Int J Game Theory (2008) 36:333–352
DOI 10.1007/s00182-007-0084-3

ORIGINAL PAPER

The dynamics of stable matchings and half-matchings for the stable marriage and roommates problems

Péter Biró · Katarína Cechlárová · Tamás Fleiner

Accepted: 19 March 2007 / Published online: 19 April 2007

Abstract We study the dynamics of stable marriage and stable roommates markets. Our main tool is the incremental algorithm of Roth and Vande Vate and its generalization by Tan and Hsueh. Beyond proposing alternative proofs for known results, we also generalize some of them to the nonbipartite case. In particular, we show that the lastcomer gets his best stable partner in both incremental algorithms. Consequently, we confirm that it is better to arrive later than earlier to a stable roommates market. We also prove that when the equilibrium is restored after the arrival of a new agent, some agents will be better off under any stable solution for the new market than at any stable solution for the original market. We also propose a procedure to find these agents.

Keywords Stable marriage problem · Stable roommates problem · Matching mechanism

K. Cechlárová was supported by VEGA grant No. 1/3001/06 and Science and Technology Assistance Agency contract No. APVT-20-004104.
T. Fleiner was supported by the Bolyai research fellowship of the Hungarian Academy of Sciences and by the K60802 OTKA grant.

P. Biró (✉) · T. Fleiner
Department of Computer Science and Information Theory,
Budapest University of Technology and Economics,
Magyar tudósok körútja 2, 1117 Budapest, Hungary
e-mail: pbiro@cs.bme.hu

T. Fleiner
e-mail: fleiner@cs.bme.hu

K. Cechlárová
Institute of Mathematics, Faculty of Science, P.J. Šafárik University,
Jesenná 5, 04154 Kosice, Slovakia
e-mail: Katarina.Cechlarova@upjs.sk

1 Introduction

The stable marriage problem was introduced and solved by Gale and Shapley (1962). In terms of graphs, this is the bipartite case of the stable matching problem, where the two sets of agents are that of men and women. The solution obtained by the authors' deferred-acceptance algorithm was proved to be optimal for men if men make proposals. This means that each man gets his best stable partner, so no man can have a better partner in any other stable matching.

The nonbipartite version, the stable roommates problem, is also defined in Gale and Shapley (1962). It is shown by an example that in a stable roommates instance a stable matching does not always exist. Irving (1985) constructed the first polynomial algorithm that finds a stable matching if one exists at all (see also the book of Gusfield and Irving (1990)). Later, Tan (1991) gave a compact characterisation of the stable roommates problem by a half-integer solution.

For the bipartite case, Knuth (1976) asked whether it is possible to obtain a stable matching by starting from an arbitrary matching and successively satisfying blocking pairs. Roth and Vande Vate (1990) gave a positive answer by a decentralized algorithm, in which pairs or single agents enter the market in a random order, and stability is achieved by a proposal-rejection process. Knuth's question for the bipartite case was also answered by Abeledo and Rothblum (1995) by a common generalization of the Roth–Vande Vate and the Gale–Shapley algorithms. Later, Diamantoudi et al. (2004) solved the same problem for the roommates case. They proved that one can always reach a stable matching, if one exists, from an arbitrary matching by successively satisfying blocking pairs. Recently, Inarra et al. (2007) generalized this result for insolvable stable roommates problems by proving the same statement for the so called, P-stable matchings instead of stable matchings.

However the original goal of Roth and Vande Vate was different, their algorithm can be used to model the dynamics of the two-sided matching market as well. In fact, they considered the situation when a new agent enters the market and the stability is restored by the natural proposal-rejection process. This mechanism also yields an algorithm to find a stable matching for a market by letting the agents enter the market in a random order. Independently, Tan and Hsueh (1995) constructed an algorithm, that finds a stable half-matching for general graphs by using a similar incremental method. In the bipartite case, the Tan–Hsueh algorithm is equivalent to the Roth–Vande Vate algorithm. In the nonbipartite case infinite repetitions can occur, these are handled by the introduction of cycles. These two algorithms are abbreviated hereafter as "incremental algorithms".

Blum et al. (1997) described the properties of a dynamic two-sided matching market. They showed that their proposed algorithm is similar to the McVitie and Wilson (1970) of the original deferred-acceptance algorithm. So, the output of the process is predictable: if some men enter the market then each man either remains matched with the same partner (if it is possible) or gets a worse (but his best) stable partner for the new market. Blum and Rothblum (2002) pointed out that these results imply that the lastcoming agent gets his best stable partner in the Roth–Vande Vate algorithm. Moreover, an agent can only benefit from entering the market later (we assume here that the others enter the market in the same order). Independently, Ma (1996) observed on an example of Knuth, that if agents enter the market successively then the Roth–Vande Vate algorithm may not find all stable matchings in general. Cechlárová (2002) strengthened Ma's result by justifying that in a stable matching output by the incremental algorithm for a bipartite graph some agent gets his best stable partner. Here we give direct proofs for the above results in the bipartite case, and we generalize most of them to general graphs with the help of our Key Lemma.

Gale and Sotomayor (1985) showed that if some man expands his preference-list then no other man is better off in the new men-optimal stable matching. This implies that the same statement is true if a number of men enter the market. Roth and Sotomayor (1990) proved that if a man arrives and becomes matched, then certain women will be better off, and some man will be worse off under any stable matching for the new market than at any stable matching for the original market. We generalize this theorem by using an improved version of a result of Pittel and Irving (1994) on the core configuration.

Our results have an economic interpretation. Matching markets are well-known applications of the stable matching problem. A detailed description of two-sided markets can be found in the book of Roth and Sotomayor (1990). An important example is job matching. Blum et al. (1997) studied the dynamics of the two-sided matching market in this context by analysing the formation of the "vacancy chains".

The dynamic formation of social and economic networks can be described by stable matching models as Jackson and Watts considered in Jackson and Watts (2002). They illustrated the occuring mechanisms with the Roth–Vande Vate algorithm in the bipartite case. We believe that the same model can be used in the nonbipartite case, where the connections between individuals might correspond to mutual "best friend" relationships. By similar reasons, Eriksson and Strimling (2005) used the same stable roommates model to analyse the mate searching processes for special preferences. Recently, the dynamics of the firm mergers was also described as a one-sided stable matching market by Angelov (2006).

Another important application of the stable roommates problem is the pairwise exchange of indivisible goods. Yuan (1996) considered the resident exchange problem in China by this approach, and the stable roommates model was also mentioned by Roth et al. (2005) as a possible solution of the kidney exchange problem. However in these one-sided matching markets the dynamic processes are not typical.

This paper is organized as follows. In Sect. 2, we define stable matchings and half-matchings. In Sect. 3, the Roth–Vande Vate and the Tan–Hsueh algorithm are described. We prove our main results in Sect. 4.

2 Stable matchings and half-matchings

Let us model the *stable matching problem* with a graph G, where the agents are represented by vertices, and two vertices are linked by an edge if the agents are both acceptable to each other. For every vertex v, let $<_v$ be a linear order on the edges incident with v. That is, every agent has strict preferences on his possibles partnerships. We say that agent v prefers edge f to e (in other words f *dominates* e at v) if $e <_v f$ holds. A *matching* M is a set of edges with pairwise distinct vertices. If an edge $e = \{u, v\}$ belongs to M, then u and v are *matched* in M, so u and v are partners in the market. An agent is *single*, if his vertex is uncovered in M, i.e. it is not incident with a matched edge.

