Int J Game Theory (2008) 36:461–471
DOI 10.1007/s00182-007-0089-y

ORIGINAL PAPER

Random paths to P-stability in the roommate problem

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Accepted: 29 March 2007 / Published online: 12 May 2007
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Abstract For solvable roommate problems with strict preferences Diamantoudi et al. (Games Econ Behav 48: 18–28, 2004) show that for any unstable matching, there exists a finite sequence of successive myopic blocking pairs leading to a stable matching. In this paper, we define P-stable matchings associated with stable partitions and, by using a proposal-rejection procedure, generalize the previous result for the *entire* class of roommate problems.

Keywords Roommate problem · Random paths to stability

JEL Classification C78

We thank Svetloslav Ivantchev for his help in programming, Ana Saracho for her participation in the early stages of the paper and Peter Biró, Marilda Sotomayor, Federico Valenciano and anonymous referees for their comments and suggestions. This research has been supported by the University of the Basque Country under project 1/UPV 00031.321-HA-7903/2000 and project GIU06/44 and by the Spanish Ministerio de Educación y Ciencia under project SEJ2006-05455, cofunded by FEDER and project BEC2000-0875. It has also benefited from the Acción Integrada HF-2006-0021/EGIDE-Picasso.

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1 Introduction

The roommate problem was introduced in 1962 by Gale and Shapley. A roommate problem is *solvable* if a stable matching exists; otherwise it is said to be *unsolvable*. The solvable problem has been investigated extensively in matching literature, see for instance Gale and Shapley (1962), Irving (1985), Roth and Sotomayor (1990) and Tan (1991). However, to the best of our knowledge few papers have analyzed unsolvable roommate problems. See for instance, Tan (1990) and Abraham et al. (2005).

Regarding the issue of paths to stability, the question is whether in the absence of a centralized procedure there exists a finite sequence of successive myopic blocking pairs leading to stable matchings. Knuth (1976) addresses this problem for a marriage problem (a special case of the roommate problem) and gives an example in which a process of decentralized decision making may cycle. Roth and Vande Vate (1990) solve this question by showing that there is a convergence path to a stable matching.[1] These authors construct a sequence of matchings associated with a sequence of increasing sets of agents without blocking pairs until a stable matching is reached.[2] There are several works on random paths to stability based on the idea of Roth and Vande Vate (1990). Chung (2000) introduces a restriction on the preferences for the roommate problem and, labeling the agents as men and women, uses the Roth-Vande Vate convergence process and extends their result. With regard to two-sided markets, Kojima and Unver (2006) study the convergence to stability in many-to-many matching problems, whereas Klaus and Klijn (2007) analyze this convergence for matching markets with couples.

Diamantoudi et al. (2004) use a different strategy for proving the convergence to stability for roommate problems with strict preferences. In particular, these authors, by fixing a stable matching, generate a path to stability that avoids cyclicity. The path gives a sequence of matchings obtained by satisfying an increasing number of blocking pairs common with pairs of the matching fixed until a stable matching is reached. This result suggests that the convergence path to a stable matching is not exclusively generated by the two-sided structure of the problem but seems to be implied by the existence of stability.

For the roommate problem, Tan (1991) obtains a necessary and sufficient condition for the existence of a stable matching under strict preferences. This author defines what is called "stable partition", which is a partition of the agents into ordered sets satisfying a notion of stability between sets and also within each set. He proves that if there exists a stable partition containing an odd ring, then there is no stable matching. In this paper, we define some specific matchings called P-stable matchings associated with stable partitions.

Following the approach used by Diamantoudi et al. (2004), we show that from any matching, there exists a path, given by a proposal-rejection procedure, that reaches a P-stable matching. Since P-stable matchings coincide with stable ones for solvable

[1] Abeledo and Rothblum (1995) derive a family of algorithms, including the Roth and Vande Vate process, that determines stable matchings for the marriage problem.

[2] Biro et al. (2006) study the dynamics of the Roth-Vande Vate mechanism and its generalization by Tan and Hsueh (1995).

roommate problems, this result is a generalization of that of Diamantoudi et al. (2004) and, by extension, of the Chung (2000) under strict preferences and of the Roth and Vande Vate (1990).

This paper is organized as follows: Sect. 2 contains the preliminaries of the paper. In Sect. 3 the notion of P-stable matching is introduced. Section 4 contains the main result, which is discussed in Sect. 5 along with some further research. Appendix contains the proofs of some Remarks and certain details of the proof of the main result.

