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# The Shapley value of exact assignment games

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**Abstract** We prove that the Shapley value of every two-sided exact assignment game lies in the core of the game.

## 1 Introduction

The Shapley value and the core are widely accepted and frequently applied solutions for cooperative transferable utility games. The Shapley value of a coalitional game is a singleton and it may be interpreted as an a priori evaluation of the game. Each element of the core of a coalitional game is stable in the sense that no coalition can improve upon this element. In view of these interpretations (see Sect. 2 for the formal definitions) it is not surprising that the Shapley value may not select a core element, even if the core is nonempty.

Indeed, for asymmetric glove games, the Shapley value does not select the unique element of the core (the unique competitive allocation). Though there exists a convergence theorem (see Shapley and Shubik 1969), our approach is rather inspired by the well-known fact that the Shapley value selects an element of the core of any convex game. In general, the Shapley value is the barycenter of the marginal contribution vectors. Now, convex games are characterized by the property that all marginal contribution vectors are elements of the core (see Shapley 1971; Ichiishi 1981).

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Within the present paper, we consider (two-sided) assignment games as introduced by Shapley and Shubik (1972). As glove games are assignment games, further conditions are needed in order to guarantee that the Shapley value is in the core of an assignment game. Two properties of a convex game are interesting: (1) the core is its unique von Neumann–Morgenstern stable set. (2) It is exact. Solymosi and Raghavan (2001) showed that every exact assignment game has a stable core. We prove that the Shapley value of an exact assignment game is in its core. Moreover, by means of an example, we show that the Shapley value may not be a member of the core of an assignment game with a stable core. For a discussion of the relation of some other solutions and the core of assignment games see Raghavan and Sudhölter (2005).

We now briefly review the contents of the paper. Section 2 recalls the definitions of the Shapley value, of assignment games, of exact games, and of the core. In Sect. 3 the main theorem, saying that the Shapley value of an assignment game is in its core, is formulated. We explain why it is possible in our context to directly apply Theorem 2 of Solymosi and Raghavan (2001) that describes simple properties of a square matrix that are necessary and sufficient to generate an exact assignment game. We use this result to derive a useful further characterization of exactness. Moreover, we present the aforementioned example, show that exactness of an assignment game is not necessary for core membership of the Shapley value, and characterize the $2 \times 2$ matrices that define assignment games whose Shapley value is a core member. Section 4 is devoted to the proof of the main result. Finally, in Sect. 5 we present an example showing that the main theorem cannot be generalized to arbitrary exact games with large cores.

## 2 Notation and definitions

A (cooperative TU) *game* is a pair $(N, v)$ such that $\emptyset \neq N$ is finite and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. Let $(N, v)$ be a game and $n = |N|$. Then the *core* of $(N, v)$, $\mathcal{C}(N, v)$, is given by

$$\mathcal{C}(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \quad \text{and} \quad x(S) \geq v(S)\ \forall S \subseteq N\},$$

where $x(S) = \sum_{i \in S} x_i$ $(x(\emptyset) = 0)$ for every $S \in 2^N$ and every $x \in \mathbb{R}^N$. In order to recall the definition of the Shapley value (see Shapley 1953), for each $s = 0, \ldots, n-1$, define $\gamma(s) = \frac{s!(n-s-1)!}{n!}$. Then the *Shapley value* of $(N, v)$, $\phi(N, v)$, is given by

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \gamma(s)(v(S \cup \{i\}) - v(S)) \quad \text{for all } i \in N, \tag{2.1}$$

where $s = |S|$. Note that

$$\sum_{S \subseteq N \setminus \{i\}} \gamma(s) = 1. \tag{2.2}$$

The game $(N, v)$ is *exact* if for any $S \subseteq N$ there exists $x \in \mathcal{C}(N, v)$ with $v(S) = x(S)$. Exact games were introduced by Shapley (1971) (see also Schmeidler 1972).

In order to recall the definition of an assignment game, we say that, for finite sets $S$ and $T$, an *assignment* of $(S, T)$ is a bijection $b : S' \to T'$ such that $S' \subseteq S$, $T' \subseteq T$, and $|S'| = |T'| = \min\{|S|, |T|\}$. We shall identify $b$ with $\{(i, b(i)) \mid i \in S'\}$. Let $\mathcal{B}(S, T)$ denote the set of assignments. A game $(N, v)$ is an *assignment game* if there exist a partition $\{P, Q\}$ of $N$ and a nonnegative real matrix $A = (a_{ij})_{i \in P, j \in Q}$ such that

$$v(S) = \max_{b \in \mathcal{B}(S \cap P, S \cap Q)} \sum_{(i,j) \in b} a_{ij} \quad \text{for all } S \subseteq N. \tag{2.3}$$

