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ORIGINAL ARTICLE

Zero-sum state constrained differential games: existence of value for Bolza problem

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Abstract We prove the existence of a lower semicontinuous value function for Bolza problem in differential games with state-constraints. As a byproduct, we obtain a new estimation of trajectories of a control system by trajectories with state constraints. This result which could be interesting by itself enables us to build a suitable strategy for constrained differential games. We also characterize the value function by means of viscosity solutions and give conditions under which the value function is locally Lipschitz continuous.

Keywords Differential games · Bolza problem · State constraints · Viability Theory

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Introduction

Our aim is to prove that Bolza problem for two-players differential games with state-constraints admits a value.

In the differential game we treat, the first player, playing with u, controls a first system

$$\begin{cases} y'(t) \ = g(y(t), u(t)), \quad u(t) \in U \\ y(t_0) = y_0 \in K_U \end{cases} \tag{1}$$

where $t_0 \in [0, T]$ and has to ensure the state constraint $y(t) \in K_U$ to be fulfilled for any $t \in [t_0, T]$. On the other hand the second player, playing with v, controls a second system

$$\begin{cases} z'(t) \ = h(z(t), v(t)), \quad v(t) \in V \\ z(t_0) = z_0 \in K_V \end{cases} \tag{2}$$

where $t_0 \in [0, T]$ and has to ensure the state constraint $z(t) \in K_V$ for any $t \in [t_0, T]$. Associated with any initial data $(t_0, x_0) = (t_0, y_0, z_0)$ and with any couple of controls $(u(\cdot), v(\cdot))$ we consider the following payoff functional:

$$J(t_0, x_0; u(\cdot), v(\cdot)) = \int_{t_0}^{T} L\big(t, x(t), u(t), v(t)\big)\mathrm{d}t + \Psi\big(x(T)\big), \tag{3}$$

where $x(t) = x[t_0, x_0; u(\cdot), v(\cdot)](t) = (y[t_0, y_0; u(\cdot)](t), z[t_0, z_0; v(\cdot)](t))$ denotes the solution of the systems (1) and (2). The function L is called *running cost* and Ψ is the *final cost*. The first player wants to maximize the functional J, while the second player's goal is to minimize J.

As is usual in differential game theory, one can define two value functions for the game, the upper one $v^\sharp$ and the lower one $v^\flat$ [see the definitions (7) and (8) below]. Here the value functions are defined through *nonanticipative strategies* [or Varayia–Roxin–Elliot–Kalton strategies, cf. Cardaliaguet et al. (2001) or preliminaries below].

The purpose of this paper is to prove that this game admits a value, i.e. to obtain the following equality:

$$\forall (t, x) \in [0, T] \times K_U \times K_V \quad v^\flat(t, x) = v^\sharp(t, x).$$

Related works and different approaches to the problem can be found in de Roquefort (1991), Bardi et al. (1995, 2000), Evans and Souganidis (1984) and Rozyev and Subbotin (1988), see also the discussion in Cardaliaguet et al. (2001). Here we follow closely the techniques developed in Cardaliaguet et al. (1999, 2001), Cardaliaguet and Plaskacz (2000), which consider the problem of existence of a value for pursuit-evasion games. The main idea amounts to reduce the qualitative game (here the Bolza problem) to a quantitative one, for which we prove an alternative result (see Proposition 2.3). This in turn gives the existence and a characterization of the value. Although this paper is strongly inspired by Cardaliaguet et al. (2001), there are important differences, that we explain now.

As is usual in state-constraint problems, the main difficulty comes from the fact that the players have to use *admissible* controls and strategies, and in particular that the set of controls allowed to a player strongly depends on its position. To overcome this difficulty, one has to be able, for two fixed initial positions of a player, to approximate a given admissible control at one position by some admissible control at the other position. This problem has been successfully handled for control problems by several authors, under various assumptions in Arisawa and Lions (1996), Frankowska and Rampazzo (2000), Loreti and Tessitore (1994). However, in the case of differential games, it is very important to build the approximating control in a nonanticipative way. Unfortunately, the constructions of the above quoted papers are all anticipative. We note also that this problem was overlooked in Cardaliaguet et al. (2001), and that, in particular, the proof of Lemma 4.4 of Cardaliaguet et al. (2001) was not completely correct because of this difficulty. The main technical result of this paper (Proposition 3.1) is the construction of such nonanticipative approximating control.

