequivalence holds in a one-stage game, we are led to consider a multi-stage game. In particular, given that the Cournot–Walras equilibrium concept has an intrinsic two-stage flavor, it seems to be natural to analyze a two-stage game where the atoms move in the first stage and the atomless sector moves in the second stage, after observing the first stage atoms' moves. Therefore, we consider the same exchange economy as in Sect. 3 and associate with it a two-stage game with observed actions (see Fudenberg and Tirole 1991), which represents a sequential reformulation of the mixed version of the Shapley's model. We provide a theorem showing that the set of the Cournot–Walras equilibrium allocations coincides with the set of a specific set of subgame perfect equilibrium allocations which we call the Pseudo–Markov perfect equilibrium allocations of this game.

The game is played in the two stages 0 and 1. Consider now the traders' actions. Let $a \in R^{l^2}$ be a vector such that $a = (a_{11}, a_{12}, \ldots, a_{ll-1}, a_{ll})$. We denote by $\mathbf{A}^0$ an action correspondence in stage 0, defined on T, such that $\mathbf{A}^0(t)$ is the singleton "do nothing," for all $t \in T_0$, and $\mathbf{A}^0(t) = \{a \in R^{l^2} : a_{ij} \geq 0,\ i, j = 1, \ldots, l;\ \sum_{j=1}^{l} a_{ij} \leq \mathbf{w}^i(t),\ i = 1, \ldots, l\}$, for all $t \in T_1$. We denote by $\mathbf{A}^1$ an action correspondence in stage 1, defined on T, such that $\mathbf{A}^1(t) = \{a \in R^{l^2} : a_{ij} \geq 0,\ i, j = 1, \ldots, l;\ \sum_{j=1}^{l} a_{ij} \leq \mathbf{w}^i(t),\ i = 1, \ldots, l\}$, for all $t \in T_0$, and $\mathbf{A}^1(t)$ is the singleton "do nothing," for all $t \in T_1$. An action selection in stage 0 is a function $\mathbf{a}^0$, defined on T, such that $\mathbf{a}^0(t) \in \mathbf{A}^0(t)$, for all $t \in T$, where ${}^1\mathbf{a}^0$ is integrable. For each $t \in T_1$, ${}^1\mathbf{a}^0(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. An action selection in stage 1 is a function $\mathbf{a}^1$, defined on T, such that $\mathbf{a}^1(t) \in \mathbf{A}^1(t)$, for all $t \in T$, where ${}^0\mathbf{a}^1$ is integrable. For each $t \in T_0$, ${}^0\mathbf{a}^1(t)$, $i, j = 1, \dots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. Let S^0 and S^1 be the sets of all action selections in stage 0 and stage 1, respectively, and let H^0 and H^1 be the sets of all stage 0 and stage 1 histories, respectively, where $H^0 = \emptyset$ and $H^1 = S^0$. In addition, let $H^2 = S^0 \times S^1$ be the set of all final histories. Given a final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$, we define the aggregate matrix $\bar{\mathbf{A}}$ as $\bar{\mathbf{A}} = (\bar{\mathbf{a}}_{ij}) = (\int_{T_0} {}^0\mathbf{a}^1_{ij}(t)\, d\mu + \int_{T_1} {}^1\mathbf{a}^0{}_{ij}(t)\, d\mu)$. Then, we can introduce the following definition (see Sahi and Yao 1989).

Definition 4 Given a final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$, a price vector p is market clearing if

$$p \in R^l_{++}, \sum_{i=1}^{l} p^i \bar{\mathbf{a}}_{ij} = p^j \left(\sum_{i=1}^{l} \bar{\mathbf{a}}_{ji} \right), \quad j = 1, \dots, l. \tag{4}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (4) if and only if $\bar{\mathbf{A}}$ is irreducible. Denote by $p(\mathbf{h}^2)$ the function which associates, to each final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ such that $\bar{\mathbf{A}}$ is irreducible, the unique, up to a scalar multiple, market clearing price vector p. Given a final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ such that p is market clearing and unique, up to a scalar multiple, consider the assignment determined as follows:

$$\begin{aligned} &\mathbf{x}^j(t, \mathbf{h}^2(t), p(\mathbf{h}^2)) = \mathbf{w}^j(t) - \textstyle\sum_{i=1}^{l} {}^0\mathbf{a}^1_{ji}(t) + \sum_{i=1}^{l} {}^0\mathbf{a}^1_{ij}(t) \frac{p^i(\mathbf{h}^2)}{p^j(\mathbf{h}^2)}, \quad \text{for all } t \in T_0, \\ &\mathbf{x}^j(t, \mathbf{h}^2(t), p(\mathbf{h}^2)) = \mathbf{w}^j(t) - \textstyle\sum_{i=1}^{l} {}^1\mathbf{a}^0_{ji}(t) + \sum_{i=1}^{l} {}^1\mathbf{a}^0_{ij}(t) \frac{p^i(\mathbf{h}^2)}{p^j(\mathbf{h}^2)}, \quad \text{for all } t \in T_1, \end{aligned} \tag{5}$$