Conversely, if $P$ and $Q$ are disjoint finite nonempty sets and if $A = (a_{ij})_{i \in P, j \in Q}$ is a nonnegative real matrix, then let $(P \cup Q, v_A)$ be the assignment game *associated* with $A$, that is, $v = v_A$ is defined by (2.3). We conclude this section by recalling a result of Shapley and Shubik (1972). Let $N = P \cup Q$. Then

$$\mathcal{C}(N, v_A) = \{x \in \mathbb{R}^N_+ \mid x(N) = v_A(N) \quad \text{and} \quad x_i + x_j \geq a_{ij} \; \forall \, (i, j) \in P \times Q\}. \tag{2.4}$$

## 3 Core membership of the Shapley value

Our main result is the following theorem.

**Theorem 3.1** *If $(N, v)$ is an exact assignment game, then $\phi(N, v) \in \mathcal{C}(N, v)$.*

We postpone the proof of the foregoing theorem to Sect. 4. This section serves to prepare the proof and to show that if exactness is relaxed, then the statement of our result is no longer valid. For assignment games, exactness is not a necessary condition for core membership of the Shapley value (see Example 3.4). However, if just four-person assignment games are considered, a simple necessary and sufficient condition on the assignment matrix exists (see Proposition 3.5).

Let $(N, v)$ be an exact assignment game and let $P$ and $Q$ be finite nonempty sets, and let $A$ be a nonnegative real $P \times Q$ matrix such that $N = P \cup Q$ and $v = v_A$. In order to prove that $\phi(N, v) \in \mathcal{C}(N, v)$, we may always assume that $A$ is a *square* matrix, that is, $p = |P| = |Q|$. Indeed, we may add zero columns or zero rows if necessary, because the Shapley value and the core both satisfy the strong null player property[1]. Also, by anonymity of the Shapley value and the core, we shall assume that

$$P = \{1, \ldots, p\}, \quad Q = \{1', \ldots, p'\}, \quad \text{and} \quad v(N) = \sum_{i \in P} a_{ii'}. \tag{3.1}$$

[1] A solution satisfies the *null player property*, if any element of the solution to a game assigns zero to any null player of the game. It satisfies the *strong null player property* if, additionally, the solution to a game $(N \cup \{k\}, w)$ that arises from $(N, v)$ by adding the null player $k$, arises from the solution to $(N, v)$ by adding a zero coordinate for the null player $k$ to any element of the solution to $(N, v)$.

If $S \subseteq N$, then we define

$$S^* = \{i' \in Q \mid i \in S\} \cup \{i \in P \mid i' \in S\}. \tag{3.2}$$

So, $S^*$ is the set of all partners of players in $S$ according to the optimal diagonal assignment of $N$. Note that $|S^*| = |S|$ and that $(S^*)^* = S$ for all $S \subseteq N$. In order to prove the following useful lemma, we first recall the following necessary and sufficient condition for the exactness of an assignment game. The matrix $A$ has a

- *dominant diagonal* if $a_{ii'} \geq a_{ij'}$ and $a_{ii'} \geq a_{ji'}$ for all $i, j \in P$;
- *doubly dominant diagonal* if $a_{ii'} + a_{jk'} \geq a_{ik'} + a_{ji'}$ for all $i, j, k \in P$.

**Theorem 3.2** (Solymosi and Raghavan 2001) *The assignment game associated with A is exact if and only if A has a dominant and a doubly dominant diagonal.*

Moreover, Solymosi and Raghavan (2001) show in their Theorem 1 that an assignment game associated with $A$ has a stable core if and only if $A$ has a dominant diagonal. The following example shows that Theorem 3.1 fails if the condition of exactness is replaced by core stability.

*Example 3.3* Let $(N, v)$ be the assignment game associated with the matrix

$$A = \begin{pmatrix} 8 & 4 & 8 \\ 4 & 4 & 1 \\ 8 & 1 & 8 \end{pmatrix}.$$

As $A$ has a dominant diagonal, the core of $(N, v)$ is stable. Also, $(N, v)$ is not exact, because $a_{11'} + a_{23'} = 8 + 1 < 8 + 4 = a_{13'} + a_{21'}$. Applying (2.1) yields $\phi(N, v) = \frac{1}{20}(82, 39, 79, 82, 39, 79)$, where the first three components refer to the "row players". As $\frac{1}{20}(39 + 39) < 4$, $\phi(N, v) \notin \mathcal{C}(N, v)$.

