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Correlated equilibrium and concave games

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Abstract This paper shows that if a game satisfies the sufficient condition for the existence and uniqueness of a pure-strategy Nash equilibrium provided by Rosen (Econometrica 33:520, 1965), then the game has a unique correlated equilibrium, which places probability one on the unique pure-strategy Nash equilibrium. In addition, it shows that a weaker condition suffices for the uniqueness of a correlated equilibrium. The condition generalizes the sufficient condition for the uniqueness of a correlated equilibrium provided by Neyman (Int J Game Theory 26:223, 1997) for a potential game with a strictly concave potential function.

JEL Classification C72

Keywords Uniqueness · Correlated equilibrium · Payoff gradient · Strict monotonicity

1 Introduction

This paper explores conditions for uniqueness of a correlated equilibrium (Aumann 1974, 1987) in a class of games where strategy sets are finite-dimensional convex sets and each player's payoff function is concave and continuously differentiable with

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respect to the player's own strategy. Liu (1996) showed that a Cournot oligopoly game with a linear demand function has a unique correlated equilibrium. Neyman (1997) studied a correlated equilibrium of a potential game (Monderer and Shapley 1996) and showed that if a potential function is concave and payoff functions are bounded, then any correlated equilibrium is a mixture of potential maximizers in Theorem 1, and that if a potential function is strictly concave and strategy sets are compact, then the potential game has a unique correlated equilibrium, which places probability one on the unique potential maximizer, in Theorem 2. The latter, which is derived from the former, generalizes the result of Liu (1996) because a Cournot oligopoly game with a linear demand function is a potential game with a strictly concave potential function (Slade 1994).

We study the correlated equilibria of a class of games examined by Rosen (1965). For a given game, consider a vector each component of which is a partial derivative of each player's payoff function with respect to the player's own strategy and call the vector the payoff gradient of the game. The payoff gradient is "strictly monotone" if the inner product of the difference of two arbitrary strategy profiles and the corresponding difference of the payoff gradients is strictly negative. Strict monotonicity of the payoff gradient implies strict concavity of each player's payoff function with respect to the player's own strategy. Theorem 2 of Rosen (1965) states that if the payoff gradient is strictly monotone and strategy sets are compact, then the game has a unique pure-strategy Nash equilibrium. The present paper shows that, under the same conditions, the game has a unique correlated equilibrium, which places probability one on the unique pure-strategy Nash equilibrium. In addition, our main result (Proposition 5) states that a weaker condition suffices for the uniqueness of a correlated equilibrium. This result generalizes Theorem 2 of Neyman (1997) because the payoff gradient of a potential game with a strictly concave potential function is strictly monotone.

To establish the main result, we first provide a sufficient condition for any correlated equilibrium to be a mixture of pure-strategy Nash equilibria, which differs from but overlaps with Theorem 1 of Neyman (1997). We then show that if the payoff gradient is strictly monotone and strategy sets are compact, then the game satisfies the sufficient condition, and thus any correlated equilibrium places probability one on the unique pure-strategy Nash equilibrium.

The organization of this paper is as follows. Preliminary definitions and results are summarized in Sect. 2. The concept of strict monotonicity for the payoff gradient is introduced in Sect. 3. The results are reported in Sect. 4.

2 Preliminaries

A game consists of a set of players $N \equiv \{1, \dots, n\}$, a measurable set of strategies $X_i \subseteq \mathbb{R}^{m_i}$ for each $i \in N$ with generic element $x_i \equiv (x_{i1}, \dots, x_{im_i})^{\mathrm{T}}$, and a measurable payoff function $u_i : X \to \mathbb{R}$ for each $i \in N$, where $X \equiv \prod_{i \in N} X_i$. We assume that X_i is a full-dimensional convex subset[1] of a Euclidean space $\mathbb{R}^{m_i}$. We write

[1] Even if X_i is not full-dimensional, we can use a reparametrization to get to the full-dimensional case.

$X_{-i} \equiv \prod_{j \neq i} X_j$ and $x_{-i} \equiv (x_j)_{j \neq i} \in X_{-i}$. We fix N and X throughout this paper and simply denote a game by $\mathbf{u} \equiv (u_i)_{i \in N}$.

A *pure-strategy Nash equilibrium* of $\mathbf{u}$ is a strategy profile $x^* \in X$ such that, for each $x_i \in X_i$ and each $i \in N$, $u_i(x^*) \geq u_i(x_i, x^*_{-i})$. A *correlated equilibrium*[2] of $\mathbf{u}$ is a probability distribution μ over X such that, for each $i \in N$ and each measurable function $\xi_i : X_i \to X_i$,

$$\int u_i(x) d\mu(x) \geq \int u_i(\xi_i(x_i), x_{-i}) d\mu(x).$$

A game $\mathbf{u}$ is *smooth* if, for each $i \in N$, u_i has continuous partial derivatives with respect to the components of x_i. In a smooth game $\mathbf{u}$, the first-order condition for a pure-strategy Nash equilibrium $x^* \in X$ is