2 Preliminaries

A *roommate problem* is a pair $(N, (\succcurlyeq_x)_{x\in N})$ where N is a finite set of agents and for each agent $x \in N$, $\succcurlyeq_x$ is a complete, transitive preference relation defined over N. Let $\succ_x$ be the strict preference associated with $\succcurlyeq_x$. In this paper, we only consider roommate problems with strict preferences, which we denote by $(N, (\succ_x)_{x\in N})$.

A *matching* μ is a one to one mapping from N onto itself such that for all $x, y \in N$ if $\mu(x) = y$, then $\mu(y) = x$. Let $\mu(x)$ denote the partner of agent x under the matching μ. If $\mu(x) = x$, then agent x is single under μ.

A pair of agents $\{x, y\} \subseteq N$ (without ruling out $x = y$) blocks the matching μ if

$$y \succ_x \mu(x) \text{ and } x \succ_y \mu(y). \tag{1}$$

That is, x and y prefer each other to their current partners at μ. If $x = y$, 1 means that agent x prefers being alone to being matched with $\mu(x)$. When 1 holds, we call $\{x, y\}$ a *blocking pair* of μ.

A matching satisfies *individual rationality* if it is not blocked by any pair $\{x, y\}$ such that $x = y$. A matching is called *stable* if it is not blocked by any pair.

Let $\{x, y\}$ be a blocking pair of μ. A matching μ' is obtained from μ by satisfying $\{x, y\}$ if $\mu'(x) = y$ and for all $z \in N\backslash\{x, y\}$,

$$\mu'(z) = \begin{cases} z & if\ \mu(z) \in \{x, y\} \\ \mu(z) & \text{otherwise.} \end{cases}$$

That is, once $\{x, y\}$ is formed, their partners (if any) at μ are alone in μ' while the remaining agents are matched as in μ.

The *abstract system associated with a roommate problem* $(N, (\succ_x)_{x\in N})$ is the pair $(\mathcal{M}, R)$ where $\mathcal{M}$ is the set of matchings and R is the binary relation defined over $\mathcal{M}$ as follows: Given $\mu, \mu' \in \mathcal{M}$, $\mu' R \mu$ if and only if μ' is obtained from μ by satisfying a blocking pair of μ. Let R^T denote the transitive closure of R. Then $\mu' R^T \mu$ if and only if there exists a finite sequence of matchings $(\mu = \mu_0, \mu_1, \ldots, \mu_k = \mu')$ such that for all $i \in \{1, \ldots, k\}$ $\mu_i R \mu_{i-1}$.

3 *P*-stable matchings

In this section, we define the P-stable matching concept associated with the notion of stable partition introduced by Tan (1991).

Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. Let $A = \{a_1, \ldots, a_k\} \subseteq N$ be an ordered set of agents. The set A is a *ring* if $k \geq 3$ and for all $i \in \{1, \ldots, k\}$, $a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i$ (subscript modulo k). The set A is a pair of mutually acceptable agents if $k = 2$ and for all $i \in \{1, 2\}$, $a_{i-1} \succ_{a_i} a_i$ (subscript modulo 2).[3] The set A is a singleton if $k = 1$.

A *stable partition* is a partition P of N such that:

(i) For all $A \in P$, the set A is a ring, a mutually acceptable pair of agents or a singleton, and
(ii) For any sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ of P (possibly $A = B$), the following condition holds:

$$\text{if } b_j \succ_{a_i} a_{i-1} \text{ then } b_{j-1} \succ_{b_j} a_i,$$

for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ such that $b_j \neq a_{i+1}$.
This condition may be interpreted as a notion of stability over partitions.

Let P be a stable partition and $A \in P$. We say that A is an odd (even) set of P if the cardinal of A is odd (even).

Remark 1 The following assertions are proved by Tan (1991):

(i) For any roommate problem $(N, (\succ_x)_{x \in N})$, there exists at least one stable partition. Furthermore, any two stable partitions have exactly the same odd sets.
(ii) Each even ring of a stable partition can be broken into pairs of mutually acceptable agents preserving stability.
(iii) A roommate problem $(N, (\succ_x)_{x \in N})$ has no stable matchings if and only if there exists a stable partition with an odd ring.

Definition 1 Let P be a stable partition. A P-stable matching is a matching μ such that for each $A = \{a_1, \ldots, a_k\} \in P$, $\mu(a_i) \in \{a_{i+1}, a_{i-1}\}$ for all $i \in \{1, \ldots, k\}$ except for a unique j where $\mu(a_j) = a_j$ if A is odd.