Note that the foregoing example is minimal in the following sense. For every assignment game with a stable core and with less than six players, the Shapley value is in the core. Indeed, if there are five or less players, then we may assume that $|P| \leq |Q|$, that is $|P| \leq 2$. By adding zero rows, if necessary, and renaming the players in such a way that the diagonal is an optimal matching, we receive a square matrix with at most 2 nonzero rows. So the matrix may only have a dominant diagonal if it has at most 2 nonzero columns as well. However, in this case the diagonal is automatically doubly dominant, because every $2 \times 2$ matrix has a doubly dominant diagonal.

The following example shows that there exists a non-exact *transitive* assignment game (a game whose symmetry group is transitive) with a stable core, whose Shapley value is in its core.

*Example 3.4* Let

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix}$$

and $v = v_A$. Note that $(N, v)$ is not exact, because $a_{11'} + a_{32'} = 2 < 4 = a_{12'} + a_{31'}$. Also, if $\pi$ is the cyclic permutation defined by $1 \mapsto 1' \mapsto 3 \mapsto 3' \mapsto 2 \mapsto 2' \mapsto 1$, then $v(\pi(S)) = v(S)$ for all $S \subseteq N$. So, applying the $k$-fold composition $\pi^k$, $k = 0, \ldots, 5$, to $N$ leaves the game unchanged and transitivity of its symmetry group is shown. As the Shapley value is anonymous and Pareto optimal, we conclude that $\phi(N, v) = (1, 1, 1, 1, 1, 1)$. By (2.4), $\phi(N, v) \in \mathcal{C}(N, v)$.

In order to present a characterization of the set of $2 \times 2$ matrices such that $v = v_A$ and $\phi(N, v) \in \mathcal{C}(N, v)$, it is useful to check the following property of the Shapley value of assignment games. Let $v = v_A$ be an assignment game associated with the $P \times Q$ matrix $A$ such that (3.1) is valid. Then for any $t \geq \max_{(i,j') \in P \times Q} \left(-a_{ij'}\right)$ we claim that, with $A + t = (a_{ij'} + t)_{i,j \in P}$,

$$\phi(N, v_{A+t}) = \frac{t}{2} + \phi(N, v_A). \tag{3.3}$$

To see this, let $w = v_{A+t}$ and observe that $w(S) = v(S) + t \min\{|S \cap P|, |S \cap Q|\}$ for any $S \subseteq N$. By $p = q$, we may conclude that, for each $i \in N$ and $S \subseteq N \setminus \{i\}$,

$$\begin{aligned} &\Big(w(S \cup \{i\}) - w(S)\Big) + \Big(w(N \setminus S) - w(N \setminus (S \cup \{i\}))\Big) \\ &\quad = t + \Big(v(S \cup \{i\}) - v(S)\Big) + \Big(v(N \setminus S) - v(N \setminus (S \cup \{i\}))\Big). \end{aligned}$$

As $\gamma(|S|) = \gamma(|N| - |S| - 1)$, our claim follows directly from (2.1).

Now we are able to characterize the set of $2 \times 2$ matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $v = v_A$ is a non-exact assignment game and $x = \phi(N, v) \in \mathcal{C}(N, v)$. By (3.1), Theorem 3.2, and anonymity we may assume that

$$b > a \quad \text{and} \quad c < d. \tag{3.4}$$

As $x \in \mathcal{C}(N, v)$, (2.4) implies that $x_1 + x_{1'} = a$. By (2.1) we may conclude that $c < a$ (otherwise $x_1 + x_{1'} > a$)[2]. Further computations show that $a = d$. Finally,

[2] Indeed, if $c \geq a$, then by (3.3), $y = x - \frac{a}{2}$ is the Shapley value of $v_{A-a}$ where $A - a = \begin{pmatrix} 0 & b-a \\ c-a & d-a \end{pmatrix}$. As $b > a$, (2.1) implies that $y_1 > 0$. Also, $y_{1'} \geq 0$ so that $x_1 + x_{1'} > a$, which is excluded for core elements.

applying the inequality $x_1 + x_{2'} \geq b$ yields $4a \geq 3b + c$. In view of Theorem 3.1 we have derived the following result.