A matching M is called *stable* if every nonmatching edge, $e \notin M$ is dominated by some matching edge, $f \in M$. Alternatively, a stable matching can be defined as a matching without a *blocking edge*: an edge $e = \{u, v\}$ is blocking for a matching M if u is either unmatched or prefers edge e to the matching edge that covers u in M, and at the same time, v is either unmatched or prefers edge e to the matching edge that covers v in M. For a matching market, the stability means that no pair of agents can benefit by leaving their actual partners and establishing a new mutual partnership.

We notice that an advantage of the graph terminology is that it can handle parallel edges that correspond to the case where two agents can make several types of partnership with each other. Moreover, this basic model can be improved easily to describe more general problems, for example the case, where an agent can have many partners (see Cechlárová and Fleiner 2005). However in this article we deal only with simple graphs,[1] we chose the graph model because the notion of half-matching also can be defined naturally in this way, that is crucial in our work.

Alternatively, stable matchings can be described with compact formulas. If M is a set of edges then let $x_M : E(G) \longrightarrow \{0, 1\}$ be its *characteristic function* i.e.

$$x_M(e) = \begin{cases} 1, & e \in M \\ 0, & e \notin M \end{cases}$$

Subset M of $E(G)$ is a stable matching if the following conditions hold:

(M) Matching:
$\sum_{v \in e} x_M(e) \leq 1$ for every vertex $v \in V(G)$

(S)Stability:
for every edge $e \in E$ there exists a vertex $v \in e$ such that $\sum_{v \in f, f \geq_v e} x_M(f) = 1$

We consider the *stable marriage* problem if the graph is bipartite, and the *stable roommates* problem if the graph is general. Gale and Shapley (1962) proved that a stable matching always exists for the marriage problem but may not exists for the roommates problem. They gave the following example to show the non-existence:

Example 1

Agents	*Preference − lists*
A :	$[B, C, D]$
B :	$[C, A, D]$
C :	$[A, B, D]$
D :	*arbitrary*

[1] A graph is *simple* if it does not have parallel edges. In this case instead of partnerships, the agents can have preferences on their possible partners, so each vertex can have a linear order on his neighbours in the graph. Then, as a widely used definition, a matching can be described equivalently by an involution μ on the set of agents, where $\mu(u) = v$ implies $\mu(v) = u$, and this means that u and v are matched ($\mu(w) = w$ corresponds to the case when w is unmatched).

Let us imagine that these agents are tennis-players, each is looking for a partner to play with for 1 h a week. For example Andy would like to play mostly with Bill, then with Cliff and finally he prefers Daniel the least. (In fact, everybody tries to avoid Daniel.) There is no stable solution. If a pair is formed from the first three players, say Andy plays with Bill, then the third one, Cliff must be matched with Daniel, but in this case Bill and Cliff block this matching.

Tan (1991) discovered, that if the agents can create half-time partnerships then a stable solution always exists in the sense that no pair of agents would simultaneously like to increase the intensity of their partnership.

Considering the above example, we suppose that Andy, Bill and Cliff agree to meet once a week and play half-time games in each formation. Thus, each of them play 1 h in sum, only Daniel remains without any tennis-partner. Stability in this case is that no pair of tennis-players want to play more time together with each other. For example Andy plays with Daniel no time at all, because Andy fills his 1-h by playing two half-hour games with better partners. Andy and Bill will not play more than a half-hour, because Bill fills the rest of his time (a half-hour) by playing with a better partner, Cliff.

A *half-matching* hM consists of matching edges M and half-weighted edges H, so that $hM = H \cup M$ and each vertex is incident either with at most one matching edge or with at most two half-weighted edges. In a matching market an agent can have at most one partner or at most two half-partners. A half-matching hM is *stable* if for each edge e not in hM there exists a vertex v, where e is dominated either by one matching edge or by two half-weighted edges, and for every half-weighted edge h there exists a vertex v, where h is dominated by another half-weighted edge. So no pair of agents wants to improve their partnership simultaneously, because for each pair of agents who are not matched, one of them fills his capacities with better partnership(s). Otherwise, if a half-matching is unstable, then a *blocking edge* is an undominated edge.

If $x_{hM} : E(G) \longrightarrow \{0, \frac{1}{2}, 1\}$ is a weight-function that describes the set of matching edges, M and the set of half-weighted edges, H so that

$$x_{hM}(e) = \begin{cases} 1, & e \in M \\ \frac{1}{2}, & e \in H \\ 0, & e \notin hM \end{cases}$$

then the same (M) and (S) inequalities preserve the half-matching and the stability-property.

The fact, that every half-weighted edge must be dominated by another half-weighted edge at one of its endvertices implies that the half-weighted edges form cycles, where the direction of the domination between two consecutive half-weighted edges is the same along the cycle. To illustrate this property in the figures, we orient each half-weighted edge to its endvertex, where it is dominated by the other half-weighted edge. Tan (1991) observed that an even-cycle can be replaced by matched pairs, but if an odd-cycle C occurs in hM then C must belong to the H-part of any stable half-matching for the given graph, so no stable matching exists. He characterized the stable half-matching[2] in the following way:

Theorem 1 (Tan) *For a stable roommates problem there always exists a stable half-matching[3] that consists of matched pairs and odd-cycles formed by half-weighted pairs. The set of agents can be partitioned into:*

(a) *unmatched (or single) agents,*
(b) *cycle-agents and*
(c) *matched agents.*

Furthermore, for any instance the same agents remain unmatched and the same odd-cycles are formed in each stable half-matching.

If for a half-matching $hM = H \cup M$ an edge $e = \{u, v\}$ is in M, then we say that the agents u and v are *partners*. If two agents can be partners in a stable half-matching we call them *stable partners*. If an edge $e = \{u, v\} \in H$ is in an odd-cycle, then u and v are *half-partners*. If u prefers v to his other half-partner, then v is the *successor* of u and u is the *predecessor* of v.

To consider the stable half-matchings of a matching market can have many motivations. First of all, if the stable half-matching does not contain any odd-cycle, then we receive a stable matching, otherwise we know the reason of the non-existence. Secondly, we can obtain a matching, by leaving one agent from each odd-cycle and forming pairs from the rest of the cycles. This matching is stable for the remaining agents. In other words every blocking edge is incident with one of the removed agents, so by compensating them somehow we can reach a kind of stability for the market.[4] Thirdly, in some real applications (like in the case of the tennis-players) the half-solutions are feasible in practice.

3 The incremental algorithms

Suppose a matching market is in an equilibrium with a stable matching. A natural question is how the situation changes if a new player enters the game and the preferences over the former partnerships are unchanged. Let the newcomer make proposals according to his preference order. If no one accepts, then everybody has a better partner, so the former matching remains stable. If somebody accepts a proposal, then a new pair is formed along the proposal. The possible left-alone partner has to leave the

[2] Originally, Tan used the term *stable partition*. We have several reasons to use this alternative notion. The expression "stable partition" is also used as a core-solution of a coalition formation game, that can be confusing. If we consider more general models (where agents can have several partners, or multiple activities are possible) definition of stable half-matching can be easily extended. Finally, the half-solution may interpret real partnerships with half-intensities.

[3] Aharoni and Fleiner (2003) showed, that the existence of the stable half-matching is a consequence of the famous theorem of Scarf (1967).