$j = 1, \dots, l$. It is easy to verify that this assignment is an allocation. Finally, given a final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$, the traders' final holdings are

$$\begin{aligned} &\mathbf{x}^j(t) = \mathbf{x}^j(t, \mathbf{h}^2(t), p(\mathbf{h}^2)) \quad \text{if } p \text{ is market clearing and unique,} \\ &\mathbf{x}^j(t) = \mathbf{w}^j(t) \quad \text{otherwise,} \end{aligned} \tag{6}$$

for all $t \in T$, $j = 1, \dots, l$.

Now, we define a strategy profile, $\mathbf{s}$, as a sequence of functions $\{\mathbf{s}^0, \mathbf{s}^1\}$, where $\mathbf{s}^0$ is defined on $T \times H^0$ and is such that $\mathbf{s}^0(t, \mathbf{h}^0) \in \mathbf{A}^0(t)$, for all $t \in T$, and $\mathbf{s}^0(\cdot, \mathbf{h}^0) \in S^0$; $\mathbf{s}^1$ is defined on $T \times H^1$ and is such that, given $\mathbf{h}^1 \in H^1$, $\mathbf{s}^1(t, \mathbf{h}^1) \in \mathbf{A}^1(t)$, for all $t \in T$, $\mathbf{s}^1(\cdot, \mathbf{h}^1) \in S^1$. For each $t \in T$, let $s(t, \cdot)$ denote the sequence of functions $\{s^0(t, \cdot), s^1(t, \cdot)\}$, where $s^0(t, \cdot) : H^0 \to \mathbf{A}^0(t)$ and $s^1(t, \cdot) : H^1 \to \mathbf{A}^1(t)$. We denote by $\mathbf{s} \setminus s(t, \cdot) = \{\mathbf{s}^0 \setminus s^0(t, \cdot), \mathbf{s}^1 \setminus s^1(t, \cdot)\}$ a strategy profile obtained by replacing $\mathbf{s}^0(t, \cdot)$ in $\mathbf{s}^0$ and $\mathbf{s}^1(t, \cdot)$ in $\mathbf{s}^1$, respectively, with the functions $s^0(t, \cdot)$ and $s^1(t, \cdot)$. With a little abuse of notation, given a strategy profile $\mathbf{s}$, we denote by ${}^1\mathbf{s}^0$ and ${}^0\mathbf{s}^1$ the functions defined, respectively, on T_1 and T_0, such that ${}^1\mathbf{s}^0(t) = {}^1\mathbf{a}^0(t) = \mathbf{s}^0(t, \mathbf{h}^0)$, for all $t \in T_1$, and ${}^0\mathbf{s}^1(t) = {}^0\mathbf{a}^1(t) = \mathbf{s}^1(t, \mathbf{h}^1)$, for all $t \in T_0$, with $\mathbf{h}^1 = \mathbf{s}^0(\cdot, \mathbf{h}^0)$. In addition, given a strategy profile $\mathbf{s}$, we define the aggregate matrix $\bar{\mathbf{S}}$ as $\bar{\mathbf{S}} = (\bar{\mathbf{s}}_{ij}) = (\int_{T_0} {}^0\mathbf{s}^1{}_{ij}(t)\, d\mu + \int_{T_1} {}^1\mathbf{s}^0{}_{ij}(t)\, d\mu)$. Then, given a strategy profile $\mathbf{s}$ such that $\bar{\mathbf{S}}$ is irreducible, we denote by $p(\mathbf{s})$ the function obtained by replacing, in Eq. (4), $\bar{\mathbf{a}}_{ij}$ with

$\bar{\mathbf{s}}_{ij}, i, j = 1, \ldots, l$. Given a strategy profile $\mathbf{s}$ such that p is market clearing and unique, up to a scalar multiple, the allocation $\mathbf{x}(t, \mathbf{s}(t), p(\mathbf{s}))$ is obtained by replacing, in (5), $\mathbf{h}^2$ with $\mathbf{s}$ and ${}^0\mathbf{a}^1$, ${}^1\mathbf{a}^0$, respectively, with ${}^0\mathbf{s}^1$, ${}^1\mathbf{s}^0$. Finally, the traders' final holdings are determined as in (6), by replacing $\mathbf{h}^2$ with $\mathbf{s}$.