**Proposition 3.5** *Let $p = 2$ and let $A = \left(a_{ij'}\right)_{i,j \in P}$ be nonnegative such that (3.1) is valid with $v = v_A$. Put $a = \min\{a_{11'}, a_{22'}\}, b = \max\{a_{12'}, a_{21'}\}$, and $c = \min\{a_{12'}, a_{21'}\}$. Then*

$$\phi(N, v) \in \mathcal{C}(N, v) \Leftrightarrow (a \geq b \text{ or } (a_{11'} = a_{22'} \text{ and } 4a \geq 3b + c)).$$

Now, we continue with our preparations of the proof of Theorem 3.1. Note that, if $(N, v_A)$ is the assignment associated with the $P \times Q$ matrix $A$ such that (3.1) is satisfied, then, by (2.3) and (3.1),

$$v_A(S \cap S^*) = \sum_{i \in P \cap S \cap S^*} a_{ii'} \quad \text{for all } S \subseteq N. \tag{3.5}$$

**Lemma 3.6** *The assignment game $(N, v)$ associated with $A$ is exact if and only if*

$$v(S) = v(S \cap S^*) + v(S \setminus S^*) \quad \text{for all } S \subseteq N. \tag{3.6}$$

*Proof* Assume that (3.6) is satisfied. Let $i \in P$ and $j, k \in P \setminus \{i\}$. By (2.3),

$$\begin{aligned} a_{ii'} + a_{jk'} &= v(\{i, j, i', k'\}) \geq a_{ik'} + a_{ji'}, \\ a_{ii'} &= v(\{i, j, i'\}) \geq a_{ji'}, \quad \text{and} \quad a_{ii'} = v(\{i, i', k'\}) \geq a_{ik'}. \end{aligned}$$

Hence, $A$ has a doubly dominant and a dominant diagonal.

Now assume that $A$ has a dominant and doubly dominant diagonal and let $S \subseteq N$ and let $T = S \cap S^*$. We shall prove (3.6) by induction on $|T| = t$. Indeed, if $t = 0$, then $T = \emptyset$ and (3.6) is true. Assume now that $t > 0$. Let $b$ be an optimal assignment for $S$, that is, $b \in \mathcal{B}(S \cap P, S \cap Q)$ and $v(S) = \sum_{(i,j') \in b} a_{ij'}$. Let $i \in T \cap P$. By the inductive hypothesis and (3.5) it remains to show that

$$v(S) \leq a_{ii'} + v(S \setminus \{i, i'\}).$$

If $(i, i') \in b$, then the proof is complete. Hence, we may assume that $(i, i') \notin b$. The following three cases may occur.

(1) There exists $j' \in S \cap Q, j \neq i$, such that $(i, j') \in b$ and $i'$ is not matched in $b$, that is, $\{(j, i') \mid j \in P\} \cap b = \emptyset$. As A has a dominant diagonal,

$$v(S) = v(S \setminus \{i, i', j'\}) + a_{ij'} \leq v(S \setminus \{i, i', j'\}) + a_{ii'} \leq v(S \setminus \{i, i'\}) + a_{ii'}.$$

(2) The case that there exists $j \in S \cap P, j \neq i$, such that $(j, i') \in b$ and $i$ is not matched in $b$, that is, $\{(i, j') \mid j \in P\} \cap b = \emptyset$, may be treated analogously to (1).

(3) If $(i, k'), (j, i') \in b$ for some $j, k \in P \setminus \{i\}$, then

$$\begin{aligned} v(S) &= v(S \setminus \{i, i', j, k'\}) + a_{ik'} + a_{ji'} \\ &\leq v(S \setminus \{i, i', j, k'\}) + a_{ii'} + a_{jk'} \leq v(S \setminus \{i, i'\}) + a_{ii'}, \end{aligned}$$

because A has a doubly dominant diagonal. □

## 4 Proof of Theorem 3.1

Let $(N, v)$ be the assignment game associated with the $P \times Q$ matrix $A$ such that (3.1) is satisfied. Let $x = \phi(N, v)$. By (2.1), $x \geq 0$. In view of (2.4) and (3.1) it suffices to show that

$$x_i + x_{i'} = a_{ii'} \quad \text{for all } i \in P, \tag{4.1}$$
$$x_i + x_{j'} \geq a_{ij'} \quad \text{for all } i, j \in P. \tag{4.2}$$

Note that (4.1) implies the well-known Pareto efficiency of $x$, that is, $x(N) = v(N)$. Let $i \in P$.