[4] This idea can be used as a heuristic to find a matching that contains as few blocking edges as possible. It is reasonable to apply such a method, since even to approximate the minimal number of the blocking pairs for general graphs is theoretically hard (see Abraham et al. 2006).

market and enter as a newcomer. Note that the same situation happen, when an agent leaves the market. If he was single, then the matching remains stable. Otherwise, if he was matched, then his partner has to leave the market and enter again as a newcomer.

The Roth–Vande Vate algorithm for the stable marriage problem

Suppose, that a bipartite graph G is built up step by step in the algorithm by adding vertices to the graph in some order. In a *phase* of the algorithm we add a new agent and restore the stability. To describe a phase, let us add a vertex v to $G - v$, where a stable matching M_v exists. Our task is to find a stable matching M for G.

If v is not incident to any blocking edge, then M_v remains stable for G, too. In this case the phase is called *inactive*.

A phase is *active* if the newcomer v is a member of some blocking pair, let $\{v, u\}$ be the best blocking pair for v. Let $v = a_0$ and $u = b_1$. If b_1 was unmatched in $M_v = M_{a_0}$, then $M_{a_0} \cup \{a_0, b_1\}$ is a stable matching for G. Otherwise, b_1 had a partner a_1 in M_{a_0}, whom he leaves after receiving a better proposal. In this case, the matching $M_{a_1} = M_{a_0} \setminus \{a_1, b_1\} \cup \{a_0, b_1\}$ is stable for $G - a_1$. So we have a similar situation as in the beginning: a_1 enters the market and makes proposals. Continuing the process, a *proposal-rejection sequence*, $S = (A|B) = a_0, b_1, a_1, \ldots$ is constructed with the following properties:

1. $M_{a_k} = M_{a_{k-1}} \setminus \{a_k, b_k\} \cup \{a_{k-1}, b_k\}$ is a stable matching for $G - a_k$.
2. a_{k-1} is a better partner for b_k than a_k and
3. b_{k+1} is a worse partner for a_k than b_k.

Note that here, $a_0, a_1, \ldots$ are from the same side, and $b_1, b_2, \ldots$ are from the other one. Property 2 is true, since b_k accepted the proposal of a_{k-1} while he left a_k. To see 3, realize that pairs (a_k, b_k) and (b_{k+1}, a_{k+1}) are in $M_{a_{k-1}}$, so 2. and the assumption that a_k prefers b_{k+1} to b_k would imply that (a_k, b_{k+1}) is a blocking pair for $M_{a_{k-1}}$.

A proposal-rejection process is illustrated in Fig. 1. In this and subsequent figures, a little arrow is directed from a dominated edge to a dominating one and thick lines correspond to matching and half-weighted edges of the current (half-)matching.

Observe that by this process, each $a_i \in A$ improves his situation and each $b_j \in B$ gets worse off. Consequently, the same agents cannot occur as new pairs. So a phase terminates in $O(m)$ time, when m denotes the number of the edges in the graph. A phase has two possible outcomes: either nobody accepts the proposals of some a_i

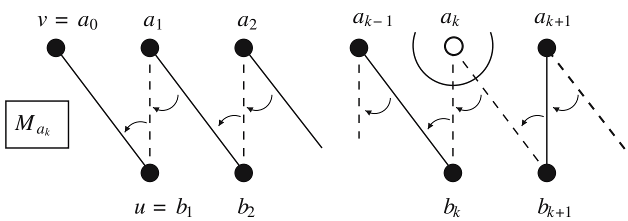

Fig. 1 Proposal-rejection sequence in the Roth–Vande Vate algorithm

(then the size of the matching remains the same) or the last b_j was unmatched, hence the size of the matching increases by one.

We illustrate with an example the mechanism of the incremental algorithm and we introduce briefly our results. The preferences of the agents on their possible partnerships in this two-sided market are the following:

Example 2

$$\begin{array}{ll} a_1 : e_1 > d_1 > f_1 & b_1 : f_3 > d_2 > n_1 > e_1 \\ a_2 : e_2 > d_2 > f_2 & b_2 : f_2 > d_1 > e_3 \\ a_3 : e_3 > d_3 > f_3 & b_3 : f_1 > d_3 > e_2 \\ a_4 : s & b_4 : s > n_2 \\ a_5 : m_1 > m_2 & b_5 : m_2 > n_3 \\ a_6 : n_1 > n_2 > n_3 > n_4 & b_6 : n_4 > m_1 \end{array}$$

Let $d = \{d_1, d_2, d_3\}$, $e = \{e_1, e_2, e_3\}$, $f = \{f_1, f_2, f_3\}$. Suppose, that at the beginning a_6 is not present in the market. Partnerships $\{e, s, m_1\}$ form a stable matching in the market. (It is the best one for every agent a_i.)(Fig. 2)

When agent a_6 enters the market, four new possibles partnerships are created. The best one for the newcomer is n_1, that is blocking the actual matching. Following the algorithm of Roth and Vande Vate let us satisfy this blocking edge: b_1 and a_6 form a new pair, and partnership e_1 terminates, so agent a_1 has to find a new partner as a newcomer. Continuing this process, the following edges will be satisfied and terminated in sequence: $d_1, e_3; d_3, e_2; d_2, n_1$. Afterwards, agent a_6 makes proposals again, that b_1 and b_4 refuse, because they prefer their partners to a_6. We will prove later, that if a new partnership is not blocking, then it cannot be present in any stable matching. In the last step of our example, a single agent b_5 accepts the proposal of a_1, and $\{d, s, m_1, n_3\}$ is a stable matching. This stable solution is the best possible one for the newcomer a_6, since the better partnerships, that were refused by his possibles partners cannot appear in any stable matching. This argument also shows that every agent that receives a partner by making a proposal during the process gets his best stable partner (Fig. 3).

Note, that if we started with the stable matching $\{f, s, m_1\}$, then the process would stop in one step, since b_5 accepts first the proposal of a_6. The obtained stable matching

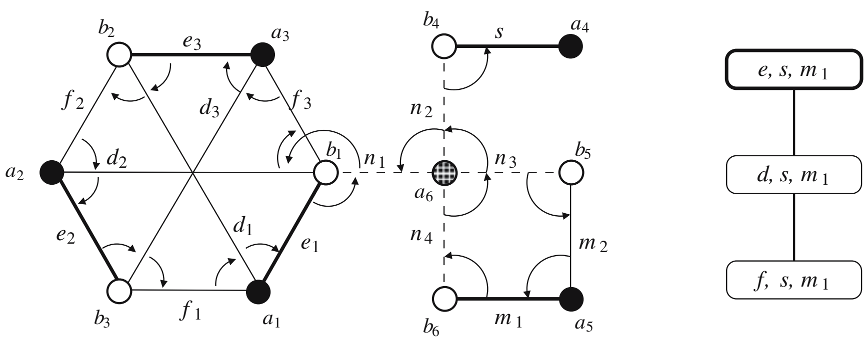

Fig. 2 A stable matching and the lattice of the stable matchings for Example 2 before the arrival of a_6

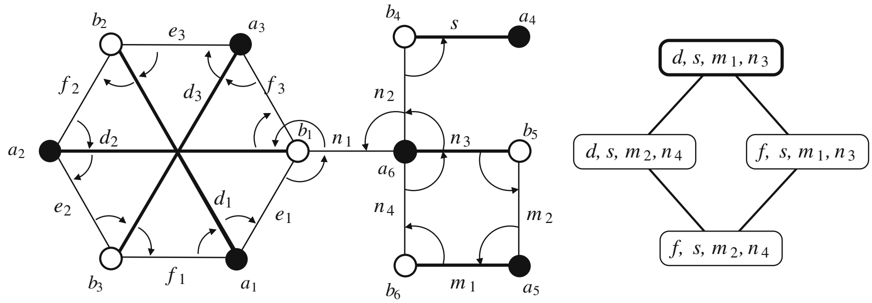

Fig. 3 The obtained stable matching, and the lattice of the stable matchings for Example 2

$\{f, s, m_1, n_3\}$ yields the best stable partner to the newcomer a_6 again, but the other agents a_i do not get necessarily their best stable partners.