We proceed now to consider the subgame represented by the stage 1 of the game outlined above, given the history $\mathbf{h}^1 \in H^1$. Given a strategy profile $\mathbf{s}$, the strategy selection $\mathbf{s}|\mathbf{h}^1$ is a function, defined on T, such that, for each $\mathbf{h}^1 \in H^1$, $\mathbf{s}|\mathbf{h}^1(t) = \mathbf{s}^1(t, \mathbf{h}^1)$, for all $t \in T$. In addition, given a history $\mathbf{h}^1 \in H^1$ and a strategy profile $\mathbf{s}$, we define the aggregate matrix $\bar{\mathbf{S}}|\mathbf{h}^1$ as $\bar{\mathbf{S}}|\mathbf{h}^1 = (\bar{\mathbf{s}}_{ij}|\mathbf{h}^1) = (\int_{T_0} {}^0\mathbf{s}_{ij}(t)|\mathbf{h}^1\, d\mu + \int_{T_1} {}^1\mathbf{h}^1{}_{ij}(t)\, d\mu)$. Then, given a history $\mathbf{h}^1 \in H^1$, and a strategy profile $\mathbf{s}$ such that $\bar{\mathbf{S}}|\mathbf{h}^1$ is irreducible, we denote by $p(\mathbf{s}|\mathbf{h}^1)$ the function obtained by replacing, in Eq. (4), $\bar{\mathbf{a}}_{ij}$ with $\bar{\mathbf{s}}_{ij}|\mathbf{h}^1$, $i, j = 1, \ldots, l$. Given a history $\mathbf{h}^1 \in H^1$ and a strategy profile $\mathbf{s}$ such that p is market clearing and unique, up to a scalar multiple, the allocation $\mathbf{x}(t, \mathbf{s}|\mathbf{h}^1(t), p(\mathbf{s}|\mathbf{h}^1))$ is obtained by replacing, in (5), $\mathbf{h}^2$ by $\mathbf{s}|\mathbf{h}^1$ and ${}^0\mathbf{a}^1$, ${}^1\mathbf{a}^0$, respectively, with ${}^0\mathbf{s}|\mathbf{h}^1$, ${}^1\mathbf{h}^1$. The traders' final holdings are determined as in (6), by replacing $\mathbf{h}^2$ with $\mathbf{s}|\mathbf{h}^1$. Finally, given a history $\mathbf{h}^1 \in H^1$, we denote by $\mathbf{s}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1$ a strategy selection obtained by replacing $\mathbf{s}(t)|\mathbf{h}^1$ in $\mathbf{s}|\mathbf{h}^1$ with $s(t)|\mathbf{h}^1 \in \mathbf{A}^1(t)$.

We are now able to define the concept of subgame perfect equilibrium for the two-stage game above.

Definition 5 A strategy profile $\hat{\mathbf{s}}$ such that $\bar{\hat{\mathbf{S}}}|\mathbf{h}^1$ is irreducible, for each $\mathbf{h}^1 \in H^1$, is a subgame perfect equilibrium if, for all $t \in T$,

$$u_t(\mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))) \geq u_t(\mathbf{x}(t, s(t, \cdot), p(\hat{\mathbf{s}} \setminus s(t, \cdot)))),$$

for all possible sequences of functions functions $s(t, \cdot)$, and, for each $\mathbf{h}^1 \in H^1$,

$$u_t(\mathbf{x}(t, \hat{\mathbf{s}}|\mathbf{h}^1(t), p(\hat{\mathbf{s}}|\mathbf{h}^1))) \geq u_t(\mathbf{x}(t, s(t)|\mathbf{h}^1, p(\hat{\mathbf{s}}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1))),$$

for all $t \in T$ and for all $s(t)|\mathbf{h}^1 \in \mathbf{A}^1(t)$.

At this point, we have to deal with the following problem. In a Cournot–Walras equilibrium, different atoms' strategies leading to the same aggregate bids generate the same prices. On the other hand, in a subgame perfect equilibrium of the two-stage Shapley's model, nothing assures that the atomless sector reacts the same way to different histories leading to the same total bids—thereby generating the same prices—even though atoms' bids affect payoffs only in the aggregate. In order to avoid this unreasonable behavior, we introduce a subgame perfect equilibrium notion characterized by the fact that, in the second stage, the atomless sector uses strategies invariant with respect to different atoms' bids summing to the same total amounts. To this purpose, we denote by $H^{1\star}(\cdot)$ the partition of H^1 such that, for each $\mathbf{h}^{1'} \in H^1$, $H^{1\star}(\mathbf{h}^{1'}) = \{\mathbf{h}^{1''} \in H^1 : \int_{T_1} \mathbf{h}^{1''}(t)\, d\mu = \int_{T_1} \mathbf{h}^{1'}(t)\, d\mu\}$. $H^{1\star}$ is a sufficient partition of the set of stage 1 histories, although it may not be the coarsest sufficient partition required to define a Markov perfect equilibrium (see Fudenberg and Tirole 1991; Maskin and Tirole 2001). For this reason, we call our equilibrium notion Pseudo–Markov perfect equilibrium. It can be formalized as follows.