**Step 1** We shall show that $x_i + x_{i'} = a_{ii'}$. Let $n = |N|$ and note that, by the definition of $\gamma(\cdot)$,

$$\gamma(s) = \gamma(n - s - 1) \quad \text{for all} \quad s = 0, \ldots, n - 1. \tag{4.3}$$

By (4.3) and (2.1) it suffices to find a bijection $f : 2^{N \setminus \{i\}} \to 2^{N \setminus \{i'\}}$ such that, for all $S \subseteq N \setminus \{i\}$,

$$n - 1 = |S| + |f(S)|, \tag{4.4}$$
$$a_{ii'} = (v(S \cup \{i\}) - v(S)) + (v(f(S) \cup \{i'\}) - v(f(S))). \tag{4.5}$$

For each $S \subseteq N \setminus \{i\}$ let $f(S) = N \setminus (S^* \cup \{i'\})$ (see (3.2) for the definition of $S^*$). We shall now verify that $f$ has the desired properties. Let $S \subseteq N \setminus \{i\}$. By definition $i' \notin f(S)$ and

$$f(S)^* = N \setminus (S \cup \{i\}). \tag{4.6}$$

Hence, $S = N \setminus (f(S)^* \cup \{i\})$. So, $f$ is a bijection with the desired range. As $|S| = |S^*|$ and $i' \notin S^*$, (4.4) is satisfied. In order to show (4.5), we distinguish two cases.

(1) $i' \in S$ (that is, $i \notin f(S)$). By Lemma 3.6 and (3.5),

$$v(S \cup \{i\}) - v(S) = a_{ii'} + v((S \setminus \{i'\}) \setminus S^*) - v(S \setminus S^*),$$
$$v(f(S) \cup \{i'\}) - v(f(S)) = v((f(S) \cup \{i'\}) \setminus f(S)^*) - v(f(S) \setminus f(S)^*).$$

Now, (4.5) follows, because, by (4.6),

$$f(S)\setminus f(S)^* = (S\cup\{i\})\setminus(S^*\cup\{i'\}) \underbrace{=}_{i\in S^*} (S\setminus\{i'\})\setminus S^*,$$
$$(f(S)\cup\{i'\})\setminus f(S)^* = (S\cup\{i\})\setminus S^* \overbrace{=} S\setminus S^*.$$

(2) In the case that $i'\notin S$ (that is, $i\in f(S)$) we may proceed analogously. Just the roles of $S$ and $f(S)$ and of $i$ and $i'$ have to be exchanged.

**Step 2** Let $j\in P$. We shall now show that $x_i+x_{j'}\geq a_{ij'}$. By the first step we may assume that $j\neq i$. By (2.2) and (4.3),

$$1=\sum_{S\subseteq N\setminus\{i\}}\gamma(|S|)=\sum_{S\subseteq N\setminus\{i,i'\}}\Big(\gamma(|S|)+\gamma(|N\setminus(S\cup\{i\})|)\Big)=2\sum_{S\subseteq N\setminus\{i,i'\}}\gamma(|S|). \tag{4.7}$$

Let $\mathcal{R}=2^{N\setminus\{i,i',j,j'\}}$. By (4.3) applied to any $R\cup\{i'\}$, $R\in\mathcal{R}$, and by (4.7),

$$\begin{aligned}1&=2\sum_{R\in\mathcal{R}}\big(\gamma(|R|)+\gamma(|R\cup\{i'\}|)+\gamma(|R\cup\{j\}|)+\gamma(|R\cup\{j'\}|)\big)\\&=\sum_{R\in\mathcal{R}}(2\gamma(|R|)+6\gamma(|R|+1)).\end{aligned}\tag{4.8}$$

For $R\subseteq N$ let $r=|R|$. By (4.8) it suffices to show that, for all $R\in\mathcal{R}$, there are $c(R)$ and $d(R)$ such that

$$x_i+x_{j'}=\textstyle\sum_{R\in\mathcal{R}}(c(R)\gamma(r)+d(R)\gamma(r+1)), \tag{4.9}$$
$$c(R)\geq 2a_{ij'}\quad\text{and}\quad d(R)\geq 6a_{ij'}. \tag{4.10}$$

By (2.1) and using $s=|S|$,

$$\begin{aligned}x_i+x_{j'}=\sum_{S\subseteq N\setminus\{i,j'\}}&\Big(g(s)\left(v(S\cup\{i\})+v(S\cup\{j'\})-2v(S)\right)\\&+\gamma(s+1)\left(2v(S\cup\{i,j'\})-v(S\cup\{j'\})-v(S\cup\{i\})\right)\Big).\end{aligned}\tag{4.11}$$

To every $R\in\mathcal{R}$ we assign three coalitions defined by

$$f_j(R)=(N\setminus R^*)\setminus\{j\},\quad f_{i'}(R)=(N\setminus R^*)\setminus\{i'\},\quad f_{ij'}(R)=(N\setminus R^*)\setminus\{i,j'\}.$$

Note that $f_j(\cdot), f_{i'}(\cdot)$, and $f_{ij'}(\cdot)$ are injective mappings on $\mathcal{R}$. Observe that for any $S\subseteq N\setminus\{i,j'\}$ there exists a unique $R\in\mathcal{R}$ such that $S$ coincides with one of the sets $R, R\cup\{j\}, R\cup\{i'\}$, and $f_{ij'}(R)$. Similarly, note that there exists a unique $R\in\mathcal{R}$ such that $S\cup\{i,j'\}$ coincides with one of the sets $N\setminus R^*, f_j(R), f_{i'}(R)$, and