$$\begin{aligned} &\lim_{t \to +0} \frac{u_i(x_i^* + t(x_i - x_i^*), x_{-i}^*) - u_i(x^*)}{t} \\ &\quad = \nabla_i u_i(x^*)^{\mathrm{T}}(x_i - x_i^*) \leq 0 \quad \text{for each } x_i \in X_i \text{ and each } i \in N, \end{aligned} \tag{1}$$

where $\nabla_i u_i \equiv \left(\partial u_i / \partial x_{i1}, \ldots, \partial u_i / \partial x_{i m_i}\right)^{\mathrm{T}}$ denotes the gradient of u_i with respect to x_i. It is straightforward to check that (1) is equivalent to

$$\sum_{i \in N} \nabla_i u_i(x^*)^{\mathrm{T}}(x_i - x_i^*) \leq 0 \quad \text{for each } x \in X. \tag{2}$$

The problem of solving this type of inequality is called the variational inequality problem.[3] The following sufficient condition for the existence of a solution is well-known.[4]

Lemma 1 *Let* $\mathbf{u}$ *be a smooth game. If* X_i *is compact for each* $i \in N$*, then there exists* $x^* \in X$ *satisfying* (2).

A game $\mathbf{u}$ is *concave* (Rosen 1965) if, for each $i \in N$, $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$ is concave for each fixed $x_{-i} \in X_{-i}$. It can be readily shown that if $\mathbf{u}$ is a smooth concave game, then the first-order condition (1) is necessary and sufficient for a pure-strategy Nash equilibrium, and thus the set of solutions to the inequality problem (2) coincides with the set of pure-strategy Nash equilibria.[5]

2 A generalized definition of a correlated equilibrium for a game with infinite strategy sets is proposed by Hart and Schmeidler (1989).

3 Let $S \subseteq \mathbb{R}^m$ be a convex set and let $F : S \to \mathbb{R}^m$ be a mapping. The variational inequality problem is to find $x^* \in S$ such that $F(x^*)^{\mathrm{T}}(x - x^*) \geq 0$ for each $x \in S$. It has been shown that a pure-strategy Nash equilibrium is a solution to the variational inequality problem with $F = (-\nabla_i u_i)_{i \in N}$ (cf. Hartman and Stampacchia 1966; Gabay and Moulin 1980).

4 See Nagurney (1993), for example.

5 Accordingly, by Lemma 1, a smooth concave game with compact strategy sets has a pure-strategy Nash equilibrium, whereas Kakutani fixed point theorem directly shows that a concave game with compact strategy sets, which is not necessarily a smooth game, has a pure-strategy Nash equilibrium if $u_i : X \to \mathbb{R}$ is continuous for each $i \in N$.

A game $\mathbf{u}$ is a *potential game* (Monderer and Shapley 1996) if there exists a function $f : X \to \mathbb{R}$ such that $u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = f(x_i, x_{-i}) - f(x_i', x_{-i})$ for each $x_i, x_i' \in X_i$, each $x_{-i} \in X_{-i}$, and each $i \in N$. This function is a *potential function*. As shown by Monderer and Shapley (1996), a smooth game $\mathbf{u}$ is a potential game with a potential function f if and only if $\nabla_i u_i = \nabla_i f$ for each $i \in N$. This implies the equivalence of the first-order condition for a pure-strategy Nash equilibrium and that for a potential maximizer $x^* \in \arg\max_{x \in X} f(x)$. From this equivalence, we can derive the following lemma (Neyman 1997) by noting that a smooth potential game with a concave potential function is a smooth concave game.[6]

Lemma 2 *In a smooth potential game with a concave potential function, a strategy profile is a pure-strategy Nash equilibrium if and only if it is a potential maximizer.*

Neyman (1997) studied a correlated equilibrium of a smooth potential game with a concave or strictly concave potential function and obtained the following two results.

Proposition 1 *Let* $\mathbf{u}$ *be a smooth potential game with bounded payoff functions. If a potential function of* $\mathbf{u}$ *is concave, then any correlated equilibrium of* $\mathbf{u}$ *is a mixture of potential maximizers.*

Proposition 2 *Let* $\mathbf{u}$ *be a smooth potential game with compact strategy sets. If a potential function of* $\mathbf{u}$ *is strictly concave, then* $\mathbf{u}$ *has a unique correlated equilibrium, which places probability one on the unique potential maximizer.*

Neyman (1997) derived Proposition 2 and Lemma 2 from Proposition 1.