Definition 6 A subgame perfect equilibrium $\hat{\mathbf{s}}$ is a Pseudo–Markov perfect equilibrium if, for all $t \in T$, $\hat{\mathbf{s}}^1(t, \mathbf{h}^{1'}) = \hat{\mathbf{s}}^1(t, \mathbf{h}^{1''})$, whenever $\mathbf{h}^{1''} \in H^{1\star}(\mathbf{h}^{1'})$.

In order to prove our equivalence theorem, we shall introduce a final assumption on the endowments and preferences of the atomless sector. We denote by L the set of commodities $\{1, \ldots, l\}$ and by $R^l_{+j>0} \subset R^l_+$ the set of vectors in R^l_+, whose j-th component is strictly positive. For each $i \in L$, we consider the set $T_i = \{t \in T_0 : \mathbf{w}^i(t) > 0\}$. Clearly, by Assumption 1, $\mu(T_i) > 0$. We say that the commodities $i, j \in L$ stand in the relation C if there is a measurable subset T_i' of T_i, with $\mu(T_i') > 0$, such that, for each trader $t \in T_i'$, $\{x \in R^l_+ : u_t(x) = u_t(y)\} \subset R^l_{+j>0}$, for all $y \in R^l_{++}$. In addition, we use the following definition provided by Codognato and Ghosal (2000a), to whom we refer for further details.

Definition 7 The set of commodities L is said to be a net if $\{\langle i, j\rangle : iCj\} \neq \emptyset$ and any pair of distinct vertices, i and j, of the directed graph $D_L(L, C)$ are connected by a path.

Then, we can introduce this final assumption.

Assumption 5 [2] The set of commodities L is a net.

We are now ready to state our equivalence theorem.

Theorem *Under Assumptions 1–5, (i) if $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ is a Cournot–Walras equilibrium with respect to the price selection $p(\mathbf{e})$, there is a Pseudo–Markov perfect equilibrium $\tilde{\mathbf{s}}$ such that $\mathbf{x}(t, p(\tilde{\mathbf{e}})) = \mathbf{x}(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))$, for all $t \in T$; (ii) if $\hat{\mathbf{s}}$ is a Pseudo–Markov perfect equilibrium, there are a strategy selection $\hat{\mathbf{e}}$ and a price selection $p(\mathbf{e})$ such that the pair $(\hat{\mathbf{e}}, \hat{\mathbf{x}})$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}})) = {}^0\mathbf{x}(t, p(\hat{\mathbf{e}}))$, for all $t \in T_0$, and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}})) = {}^1\mathbf{x}(t, \hat{\mathbf{e}}(t), p(\hat{\mathbf{e}}))$, for all $t \in T_1$, is a Cournot–Walras equilibrium with respect to the price selection $p(\mathbf{e})$.*

Proof (i) Let $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ be a Cournot–Walras equilibrium with respect to the price selection $p(\mathbf{e})$. Let $p(\mathbf{h}^1)$ denote a function obtained by replacing, in the price selection $p(\mathbf{e})$, each strategy selection $\mathbf{e}$ with a history $\mathbf{h}^1$ such that $\mathbf{h}^1(t) = \mathbf{e}(t)$, for all $t \in T_1$. Consider now a history $\mathbf{h}^1 \in H^1$. As, by assumption, $p(\mathbf{h}^1) \gg 0$, Assumption 2 implies that $p(\mathbf{h}^1){}^0\mathbf{x}(t, p(\mathbf{h}^1)) = p(\mathbf{h}^1)\mathbf{w}(t)$, for all $t \in T_0$. But then, by Lemma 5 in Codognato and Ghosal (2000a), for all $t \in T_0$, there exist $\lambda^j \geq 0$, $j = 1, \ldots, l$, $\sum_{j=1}^l \lambda^j = 1$, such that

$$^0\mathbf{x}^j(t, p(\mathbf{h}^1)) = \lambda^j \frac{\sum_{j=1}^l p^j(\mathbf{h}^1)\mathbf{w}^j(t)}{p^j(\mathbf{h}^1)}, \quad j = 1, \ldots, l.$$

[2] This is the weakest assumption which allows all traders to have boundary endowments and indifference curves that intersect the boundary of the consumption set, and which guarantees that, with an atomless continuum of traders, the set of the Cournot–Nash equilibrium allocations of the Shapley's model and the set of the Walras equilibrium allocations coincide (for a proof, see Codognato and Ghosal 2000a). It is related to the irreducibility assumption on traders' endowments and preferences used by Gale (1960) in the more specific framework of linear exchange economies.