$R \cup \{i, j'\}$. Let $R \in \mathcal{R}$ and

$$\begin{aligned} c(R) &= v(R \cup \{i\}) + v(R \cup \{j'\}) - 2v(R) \\ &\quad + 2v(N \setminus R^*) - v((N \setminus R^*) \setminus \{i\}) - v((N \setminus R^*) \setminus \{j'\}), \end{aligned}$$

let

$$\begin{aligned} d_j(R) &= v(R \cup \{i,j\}) + v(R \cup \{j,j'\}) - 2v(R \cup \{j\}) \\ &\quad + 2v(f_j(R)) - v(f_j(R) \setminus \{i\}) - v(f_j(R) \setminus \{j'\}), \\ d_{i'}(R) &= v(R \cup \{i,i'\}) + v(R \cup \{i',j'\}) - 2v(R \cup \{i'\}) \\ &\quad + 2v(f_{i'}(R)) - v(f_{i'}(R) \setminus \{i\}) - v(f_{i'}(R) \setminus \{j'\}), \\ d_{ij'}(R) &= v(f_{ij'}(R) \cup \{i\}) + v(f_{ij'}(R) \cup \{j'\}) - 2v(f_{ij'}(R)) \\ &\quad + 2v(R \cup \{i,j'\}) - v(R \cup \{j'\}) - v(R \cup \{i\}), \end{aligned}$$

and put $d(R) = d_j(R) + d_{i'}(R) + d_{ij'}(R)$. Thus, (4.9) is implied by (4.11) and (4.3). Let $\widetilde{R} = R \setminus R^*$. Lemma 3.6 together with (3.5) yield

$$\begin{aligned} c(R) &= a_{ii'} + a_{jj'} + v(\widetilde{R} \cup \{i\}) - v(\widetilde{R} \cup \{i'\}) - v(\widetilde{R} \cup \{j\}) + v(\widetilde{R} \cup \{j'\}), \\ d_j(R) &= a_{ii'} + a_{jj'} - 2v(\widetilde{R} \cup \{j\}) + 2v(\widetilde{R} \cup \{j'\}) + v(\widetilde{R} \cup \{i,j\}) - v(\widetilde{R} \cup \{i',j'\}), \\ d_{i'}(R) &= a_{ii'} + a_{jj'} + 2v(\widetilde{R} \cup \{i\}) - 2v(\widetilde{R} \cup \{i'\}) - v(\widetilde{R} \cup \{i,j\}) + v(\widetilde{R} \cup \{i',j'\}), \\ d_{ij'}(R) &= a_{ii'} + a_{jj'} - v(\widetilde{R} \cup \{i\}) + v(\widetilde{R} \cup \{i'\}) + v(\widetilde{R} \cup \{j\}) - v(\widetilde{R} \cup \{j'\}) \\ &\quad + 2v(\widetilde{R} \cup \{i,j'\}) - 2v(\widetilde{R} \cup \{i',j\}). \end{aligned}$$

We conclude that

$$\begin{aligned} d(R) &= 3a_{ii'} + 3a_{jj'} + v(\widetilde{R} \cup \{i\}) - v(\widetilde{R} \cup \{i'\}) - v(\widetilde{R} \cup \{j\}) + v(\widetilde{R} \cup \{j'\}) \\ &\quad + 2v(\widetilde{R} \cup \{i,j'\}) - 2v(\widetilde{R} \cup \{i',j\}). \end{aligned}$$

Therefore, in order to verify (4.10) it suffices to prove that

$$a_{ij'} \le a_{ii'} + v(\widetilde{R} \cup \{j'\}) - v(\widetilde{R} \cup \{i'\}), \tag{4.12}$$

$$a_{ij'} \le a_{jj'} + v(\widetilde{R} \cup \{i\}) - v(\widetilde{R} \cup \{j\}), \tag{4.13}$$

$$2a_{ij'} \le a_{ii'} + a_{jj'} + v(\widetilde{R} \cup \{i,j'\}) - v(\widetilde{R} \cup \{j,i'\}). \tag{4.14}$$