Another new aspect of the present paper is the regularity property and the characterization of the value function. We prove, under suitable regularity conditions on the running cost and the final cost, that the value function is Lipschitz continuous and that it is the unique viscosity solution of some Hamilton–Jacobi equation with a discontinuous Hamiltonian.

To obtain our results we need several assumptions on our system. Some comments on these conditions are now in order. As explained above, we suppose that the dynamics of the players are separated and that each player has to ensure his own state-constraint to be fulfilled. This is a natural assumptions concerning the applications [most games in Isaacs (1965) are of this form]. Moreover if the system has not this structure, one has to decide which player is penalized when both constraints are simultaneously violated. Then the game becomes a *non-zerosum* game, for which little is known.

We also assume that the constraints K_U and K_V are smooth and that some classical transversality conditions of the vector fields at the boundary of the constraints hold [see assumptions (4) below]. Although the smoothness of the constraints could be overcome (see Bettiol and Frankowska 2006), the transversality condition is necessary for the construction of Sect. 3 and the regularity of the value function. This is already the case for control systems (see Soner 1986). Moreover such conditions are often fulfilled in practice.

The other assumptions are more technical: we first need that, either the running cost does not depend on the controls, or that both running cost and dynamics are affine with respect to the controls and that there is a one-to-one correspondence between velocities and controls [see condition 2 in (6)]. The reason is the following: the presence of constraints on the state naturally imposes some constraints on the velocities, which in turn restrict the possible controls. When the running cost depends on the controls, then it becomes very sensitive of the way the controls depend on the velocity. Without condition 2 we are unable to show the regularity of the value functions in Sect. 4. We are not even sure that the "alternative" holds.

The last assumption upon which we wish to comment is the convexity of the sets $g(y, U)$ and $h(z, V)$ [assumption (iii) in (4)]. This assumption is necessary in our approach with discriminating kernels: it plays a central role in one of the characterizations of the discriminating domains. It might not be required for the construction of the approximated strategy in Sect. 3, but it simplifies the – already complicated – proof. When the terminal payoff is Lipschitz continuous, both existence and characterization of the value function probably hold without this assumption, but some other approach is needed.

The paper is organized in the following way. Section 1 is devoted to the statement of the main result (existence of a value) and to its proof, based on the characterization of the value functions in terms of the so-called discriminating kernel, which is a victory domain in the qualitative game. This characterization is based on a result usually called "alternative" (Proposition 2.3), which is proved in Sect. 4. As explained above, such a proof requires a technical tool allowing to compare the sets of admissible controls at different points. This is the aim of Sect. 3. Finally, Sect. 5 is devoted to the regularity and characterization of the value function as a viscosity solutions of some Hamilton–Jacobi equation.

1 Preliminaries

1.1 Notations and assumptions

We first introduce some notations. Throughout this paper, $|\cdot|$ denotes the euclidean norm of R^N. If K is a subset of R^N, $d_K(x)$ denotes the distance from x to K, i.e., $d_K(x) = \inf_{y\in K} |y - x|$. We also denote by B_N the closed unit ball of R^N (and we write simply B when the dimension of the ball is understood). If K is a subset of R^N and $r > 0$, we denote by $K + rB$ the set of points $x \in R^N$ such that $d_K(x) \le r$.

In the following lines we summarize all the assumptions concerning with the vector fields of the dynamics:

$$\begin{cases} \text{(i)} & U \text{ and } V \text{ are compact subsets of some finite} \\ & \text{dimensional spaces;} \\ \text{(ii)} & f : R^n \times U \times V \to R^n \text{ is continuous and} \\ & \text{Lipschitz continuous (with Lipschitz constant } M) \\ & \text{with respect to } x \in R^n; \\ \text{(iii)} & \bigcup_u f(x,u,v) \text{ and } \bigcup_v f(x,u,v) \text{ are convex for any } x; \\ \text{(iv)} & K_U = \{y \in R^l,\ \phi_U(y) \le 0\} \text{ with } \phi_U \in \mathcal{C}^2(R^l; R); \\ & \nabla\phi_U(y) \ne 0 \text{ if } \phi_U(y) = 0; \\ \text{(v)} & K_V = \{z \in R^m,\ \phi_V(z) \le 0\} \text{ with } \phi_V \in \mathcal{C}^2(R^m; R), \\ & \nabla\phi_V(z) \ne 0 \text{ if } \phi_V(z) = 0; \\ \text{(vi)} & \forall y \in \partial K_U,\ \exists u \in U \text{ such that } < \nabla\phi_U(y), g(y,u) > < 0; \\ \text{(vii)} & \forall z \in \partial K_V,\ \exists v \in V \text{ such that } < \nabla\phi_V(z), h(z,v) > < 0; \end{cases} \tag{4}$$