Define now a function $\boldsymbol{\lambda} : T_0 \to R^l_+$ such that $\boldsymbol{\lambda}^j(t) = \lambda^j(t)$, $j = 1, \ldots, l$, for all $t \in T_0$. Let $\tilde{\mathbf{s}}|\mathbf{h}^1$ denote a function, defined on T, such that $\tilde{\mathbf{s}}(t)|\mathbf{h}^1 \in \mathbf{A}^1(t)$, for all $t \in T$, and ${}^0\tilde{\mathbf{s}}_{ij}(t)|\mathbf{h}^1 = \mathbf{w}^i(t)\boldsymbol{\lambda}^j(t)$, $i, j = 1, \ldots, l$, for all $t \in T_0$. It is straightforward to show that the function ${}^0\tilde{\mathbf{s}}|\mathbf{h}^1$ is integrable. We want now to show that the matrix $\bar{\tilde{\mathbf{S}}}|\mathbf{h}^1 = (\bar{\tilde{\mathbf{s}}}_{ij}|\mathbf{h}^1) = (\int_{T_0} {}^0\mathbf{s}^1{}_{ij}(t)|\mathbf{h}^1\, d\mu + \int_{T_1} {}^1\mathbf{h}^1_{ij}(t)\, d\mu)$ is irreducible. Let $i, j \in L$ be two commodities which stand in the relation C. Consider a trader $t \in T_i'$. First, observe that $p(\mathbf{h}^1)\mathbf{w}(t) > 0$ since, by assumption, $p(\mathbf{h}^1) \gg 0$. This, together with Assumption 2, implies that ${}^0\mathbf{x}(t, p(\mathbf{h}^1)) > 0$ and, given that the commodities i and j stand in the relation C, that ${}^0\mathbf{x}^j(t, p(\mathbf{h}^1)) > 0$. Consider now the matrix $\bar{\mathbf{S}}^L|\mathbf{h}^1 = (\bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1)$ such that $\bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1 = \int_{T_i'} \mathbf{w}^i(t)\boldsymbol{\lambda}^j(t)\, d\mu$, if iCj, and $\bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1 = 0$, otherwise. If iCj, $\bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1 > 0$, since, for each $t \in T_i'$, $\mathbf{w}^i(t) > 0$ and $\boldsymbol{\lambda}^j(t) > 0$. But then, the matrix $\bar{\tilde{\mathbf{S}}}|\mathbf{h}^1$ is irreducible as it is such that $\bar{\tilde{\mathbf{s}}}_{ij}|\mathbf{h}^1 \geq \bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1$, $i, j = 1, \ldots, l$, and the matrix $\bar{\mathbf{S}}^L|\mathbf{h}^1$, by Assumption 5 and by the argument used in the proof of Theorem 2 in Codognato and Ghosal (2000a), is irreducible. Since it is easy to verify that

$$ {}^0\mathbf{x}^j(t, p(\mathbf{h}^1)) = \mathbf{w}^j(t) - \sum_{i=1}^{l} \tilde{\mathbf{s}}_{ji}(t)|\mathbf{h}^1 + \sum_{i=1}^{l} \tilde{\mathbf{s}}_{ij}(t)|\mathbf{h}^1 \frac{p^i(\mathbf{h}^1)}{p^j(\mathbf{h}^1)}, $$

for all $t \in T_0$, $j = 1, \ldots, l$, and as $p(\mathbf{h}^1)$ satisfies Eq. (1), we have

$$ \begin{aligned} &\int_{T_0} \mathbf{w}^j(t)\, d\mu - \sum_{i=1}^{l} \int_{T_0} \tilde{\mathbf{s}}_{ji}(t)|\mathbf{h}^1\, d\mu + \sum_{i=1}^{l} \int_{T_0} \tilde{\mathbf{s}}_{ij}(t)|\mathbf{h}^1\, d\mu \frac{p^i(\mathbf{h}^1)}{p^j(\mathbf{h}^1)} \\ &\quad + \sum_{i=1}^{l} \int_{T_1} \mathbf{h}^1_{ij}(t)\, d\mu \frac{p^i(\mathbf{h}^1)}{p^j(\mathbf{h}^1)} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{h}^1_{ji}(t)\, d\mu, \end{aligned} $$