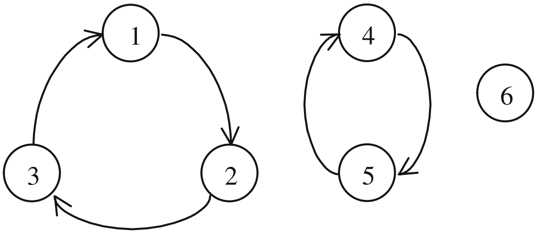
We illustrate the notion of P-stable matching with the following example.

Example 1 Consider the following 6-agent roommate problem:

$$\begin{array}{c}
2 \succ_1 3 \succ_1 1 \succ_1 4 \succ_1 5 \succ_1 6 \\
3 \succ_2 1 \succ_2 2 \succ_2 4 \succ_2 5 \succ_2 6 \\
1 \succ_3 2 \succ_3 3 \succ_3 4 \succ_3 5 \succ_3 6 \\
5 \succ_4 4 \succ_4 1 \succ_4 2 \succ_4 3 \succ_4 6 \\
4 \succ_5 5 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 6 \\
6 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 4 \succ_6 5
\end{array}$$

It is easy to verify that $P = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$ is a stable partition where $A_1 = \{1, 2, 3\}$ is an odd ring, $A_2 = \{4, 5\}$ is a pair of mutually acceptable agents and $A_3 = \{6\}$ is a singleton. Partition P can be represented graphically as follows:

[3] Hereafter we omit subscript modulo k.



The P-stable matchings associated with the stable partition P are:

$$\mu_1 = [\{1\}, \{2, 3\}, \{4, 5\}, \{6\}]$$
$$\mu_2 = [\{2\}, \{1, 3\}, \{4, 5\}, \{6\}]$$
$$\mu_3 = [\{3\}, \{1, 2\}, \{4, 5\}, \{6\}].$$

Remark 2 If μ is a P-stable matching, then the matching that results if the single agents from odd rings are excluded from μ is stable.[4]

Remark 3 For a solvable roommate problem $(N, (\succ_x)_{x\in N})$ the set of P -stable matchings for all stable partitions coincides with the set of stable matchings.[5]

4 Random paths to P-stable matchings

For solvable roommate problems with strict preferences Diamantoudi et al. (2004) prove that "*for any matching μ, there exists a finite sequence of matchings $(\mu = \mu_0, \mu_1, \ldots, \mu_m = \overline{\mu})$ such that for all $i \in \{1, \ldots, m\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair of μ_{i-1} and $\overline{\mu}$ is a stable matching*". We generalize the previous result by proving the following:

Theorem 1 *Let $(N, (\succ_x)_{x\in N})$ be a roommate problem. Then, for any matching μ, there exists a finite sequence of matchings $(\mu = \mu_0, \mu_1, \ldots, \mu_m = \overline{\mu})$ such that for all $i \in \{1, \ldots, m\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair of μ_{i-1} and $\overline{\mu}$ is a P-stable matching for some stable partition P.*

Proof Let μ be an arbitrary matching. Suppose that μ is not a P -stable matching for any stable partition P (if μ is a P -stable matching for some stable partition P, $m = 0$ and we are done). We prove that there exists a P-stable matching $\overline{\mu}$ such that $\overline{\mu} R^T \mu$.

Fix a stable partition P^*.[6] Given any $A^* = \{a_1^*, \ldots, a_k^*\} \in P^*$, let $N_{A^*}(\mu)$ denote the set of agents $a_i^* \in A^*$ such that $\mu(a_i^*) \in \{a_{i+1}^*, a_{i-1}^*\}$ or $\mu(a_i^*) = a_i^*$ if $\mu(a_j^*) \in \{a_{j+1}^*, a_{j-1}^*\}$ for all $j \neq i$. Let $n(\mu)$ be the number of pairs (including singletons) matched under μ and contained in $N_{A^*}(\mu)$ for some $A^* \in P^*$.

It suffices to prove the following:

4 See Tan (1990).

5 See the proof in Appendix.

6 From Remark 1, we can assume, without loss of generality, that P^* has no even rings.

Claim For any matching μ which is not a P-stable matching for any stable partition P, there exists a matching μ' such that $\mu' R^T \mu$ and $n(\mu') \geq n(\mu) + 1$.