For any $x = (y, z) \in R^n$, we set

$$U(y) := \{u \in U |\ g(y, u) \in T_{K_U}(y)\}$$

where $T_{K_U}(y)$ is the tangent half-space to the set K_U. Notice that under assumptions (4) the set-valued map $y \rightsquigarrow g(y, U(y))$ is lower semicontinuous with convex compact values (for definitions and properties, see Aubin and Frankowska 1990).

For any starting point $x_0 = (y_0, z_0) \in K_U \times K_V$, for any initial time $t_0 \in [0, T]$ and for any measurable controls $(u(\cdot), v(\cdot)) : [t_0, T] \to U \times V$, we denote by $x[t_0, x_0; u(\cdot), v(\cdot)](t) = (y[t_0, y_0; u(\cdot)](t), z[t_0, z_0; v(\cdot)](t))$ the solution of system (1–2).

The first player, controlling $u(\cdot)$, has to ensure that $y(t) \in K_U$ for any $t \geq t_0$, while the second player, by using $v(\cdot)$, has to ensure the state constraint $z(t) \in K_V$ for any $t \geq t_0$.

We need to introduce the notions of admissible controls: $\forall y_0 \in K_U, \forall z_0 \in K_V$ and $\forall t_0 \in [0, T]$ we define

$$\mathcal{U}(t_0, y_0) := \{u(\cdot) : [t_0, +\infty) \to U \text{ measurable} \mid y[t_0, y_0; u(\cdot)](t) \in K_U \quad \forall t \geq t_0\};$$
$$\mathcal{V}(t_0, z_0) := \{v(\cdot) : [t_0, +\infty) \to V \text{ measurable} \mid z[t_0, z_0; v(\cdot)](t) \in K_V \quad \forall t \geq t_0\}.$$

Under the assumptions (4), the Viability Theorem (see Aubin 1991; or Aubin and Frankowska 1990) assures that for all $x_0 = (y_0, z_0) \in K_U \times K_V$

$$\mathcal{U}(t_0, y_0) \neq \emptyset \quad \text{and} \quad \mathcal{V}(t_0, z_0) \neq \emptyset.$$

Let $\mathcal{X}(t_0)$ and $\mathcal{Y}(t_0)$ be two spaces of time-measurable functions defined on $[t_0, +\infty)$. A map $\gamma : \mathcal{X}(t_0) \to \mathcal{Y}(t_0)$ is called *non-anticipative* if, for any $\tau > 0$ and any $x_1(\cdot), x_2(\cdot) \in \mathcal{X}(t_0)$ such that $x_1(\cdot) = x_2(\cdot)$ almost everywhere on $[t_0, t_0 + \tau]$, we have $\gamma(x_1(\cdot)) = \gamma(x_2(\cdot))$ almost everywhere on $[t_0, t_0 + \tau]$. In particular, both players play *non-anticipative strategies*. A map $\alpha : \mathcal{V}(t_0, z_0) \to \mathcal{U}(t_0, y_0)$ is a non-anticipative strategy (for the first player Ursula and for the point $(t_0, x_0) := (t_0, y_0, z_0) \in R^+ \times K_U \times K_V$) if, for any $\tau > 0$, for all controls $v_1(\cdot)$ and $v_2(\cdot)$ belonging to $\mathcal{V}(t_0, z_0)$, which coincide a.e. on $[t_0, t_0 + \tau]$, $\alpha(v_1(\cdot))$ and $\alpha(v_2(\cdot))$ coincide almost everywhere on $[t_0, t_0 + \tau]$. The non-anticipative strategies β for the second player Victor are symmetrically defined. For any point $x_0 \in K_U \times K_V$ and $\forall t_0 \in [0, T]$ we denote by $S_U(t_0, x_0)$ and by $S_V(t_0, x_0)$ the sets of the non-anticipative strategies for Ursula and Victor respectively.