3 Strict monotonicity of the payoff gradient

Let $S \subseteq \mathbb{R}^m$ be a convex set and let $F : S \to \mathbb{R}^m$ be a mapping. A mapping F is *strictly monotone* if $(F(x) - F(y))^{\mathrm{T}}(x - y) > 0$ for each $x, y \in S$ with $x \neq y$. The following sufficient condition for strict monotonicity is well-known.[7]

Lemma 3 *If a mapping* $F : S \to \mathbb{R}^m$ *is continuously differentiable and the Jacobian matrix of* F *is positive definite for each* $x \in S$*, then* F *is strictly monotone.*

Let us call $(\nabla_i u_i)_{i \in N}$ the *payoff gradient* of a smooth game $\mathbf{u}$. We say that, with some abuse of language, the payoff gradient of $\mathbf{u}$ is *strictly monotone* if the mapping $x \mapsto (-\nabla_i u_i(x))_{i \in N}$ is strictly monotone, i.e.,

$$\sum_{i \in N} (\nabla_i u_i(x) - \nabla_i u_i(y))^{\mathrm{T}}(x_i - y_i) < 0 \quad \text{for each } x, y \in X \text{ with } x \neq y. \tag{3}$$

[6] If a potential function f is concave, then $f(tx_i + (1-t)x_i', x_{-i}) - f(x_i', x_{-i}) \geq t(f(x_i, x_{-i}) - f(x_i', x_{-i}))$. Hence, $u_i(tx_i + (1-t)x_i', x_{-i}) - u_i(x_i', x_{-i}) \geq t(u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}))$, which implies that $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$ is concave.

[7] See Nagurney (1993), for example.

Let $c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_{++}$ and call $(c_i \nabla_i u_i)_{i \in N}$ the *c-weighted payoff gradient* of **u**. The c-weighted payoff gradient of **u** is *strictly monotone* if the mapping $x \mapsto (-c_i \nabla_i u_i(x))_{i \in N}$ is strictly monotone,[8] i.e.,

$$\sum_{i \in N} c_i (\nabla_i u_i(x) - \nabla_i u_i(y))^{\mathrm{T}} (x_i - y_i) < 0 \quad \text{for each } x, y \in X \text{ with } x \neq y. \tag{4}$$

Note that if $c_i = c_j$ for each $i, j \in N$, then (4) implies (3).

Let $\gamma \equiv (\gamma_i)_{i \in N}$ with $\gamma_i : X_i \to \mathbb{R}_{++}$ and call $(\gamma_i \nabla_i u_i)_{i \in N}$ the *γ-weighted payoff gradient* of **u**. The γ-weighted payoff gradient of **u** is *strictly monotone* if the mapping $x \mapsto (-\gamma_i(x_i) \nabla_i u_i(x))_{i \in N}$ is strictly monotone, i.e.,

$$\sum_{i \in N} (\gamma_i(x_i) \nabla_i u_i(x) - \gamma_i(y_i) \nabla_i u_i(y))^{\mathrm{T}} (x_i - y_i) < 0 \quad \text{for each } x, y \in X \text{ with } x \neq y. \tag{5}$$

Note that if $\gamma_i(x_i) = c_i \in \mathbb{R}_{++}$ for each $x_i \in X_i$ and each $i \in N$, then (5) implies (4).

Rosen (1965) showed that strict monotonicity of the c-weighted payoff gradient leads to the uniqueness of a pure-strategy Nash equilibrium.

Proposition 3 *Let* **u** *be a smooth game with compact strategy sets. If there exists $c \in \mathbb{R}^N_{++}$ such that the c-weighted payoff gradient of* **u** *is strictly monotone, then* **u** *has a unique pure-strategy Nash equilibrium. Especially, if the payoff gradient of* **u** *is strictly monotone, then* **u** *has a unique pure-strategy Nash equilibrium.*

In the next section, we show that strict monotonicity of the γ-weighted payoff gradient leads to the uniqueness of a correlated equilibrium.

Before closing this section, we discuss two implications of strict monotonicity.[9]

Lemma 4 *Let* **u** *be a smooth potential game. A potential function of* **u** *is strictly concave if and only if the payoff gradient of* **u** *is strictly monotone.*

Proof Let f be a potential function and suppose that f is strictly concave. For each $x, y \in X$ with $x \neq y$, $\sum_{i \in N} \nabla_i f(x)^{\mathrm{T}} (y_i - x_i) > f(y) - f(x)$ and $\sum_{i \in N} \nabla_i f(y)^{\mathrm{T}} (x_i - y_i) > f(x) - f(y)$. Adding these two inequalities, we have

$$\sum_{i \in N} (\nabla_i f(x) - \nabla_i f(y))^{\mathrm{T}} (x_i - y_i) = \sum_{i \in N} (\nabla_i u_i(x) - \nabla_i u_i(y))^{\mathrm{T}} (x_i - y_i) < 0$$

since $\nabla_i f = \nabla_i u_i$. Therefore, the payoff gradient of **u** is strictly monotone.

Conversely, suppose that the payoff gradient of **u** is strictly monotone. Fix $x, y \in X$ with $x \neq y$. Let $\phi(t) = f(x + t(y - x))$ for each $t \in [0, 1]$. Then, ϕ is differentiable

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[8] Rosen (1965) called this property "diagonal strict concavity".