The Tan–Hsueh algorithm for the stable roommates problem

Tan and Hsueh (1995) proposed an incremental algorithm to find a stable half-matching. In this more general setting we use the terminology of the Roth–Vande Vate algorithm. The only difference is that G is not bipartite, so instead of a matching, we maintain a half-matching hM_v for $G - v$.

Hereafter, we suppose that the stable half-matchings have no even-cycles. As we mentioned before, an even-cycle can be always separated into matching pairs, moreover, as we will see later, incremental algorithm does not create even-cycles. By Theorem 2 we know, that for a fixed stable roommates problem, the same odd-cycles are present in each stable half-matching $hM^i = H \cup M^i$, so H is determined, only the M^i part, the matching pairs can differ for two stable half-matchings for a given graph. In fact, H can be considered as a disjoint union of half-weighted cycles, so whenever we modify a stable half-matching during the processes, we will only add or remove matching edges or half-weighted odd-cycles.

If nobody accepts the newcomer's proposal, then the phase is called *inactive* again and the current stable half-matching is unchanged.

If some agent u accepts the proposal of v then three cases are possible:

(a) If u is unmatched in hM_v, then $hM = hM_v \cup \{v, u\}$ is a stable half-matching for G.

(b) If u is a cycle-vertex in hM_v, so $u = c_0$ for some cycle $C = (c_0, c_1, \ldots, c_{2k-1}, c_{2k})$, then $hM = hM_v \setminus C \cup \{v, u\} \cup \{c_1, c_2\} \cup, \ldots, \cup \{c_{2k-1}, c_{2k}\}$ is a stable half-matching for G (i.e. we remove the half-weighted cycle C and we add some matching edges).

(c) If u is matched with x in hM_v, then $hM_x = hM_v \setminus \{u, x\} \cup \{v, u\}$ is a stable half-matching for $G - x$.

The current phase ends in cases (a) and (b). Here, unlike in the bipartite case, it can happen that an agent, that made a proposal earlier can receive a proposal later during the same phase. So the proposal-rejection sequence might never end. One result of Tan and Hsueh (1995) was that a repetition always occurs along an odd-cycle.

Theorem 2 (Tan–Hsueh) *If* $S = (A|B) = a_0, b_1, a_1, \ldots$ *is a proposal-rejection sequence and* $a_i = b_k$ $(i < k)$ *is the first return, then this proposal-rejection sequence can be extended in such a way that it will return to* a_k *at* b_{k+m+1}*, and the following properties are true:* $\{a_k, b_{k+1}, \ldots, b_{k+m}, a_{k+m}\}$ *are distinct vertices, and in the inverse order they form an odd-cycle* C*, and* $hM = hM_{a_k} \setminus \{a_{k+1}, b_{k+1}\} \setminus \cdots \setminus \{a_{k+m}, b_{k+m}\} \cup C$ *is a stable half-matching.*

Example in Figs. 4 and 5 illustrate the Tan–Hsueh algorithm: here, vertex v enters. The first vertex accepting v's proposal is u, and u's previous partner x is left alone. Figure 4 shows the stable half-matching hM_x for $G - x$. In the next step, x makes proposals. Figure 5 illustrates the termination of this phase by obtaining an odd cycle, namely the three-cycle containig vertex x and edges c_1 and c_2.

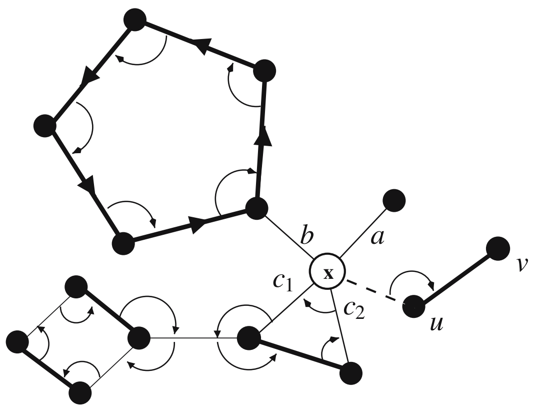

Fig. 4 The Tan–Hsueh algorithm in an example

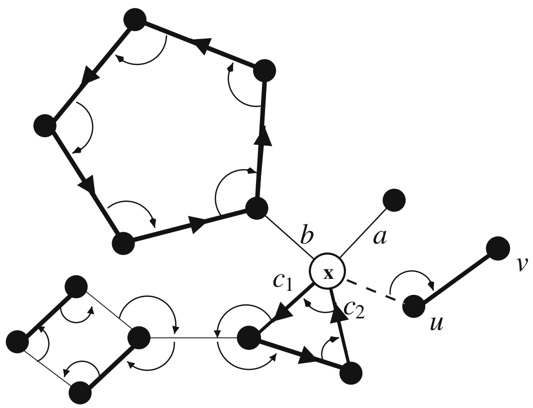

Fig. 5 The obtained stable half-matching

4 Properties of the dynamic solutions

In this section we prove our results.

Getting the best stable partner by making proposals

Lemma 3 (Key Lemma) *If hM_v is a stable half-matching for $G - v$, and edge $\{v, u\}$ is not blocking hM_v, then v and u cannot be matched in a stable half-matching for G.*

Proof Let us suppose that $\{v, u\}$ is not blocking hM_v but there is a stable half-matching hM of G, where v and u are matched. Let $v = a_0$ and $u = b_1$. First we consider the case where none of hM and hM_v contains an odd-cycle. Then b_1 has a partner in M_v (say a_1), who is better than a_0. So $\{a_0, b_1\} <_{b_1} \{a_1, b_1\}$, where $\{a_0, b_1\} \in M \setminus M_v$. Since M cannot dominate $\{a_1, b_1\}$ at b_1, this edge must be dominated at a_1 by some edge $\{a_1, b_2\}$ of M. As $\{a_1, b_2\}$ is not in M_v, it must be dominated at b_2 by an edge $\{a_2, b_2\}$ of M_v, and so on. The alternating sequence $(a_0, b_1, a_1, b_2, \dots)$ has the following property: $\{a_{i-1}, b_i\} \in M \setminus M_v$ and $\{b_i, a_i\} \in M_v \setminus M$, furthermore the domination is also in sequence: $\{a_{i-1}, b_i\} <_{b_i} \{a_i, b_i\}$ and $\{a_i, b_i\} <_{a_i} \{a_i, b_{i+1}\}$ for every i. We call this sequence *alternating preference sequence*. Because a_0 is not covered by the stable matching M_v, the sequence can return neither to a_0, nor to any other vertex. Otherwise, the first such a repeted vertex would be covered by two matching edges, a contradiction. (This part of the proof already confirms the bipartite case.)