$j = 1, \ldots, l$. This implies that

$$ \sum_{i=1}^{l} p^i(\mathbf{h}^1)\bar{\tilde{\mathbf{s}}}_{ij}|\mathbf{h}^1 = p^j(\mathbf{h}^1)\left(\sum_{i=1}^{l} \bar{\tilde{\mathbf{s}}}_{ji}|\mathbf{h}^1\right), \quad j = 1, \ldots, l, $$

and, consequently, by Eq. (4), that $p(\mathbf{h}^1) = p(\tilde{\mathbf{s}}|\mathbf{h}^1)$. It is then straightforward to verify that ${}^0\mathbf{x}^j(t, p(\mathbf{h}^1)) = \mathbf{x}^j(t, \tilde{\mathbf{s}}(t)|\mathbf{h}^1, p(\tilde{\mathbf{s}}|\mathbf{h}^1))$, for all $t \in T_0$, $j = 1, \ldots, l$, ${}^1\mathbf{x}^j(t, \mathbf{h}^1(t), p(\mathbf{h}^1)) = \mathbf{x}^j(t, \tilde{\mathbf{s}}(t)|\mathbf{h}^1, p(\tilde{\mathbf{s}}|\mathbf{h}^1))$, for all $t \in T_1$, $j = 1, \ldots, l$. It remains now to show that no trader $t \in T$, in stage 1, has an advantageous deviation from $\tilde{\mathbf{s}}(t)|\mathbf{h}^1$. This is trivially true for all $t \in T_1$. Suppose now that there exist a trader $t \in T_0$ and an action $s(t)|\mathbf{h}^1 \in \mathbf{A}^1(t)$ such that

$$ u_t(\mathbf{x}(t, s(t)|\mathbf{h}^1, p(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1))) > u_t(\mathbf{x}(t, \tilde{\mathbf{s}}(t)|\mathbf{h}^1, p(\mathbf{s}|\mathbf{h}^1))). $$

Since, as an immediate consequence of Definition 4, $p(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1) = p(\tilde{\mathbf{s}}|\mathbf{h}^1)$, the last inequality implies that

$$u_t(\mathbf{x}(t, s(t)|\mathbf{h}^1, p(\mathbf{h}^1))) > u_t({}^0\mathbf{x}(t, p(\mathbf{h}^1))).$$

As $p(\mathbf{h}^1)\mathbf{x}(t, s(t)|\mathbf{h}^1, p(\mathbf{h}^1)) = p(\mathbf{h}^1)\mathbf{w}(t)$, this implies that ${}^0\mathbf{x}(t, p(\mathbf{h}^1)) \notin \mathbf{\Delta}_{p(\mathbf{h}^1)}(t) \cap \mathbf{\Gamma}_{p(\mathbf{h}^1)}(t)$, a contradiction. Let now $\tilde{\mathbf{h}}^1$ be a history such that $\tilde{\mathbf{h}}^1(t) = \tilde{\mathbf{e}}(t)$, for all $t \in T_1$, and let $\tilde{\mathbf{s}}$ be a strategy profile such that, for all $t \in T$, $\tilde{\mathbf{s}}^0(t, \mathbf{h}^0) = \tilde{\mathbf{h}}^1(t)$ and $\tilde{\mathbf{s}}^1(t, \mathbf{h}^1) = \tilde{\mathbf{s}}(t)|\mathbf{h}^1$, for each $\mathbf{h}^1 \in H^1$. Then, $p(\tilde{\mathbf{e}}) = p(\tilde{\mathbf{s}})$ and ${}^0\mathbf{x}^j(t, p(\tilde{\mathbf{e}})) = \mathbf{x}^j(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))$, for all $t \in T_0$, $j = 1, \ldots, l$, ${}^1\mathbf{x}^j(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}})) = \mathbf{x}^j(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))$, for all $t \in T$, $j = 1, \ldots, l$. Moreover, since $p(\mathbf{h}^1)$ is a price selection, it follows that $p(\mathbf{h}^{1'}) = p(\mathbf{h}^{1''})$ whenever $\mathbf{h}^{1''} \in H^{1\star}(\mathbf{h}^{1'})$. This implies that, for all $t \in T$, $\mathbf{s}^1(t, \mathbf{h}^{1'}) = \mathbf{s}^1(t, \mathbf{h}^{1''})$, whenever $\mathbf{h}^{1''} \in H^{1\star}(\mathbf{h}^{1'})$. In order to show that $\tilde{\mathbf{s}}$ is a Pseudo–Markov perfect equilibrium, it remains now to show that no trader $t \in T$ has an advantageous deviation from $\tilde{\mathbf{s}}$. As, for each trader $t \in T_0$, $p(\tilde{\mathbf{s}} \setminus s(t, \cdot)) = p(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1 \setminus s(t, \tilde{\mathbf{h}}^1)|\tilde{\mathbf{h}}^1)$, it is straightforward consequence of the previous discussion that no trader $t \in T_0$ has an advantageous deviation from $\tilde{\mathbf{s}}$. Suppose that there exists a trader $t \in T_1$ and a sequence of functions $s(t, \cdot)$ such that

$$u_t(\mathbf{x}(t, \tilde{\mathbf{s}} \setminus s(t, \cdot), p(\tilde{\mathbf{s}} \setminus s(t, \cdot)))) > u_t(\mathbf{x}(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))).$$