[9] I thank a referee for pointing out the next two lemmas with proofs.

and, by the mean-value theorem, there exist $0 < \theta_1 < 1/2 < \theta_2 < 1$ such that $\phi(1/2) - \phi(0) = \phi'(\theta_1)/2$ and $\phi(1) - \phi(1/2) = \phi'(\theta_2)/2$, which are rewritten as

$$f((x+y)/2) - f(x) = \sum_{i \in N} \nabla_i f(x + \theta_1(y-x))^{\mathrm{T}}(y_i - x_i)/2, \tag{6}$$

$$f(y) - f((x+y)/2) = \sum_{i \in N} \nabla_i f(x + \theta_2(y-x))^{\mathrm{T}}(y_i - x_i)/2. \tag{7}$$

On the other hand, since the payoff gradient of $\mathbf{u}$ is strictly monotone,

$$\sum_{i \in N} (\nabla_i u_i(x + \theta_2(y-x)) - \nabla_i u_i(x + \theta_1(y-x)))^{\mathrm{T}}(\theta_2 - \theta_1)(y_i - x_i) < 0.$$

Thus, since $\theta_2 - \theta_1 > 0$ and $\nabla_i u_i = \nabla_i f$,

$$\sum_{i \in N} \nabla_i f(x + \theta_2(y-x))^{\mathrm{T}}(y_i - x_i) < \sum_{i \in N} \nabla_i f(x + \theta_1(y-x))^{\mathrm{T}}(y_i - x_i).$$

This inequality, (6), and (7) imply that $f((x+y)/2) > (f(x) + f(y))/2$. Therefore, by the continuity of f, f is strictly concave. □

Lemma 5 *Let* $\mathbf{u}$ *be a smooth game. If there exists* $c \in \mathbb{R}^N_{++}$ *such that the c-weighted payoff gradient of* $\mathbf{u}$ *is strictly monotone, then, for each* $i \in N$ *and each* $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$ *is strictly concave.*

Proof Fix arbitrary $i \in N$ and $x_{-i} \in X_{-i}$, and consider a game with a singleton player set $\{i\}$, strategy set X_i, and payoff function $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$. This game is trivially a potential game with a potential function $u_i(\cdot, x_{-i})$. The payoff gradient of this game is strictly monotone. Thus, by Lemma 4, the potential function is strictly concave. This implies that $u_i(\cdot, x_{-i})$ is strictly concave. □

4 Results

We provide a sufficient condition for any correlated equilibrium to be a mixture of pure-strategy Nash equilibria.

Proposition 4 *Let* $\mathbf{u}$ *be a smooth game with bounded payoff functions. Assume that there exist a pure-strategy Nash equilibrium* $x^* \in X$ *and a bounded measurable function* $\gamma_i : X_i \to \mathbb{R}_{++}$ *for each* $i \in N$ *such that:*

(i) $\sum_{i \in N} \gamma_i(x_i) \nabla_i u_i(x)^{\mathrm{T}}(x_i^* - x_i) \begin{cases} \geq 0 & \text{for each } x \in X, \\ > 0 & \text{if } x \text{ is not a pure-strategy Nash equilibrium,} \end{cases}$

(ii) $\inf_{(x,t) \in X \times (0,1]} \dfrac{u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)}{t} > -\infty$ *for each* $i \in N$.

Then, any correlated equilibrium of $\mathbf{u}$ *is a mixture of pure-strategy Nash equilibria.*

Proof Let μ be a probability distribution over X such that $\mu(Y) > 0$ for some measurable set $Y \subseteq X$ containing no pure-strategy Nash equilibria. It is enough to show that μ is not a correlated equilibrium. By (i),

$$\int \sum_{i \in N} \gamma_i(x_i) \nabla_i u_i(x)^{\mathrm{T}}(x_i^* - x_i) d\mu(x) > 0.$$

Thus, there exists $i \in N$ such that

$$\int \gamma_i(x_i) \nabla_i u_i(x)^{\mathrm{T}}(x_i^* - x_i) d\mu(x) > 0.$$

By (ii) and since γ_i is bounded, $\inf_{(x,t) \in X \times (0,1]} \gamma_i(x_i)(u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x))/t > -\infty$. Thus, by the Lebesgue–Fatou Lemma,

$$\begin{aligned} &\liminf_{t \to +0} \int \gamma_i(x_i) \frac{u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)}{t} d\mu(x) \\ &\geq \int \liminf_{t \to +0} \gamma_i(x_i) \frac{u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)}{t} d\mu(x) \\ &= \int \gamma_i(x_i) \nabla_i u_i(x)^{\mathrm{T}}(x_i^* - x_i) d\mu(x) > 0. \end{aligned}$$

Therefore, there exists $t > 0$ such that

$$\int \gamma_i(x_i)\big(u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)\big) d\mu(x) > 0. \tag{8}$$

Set $\xi_i(x_i) = x_i + t(x_i^* - x_i)$ for each $x_i \in X_i$.