Without loss of generality, we introduce two assumptions.

S1 The matching μ is individually rational.

Otherwise, there exists an individually rational matching $\widetilde{\mu}$ such that $\widetilde{\mu} R^T \mu$ and $n(\widetilde{\mu}) \geq n(\mu)$.

Let $N(\mu)$ denote the set of agents that belong to some set $A^* \in P^*$ such that $A^* = N_{A^*}(\mu)$ and let $N'(\mu) = N \backslash N(\mu)$.

S2 The matching μ is blocked by a pair $\{x, y\} \nsubseteq N(\mu)$.

Otherwise, there exists a matching $\widetilde{\mu}$ verifying S2 such that $\widetilde{\mu} R^T \mu$ and $n(\widetilde{\mu}) = n(\mu)$.[7]

To prove the claim we distinguish two cases:

Case 1 There is an agent in $N'(\mu)$ who is alone under μ.

In this case we give a proposal-rejection procedure intuitively described as follows. Let $y \in N'(\mu)$ who is alone under μ and let $y_0 = y$ and $A_0^* \in P^*$ such that $y_0 \in A_0^*$. Let x_1 denote the predecessor of y_0 in A_0^*, $y_1 = \mu(x_1)$ and $A_1^* \in P^*$ such that $y_1^* \in A_1^*$. As agent y_0 prefers x_1 to being alone, y_0 proposes x_1. If x_1 accepts the proposal (that is, x_1 prefers y_0 to his partner under μ) the pair $\{y_0, x_1\}$ blocks μ and the procedure concludes. Otherwise, let x_2 be the predecessor of y_1 in A_1^*, $y_2 = \mu(x_2)$ and $A_2^* \in P^*$ such that $y_2 \in A_2^*$. Since agent x_1 prefers y_1 to y_0, then, by stability of P^*, agent y_1 prefers x_2 to x_1. So y_1 becomes a new proposer in the process and offers x_2 the possibility of forming a new pair. Then, if x_2 accepts the proposal, the pair $\{y_1, x_2\}$ blocks μ and the procedure concludes. Otherwise, it may continue iteratively in this manner.

Formally, the procedure described above considers a sequence of pairs, $\{x_t, y_t\}_{t=0}^{\infty}$, that are matched under μ and a sequence of sets of P^*, $\{A_t^*\}_{t=0}^{\infty}$, defined inductively as follows:

(i) for $t = 0$, $x_0 = \mu(y)$, $y_0 = y$ and $A_0^* \in P^*$ such that $y_0 \in A_0^*$.

(ii) for $t \geq 1$, x_t is the predecessor of y_{t-1} in A_{t-1}^*, $y_t = \mu(x_t)$ and $A_t^* \in P^*$ such that $y_t \in A_t^*$.

Given that N is finite there exists a $r \in \mathbb{N}$ such that $y_t \succ_{x_t} y_{t-1}$ for all $t = 1, \ldots, r-1$ and $y_{r-1} \succ_{x_r} y_r$. Then the procedure generates the blocking pair $\{y_{r-1}, x_r\}$ of μ which induces a matching μ_1 for which $\{y_{r-1}, x_r\} \subseteq N_{A_{r-1}^*}(\mu_1)$. If $n(\mu_1) \geq n(\mu)+1$, then the claim follows. Otherwise, $n(\mu_1) = n(\mu)$ and $\{y_{r-1}, x_r\}$ breaks $\{x_r, y_r\} \subseteq N_{A_{r-1}^*}(\mu)$. We distinguish two cases: (a) If $r \geq 2$ the procedure applied to μ_1 and y_0 generates the blocking pair $\{y_{r-2}, x_{r-1}\}$ of μ_1, which induces a matching μ_2 for which $\{y_{r-2}, x_{r-1}\} \subseteq N_{A_{r-2}^*}(\mu_2)$. Since $\mu_1(x_{r-1}) = x_{r-1}$ then $n(\mu_2) \geq n(\mu_1) + 1$ and therefore $n(\mu_2) \geq n(\mu) + 1$. (b) If $r = 1$, we have $\{x_1, y_1\} \subseteq N_{A_0^*}(\mu)$ where $y_1 = \mu(x_1)$. As $\mu_1(y_1) = y_1$ and $y_1 \in N'(\mu_1)$, we can apply the procedure to μ_1 and y_1. Now, if after applying the procedure to μ_1 and y_1 we are again in case (b) and so on successively, then it is easy to verify that $A_0^* = N_{A_0^*}(\mu)$. Hence $y_0 \in N(\mu)$, which is not possible since $y_0 \in N'(\mu)$. Consequently, the claim is also satisfied in this case.