The other case is, when hM_v or hM may contain odd-cycles. The properties of the alternating preference sequence remain the same, the difference is that the edges can be half-weighted edges as well. To avoid repetition, the idea is the following: when an edge $\{a_i, b_i\} \in hM_v$ is dominated at a_i in hM by two edges (so a_i is in a cycle in hM), then we chose for b_{i+1} the predecessor of a_i. Edge $\{a_i, b_{i+1}\}$ is still not in hM_v, so it must be dominated at b_{i+1}. But then the edge(s) that dominate(s) $\{a_i, b_{i+1}\}$ is (are) better than either of the edges that cover b_{i+1} in hM, so they are not in hM. This is why every new edge in this sequence will be alternately in $hM \setminus hM_v$ and $hM_v \setminus hM$ (Fig. 6).

We have a finite number of agents, so the alternating preference sequence must return. We consider the first such repetition. If $a_k = a_i$ for some $k \neq i$ then $\{b_k, a_i\}$ and $\{b_i, a_i\}$ would be in the same odd-cycle in hM_v, but a_i would be the predecessor of both b_i and b_k by the inductive definition of the sequence, that is impossible. In the other case, assume that $a_k = b_i$ for some $k \neq i$. This means that $\{b_k, b_i\}$ and $\{b_i, a_i\}$ are in the same odd-cycle in hM_v. By definition, a_i is the predecessor of b_i, so b_k should be the successor of b_i, that would imply $\{a_i, b_i\} <_{b_i} \{b_k, b_i\}$. On the other hand, since $\{b_k, b_i\} \in hM_v \setminus hM$, it must be dominated at b_i in hM. By the inductive rules $\{a_{i-1}, b_i\} \in hM$, this means $\{b_k, b_i\} <_{b_i} \{a_{i-1}, b_i\} <_{b_i} \{a_i, b_i\}$, a contradiction.

Similarly, If $b_k = b_i$ for some $k \neq i$ then $\{a_k, b_i\}$ and $\{a_i, b_i\}$ would be in the same odd-cycle in hM, but b_i would be the predecessor of both a_i and a_k by the inductive definition of the sequence, that is impossible. Finally, assume that $b_k = a_i$ for some $k \neq i$. This means that $\{a_k, a_i\}$ and $\{a_i, b_i\}$ are in the same odd-cycle in hM_v. By definition, b_i is the predecessor of a_i, so a_k should be the successor of a_i, that

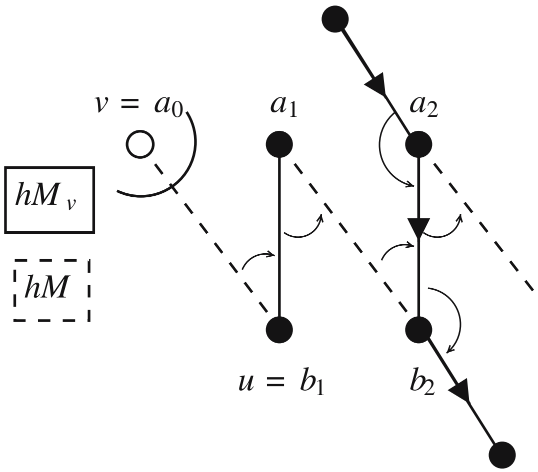

Fig. 6 Alternating preference sequence with half-weighted edges

would imply $\{b_i, a_i\} <_{a_i} \{a_k, a_i\}$. On the other hand, since $\{a_k, a_i\} \in hM \setminus hM_v$, it must be dominated at a_i in hM_v. By the inductive rules $\{b_{i-1}, a_i\} \in hM_v$, this means $\{a_k, a_i\} <_{a_i} \{b_{i-1}, a_i\} <_{a_i} \{b_i, a_i\}$, a contradiction. □

The following Lemma is well-known.

Lemma 4 *If v is the best stable partner for u then u is the worst stable partner for v.*

Proof If indirectly, v and u are matched in a stable half-matching hM, but v has an even worse partner u' in a stable half-matching hM', then u would have some other partner v' worse than v, because v was u's best stable partner. So $\{u, v\}$ would be a blocking edge for hM', contradiction. □

To generalize the results of Blum et al. (1997) we prove that a newcomer gets his best stable partner in the output of the incremental algorithm in the nonbipartite case as well.

Theorem 5 *Suppose that an agent v enters the market and the stability is restored by a proposal-rejection process along the sequence $S = (A|B)$. Then each agent $a \in A$, who became matched by making (accepting) a proposal gets his best (worst) stable partner in the obtained stable half-matching.*

Proof If an agent a is matched in the output, and receives a partner by making a proposal, then later he cannot accept any proposal because then he would be a cycle-agent. The last time when agent a makes a proposal during the process he does not prefer his last partner only to some agents that refused him. Because of the Key Lemma, no one of these agents can be a partner of a in a stable solution, so obviously agent a received his best stable partner. Similarly, each matched agent $b \in B$ gets his worst stable partner by Lemma 4. □

Corollary 6 *If an agent enters the market last and becomes matched, then he gets his best stable partner.* □

If a phase is inactive in the incremental algorithm, then each stable half-matching of the extended graph is also a stable half-matching in the original. That is, if hM is a stable half-matching for G not covering some vertex x, then hM is a stable half-matching for $G - x$ too, because after deleting x from G no blocking edge can appear. So, by using the Key Lemma, we can confirm our main result:

Theorem 7 *Each matched agent, that gets a partner in the last active phase by making (accepting) a proposal, receives his best (worst) stable partner in the stable solution output by the incremental algorithm.* □

Remark The vertices that remained uncovered in the last active phase or entered later in an inactive phase, will still be uncovered at the end of the algorithm, just like they are in every stable matching. The vertices that form an odd-cycle in the last active phase will form an odd-cycle at the end of the algorithm, just like they do in every stable half-matching. Hence these agents also get their best stable partners in this sense.

Corollary 8 *A stable matching, where no matched agent gets his best stable partner, cannot be output by the incremental algorithm.* □

Let us remark that we did not prove that any stable matching where somebody gets his best stable partner or contains an odd-cycle can be obtained with an incremental algorithm. Our result gives only a necessary condition not a sufficient one.

Blum et al. (1997) proved, that if a man m enters the market and another man m' was matched with w' in M_m, then m' and w' remain matched in the obtained stable matching M for the new market if and only if they are stable partners for the new market. Otherwise m' and w' get those agents to whom they are matched in the men-optimal stable matching of the new market. (So m' receives his best stable partner, and w' receives her worst stable partner in this case.) Below, we generalize this statement for the nonbipartite case.

Theorem 9 *Suppose that w and u are matched in a stable half-matching hM_v for $G - v$. They remain matched in the stable half-matching hM, obtained by the proposal-rejection process after the arrival of v if and only if they are stable partners for G as well. Otherwise, if they are not involved in a cycle, then one of them gets a better partner than he had in hM_v but receives his worst stable partner, the other one becomes single or gets a worse partner than he had in hM_v but receives his best stable partner in hM .*

Proof If w and u are not involved in the proposal-rejection process, then obviously they remain matched. Otherwise, if $S = (A|B)$ is the proposal-rejection sequence, then one of them, w is in A and the other one must be in B. As they are not involved in a cycle, u improves his situation and w gets worse off during the process, and finally (by Theorem 7) u gets his worst stable partner (better than w) and w gets his best stable partner (worse than u), so u and w cannot be stable partners in the output. □

Improving the situation by accepting proposals

Our next goal is to generalize the following result of (Roth and Sotomayor 1990, Theorem 2.26). First we give its proof implied by the previous results of this paper.