For a measurable function $f : X \to \mathbb{R}$, let $E_{\mu(x)}[f(x)|x_i]$ denote the conditional expected value of $f(x)$ given $x_i \in X_i$ with respect to μ. Define the measurable set

$$S_i = \{x_i \in X_i \mid E_{\mu(x)}[u_i(\xi_i(x_i), x_{-i}) - u_i(x)|x_i] \geq 0\}$$

and write $1_{S_i} : X_i \to \{0, 1\}$ for its indicator function. Let $\bar{\gamma}_i = \sup_{x_i \in S_i} \gamma_i(x_i) < \infty$. Then,

$$\begin{aligned} &E_{\mu(x)}[1_{S_i}(x_i)(u_i(\xi_i(x_i), x_{-i}) - u_i(x))|x_i] \\ &\geq \frac{\gamma_i(x_i)}{\bar{\gamma}_i} E_{\mu(x)}[1_{S_i}(x_i)(u_i(\xi_i(x_i), x_{-i}) - u_i(x))|x_i] \\ &\geq \frac{\gamma_i(x_i)}{\bar{\gamma}_i} E_{\mu(x)}[u_i(\xi_i(x_i), x_{-i}) - u_i(x)|x_i] \\ &= \frac{1}{\bar{\gamma}_i} E_{\mu(x)}[\gamma_i(x_i)(u_i(\xi_i(x_i), x_{-i}) - u_i(x))|x_i]. \end{aligned}$$

This and (8) imply that

$$\begin{aligned}&\int 1_{S_i}(x_i)\big(u_i(\xi_i(x_i), x_{-i}) - u_i(x)\big)d\mu(x)\\ &\geq \frac{1}{\bar{\gamma}_i}\int \gamma_i(x_i)\big(u_i(\xi_i(x_i), x_{-i}) - u_i(x)\big)d\mu(x) > 0.\end{aligned}$$

Let $\xi_i' : X_i \to X_i$ be such that $\xi_i'(x_i) = \xi_i(x_i)$ if $x_i \in S_i$ and $\xi_i'(x_i) = x_i$ otherwise. Then,

$$\int \big(u_i(\xi_i'(x_i), x_{-i}) - u_i(x)\big)d\mu(x) = \int 1_{S_i}(x_i)\big(u_i(\xi_i(x_i), x_{-i}) - u_i(x)\big)d\mu(x) > 0,$$

and thus μ is not a correlated equilibrium. □

As the next lemma shows, a smooth potential game with bounded payoff functions satisfies the sufficient condition for any correlated equilibrium to be a mixture of pure-strategy Nash equilibria given by Proposition 4 if its potential function is concave and a potential maximizer exists. On the other hand, Proposition 1 does not assume the existence of a potential maximizer *a priori*: it asserts that if a correlated equilibrium exists, then a potential maximizer also exists, and any correlated equilibrium is a mixture of potential maximizers, i.e., pure-strategy Nash equilibria. In this sense, Proposition 4 and the following lemma together partially explain Proposition 1.

Lemma 6 *Let* $\mathbf{u}$ *be a smooth potential game with bounded payoff functions. If a potential function of* $\mathbf{u}$ *is concave and a potential maximizer exists, then, for a potential maximizer* $x^* \in X$ *and* $\gamma_i : X_i \to \mathbb{R}_{++}$ *with* $\gamma_i(x_i) = 1$ *for each* $x_i \in X_i$ *and each* $i \in N$*, conditions* (i) *and* (ii) *in Proposition 4 are true.*

Proof Let f be a potential function and write $X^* = \arg\max_{x\in X} f(x)$. The set X^* is non-empty by assumption and, by Lemma 2, it coincides with the set of pure-strategy Nash equilibria. Let $x^* \in X^*$. Then, by the concavity of f,

$$\sum_{i\in N} \nabla_i u_i(x)^{\mathrm{T}}(x_i^* - x_i) = \sum_{i\in N} \nabla_i f(x)^{\mathrm{T}}(x_i^* - x_i) \geq f(x^*) - f(x) \geq 0$$

for each $x \in X$. If $x \notin X^*$ then $\sum_{i\in N} \nabla_i u_i(x)^{\mathrm{T}}(x_i^* - x_i) \geq f(x^*) - f(x) > 0$, which establishes (i). Next, since f is concave, $(u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x))/t = (f(x_i + t(x_i^* - x_i), x_{-i}) - f(x))/t$ is decreasing in $t \in (0, 1]$. Thus, since u_i is bounded,

$$\inf_{(x,t)\in X\times(0,1]} \frac{u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)}{t} \geq \inf_{x\in X}\big(u_i(x_i^*, x_{-i}) - u_i(x)\big) > -\infty,$$

which establishes (ii). □

Using Proposition 4, we show that strict monotonicity of the γ-weighted payoff gradient leads to the uniqueness of a correlated equilibrium.

Proposition 5 *Let* $\mathbf{u}$ *be a smooth game with compact strategy sets. If, for each* $i \in N$, *there exists a bounded measurable function* $\gamma_i : X_i \to \mathbb{R}_{++}$ *such that the* γ*-weighed payoff gradient of* $\mathbf{u}$ *is strictly monotone, then* $\mathbf{u}$ *has a unique correlated equilibrium, which places probability one on a unique pure-strategy Nash equilibrium. Especially, if the payoff gradient of* $\mathbf{u}$ *is strictly monotone, then* $\mathbf{u}$ *has a unique correlated equilibrium.*

Proposition 5 generalizes Proposition 2 because, by Lemma 4, the payoff gradient of a smooth potential game with a strictly concave potential function is strictly monotone. Proposition 5 also generalizes Proposition 3 because the c-weighted payoff gradient is a special case of the γ-weighted payoff gradient.