[7] See the proof in Appendix.

Remark 4 If $N'(\mu)$ has s agents $(z_1, \ldots, z_s)$ that are single under μ with at most two of them belonging to the same odd ring, such that $\{z_i, z_j\}$ is not a pair of mutually acceptable agents of P^* for all $i, j \in \{1, \ldots, s\}$, then there exists a matching μ' such that $\mu' R^T \mu$ and $n(\mu') \geq n(\mu) + s$.[8]

Case 2 There is no agent in $N'(\mu)$ who is alone under μ.

We consider two cases:

(i) If the matching μ restricted to $N'(\mu)$ is not stable, then μ is blocked by a pair $\{x, y\} \subseteq N'(\mu)$ which induces a matching μ_1 such that $n(\mu_1) \geq n(\mu) - 1$, since by the stability of P^* at most there exists a $z \in \{x, y\}$ such that $\{z, \mu(z)\} \subseteq N_{A^*}(\mu)$ for some $A^* \in P^*$. Now, if $\{\mu(x), \mu(y)\} \subseteq N(\mu_1)$ then $n(\mu_1) \geq n(\mu) + 1$. If $\mu(x) \in N(\mu_1)$ and $\mu(y) \notin N(\mu_1)$ then $n(\mu_1) \geq n(\mu)$ and as $\mu(y)$ is alone under μ_1, by applying the proposal-rejection procedure (given in Case 1), the claim follows. If $\mu(x) \notin N(\mu_1)$ and $\mu(y) \in N(\mu_1)$, the same argument applies. If $\{\mu(x), \mu(y)\} \cap N(\mu_1) = \varnothing$ then, if $\{\mu(x), \mu(y)\}$ is a pair of mutually acceptable agents of P^*, it blocks μ_1 and induces a matching μ_2 with $n(\mu_2) \geq n(\mu) + 1$. Otherwise, from Remark 4, the claim is implied.

(ii) If the matching μ restricted to $N'(\mu)$ is stable, by S2 the matching μ is blocked by a pair $\{x, y\} \nsubseteq N(\mu)$ which induces a matching μ_1. Now, by stability of μ in $N'(\mu)$, $z \in N(\mu)$ for some $z \in \{x, y\}$. Suppose, without loss of generality, that $z = x$. Then we have $x \in N(\mu)$ and $y \notin N(\mu)$. From Remark 1 (iii), all odd sets of P^* are contained in $N(\mu)$. Hence $\mu(y) \notin N(\mu_1)$. Let $A^* \in P^*$ such that $x \in A^*$. Then $\{x, y\}$ breaks the pair $\{x, \mu(x)\} \subseteq N_{A^*}(\mu)$. If A^* is even we have $n(\mu_1) = n(\mu) - 1$. But, as $\mu(x), \mu(y) \notin N(\mu_1)$ and they are alone under μ_1, by Remark 4, the claim is satisfied. So, we can assume that A^* is odd. Let z denote the agent belonging to A^* such that $\mu(z) = z$. We distinguish two cases: (a) If $z = x$ we have $n(\mu_1) = n(\mu) - 1$. Applying the proposal-rejection procedure to μ_1 and $\mu(y)$ there exists a matching $\widetilde{\mu}$ such that $\widetilde{\mu} R^T \mu_1$ and $n(\widetilde{\mu}) \geq n(\mu_1) + 1$. Hence $n(\widetilde{\mu}) \geq n(\mu)$. If $\mu(y) \notin N(\widetilde{\mu})$ applying the proposal-rejection procedure to $\widetilde{\mu}$ and $\mu(y)$ one more time, the claim follows. Otherwise, it is easy to verify that $\widetilde{\mu}$ satisfies the condition of Case 1 (the agent $\mu_1(u) \in N'(\widetilde{\mu})$ is alone under $\widetilde{\mu}$, where u is the predecessor of $\mu(y)$ in P^*), hence the claim is implied. (b) If $z \neq x$ we have that $\{x, y\}$ breaks the singleton $\{z\} \subseteq N(\mu)$. Hence, $n(\mu_1) = n(\mu) - 2$. But, as in $N'(\mu_1)$ there are three single agents ($\mu(x)$, $\mu(y)$ and z) under μ_1, from Remark 4, the claim follows. □

In what follows we introduce two examples of roommate problems that illustrate the proposal-rejection procedure for solvable and unsolvable roommate problems respectively.