Theorem 10 (Roth–Sotomayor) *Suppose a woman w is added to the market $G - w$. Let M^W be the woman-optimal stable matching for the new market, G and let M_w^M be the man-optimal stable matching for $G - w$. If w is not single in M^W, then there exists a nonempty subset of men, S, such that each man in S is better off, and each woman in S' is worse off under any stable matching for the new market than at any stable matching for the original market, when S' denotes the partners of men in S under matching M_w^M.*

Proof After adding w to the market during the proposal-rejection process each man that gets a partner by accepting a proposal gets his worst possible partner at the end of the process by Theorem 7. So they get the same partners as in M^W. But these partners are strictly better than their original partners in M_w^M, that were actually their best stable partners for $G - w$. Similarly, each woman that gets a new partner during the process by making a proposal gets her best stable partner for G, so they get the same partners as in M^W. But these partners are strictly worse than their original partners in M_w^M, that were actually their worse stable partners for $G - w$. □

Pittel and Irving (1994) considered the following situation. A new agent v enters the market, and a perfect stable matching (i.e. a stable matching where no agent is single) is achieved in such a way that the proposal-rejection sequence is as short as possible. They called this special half-matching with the associated alternating sequence a *core configuration relative to v*. Pittel and Irving (1994) proved the following interesting property.

Theorem 11 (Irving–Pittel) *If hM_v is a core configuration relative to v, then the associated proposal-rejection sequence $v = a_0, b_1, a_1, \ldots, a_{k-1}, b_k$ consists of $2k$ distinct persons, it is uniquely defined, and for every $i = 1, \ldots, k-1$*

1. *b_i is the worst stable partner of a_i for $G - v$;*
2. *a_i is the best stable partner of b_i for $G - v$.*

We generalize Theorem 11 by extending the notion of core configuration. A stable half-matching hM_v is a *core configuration* relative to v if after adding v to the graph, the associated proposal-rejection sequence $S(hM_v)$ is as short as possible, by assuming that in case of cycling the sequence is restricted till b_k, where $a_i = b_k$ is the first return.

Theorem 12 *If hM_v is a core configuration relative to v, then the associated proposal-rejection sequence $a_0(= v), b_1, a_1, \ldots, a_{k-1}, b_k(, a_k)$ consists of distinct persons, it is uniquely defined, and for every agent in the sequence, who is matched for G, the following properties are true:*

(a) *b_i is the worst stable partner of a_i for $G - v$ and b_{i+1} is the best stable partner of a_i for G;*
(b) *a_i is the best stable partner of b_i for $G - v$ and a_{i-1} is the worst stable partner of b_i for G.*

Proof In our proof we construct a core configuration. Suppose that hM^0 is an arbitrary stable half-matching for G. Let a new agent u enter the market in such a way that u

is acceptable only for v and u is the most preferred partner for v. Let us denote the proposal-rejection sequence by $S(hM^0)$ and the output stable half-matching for $G+u$ by hM^0_{+u}. Obviously, u and v are partners in any stable half-matching hM'_{+u} for $G+u$, moreover hM'_{+u} is a stable half-matching for $G+u$ if and only if $hM'_v = hM'_{+u} \setminus \{u, v\}$ is a stable half-matching for $G - v$. So, by deleting $\{u, v\}$ from hM^0_{+u} we get a stable half-matching, say hM_v for $G - v$. We prove that hM_v is a core configuration relative to v. (We denote the associated proposal-rejection sequence by $S(hM_v)$ and the output stable half-matching for G by hM.)

To prove that $S(hM_v)$ is as short as possible we show that each agent that is involved in $S(hM_v)$ must be involved in any other proposal-rejection sequence as well, and each agent occurs exactly once in $S(hM_v)$ (unless a new odd-cycle is created, when $a_i = b_k$ occurs twice.)

First, we prove that if $x \in S(hM_v)$ then $x \in S(hM'_v)$ for any stable half-matching hM'_v for $G - v$. We consider the cases according the status of x (unmatched, cycle-agent or matched) in the stable half-matchings for $G - v$ and G.

1–2. No agent can be unmatched for $G - v$ and a cycle-agent for G, similarly no agent can be a cycle-agent for $G - v$ and unmatched for G.

3–4. If an agent is unmatched/cycle-agent for $G - v$ and remains unmatched/cycle-agent for G then he cannot be involved in any proposal-rejection sequence.

5. If x is matched for $G - v$ and becomes unmatched for G then $x = a_k$, so x is the last agent in $S(hM_v)$ (nobody accepts his proposal) and obviously x must be the last agent in any other $S(hM'_v)$ as well.
6. If x is unmatched for $G - v$ and becomes matched for G then $x = b_k$, so x is the last agent in $S(hM_v)$ (he accepts the last proposal) and obviously x must be the last agent in any other $S(hM'_v)$ as well.
7. If x is a cycle-agent for $G - v$ and becomes matched for G then $x = b_k$, so x is the last agent in $S(hM_v)$ (he accepts the last proposal). We prove that for any stable half-matching hM'_v x is the last agent in $S(hM'_v)$ as well. Let $C = (c_0, c_1, \ldots, c_{2k})$ be the cycle that eliminates when v enters the market. We suppose indirectly that two different cycle-agents $x = c_0$ and c_i accept the last proposals, made by y and y' in $S(hM_v)$ and $S(hM'_v)$, respectively. Obviously, the agent who made the final proposal is better than the predecessor of that cycle-agent who accepts it, (so $y >_{c_0} c_{2k}$ and $y' >_{c_i} c_{i-1}$). From Theorem 7, we also know that c_0 and c_i get their worst stable partners in hM and hM', respectively. This is a contradiction, because if i is even then c_i would be matched with c_{i-1} in hM and if i is odd then c_0 would be matched with c_{2k} in hM'.
8. If x is matched for $G - v$ and became a cycle-agent for G then x must occur in any proposal-rejection sequence until the first return, since Tan and Hsueh (1995) proved that no new agent occurs in the sequence after the first return.
9. Finally we consider the case where x is matched for $G - v$ and for G as well. Let us denote x's partners by y^0, y_v and y in hM^0, hM_v and hM, respectively.

(a) If $y <_x y_v$, then x must receive y during $S(hM_v)$ by making a proposal, so from Theorem 7 y is the best stable partner of x for G. Thus, $y^0 \leq y$ implies $y^0 <_x y_v$, it means that x must receive y_v during $S(hM^0)$ by accepting a proposal, so y_v

is the worst stable partner of x for $G - v$. It is obvious now that x gets a worse partner under any stable half-matching for G than at any stable half-matching for $G - v$, so x must be involved in any proposal-rejection sequence.

(b) Similarly, if $y >_x y_v$, then x must receive y during $S(hM_v)$ by accepting a proposal, so from Theorem 7 y is the worst stable partner of x for G. Thus, $y^0 \geq y$ implies $y^0 >_x y_v$, it means that x must receive y_v during $S(hM^0)$ by making a proposal, so y_v is the best stable partner of x for G. It is obvious now that x gets a better partner under any stable half-matching for G than at any stable half-matching for $G - v$, so x must be involved in any proposal-rejection sequence.

(c) If $y = y_v$, then x cannot be involved in $S(hM_v)$.