To prove Proposition 5, we first show the existence and uniqueness of a pure-strategy Nash equilibrium.

Lemma 7 *Let* $\mathbf{u}$ *be a smooth game with compact strategy sets. If, for each* $i \in N$, *there exists a function* $\gamma_i : X_i \to \mathbb{R}_{++}$ *such that the* γ*-weighed payoff gradient of* $\mathbf{u}$ *is strictly monotone, then* $\mathbf{u}$ *has a unique pure-strategy Nash equilibrium.*

Proof First, we show that $\mathbf{u}$ has a pure-strategy Nash equilibrium. By Lemma 1, there exists $x^* \in X$ satisfying (2), which is equivalent to (1). Thus, it is enough to show that x^* is a pure-strategy Nash equilibrium. Fix $i \in N$ and $x_i \neq x_i^*$. Since the γ-weighed payoff gradient of $\mathbf{u}$ is strictly monotone, (5) holds. Especially, when $x = x^*$ and $y = (x_i + t(x_i^* - x_i), x_{-i}^*)$ in (5), we have

$$\begin{aligned}&(\gamma_i(x_i^*)\nabla_i u_i(x^*) - \gamma_i(x_i + t(x_i^* - x_i))\\&\quad \times \nabla_i u_i(x_i + t(x_i^* - x_i), x_{-i}^*))^{\mathrm{T}}(1-t)(x_i^* - x_i) < 0\end{aligned}$$

for each $t \in [0, 1)$. Hence, by (1),

$$\begin{aligned}\nabla_i u_i(x_i + t(x_i^* - x_i), x_{-i}^*)^{\mathrm{T}}(x_i^* - x_i) &> \frac{\gamma_i(x_i^*)}{\gamma_i(x_i + t(x_i^* - x_i))}\nabla_i u_i(x^*)^{\mathrm{T}}(x_i^* - x_i)\\&\geq 0,\end{aligned}$$

and thus

$$\frac{d}{dt}u_i(x_i + t(x_i^* - x_i), x_{-i}^*) = \nabla_i u_i(x_i + t(x_i^* - x_i), x_{-i}^*)^{\mathrm{T}}(x_i^* - x_i) > 0$$

for each $t \in [0, 1)$. Therefore, $u_i(x^*) \geq u_i(x_i, x_{-i}^*)$. Since $x_i \in X_i$ and $i \in N$ were chosen arbitrarily, x^* is a pure-strategy Nash equilibrium.

Next, we show that a pure-strategy Nash equilibrium is unique. Let $x^*, y^* \in X$ be two pure-strategy Nash equilibria. By (1), for each $i \in N$, $\gamma_i(x_i^*)\nabla_i u_i(x^*)^{\mathrm{T}}(y_i^* - x_i^*) \leq 0$ and $\gamma_i(y_i^*)\nabla_i u_i(y^*)^{\mathrm{T}}(x_i^* - y_i^*) \leq 0$. By adding them, for each $i \in N$, $(\gamma_i(x_i^*)\nabla_i u_i(x^*) - \gamma_i(y_i^*)\nabla_i u_i(y^*))^{\mathrm{T}}(x_i^* - y_i^*) \geq 0$. Therefore, $\sum_{i \in N}(\gamma_i(x_i^*)$

$\nabla_i u_i(x^*) - \gamma_i(y_i^*)\nabla_i u_i(y^*))^{\mathrm{T}}(x_i^* - y_i^*) \geq 0$. On the other hand, if $x^* \neq y^*$, then, by strict monotonicity, $\sum_{i \in N}(\gamma_i(x_i^*)\nabla_i u_i(x^*) - \gamma_i(y_i^*)\nabla_i u_i(y^*))^{\mathrm{T}}(x_i^* - y_i^*) < 0$. Thus, x^* and y^* coincide. □

We are now ready to prove Proposition 5.

Proof of Proposition 5 We show that **u** satisfies the sufficient condition for any correlated equilibrium to be a mixture of pure-strategy Nash equilibria given by Proposition 4. By Lemma 7, **u** has a unique pure-strategy Nash equilibrium $x^* \in X$. For each $x \neq x^*$, by strict monotonicity,

$$\sum_{i \in N}(\gamma_i(x_i^*)\nabla_i u_i(x^*) - \gamma_i(x_i)\nabla_i u_i(x))^{\mathrm{T}}(x_i^* - x_i) < 0.$$

Thus, by (1),

$$\sum_{i \in N}\gamma_i(x_i)\nabla_i u_i(x)^{\mathrm{T}}(x_i^* - x_i) > \sum_{i \in N}\gamma_i(x_i^*)\nabla_i u_i(x^*)^{\mathrm{T}}(x_i^* - x_i) \geq 0,$$

which establishes (i).