[8] See the proof in Appendix.

Example 2 Consider the 9-agents example given in Diamantoudi et al. (2004)

$$2 \succ_1 7 \succ_1 6 \succ_1 3 \succ_1 4 \succ_1 8 \succ_1 5 \succ_1 1 \succ_1 9$$
$$5 \succ_2 3 \succ_2 1 \succ_2 4 \succ_2 8 \succ_2 7 \succ_2 6 \succ_2 2 \succ_2 9$$
$$4 \succ_3 2 \succ_3 7 \succ_3 5 \succ_3 6 \succ_3 1 \succ_3 8 \succ_3 3 \succ_3 9$$
$$8 \succ_4 5 \succ_4 3 \succ_4 6 \succ_4 1 \succ_4 2 \succ_4 7 \succ_4 4 \succ_4 9$$
$$6 \succ_5 4 \succ_5 2 \succ_5 8 \succ_5 7 \succ_5 3 \succ_5 1 \succ_5 5 \succ_5 9$$
$$1 \succ_6 8 \succ_6 5 \succ_6 7 \succ_6 3 \succ_6 4 \succ_6 2 \succ_6 6 \succ_6 9$$
$$3 \succ_7 1 \succ_7 8 \succ_7 2 \succ_7 5 \succ_7 6 \succ_7 4 \succ_7 7 \succ_7 9$$
$$7 \succ_8 6 \succ_8 4 \succ_8 1 \succ_8 2 \succ_8 5 \succ_8 3 \succ_8 8 \succ_8 9$$
$$9 \succ_9 1 \succ_9 2 \succ_9 3 \succ_9 4 \succ_9 5 \succ_9 6 \succ_9 7 \succ_9 8$$

In this example, $P^* = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9\}\}$ is a stable partition. Since P^* has no odd rings, then there exists a stable matching. Consider the following matching $\mu = [\{2,3\},\{4,5\},\{6,1\},\{7,8\},\{9\}]$, which is unstable. Then $n(\mu) = 2$, $N(\mu) = \{7,8,9\}$ and $N'(\mu) = \{1,2,3,4,5,6\}$. Since there is no agent in $N'(\mu)$ who is alone under μ we are in Case 2. This matching is blocked only by $\{1,7\}$, which induces $\mu_1 = [\{2,3\},\{4,5\},\{6\},\{1,7\},\{8\},\{9\}]$. Hence, we are in Case 2 (ii). Notice that $\{1,7\}$ breaks $\{7,8\} \subseteq N(\mu)$, hence $n(\mu_1) = 1 = n(\mu) - 1$. Now, under μ_1, agents 6 and 8 are alone. Applying the proposal-rejection procedure (given in Case 1) to μ_1 and agent 8, the procedure considers the following sequence of pairs that are matched under μ_1: $\{8\}$, $\{7,1\}$, $\{2,3\}$, $\{4,5\}$ and $\{6\}$ and generates the blocking pair $\{5,6\}$, which induces $\mu_2 = [\{2,3\},\{4\},\{5,6\},\{1,7\},\{8\},\{9\}]$. Since under μ_2 agent 8 is alone, the procedure applied to μ_2 and 8 generates the blocking pair $\{3,4\}$ and $\mu_3 = [\{2\},\{3,4\},\{5,6\},\{1,7\},\{8\},\{9\}]$ is reached. For this matching $n(\mu_3) = 3 = n(\mu) + 1$ as the claim states.

To complete the sequence that leads to $\overline{\mu}$, since agent 8 is still alone under μ_3, we apply the procedure to them. Then μ_3 is blocked by $\{1,2\}$, which generates the matching: $\mu_4 = [\{1,2\},\{3,4\},\{5,6\},\{7\},\{8\},\{9\}]$ and μ_4 is blocked by $\{7,8\}$, inducing the stable matching: $\mu_5 = [\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9\}]$.