In order to show (4.12) we distinguish two cases. If $i'$ is not matched in an optimal assignment of $\widetilde{R} \cup \{i'\}$, that is, if $v(\widetilde{R} \cup \{i'\}) = v(\widetilde{R})$, then the desired inequality is immediately implied, because $A$ has a dominant diagonal. If $i'$ is matched to some $k \in P \cap \widetilde{R}$ in an optimal assignment, that is, if $v(\widetilde{R} \cup \{i'\}) = a_{ki'} + v(\widetilde{R} \setminus \{k\})$, then $v(\widetilde{R} \cup \{j'\}) \ge a_{kj'} + v(\widetilde{R} \setminus \{k\})$ implies that

$$a_{ii'} + v(\widetilde{R} \cup \{j'\}) - v(\widetilde{R} \cup \{i'\}) \ge a_{ii'} + a_{kj'} - a_{ki'}$$

and, so (4.12) is valid, because $A$ has a doubly dominant diagonal. In a completely analogous way we may show (4.13).

In order to prove (4.14), put

$$\beta = a_{ii'} + a_{jj'} + v(\widetilde{R} \cup \{i, j'\}) - v(\widetilde{R} \cup \{j, i'\}).$$

Let $b \in \mathcal{B}(P \cap (\widetilde{R} \cup \{j\}), Q \cap (\widetilde{R} \cup \{i'\}))$ be an optimal assignment for $\widetilde{R} \cup \{j, i'\}$ (see (2.3)). Four cases may occur.

*Case 1* $(j, i') \in b$. So, $v(\widetilde{R} \cup \{j, i'\}) = v(\widetilde{R}) + a_{ji'}$. As $v(\widetilde{R} \cup \{i, j'\}) \geq v(\widetilde{R}) + a_{ij'}$, we may conclude that $\beta \geq a_{ii'} + a_{jj'} + a_{ij'} - a_{ji'} \geq 2a_{ij'}$, because $A$ has a dominant diagonal.

*Case 2* There exists some $\ell \in P \cap \widetilde{R}$ such that $(\ell, i') \in b$ and $j$ is not matched in $b$. Then $v(\widetilde{R} \cup \{j, i'\}) = a_{\ell i'} + v(\widetilde{R} \setminus \{\ell\})$ and $v(\widetilde{R} \cup \{i, j'\}) \geq a_{\ell j'} + v(\widetilde{R} \setminus \{\ell\})$. So,

$$\beta \geq a_{ii'} + a_{jj'} + a_{\ell j'} - a_{\ell i'} \geq a_{jj'} + a_{ij'} + a_{\ell i'} - a_{\ell i'} \geq 2a_{ij'},$$

where the second inequality is true, because $A$ has a doubly dominant diagonal, and the third inequality holds, because $A$ has a dominant diagonal.

*Case 3* There exists $k' \in Q \cap \widetilde{R}$ such that $(j, k') \in b$ and $i'$ is not matched in $b$. Then $v(\widetilde{R} \cup \{j, i'\}) = a_{jk'} + v(\widetilde{R} \setminus \{k'\})$, $v(\widetilde{R} \cup \{i, j'\}) \geq a_{ik'} + v(\widetilde{R} \setminus \{k'\})$ and, analogously to Case 2,

$$\beta \geq a_{ii'} + a_{jj'} + a_{ik'} - a_{jk'} \geq a_{ii'} + a_{jk'} + a_{ij'} - a_{jk'} \geq 2a_{ij'}.$$

*Case 4* There exist $k' \in Q \cap \widetilde{R}$ and $\ell \in P \cap \widetilde{R}$ such that $(j, k'), (\ell, i') \in b$. Then $v(\widetilde{R} \cup \{j, i'\}) = a_{jk'} + a_{\ell i'} + v(\widetilde{R} \setminus \{\ell, k'\})$ and $v(\widetilde{R} \cup \{i, j'\}) \geq a_{ik'} + a_{\ell j'} + v(\widetilde{R} \setminus \{\ell, k'\})$. Applying the fact that $A$ has a doubly dominant diagonal twice yields,

$$\beta \geq a_{ii'} + a_{jj'} + a_{ik'} + a_{\ell j'} - a_{jk'} - a_{\ell i'} \geq a_{ij'} + a_{\ell i'} + a_{jk'} + a_{ij'} - a_{jk'} - a_{\ell i'} = 2a_{ij'}.$$

*Remark 4.1* (1) The foregoing proof does not use Iñarra and Usategui (1993) who presented a condition that is equivalent to $\phi(N, v) \in \mathcal{C}(N, v)$ in their Theorem 1. In order to verify this condition for a general TU game, $2^{|N|-1}$ inequalities have to be checked. Hence, Proposition 3.5 significantly simplifies the foregoing condition for $2 \times 2$ assignment games.