Fix $i \in N$. By the mean-value theorem, for each $x \in X$ and each $t \in (0, 1]$, there exists $\theta \in (0, t)$ such that $(u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x))/t = \nabla_i u_i(x_i + \theta(x_i^* - x_i), x_{-i})^{\mathrm{T}}(x_i^* - x_i)$. Thus, since X is compact and $\nabla_i u_i$ is continuous,

$$\begin{aligned} &\inf_{(x,t)\in X\times(0,1]} \frac{u_i(x_i + t(x_i^* - x_i), x_{-i}) - u_i(x)}{t} \\ &\qquad \geq \min_{(x,\theta)\in X\times[0,1]} \nabla_i u_i(x_i + \theta(x_i^* - x_i), x_{-i})^{\mathrm{T}}(x_i^* - x_i) > -\infty, \end{aligned}$$

which establishes (ii).

Therefore, by Proposition 4, any correlated equilibrium of **u** places probability one on the unique pure-strategy Nash equilibrium x^*. □

Using Lemma 3, we can obtain a sufficient condition for strict monotonicity of the γ-weighed payoff gradient which is in some cases easier to verify than (5) if payoff functions are twice continuously differentiable. By considering a special case with $X_i \subseteq \mathbb{R}$ for each $i \in N$, we have the following corollary of Proposition 5.

Corollary 6 *Let* **u** *be a smooth game. Suppose that, for each* $i \in N$, $X_i \subseteq \mathbb{R}$ *is a closed bounded interval and that payoff functions are twice continuously differentiable. If, for each* $i \in N$, *there exists a continuously differentiable function* $\gamma_i : X_i \to \mathbb{R}_{++}$ *such that the matrix*

$$\left[\delta_{ij}\frac{d\gamma_i(x_i)}{dx_i}\frac{\partial u_i(x)}{\partial x_i}\right] + \left[\gamma_i(x_i)\frac{\partial^2 u_i(x)}{\partial x_i \partial x_j}\right] \tag{9}$$

is negative definite for each $x \in X$ (where δ_{ij} is the Kronecker delta), then **u** *has a unique correlated equilibrium, which places probability one on a unique pure-strategy Nash equilibrium. Especially, if the matrix $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ is negative definite for each $x \in X$, then* **u** *has a unique correlated equilibrium.*

Proof It is enough to show that the γ-weighted payoff gradient of **u** is strictly monotone. Note that, for each $i \in N$, $\gamma_i : X_i \to \mathbb{R}_{++}$ is a bounded measurable function. Consider the mapping $x \mapsto (-\gamma_i(x_i)\nabla_i u_i(x))_{i \in N}$. Then, (9) is the Jacobian matrix multiplied by -1. Thus, if (9) is negative definite for each $x \in X$, then, by Lemma 3, the mapping is strictly monotone. Therefore, the γ-weighted payoff gradient of **u** is strictly monotone. □

As shown by Monderer and Shapley (1996), if the matrix $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ is symmetric for each $x \in X$, then **u** is a potential game and $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ coincides with the Hessian matrix of a potential function. Thus, if $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ is symmetric and negative definite for each $x \in X$, then **u** is a smooth potential game with a strictly concave potential function, and thus, by Proposition 2, a correlated equilibrium of **u** is unique. Corollary 6 says that $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ need not be symmetric for the uniqueness of a correlated equilibrium.

Finally, we discuss two examples.

Example 1 Consider a Cournot oligopoly game with differentiated products in which a strategy of firm $i \in N$ is a quantity of differentiated product $i \in N$ to produce. For each $i \in N$, let $X_i \subseteq \mathbb{R}_+$ be a closed bounded interval. The inverse demand function for product i is denoted by $p_i : X \to \mathbb{R}_+$ and the cost function of firm i is denoted by $c_i : X_i \to \mathbb{R}_+$. It is assumed that both functions are twice continuously differentiable and that $d^2 c_i(x_i)/dx_i^2 \geq 0$ for each $x_i \in X_i$. The payoff function $u_i : X \to \mathbb{R}$ of firm i is given by $u_i(x) = p_i(x)x_i - c_i(x_i)$.

The matrix (9) is calculated as

$$\left[\delta_{ij} \frac{d\gamma_i(x_i)}{dx_i} \frac{\partial p_i(x)x_i}{\partial x_i}\right] + \left[\gamma_i(x_i) \frac{\partial^2 p_i(x)x_i}{\partial x_i \partial x_j}\right] - \left[\delta_{ij} \frac{d}{dx_i}\left(\gamma_i(x_i) \frac{dc_i(x_i)}{dx_i}\right)\right].$$

If $\gamma(x_i) = 1$ for each $i \in N$, then the above reduces to

$$\left[\frac{\partial^2 p_i(x)x_i}{\partial x_i \partial x_j}\right] - \left[\delta_{ij} \frac{d^2 c_i(x_i)}{dx_i^2}\right].$$

Since the matrix $\left[\delta_{ij} d^2 c_i(x_i)/dx_i^2\right]$ is positive semidefinite for each $x \in X$, if the matrix $\left[\partial^2 p_i(x)x_i/\partial x_i \partial x_j\right]$ is negative definite for each $x \in X$, then, by Corollary 6, **u** has a unique correlated equilibrium. As a special case, consider a linear inverse demand function $p_i(x) = \sum_{j \in N} a_{ij}x_j + b_i$ for each $i \in N$. Then, $\partial^2 p_i(x)x_i/\partial x_i^2 = 2a_{ii}$ and $\partial^2 p_i(x)x_i/\partial x_i \partial x_j = a_{ij}$ for $i \neq j$. Thus, if the matrix $[(1 + \delta_{ij})a_{ij}]$ is negative definite, then **u** has a unique correlated equilibrium. Note that if $[(1 + \delta_{ij})a_{ij}]$ is symmetric, i.e., $a_{ij} = a_{ji}$ for each $i, j \in N$, then $[\partial^2 u_i(x)/\partial x_i \partial x_j]$ is symmetric, and thus **u** is a potential game.