We consider costs with the following assumptions:

$$\begin{cases} \text{i) } L : [0, T] \times R^n \times U \times V \longrightarrow R \text{ is a bounded and Lipschitz} \\ \quad \text{continuous with respect to all the variables of constant } M; \\ \text{ii) } \Psi : R^n \longrightarrow R \text{ is bounded and lower semicontinuous.} \end{cases} \tag{5}$$

We also have to assume some structure conditions on g, h and L; namely,

one of the following conditions holds:

Condition 1– $L := L(t,x)$ (L does not depend on u and v)

Condition 2– $L := L(t,x,u,v) = L_0(t,x) + L_1(t,x)u + L_2(t,x)v,$ (6)
$g(y,u) = g_1(y)u + g_2(y), \quad h(z,v) = h_1(z)v + h_2(z)$
where $g_1(y)$ and $h_1(z)$ are invertible bounded matrices with inverse Lipschitz continuous w.r. to x.

We are now ready to define the value functions of the game. The lower value $v^\flat$ is defined by:

$$v^\flat(t_0,x_0) := \inf_{\beta\in S_V(t_0,x_0)} \sup_{u(\cdot)\in\mathcal{U}(t_0,y_0)} J(t_0,x_0;u(\cdot),\beta(u(\cdot))), \tag{7}$$

where J is defined by (3). On the other hand we define the upper value function as follows:

$$v^\sharp(t_0,x_0) := \lim_{\varepsilon\to 0^+} \sup_{\alpha\in S_U(t_0,x_0)} \inf_{v(\cdot)\in\mathcal{V}(t_0,z_0)} J_\varepsilon(t_0,x_0;\alpha(v(\cdot)),v(\cdot)) \tag{8}$$

with

$$J_\varepsilon(t_0,x_0;u(\cdot),v(\cdot)) := \int_{t_0}^{T} L\big(t,x(t),u(t),v(t)\big)\mathrm{d}t + \Psi_\varepsilon\big(x(T)\big),$$

where $x(t) = x[t_0,x_0;u(\cdot),v(\cdot)](t)$ and Ψ_ε is the lower semicontinuous function defined by

$$\Psi_\varepsilon(x) := \inf\left\{\rho\in R \mid \exists y\in R^n \text{ with } |(y,\rho)-(x,\Psi(x))| = \varepsilon\right\}.$$

Remark 1.1 When the function Ψ is continuous then it turns out that the upper value function is also given by:

$$v^\sharp(t_0,x_0) = \sup_{\alpha\in S_U(t_0,x_0)} \inf_{v(\cdot)\in\mathcal{V}(t_0,z_0)} J(t_0,x_0;\alpha(v(\cdot)),v(\cdot)). \tag{9}$$

However when Ψ is not continuous, we cannot expect that the game has a value when $v^\sharp$ is defined by the above formula [see for instance the counterexample given in Cardaliaguet et al. (2001)]. The correct formulation for $v^\sharp$ is (8).

Proof of Remark 1.1. Let $v^\sharp$ be the value function given by (8) and w the function given by (9). Fix (t_0,x_0) and $\varepsilon'\in(0,1)$. From (4), there exists $M_1>0$ such that for each control $u(\cdot)$ and $v(\cdot)$, $x[t_0,x_0;u(\cdot),v(\cdot)]([t_0,T])\subset M_1B_n$. Since Ψ is uniformly continuous on $(M+1)B_n$, there exists $\varepsilon>0$ such that

$$\forall x \in (M+1)B,\ \forall a \in B,\ |\Psi(x+\varepsilon a) - \Psi(x)| \leq \varepsilon'.$$

Thus

$$\begin{aligned}\Psi(x) - 2\varepsilon' &\leq \inf_{a \in B_n} \Psi(x+\varepsilon a) - \varepsilon' \\ &\leq \inf_{(a,b) \in B_{n+1}} \Psi(x+\varepsilon a) + \varepsilon b \leq \Psi_\varepsilon(x).\end{aligned}$$