Example 3 Consider the following 9-agent roommate problem:

$$2 \succ_1 3 \succ_1 1 \succ_1 4 \succ_1 5 \succ_1 6 \succ_1 7 \succ_1 8 \succ_1 9$$
$$4 \succ_2 1 \succ_2 2 \succ_2 3 \succ_2 5 \succ_2 6 \succ_2 7 \succ_2 8 \succ_2 9$$
$$1 \succ_3 4 \succ_3 5 \succ_3 3 \succ_3 2 \succ_3 6 \succ_3 7 \succ_3 8 \succ_3 9$$
$$6 \succ_4 3 \succ_4 2 \succ_4 4 \succ_4 1 \succ_4 5 \succ_4 7 \succ_4 8 \succ_4 9$$
$$3 \succ_5 6 \succ_5 5 \succ_5 1 \succ_5 2 \succ_5 4 \succ_5 7 \succ_5 8 \succ_5 9$$
$$5 \succ_6 8 \succ_6 4 \succ_6 6 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 7 \succ_6 9$$
$$8 \succ_7 9 \succ_7 7 \succ_7 1 \succ_7 2 \succ_7 3 \succ_7 4 \succ_7 5 \succ_7 6$$
$$9 \succ_8 6 \succ_8 7 \succ_8 8 \succ_8 1 \succ_8 2 \succ_8 3 \succ_8 4 \succ_8 5$$
$$7 \succ_9 8 \succ_9 9 \succ_9 1 \succ_9 2 \succ_9 3 \succ_9 4 \succ_9 5 \succ_9 6$$

$P_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}\}$ and $P_2 = \{\{1, 3\}, \{2, 4\}, \{5, 6\}, \{7, 8, 9\}\}$ are the stable partitions for this roommate problem that do not contain any even ring.[9] Since they have an odd ring, then there is no stable matching. Fix $P^* = P_1$ and consider the matching $\mu = [\{1, 2\}, \{3, 5\}, \{4, 6\}, \{7, 9\}, \{8\}]$ which is not a P -stable matching for any stable partition P. Then $n(\mu) = 3$, $N(\mu) = \{1, 2, 7, 8, 9\}$ and $N'(\mu) = \{3, 4, 5, 6\}$. The matching μ is blocked by $\{6, 8\} \not\subseteq N(\mu)$ which induces $\mu_1 = [\{1, 2\}, \{3, 5\}, \{4\}, \{7, 9\}, \{6, 8\}]$ with $n(\mu_1) = 2 = n(\mu) - 1$. Hence, we are in Case 2 (ii)(a). Then we apply the proposal-rejection procedure to μ_1 and agent 4. The procedure generates the blocking pair $\{3, 4\}$ and the matching $\mu_2 = [\{1, 2\}, \{3, 4\}, \{5\}, \{7, 9\}, \{6, 8\}]$ for which $n(\mu_2) = 3 = n(\mu)$, $N(\mu_2) = \{1, 2, 3, 4\}$ and $N'(\mu_2) = \{5, 6, 7, 8, 9\}$. Since agent 5 is alone under μ_2, applying the proposal-rejection procedure again now to μ_2 and 5, we obtain $\mu_3 = [\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 9\}, \{8\}]$, which is a P_1-stable matching.

Notice that starting from matching $\mu = [\{1, 3\}, \{2, 4\}, \{5\}, \{6\}, \{7, 9\}, \{8\}]$, and applying the proposal-rejection procedure, a P_2-stable matching is reached, but never a P_1-stable matching.

5 Concluding remarks

In this paper, we have generalized the result of Diamontoudi et al. (2004). In particular, Theorem 1 establishes that the set of P-stable matchings has the property of "outer stability" in the following sense. If μ is not a P-stable matching for any stable partition, then there exists a P-stable matching $\overline{\mu}$ such that $\overline{\mu} R^T \mu$. On the other hand, Example 3 allows to see the interest of P -stable matchings for unsolvable roommate problems.

In this example, the set of the P-stable matchings is the union of two disjoint sets: the set of the P_1-stable matchings and the set of the P_2-stable matchings, associated with the stable partitions P_1 and P_2 respectively. It is easy to verify that the matchings of each such set are symmetrically connected by the relation R^T, that is, given any two matchings of the same set, there is a path from one to another. However, any two matchings belonging to two distinct sets are not R^T comparable, that is, there is not a path from a P_1-stable matching to a P_2 -stable matching, and conversely. Moreover, notice that from any matching of the set of P-stable matchings, there is not a path to any other matching outside this set. These ideas will be addressed in further research.

Appendix

Proof of Remark 3 If a stable matching exists, then by Remark 1 (iii), no stable partition P contains odd rings. Hence, by Remark 2, if μ is a P-stable matching, then μ is a stable matching. Conversely, if μ is a stable matching, then μ is a P-stable matching where the partition P is formed by all pairs matched under μ. □

9 $P_3 = [\{1, 2, 4, 3\}, \{5, 6\}, \{7, 8, 9\}]$ is also a stable partition.