Now, we prove that each agent occurs exactly once in $S(hM_v)$. Let us consider the above sequence with an extra stopping rule: if a_j looks for a new partner let choose the best one among those that either form a blocking pair with a_j or a b_i for $i < j$ such that b_i prefers a_j to a_i (and not to his actual partner a_{i-1}). Assume that the first repetition (according to the extra stopping rule) would occur at b_{j+1}.

Case 1 If $b_i = b_{j+1}$ for some $i < j$ then let hM_{a_j} be the actual stable half-matching for $G - a_j$. We construct a new stable partition for $G - v$: $hM'_v = hM_{a_j} \cup \{a_j, b_i\} \setminus \{\{a_{p-1}, b_p\}, 1 \leq p \leq i\} \cup \{\{a_p, b_p\}, 1 \leq p \leq i-1\}$. It is stable, because by compairing with hM_v only agents $\{a_q, i \leq q \leq j\}$ get worse partners, but the extra stopping rule preserves that no edge $\{\{a_q, b_p\}, 1 \leq p < i \leq q \leq j\}$ can block hM'_v (and obviously no other edge).

Since in hM'_v every agent $\{b_q, i \leq q \leq j\}$ gets a better partner than in hM_v, and every agent $\{a_q, i \leq q \leq j\}$ gets a worse partner than in hM_v. If some of these agents is matched for $G - v$ and G as well, then it is a contradiction, because in hM_v they are matched with their best/worst stable partners, respectively.

The last case that we have to consider, that all of these agents are matched for $G - v$ and become a cycle-agent for G. These agents are obviously in the same cycle [let say $(c_0, c_1, \ldots, c_{2k})$] in hM^0 as well. So, when $S(hM^0)$ ends at c_0 by eliminating this cycle, each of these agents becomes matched in hM_v to either with his successor or with his predecessor (so $\{c_{2i-1}, c_{2i}\} \in hM_v$ for all $1 \leq i \leq k$). We show that a_{i-1} must also be a cycle-agent for G. Otherwise a_{i-1} must receive a worse partner than b_i in hM, and for b_i his predecessor is also worse than a_{i-1} (that is why b_i accepted the proposal of a_{i-1}), so a_{i-1} and b_i would block hM. By continuing this argument, for some $p < i$, a_p must be c_0, [the cycle-agent in hM_v that accepted the last proposal in $S(hM^0)$]. But then b_{p+1} must be the predecessor of c_0: c_{2k}. Otherwise, if for some $1 \leq r < 2k$, $c_r = b_{p+1}$ then $c_{2k} <_{c_0} c_r$ (since c_{2k} is matched with c_{2k-1} in hM_v, so he would accept the proposal of c_0) and $c_{r-1} <_{c_r} c_0$ (since c_r accepted the proposal of c_0), so c_0 and c_r would form a blocking pair in hM. Similarly, we can prove that the sequence goes along this odd-cycle, so for each d $(0 < d < j - p)$ $a_{p+d} = c_{2(k-d)+1}$ and $b_{p+d} = c_{2(k-d)}$. Finally, $b_i = b_{j+1}$ cannot be the predecessor of a_j in hM, a contradiction.

Case 2 If the first repetition is such that $a_i = b_{j+1}$, then the extra stopping rule was not used. This proves that a new odd-cycle can be created, so $hM = hM_{a_j} \setminus$

$\{\{a_q, b_{q+1}\}, i \leq q \leq j\} \cup (a_i, a_j, b_j, a_{j-1}, \ldots, a_{i+1}, b_{i+1})$ is the output stable half-matching for G. □

Theorem 12 implies the following generalization of Theorem 10.

Theorem 13 *Suppose that a new agent is added to the market and a new stable solution is reached by the proposal-rejection process. There may exist some agents that are better off, and some other agents that are worse off under any stable half-matching for the new market than at any stable half-matching for the original market. We can find all of these agents algorithmically.* □

The arrival order determines the benefits

If we suppose that a centralized matching program uses the incremental algorithm, or if we model the dynamics of the matching market by the natural proposal-rejection process, then the received solutions are determined by the arrival order of the agents. We discuss here, as a consequence of the theorems from the last subsections, how the benefits of an agents depend on the arrival orders.

Blum and Rothblum (2002) realized that an agent can only benefit by arriving later to the market in the Roth–Vande Vate algorithm. By a similar argument, we can generalize this result for the nonbipartite case.

Lemma 14 *Let us suppose that u is a matched agent for both $G - v$ and G, and let hM_v and hM'_v be two half-matchings for $G - v$ such that u gets at least as good partner in hM_v as in hM'_v(denoted by $hM_v \geq_u hM'_v$). Let hM and hM' be the outputs received by the proposal-rejection process after the arrival of v, respectively. Then u gets at least as good partner in hM as in hM' (so $hM \geq_u hM'$).*

Proof Indirectly, assume that u gets a better partner in hM' than in hM, so $hM <_u hM'$. This implies $hM_v >_u hM$ or $hM'_v <_u hM'$. In the first case u gets a worse partner by the proposal-rejection process, so by the Theorem 9 u gets his best stable partner in hM, a contradiction. Similarly, in the second case u gets a better partner by the proposal-rejection process, so by Theorem 9 u gets his worst stable partner in hM', a contradiction. □

Lemma 15 *Let hM_v and hM'_v be two half-matchings for $G - v$ such that u is in the same situation in hM_v as in hM'_v, so u gets the same partner or u is unmatched or a cycle-agent. Let hM and hM' be the outputs received by the proposal-rejection process after the arrival of v, respectively. Then u is in the same situation in hM' as in hM, so u gets the same partner if he is matched for G.*

Proof If u is matched for $G - v$, then Theorem 9 preserves the above property. If u is unmatched or a cycle agent for $G - v$, then the statement is an easy consequence of the points 6 and 7 from the proof of Theorem 12, respectively. □

Theorem 16 *Let in the incremental algorithm two arrival orders O and O' differ only in one agent v in such a way that v arrives later in O. Let hM and hM' be the outputs of the algorithm realized with the orders O and O', respectively. If v is a matched agent, then he gets at least as good partner in hM as in hM'.*

Proof Consider the market at the moment when v arrives according to O. Now, the same agents are present in the market according both arrival orders. Since v is the lastcomer according O, Theorem 7 implies that after the proposal-rejection processes v cannot be better off in the stable solution according to O'. Afterwards, during the incremental algorithm, the same agents enter the market in each phase, so by the above Lemmas 14 and 15 it follows, that v cannot be better off in the stable solutions according to O' anymore. □

Let us define two relations, B^* and W^* between the agents in the following way: we denote by uB^*v if agent v occurs as an agent that makes a proposal in $S(hM_u)$ (the proposal-rejection sequence according to the core configuration relative to u). As an easy consequence of Theorem 12, uB^*v implies that v gets his best stable partner if u enters the market last, moreover the same Theorem says, that v gets an even better partner if u does not enter the market at all. So v can only benefit if u is out of the market, or he arrives as late as possible. That is why we may regard u as a *nightmare-agent* of v.

Similarly, we denote by uW^*v if agent v occurs as an agent that receives a proposal in $S(hM_u)$. Here, uW^*v implies that v gets his worst stable partner if u enters the market last, but v gets an even worse partner if u is not present in the market. So v can only benefit if u is in the market, and he arrives as soon as possible. We call u as a *dream-agent* of v in this case.

Obviously, neither B^* nor W^* is symmetric. Moreover, the following Lemma proves, that uB^*v implies that vB^*u cannot be true, so the relation B^* is antisymmetric.