Consequently, for any control $u(\cdot)$ and $v(\cdot)$, we have

$$|J(t_0, x_0; \alpha(v(\cdot)), v(\cdot)) - J_\varepsilon(t_0, x_0; \alpha(v(\cdot)), v(\cdot))| \leq 2\varepsilon'$$

which entails that $|v^\sharp(t_0, x_0) - w(t_0, x_0)| \leq 2\varepsilon'$. Whence the desired result, ε' being arbitrary. □

1.2 Discriminating domains and kernels

We now introduce the notion of discriminating domain and kernel for a Hamiltonian function H [the original definition is given in Aubin (1991), cf. also (Cardaliaguet et al. 1999)]. This notion is fundamental in our approach because we will prove that the epigraphs of the values are discriminating kernels of suitably defined Hamiltonians.

Definition 1.2 *A closed subset D of R^N is a discriminating domain for $H : R^N \times R^N \to R$ if and only if*

$$\forall x \in D, \quad \forall p \in NP_D(x), \quad H(x; p) \leq 0,$$

where $NP_D(x)$ denotes the set of proximal normal to D at the point x, i.e., the set of $p \in R^N$ such that the distance of $x + p$ to D is equal to $|p|$.

Note that, if K is the closure of an open set with $\mathcal{C}^2$ boundary, then the above definition reduces to the condition:

$$\forall x \in \partial K, \quad H(x; \nu_x) \leq 0,$$

where ν_x denotes the outward unit normal to K at x. Next we introduce the notion of discriminating kernel: for a given closed set K – which is not a discriminating domain in general – it is possible to define a largest discriminating domain contained in K. This is the discriminating kernel of K.

Proposition 1.3 (Cardaliaguet 1997) *Let $H : R^N \times R^N \to R$ be a lower semicontinuous map. If K is a subset of R^N, then K contains the largest* (*for the inclusion*) *closed discriminating domain for H. This set, denoted by* $\mathrm{Disc}_H(K)$, *is called the discriminating kernel of K for H.*

2 Value functions

In this section we state the main result of this paper, namely the existence of a value for our game. For this we have to consider separately the case of a running cost independent of the control, i.e., $L = L(t, x)$, and the more difficult situation where $L = L(t, x, u, v)$.

Theorem 2.1 *Assume that* (4, 5, 6) *hold. Then the game has a value, and this value is lower semi-continuous:*

$$\forall (t,x) \in [0,T] \times K_U \times K_V \quad v^\flat(t,x) = v^\sharp(t,x).$$

The proof of this result is obtained by identifying the epigraph of the value function as the discriminating kernel of some closed set for a suitable dynamics. We now explain this identification.

Let us define an "extended" dynamics function $\tilde{f} : R^{n+2} \times U \times V \to R^{n+2}$

$$\tilde{f}\big((t,x,\rho),u,v\big) := \Big(1, f(x,u,v), -L(t,x,u,v)\Big) \tag{10}$$

and a corresponding Hamiltonian:

$$\widetilde{H}(t,x,\rho;p_t,p_x,p_\rho) := \begin{cases} \sup_{u\in U(y)} \inf_{v\in V} \langle \tilde{f}, p\rangle & \text{if } (t,x,\rho) \notin \mathcal{E} \\ \min\Big\{0; \sup_{u\in U(y)} \inf_{v\in V} \langle \tilde{f}, p\rangle\Big\} & \text{if } (t,x,\rho) \in \mathcal{E}, \end{cases} \tag{11}$$

where

$$\mathcal{E} := \{(t,x,\rho) \in [0,T] \times K_U \times K_V \times R \mid t = T,\ \rho \geq \Psi(x)\} = \{T\} \times \mathcal{E}pi\Psi.$$

We also define the set $\mathcal{K} := [0,T] \times K_U \times K_V \times R \subset R^{n+2}$. Writing explicitly the scalar product term, we get

$$\langle \tilde{f}, p\rangle = p_t + \langle f(x,u,v), p_x\rangle - p_\rho L(t,x,u,v) \quad \forall p = (p_t,p_x,p_\rho) \in R^{n+2}.$$

Hereafter, in order to simplify the notations, we will write X for the triple (t,x,ρ), i.e. $X = (t,x,\rho) \in R^N$ where $N := n+2$. In particular, once $X_0 = (t_0,x_0,\rho_0)$ and $(u(\cdot),v(\cdot)) \in \mathcal{U}(t_0,y_0) \times \mathcal{V}(t_0,z_0)$ are fixed, we denote by $X[X_0;u(\cdot),v(\cdot)](s)$ the solution of the Cauchy problem

$$X'(s) = \tilde{f}(X(s),u(s),v(s)), \quad X(t_0) = X_0. \tag{12}$$

Let us point out that the t-component of X, denoted $t(s)$ is always equal to s.