Example 2 For each $i \in N$, let $X_i \subseteq \mathbb{R}$ be a closed bounded interval, and let $\mathbf{u}$ be a smooth game such that the payoff gradient of $\mathbf{u}$ is strictly monotone. Consider another game $\mathbf{v} = (v_i)_{i \in N}$ such that, for each $x \in X$ and each $i \in N$,

$$v_i(x) = w_i(x_i)u_i(x) - \int_{c_i}^{x_i} \frac{dw_i(t)}{dt} u_i(t, x_{-i})dt + z_i(x_{-i}), \tag{10}$$

where $w_i : X_i \to \mathbb{R}_{++}$ is a continuously differentiable function, $z_i : X_{-i} \to \mathbb{R}$ is a bounded measurable function, and $c_i \in X_i$. Then, $\nabla_i v_i(x) = w_i(x_i)\nabla_i u_i(x)$ for each $x \in X$ and each $i \in N$. Since the mapping $x \mapsto (-\nabla_i u_i(x))_{i \in N}$ is strictly monotone, so is the mapping $x \mapsto (-\nabla_i v_i(x)/w_i(x_i))_{i \in N}$. This implies that the γ-weighted payoff gradient of $\mathbf{v}$ is strictly monotone with $\gamma_i(x_i) = 1/w_i(x_i)$ for each $x_i \in X_i$ and each $i \in N$. Therefore, by Proposition 5, not only $\mathbf{u}$ but also $\mathbf{v}$ have a unique correlated equilibrium.

For example, assume that $\min X_i > 0$ and let $w_i(x_i) = x_i$ for each $x_i \in X_i$ and each $i \in N$. Then, (10) is rewritten as

$$v_i(x) = x_i u_i(x) - \int_{c_i}^{x_i} u_i(t, x_{-i})dt + z_i(x_{-i}). \tag{11}$$

Furthermore, let $u_i(x) = -\partial f_i(x)/\partial x_i$ and $z_i(x_{-i}) = f_i(c_i, x_{-i})$ for each $x \in X$ and each $i \in N$, where $f_i : X \to \mathbb{R}$ is a twice continuously differentiable function. Then, (11) is rewritten as

$$v_i(x) = f_i(x) - x_i \frac{\partial f_i(x)}{\partial x_i}.$$

One possible interpretation is that $x_i \in X_i$ is a quantity of a good consumed by player i, $f_i(x)$ is player i's benefit of consumption, where there exists a consumption externality, and $x_i(\partial f_i(x)/\partial x_i)$ is player i's consumption expenditure when the price of the good is set at the marginal benefit of consumption and player i knows that the price depends on x_i. In the game $\mathbf{v}$, each player chooses his consumption to maximize the benefit minus the cost, whereas, in the game $\mathbf{u} = (-\partial f_i/\partial x_i)_{i \in N}$, each player chooses his consumption to minimize the marginal benefit. By Proposition 5, if the payoff gradient of $\mathbf{u}$ is strictly monotone, then not only $\mathbf{u}$ but also $\mathbf{v}$ have a unique correlated equilibrium.

In general, if $X_i \subseteq \mathbb{R}$ for each $i \in N$, then, for each game $\mathbf{v}$ and $\gamma = (\gamma_i)_{i \in N}$ with $\gamma_i : X_i \to \mathbb{R}_{++}$, there exists a game $\mathbf{u}$ such that $\nabla_i u_i(x) = \gamma_i(x_i)\nabla_i v_i(x)$ for each $x \in X$ and each $i \in N$. In this case, if the γ-weighted payoff gradient of $\mathbf{v}$ is strictly monotone, then the payoff gradient of $\mathbf{u}$ is strictly monotone. In other words, for each game $\mathbf{v}$ of which γ-weighted payoff gradient is strictly monotone, there exists a game $\mathbf{u}$ of which payoff gradient is strictly monotone such that $\nabla_i u_i(x) = \gamma_i(x_i)\nabla_i v_i(x)$

for each $x \in X$ and each $i \in N$.[10] It should be noted that this is not always true if $X_i \subseteq \mathbb{R}^{m_i}$ with $m_i \geq 2$: in this case, for given $\mathbf{v}$ and γ, there may not exist $\mathbf{u}$ such that $\nabla_i u_i(x) = \gamma_i(x_i)\nabla_i v_i(x)$.

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[10] It can be readily shown that $\mathbf{u}$ and $\mathbf{v}$ have the same best-response correspondence. See Morris and Ui (2004).