Proof of S2 Consider the partition P of N such that for each $A \in P$, $A = A^*$ for some $A^* \in P^*$ such that $A^* = N_{A^*}(\mu)$ or $A = \{x, \mu(x)\}$ where $x \notin N(\mu)$. Since μ is not a P -stable matching, P is not stable, hence there exist two sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ of P such that

$$b_j \succ_{a_i} a_{i-1} \text{ and } a_i \succ_{b_j} b_{j-1} \tag{2}$$

for some $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$. If $a_i, b_j \notin N(\mu)$, then $a_{i-1} = \mu(a_i)$ and $b_{j-1} = \mu(b_j)$. Hence, by (2) the pair $\{a_i, b_j\} \nsubseteq N(\mu)$ blocks μ. So, we can assume that a_i or $b_j \in N(\mu)$. Now, by stability of P^*, only one of them (a_i or b_j) belongs to $N(\mu)$. Suppose, without loss of generality, that $a_i \in N(\mu)$ and $b_j \notin N(\mu)$. Then $b_{j-1} = \mu(b_j)$. As $a_i \in N(\mu)$ and $a_i \in A$ we have $A = A^*$ for some $A^* \in P^*$ such that $A^* = N_{A^*}(\mu)$. Thus, $\mu(a_i) \in \{a_{i-1}, a_i, a_{i+1}\}$. Now, if $\mu(a_i) \in \{a_{i-1}, a_i\}$ or $\mu(a_i) = a_{i+1}$ and $b_j \succ_{a_i} a_{i+1}$ then by [2] the pair $\{a_i, b_j\}$ blocks μ. Thus, we assume that $\mu(a_i) = a_{i+1}$ and $a_{i+1} \succ_{a_i} b_j$. But then A^* is an odd ring, hence there exists a $j \in \{1, \ldots, k\}$ such that $\mu(a_j) = a_j$. Now, $a_{j-1} \succ_{a_j} a_j$. As $\mu(a_{j-1}) = a_{j-2}$ and $a_j \succ_{a_{j-1}} a_{j-2}$, then the pair $\{a_j, a_{j-1}\}$ blocks μ, which induces a matching μ_1 for which $\mu_1(a_{j-2}) = a_{j-2}$. If $i = j - 2$ we have μ_1, which verifies S2, $\mu_1 R \mu$ and $n(\mu_1) = n(\mu)$. Otherwise, by reasoning in a similar way for μ_1 and so on we conclude that there exists a matching $\widetilde{\mu}$ verifying S2 such that $\widetilde{\mu} R^T \mu$ and $n(\widetilde{\mu}) = n(\mu)$. □

Proof of Remark 4 As $\mu(z_1) = z_1$ and $z_1 \in N'(\mu)$ we apply the proposal-rejection procedure to μ and z_1. We can assume, without loss of generality, that we are not in Case 1b. Then, there exists a matching $\widetilde{\mu}$ such that $\widetilde{\mu} R^T \mu$ and $n(\widetilde{\mu}) \geq n(\mu) + 1$. (If there is a $r \in \{1, \ldots, s\}$ such that $\{z_1, z_r\} \subseteq A^*$ for some odd ring $A^* \in P^*$, and z_r is the predeccesor of z_1 in A^*, then we apply the procedure to μ and z_r.) Now, if $s = 1$ the result follows. Otherwise, it is easy to see that at least $(s - 1)$ agents of $\{z_1, \ldots, z_s\}$ are single under $\widetilde{\mu}$. Let k be the number of agents of $\{z_1, \ldots, z_s\}$ that are single under $\widetilde{\mu}$ and contained in $N(\widetilde{\mu})$. Then, we have $n(\widetilde{\mu}) \geq n(\mu) + 1 + k$. Hence, if $k \geq s - 1$, $n(\widetilde{\mu}) \geq n(\mu) + s$ and we are done. If $k < s - 1$, then $N'(\widetilde{\mu})$ contains at least $(s - 1 - k)$ agents of $\{z_1, \ldots, z_s\}$ that are single under $\widetilde{\mu}$. Hence by reasoning in a similar way for $\widetilde{\mu}$ and so on we conclude that there exists a matching μ' such that $\mu' R^T \mu$ and $n(\mu') \geq n(\mu) + s$. □

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