Theorem 2.2 *Let $\widetilde{H}$ be defined as in* (11) *and assume* (4, 5, 6) *hold. Then*

$$\mathcal{E}pi(v^\flat) = \mathrm{Disc}_{\widetilde{H}}(\mathcal{K}) \quad \text{and} \quad \mathcal{E}pi(v^\sharp) = \mathrm{Disc}_{\widetilde{H}}(\mathcal{K}).$$

From Theorem 2.2 we can easily deduce the proof of Theorem 2.1.

Proof of Theorem 2.1 Since the epigraph of $v^\flat(t,x)$ and $v^\sharp(t,x)$ coincide, both functions are equal. Furthermore, since the discriminating kernel is a closed set, their epigraph is closed, and therefore they are lower semicontinuous. □

We now turn to the proof of Theorem 2.2, which is crucially related to the following interpretation result of discriminating kernels:

Proposition 2.3 *Suppose that assumptions of Theorem 2.2 hold true. Then the Discriminating Kernel* $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ *can be characterized as follows*

(i) $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ *is the set of all the starting points* $X_0 = (t_0, x_0, \rho_0) \in [0,T] \times K_U \times K_V \times R$ *for which there exists an admissible nonanticipative strategy* $\beta : \mathcal{U}(t_0, y_0) \longrightarrow \mathcal{V}(t_0, z_0)$ *such that, for any admissible control* $u(\cdot) \in \mathcal{U}(t_0, y_0)$*, the solution* $X[X_0; u(\cdot), \beta(u(\cdot))]$ *remains in* $\mathcal{K}$ *as long as it does not reach* $\mathcal{E}$*;*
(ii) *The set* $([0,T] \times K_U \times K_V \times R) \setminus \mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ *is the set of all the starting points* $X_0 = (t_0, x_0, \rho_0) \in [0,T] \times K_U \times K_V \times R$ *for which there exists* $\varepsilon > 0$ *and an admissible nonanticipative strategy* $\alpha : \mathcal{V}(t_0, z_0) \longrightarrow \mathcal{U}(t_0, y_0)$ *such that, for any admissible control* $v(\cdot) \in \mathcal{V}(t_0, z_0)$*, the solution* $X[X_0; \alpha(v(\cdot)), v(\cdot)](s)$ *avoids* $\mathcal{E} + \varepsilon B$ *for any time s.*

This result can be viewed as an alternative Theorem [cf. Cardaliaguet (1996); Krasovskii and Subbotin (1998)]. After some estimation results in Sect. 3, the fourth section will be devoted to the proof of Proposition 2.3.

Proof of Theorem 2.2 Let us first check that $\mathcal{E}pi(v^\flat) = \mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$. By *i*) of Proposition 2.3 we get

$$\mathrm{Disc}_{\widetilde{H}}(\mathcal{K}) = \Big\{ (t_0, x_0, \rho_0) \in \mathcal{K} \text{ for which } \exists \beta \in S_V(t_0, x_0) \quad \text{s. t. } \forall u(\cdot) \in \mathcal{U}(t_0, y_0) \text{ the trajectory } X[X_0, u(\cdot), \beta(u(\cdot))](\cdot) \text{ remains in } \mathcal{K} \text{ until it reaches } \mathcal{E} \Big\}.$$

Suppose that $(t_0, x_0, \rho_0) \in \mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$. By the above characterization of $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ we can choose a strategy $\beta \in S_V(t_0, x_0)$ such that, for any $u(\cdot) \in \mathcal{U}(t_0, y_0)$, the solution $X(s) = (t(s), x(s), \rho(s))$ of the system

$$\begin{cases} t'(s) &= 1 \\ x'(s) &= f(x(s), u(s), \beta(u(\cdot))(s)), \\ \rho'(s) &= -L(s, x(s), u(s), \beta(u(\cdot))(s)), \\ t(t_0) &= t_0, \; x(t_0) = x_0, \; \rho(t_0) = \rho_0 \end{cases} \tag{13}$$

satisfies

$$\rho(T) = \rho_0 - \int_{t_0}^{T} L(r, x(r), u(r), \beta(u(\cdot))(r)) \mathrm{d}r \geq \Psi(x(T)).$$