(2) It can easily be verified that an exact assignment game $(N, v)$ is a *partially average convex* game as introduced in the aforementioned paper if and only if the underlying matrix is a diagonal matrix, that is, $(N, v)$ is convex, provided that $p \geq 3$.

## 5 An exact game with a large core

In this section, we shall present an exact TU game $(N, v)$ whose core (a) does not contain the Shapley value and (b) is *large*, that is, if $y \in \mathbb{R}^N$ satisfies $y(S) \geq v(S)$

for all $S \subseteq N$, then there exists $x \in \mathcal{C}(N,v)$ such that $x \leq y$. (Sharkey (1982) introduced largeness of the core and showed that this property implies core stability.) Thus, this example complements Theorem 3.1: Indeed, according to Solymosi and Raghavan (2001), an assignment game is exact if and only if it has a large core.

Let $N = \{1,\ldots,5\}$, let $\lambda^1,\lambda^2,\lambda^3 \in \mathbb{R}^N$ be given by

$$\lambda^1 = (1,1,1,0,0), \quad \lambda^2 = (1,1,0,1,0), \quad \lambda^3 = (0,0,1,1,1),$$

and let $v$ be defined by

$$v(S) = \min_{r=1,2,3} \lambda^r(S) \quad \text{for all } S \subseteq N.$$

As $\lambda^r(N) = 3$ for all $r = 1,2,3$, the game $(N,v)$ is exact. By (2.1),

$$\phi(N,v) = \phi = \frac{1}{60}(36,36,41,41,26).$$

(In order to determine $\phi$, just $\phi_1$ and $\phi_5$ have to be computed via (2.1), because players 1 and 2 are substitutes as well as 3 and 4 are.) Hence,

$$v(\{2,3,4\}) = 2 > \frac{118}{60} = \phi(\{2,3,4\}).$$

So, $\phi(N,v) \notin \mathcal{C}(N,v)$. We are left to prove the following lemma.

**Lemma 5.1** *The game $(N,v)$ has a large core.*

*Proof* Let $X = \{x \in \mathbb{R}^N \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$. Taking into account that $v$ is monotonic ($v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$), that 1 and 2 as well as 3 and 4 are substitutes, and a careful inspection of the coalition function $v$ yields that $x \in X$ if and only if

$$x \geq 0; \tag{5.1}$$
$$x_3 + x_4 \geq 1; \tag{5.2}$$
$$x_i + x_j \geq 1 \quad \text{for all } i \in \{1,2\} \quad \text{and} \quad j \in \{3,4,5\}; \tag{5.3}$$
$$x_i + x_3 + x_4 \geq 2 \quad \text{for all } i \in \{1,2\}. \tag{5.4}$$

Note that (5.3) and (5.4) imply $x(N) \geq 3 = v(N)$. Let $x \in X$ and $Y = \{y \in X \mid y \leq x\}$. Then $Y$ is a nonempty compact (convex polyhedral) set. As $y \mapsto y(N)$, $y \in Y$, is continuous, there exists $z \in Y$ such that $z(N) \leq y(N)$ for all $y \in Y$. It remains to show that $z(N) = v(N) = 3$. By (5.1) one of the following 4 cases occurs.

*Case 1* $z_1 = 0$. Then, by (5.3), $z_i \geq 1$ for all $i = 3,4,5$, and, by (5.1), $z \geq \lambda^3$. As $\lambda^3 \in X$, $z = \lambda^3$ and the proof is complete.

*Case 2* The case $z_2 = 0$ may be treated analogously.

*Case 3* $z_5 = 0$. Then (5.3) implies $z_1, z_2 \geq 1$ and (5.2) implies that $z_3 + z_4 \geq 1$. As $z(N)$ is minimal, $z \geq 0$ implies that $z_3 + z_4 = 1 = z_1 = z_2$. Hence, $z(N) = 3$.

*Case 4* $z_1, z_2, z_5 > 0$. Then $z_1 = z_2 \leq 2$ by minimality of $z(N)$. In view of (5.2), $z_1 = z_2 \leq 1$. Again minimality implies that $z_1 + z_5 = z_2 + z_5 = 1$. By (5.4), $z_3 + z_4 \geq 2 - z_1$. So it remains to show that $z_3 + z_4 = 2 - z_1$. Assume the contrary. As 3 and 4 are substitutes, we may assume that $z_3 \geq z_4$. Then $z_3 > 1 - \frac{z_1}{2}$. As $z_1 \leq 2$, there exists $\epsilon > 0$ such that $z_3 - \epsilon \geq 0$, $z_1 + z_3 - \epsilon \geq 1$ and $z_3 + z_4 - \epsilon \geq 2 - z_1$ and a contradiction to the minimality of $z(N)$ has been obtained. □

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