Let $\tilde{\mathbf{h}}^1 \setminus h(t)$ be a history obtained by replacing $\tilde{\mathbf{h}}^1(t)$ in $\tilde{\mathbf{h}}^1$ with $h(t) = s^0(t, \mathbf{h}^0)$ and let $\tilde{\mathbf{e}} \setminus e(t)$ be a strategy selection obtained by replacing $\tilde{\mathbf{e}}(t)$ in $\tilde{\mathbf{e}}$ by $e(t) = s^0(t, \mathbf{h}^0)$. As $p(\tilde{\mathbf{e}} \setminus e(t)) = p(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1 \setminus h(t)) = p(\tilde{\mathbf{s}} \setminus s(t, \cdot))$, the last inequality implies that

$$\begin{aligned} u_t({}^1\mathbf{x}(t, e(t), p(\tilde{\mathbf{e}} \setminus e(t)))) = u_t(\mathbf{x}(t, s(t, \cdot), p(\tilde{\mathbf{s}} \setminus s(t, \cdot)))) > \\ u_t(\mathbf{x}(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))) = u_t({}^1\mathbf{x}(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))), \end{aligned}$$

a contradiction. (ii) Let $\hat{\mathbf{s}}$ be a Pseudo–Markov perfect equilibrium. Consider a history $\mathbf{h}^1 \in H^1$. First, it is straightforward to show that, for all $t \in T_0$, $p(\hat{\mathbf{s}}|\mathbf{h}^1)\mathbf{x}(t, \hat{\mathbf{s}}|\mathbf{h}^1(t), p(\hat{\mathbf{s}}|\mathbf{h}^1)) = p(\hat{\mathbf{s}}|\mathbf{h}^1)\mathbf{w}(t)$. We want now to show that, for all $t \in T_0$, $\mathbf{x}(t, \hat{\mathbf{s}}|\mathbf{h}^1(t), p(\hat{\mathbf{s}}|\mathbf{h}^1)) = {}^0\mathbf{x}(t, p(\hat{\mathbf{s}}|\mathbf{h}^1))$. Suppose that this is not the case for a trader $t \in T_0$. Then, by Assumption 2, there is a bundle $z \in \{x \in R^l_+ : p(\hat{\mathbf{s}}|\mathbf{h}^1)x = p(\hat{\mathbf{s}}|\mathbf{h}^1)\mathbf{w}(t)\}$ such that $u_t(z) > u_t(\mathbf{x}(t, \hat{\mathbf{s}}|\mathbf{h}^1(t), p(\hat{\mathbf{s}}|\mathbf{h}^1))$. By Lemma 5 in Codognato and Ghosal (2000a), there exist $\lambda^j \geq 0$, $j = 1, \ldots, l$, $\sum_{j=1}^l \lambda^j = 1$, such that

$$z^j = \lambda^j \frac{\sum_{j=1}^l p^j(\hat{\mathbf{s}}|\mathbf{h}^1)\mathbf{w}^j(t)}{p^j(\hat{\mathbf{s}}|\mathbf{h}^1)}, \quad j = 1, \ldots, l.$$

Let $s_{ij}(t) = \mathbf{w}^i(t)\lambda^j$, $i, j = 1, \ldots, l$. Since, as an immediate consequence of Definition 4, $p(\hat{\mathbf{s}}|\mathbf{h}^1) = p(\hat{\mathbf{s}}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1)$, it is easy to verify that

$$z^j = \mathbf{x}^j(t, s(t), p(\hat{\mathbf{s}}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1)), \quad j = 1, \ldots, l.$$

But then, we have

$$u_t(\mathbf{x}(t, s(t), p(\hat{\mathbf{s}}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1))) = u_t(z) > u_t(\mathbf{x}(t, \hat{\mathbf{s}}|\mathbf{h}^1(t), p(\hat{\mathbf{s}}|\mathbf{h}^1))),$$

a contradiction. As the function $\mathbf{x}(\cdot, \mathbf{h}^1(\cdot), p(\hat{\mathbf{s}}|\mathbf{h}^1))$ is an allocation, we obtain that

$$\int_{T_0} {}^0\mathbf{x}^j(t, p(\hat{\mathbf{s}}|\mathbf{h}^1))\, d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{h}^1_{ij}(t)\, d\mu \frac{p^i(\hat{\mathbf{s}}|\mathbf{h}^1)}{p^j(\hat{\mathbf{s}}|\mathbf{h}^1)} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{h}^1_{ji}(t)\, d\mu. \tag{7}$$