Lemma 17 *If uB^*v, then $S(hM_v)$ is the restriction of $S(hM_u)$.*

The proof is trivial from the proof of Theorem 12, so it is omitted.

Corollary 18 *The relation B^* is transitive, so uB^*v and vB^*w imply uB^*w. Moreover, uB^*v and vW^*w imply uW^*w.*

To show, that the relation B^* is not a linear order, one can easily find an example, where uB^*w and vB^*w, but there is no B^* relation between u and v. We believe that there should be some further relevant questions to consider about these relations.

The increasing side gets worse off

Finally, we give alternative proofs for some special results that have so far been known only for two-sided matching markets. Lemma 19 is a straightforward consequence of Theorem 2 in Gale and Sotomayor (1985).

Lemma 19 *If a man enters the market then no man can have better partner in the new men-optimal stable matching than in the former men-optimal stable matching.*

Proof Let m be the man that enters the market last. We shall prove that if a man m' gets w' in the men-optimal stable matching M^M, then m' cannot have a worse partner in the men-optimal stable matching M_m^M for $G-m$. If m is unmatched in M^M, then M^M

is also stable for $G - m$. If $\{m, w\} \in M^M$, then $M^M \setminus \{m, w\}$ is stable for $G - \{m, w\}$. After w reenters the market, during the proposal-rejection process m' either remains matched with w' or receives a proposal from a better woman for him. □

Theorem 20 *If some men enter the market one after another then at the end of the proposal-rejection process they all get their best stable partners in the resulting stable matching.*

Proof Suppose that a man m' is matched with his best stable partner w' before a new man, m enters. If m' remains matched with w' in the new obtained matching, then by Lemma 19 w' is still his best stable partner. If m' gets a new partner during the phase, then he must receive her by making a proposal, so Theorem 7 proves that m' gets his best stable partner again. □

The following theorem of Blum et al. (1997) can be proved in the same way by using Theorem 9.

Theorem 21 *If some men enter the market then any other man m either remains matched with his original partner w if w is still a stable partner for m or m receives his best stable partner in the output.* □

If the arrival order is such that women enter the market first and men follow after that, then the output will be the same as the output of the deferred-acceptance algorithm with men proposing by Gale and Shapley (1962). Theorem 20 shows alternatively, that the received stable matching is optimal for the men.

Conclusion and further questions

We have studied matching markets, where agents enter and leave one after another, and they are able to terminate and build new partnerships without restrictions. By this assumption, a new stable state is created for the market by a natural decentralized process if such an equilibrium exists. For a two-sided market a new stable matching, for a general market a new stable half-matching can always be obtained this way.

The main lesson of our study is that an agent can benefit if he enters the market as late as possible. This fact may encourage an agent to leave the market and enter again with the hope of getting a better partner. We can avoid this kind of instability if and only if the stable solution is unique.

Accepting a proposal always means an improvement for the agent. Moreover, among the agents that accept proposals during the process, some are strictly better off under any stable solution for the new market than at any stable solution for the former one. Finally, if in a two-sided market the number of men increases then the best stable partner for each man gets worse.

To generalize these results further, it is reasonable to consider the cases, where an agent can be matched with more than one partners. Cantala (2004) studied many-to-one matching markets under *q-substitutable* preferences, Kojima and Ünver (2007) considered many-to-many matching markets under *categorywise-responsive* preferences. Cantala used the idea of Blum et al. (1997) to analyse the restabilizing mechanism of that market, Kojima and Ünver proved that a pairwise stable matching can be obtained by successively satisfying blocking edges by an algorithm similar to the one of Roth and Vande Vate (1990). A natural question is the study of nonbipartite versions of these dynamic matching markets.

In this paper we study the automatism of the dynamic matching market, where the processes are predicted by the preferences of the agents and their arrival order. But we do not consider the strategic issues, i.e. whether an agent can benefit by not acting according his true preference. This is a relevant question that can become the subject of a further research.

Acknowledgement We thank Elena Molis for pointing out some mistaces of an earlier draft. We thank also the anonymous referee for his careful reading and constructive suggestions.

References

Abeledo H, Rothblum UG (1995) Paths to marriage stability. Discret Appl Math 63:1–12
Abraham DJ, Biró P, Manlove DF (2006) "Almost stable" matchings in the roommates problem. In: Proceedings of WAOA 2005: the 3rd workshop on approximation and online algorithms. Lecture notes in computer science, vol 3879, pp 1–14
Aharoni R, Fleiner T (2003) On a lemma of Scarf. J Comb Theory Ser B 87:72–80
Angelov N (2006) Modelling firm mergers as a roommates problem (preprint)
Blum Y, Rothblum UG (2002) "Timing is everything" and marital bliss. J Econ Theory 103:429-443
Blum Y, Roth AE, Rothblum UG (1997) Vacancy chains and equilibration in senior-level labor markets. J Econ Theory 76:362–411
Cantala D (2004) Restabilizing matching markets at senior level. Games Econ Behav 48:1–17
Cechlárová K (2002) Randomized matching mechanism revisited (preprint)
Cechlárová K, Fleiner T (2005) On a generalization of the stable roommates problem. ACM Trans Algorithms 1(1):143–156
Diamantoudi E, Miyagawa E, Xue L (2004) Random paths to stability in the roommates problem. Games Econ Behav 48:18–28
Eriksson K, Strimling P (2005) How unstable are matchings from the decentralized mate search? (preprint)
Gale D, Shapley LS (1962) College admissions and stability of marriage. Am Math Mon 69:9–15
Gale D, Sotomayor M (1985) Some remarks on the stable matching problem. Discret Appl Math 11:223–232
Gusfield D, Irving RW (1990) The stable marriage problem: structure and algorithms. MIT, Cambridge
Inarra E, Larrea C, Molis E (2007) Random paths to P-stability in the roommate problem. Int J Game Theory (forthcoming)
Irving RW (1985) An efficient algorithm for the "stable roommates" problem. J Algorithms 6:577–595
Jackson MO, Watts A (2002) The evolution of social and economic networks. J Econ Theory 106:265–295
Knuth DE (1976) Mariages stables. Les Presses de l'Universite de Montreal, Montreal
Kojima F, Ünver U (2007) Random paths to pairwise stability in many-to-many matching problems: a study on market equilibration. Int J Game Theory (forthcoming)
Ma J (1996) On randomised matching mechanism. Econ Theory 8(2):377–381
McVitie D, Wilson LB (1970) Stable marriage assignment for unequal sets. BIT 10:295–309
Pittel BG, Irving RW (1994) An upper bound for the solvability of a random stable roommates instance. Random Struct Algorithms 5:465–486
Roth AE, Sotomayor M (1990) Two-sided matching: a study in game-theoretic modeling and analysis. In: Econometric society monograph series. Cambridge University Press, London
Roth AE, Vande Vate JH (1990) Random paths to stability in two-sided matching. Econometrica 58:1475–1480
Roth AE, Sönmez T, Ünver U (2005) Pairwise kidney exchange. J Econ Theory 125(2):151–188
Scarf HE (1967) The core of an N person game. Econometrica 35:50–69
Tan JJM (1991) A necessary and sufficient condition for the existence of a complete stable matching. J Algorithms 12:154–178
Tan JJM, Hsueh JC (1995) A generalisation of the stable matching problem. Discret Appl Math 59:87–102
Yuan Y (1996) Residence exchange wanted: a stable residence exchange problem. Eur J Oper Res 90:536–546