Let now $p(\mathbf{e})$ be a function which associates, with each $\mathbf{e}$, the price vector $p(\hat{\mathbf{s}}|\mathbf{h}^1)$, where $\mathbf{h}^1$ is such that $\mathbf{h}^1(t) = \mathbf{e}(t)$, for all $t \in T_1$. First, let us notice that, since $\hat{\mathbf{s}}$ is a Pseudo–Markov perfect equilibrium, we have $p(\hat{\mathbf{s}}|\mathbf{h}^{1'}) = p(\hat{\mathbf{s}}|\mathbf{h}^{1''})$ if $\int_{T_1} \mathbf{h}^{1'}(t)\, d\mu = \int_{T_1} \mathbf{h}^{1''}(t)\, d\mu$ and then $p(\mathbf{e}') = p(\mathbf{e}'')$, if $\int_{T_1} \mathbf{e}'(t)\, d\mu = \int_{T_1} \mathbf{e}''(t)\, d\mu$. Moreover, if we replace, in Eq. (7), the history $\mathbf{h}^1$ with a strategy selection $\mathbf{e}$ such that $\mathbf{e}(t) = \mathbf{h}^1(t)$, for all $t \in T_1$, and the price $p(\hat{\mathbf{s}}|\mathbf{h}^1)$ with the price $p(\mathbf{e})$, it follows immediately that the function $p(\mathbf{e})$ satisfies Eq. (1). Therefore, we can conclude that $p(\mathbf{e})$ is a price selection. It also follows immediately from the above argument that, for each history $\mathbf{h}^1 \in H^1$, $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}})) = {}^0\mathbf{x}(t, p(\hat{\mathbf{e}}))$, for all $t \in T_0$, and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}})) = {}^1\mathbf{x}(t, \hat{\mathbf{e}}(t), p(\hat{\mathbf{e}}))$, for all $t \in T_1$, where $\mathbf{e}$ is a strategy selection such that $\mathbf{e}(t) = \mathbf{h}^1(t)$, for all $t \in T_1$. Let now $\hat{\mathbf{h}}^1$ be a history such that $\hat{\mathbf{h}}^1(t) = \hat{\mathbf{s}}^0(t, \mathbf{h}^0)$, for all $t \in T$, and let $\hat{\mathbf{e}}$ be a strategy selection such that $\hat{\mathbf{e}}(t) = \hat{\mathbf{h}}^1(t)$, for all $t \in T_1$. As $p(\hat{\mathbf{s}}) = p(\hat{\mathbf{s}}|\hat{\mathbf{h}}^1)$, $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}})) = {}^0\mathbf{x}(t, p(\hat{\mathbf{e}}))$, for all $t \in T_0$, and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}})) = {}^1\mathbf{x}(t, \hat{\mathbf{e}}(t), p(\hat{\mathbf{e}}))$, for all $t \in T_1$. But then, in order to show that the pair $(\hat{\mathbf{e}}, \hat{\mathbf{x}})$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}})) = {}^0\mathbf{x}(t, p(\hat{\mathbf{e}}))$, for all $t \in T_0$, and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}})) = {}^1\mathbf{x}(t, \hat{\mathbf{e}}(t), p(\hat{\mathbf{e}}))$, for all $t \in T_1$, is a Cournot–Walras equilibrium with respect to the price selection $p(\mathbf{e})$, it remains to show that no trader $t \in T_1$ has an advantageous deviation from the strategy selection $\hat{\mathbf{e}}$. Suppose, on the contrary, that there exists a trader $t \in T_1$ and a strategy $e(t) \in \mathbf{E}(t)$ such that

$$u_t({}^1\mathbf{x}(t, e(t), p(\hat{\mathbf{e}} \setminus e(t)))) > u_t({}^1\mathbf{x}(t, \hat{\mathbf{e}}(t), p(\hat{\mathbf{e}}))).$$

Let $\hat{\mathbf{h}}^1 \setminus h(t)$ be a history obtained by replacing $\hat{\mathbf{h}}^1(t)$ in $\hat{\mathbf{h}}^1$ with $h(t) = e(t)$ and let $\hat{\mathbf{s}} \setminus s(t)$ be a strategy profile obtained by replacing $\hat{\mathbf{s}}^0(t, \cdot)$ in $\hat{\mathbf{s}}^0$ with $s^0(t) = h(t)$. As $p(\hat{\mathbf{s}} \setminus s(t)) = p(\hat{\mathbf{s}}|\hat{\mathbf{h}}^1 \setminus h(t)) = p(\hat{\mathbf{e}} \setminus e(t))$, the last inequality implies that

$$u_t({}^1\mathbf{x}(t, s(t), p(\hat{\mathbf{s}} \setminus s(t)))) = u_t({}^1\mathbf{x}(t, e(t), p(\hat{\mathbf{e}} \setminus e(t)))) > u_t({}^1\mathbf{x}(t, \hat{\mathbf{e}}(t), p(\hat{\mathbf{e}}))) = u_t({}^1\mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))),$$

a contradiction. □

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