Hence, we have

$$v^\flat(t_0,x_0) \le \sup_{u(\cdot)\in\mathcal{U}(t_0,y_0)} J(t_0,x_0;u(\cdot),\beta(u(\cdot))) \le \rho_0$$

and so $(t_0,x_0,\rho_0) \in \mathcal{E}pi(v^\flat)$.

On the other hand suppose that the point (t_0,x_0) belongs to the domain of $v^\flat$, and let $\rho_0 > v^\flat(t_0,x_0)$. There exists a nonanticipative strategy $\beta \in S_V(t_0,x_0)$ such that

$$v^\flat(t_0,x_0) \le \sup_{u(\cdot)\in\mathcal{U}(t_0,y_0)} J(t_0,x_0;u(\cdot),\beta(u(\cdot))) \le \rho_0 .$$

Hence for any $u(\cdot) \in \mathcal{U}(t_0,y_0)$ we have

$$v^\flat(t_0,x_0) \le \int_{t_0}^{T} L(r,x(r),u(r),\beta(u(\cdot))(r))\mathrm{d}r + \Psi(x(T)) \le \rho_0,$$

and the ρ-component of $X[X_0,u(\cdot),\beta(u(\cdot))](\cdot)$ – denoted by $\rho(\cdot)$ – satisfies

$$\rho(T) = \rho_0 - \int_{t_0}^{T} L(r,x(r),u(r),\beta(u(\cdot))(r))\mathrm{d}r \ge \Psi(x(T)).$$

Therefore the trajectory $X[X_0,u(\cdot),\beta(u(\cdot))](\cdot)$ remains in $\mathcal{K}$ until it reaches $\mathcal{E}$. Since this holds true for all $u(\cdot) \in \mathcal{U}(t_0,y_0)$, thanks to i) of Proposition 2.3, we have $(t_0,x_0,\rho_0) \in \mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ for any $\rho_0 > v^\flat(t_0,x_0)$. $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ being closed, we obtain

$$\mathcal{E}pi(v^\flat) \subset \mathrm{Disc}_{\widetilde{H}}(\mathcal{K}).$$

Now, we prove that $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K}) = \mathcal{E}pi(v^\sharp)$. Suppose that $X_0 = (t_0,x_0,\rho_0) \in \mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$. Then, by (ii) of Proposition 2.3, for any $\varepsilon > 0$ and for any nonanticipative strategy $\alpha(\cdot) \in S_U(t_0,x_0)$ there exists a control of Victor $v(\cdot) \in \mathcal{V}(t_0,z_0)$ such that for some $\tau \in [t_0,T]$ $X(\tau) = X[X_0,\alpha(v(\cdot))(\cdot),v(\cdot)](\tau) \in \mathcal{E}+\varepsilon B$. Therefore we get the following two facts: $T-\tau \le \varepsilon$ and $\big(x(\tau),\rho(\tau)\big) \in \mathcal{E}pi(\Psi_\varepsilon)$. On the other hand we have: $|\rho(T)-\rho(\tau)| \le M_L\varepsilon$, $|x(T)-x(\tau)| \le M_f\varepsilon$ where M_L and M_f are the upper bounds of the function L and f respectively. So, since $\mathrm{d}_{\mathcal{E}pi(\Psi)}\big(x(\tau),\rho(\tau)\big) \le \varepsilon$, we obtain:

$$\begin{aligned}\mathrm{d}_{\mathcal{E}pi(\Psi)}\Big(\big(x(T),\rho(T)\big)\Big) &\le \mathrm{d}_{\mathcal{E}pi(\Psi)}\Big(\big(x(\tau),\rho(\tau)\big)\Big) + |x(T)-x(\tau)| + |\rho(T)-\rho(\tau)| \\ &\le (M_L+M_f+1)\varepsilon\end{aligned}$$

which implies, calling $\varepsilon' := (M_L + M_f + 1)\varepsilon$,

$$\rho(T) \geq \Psi_{\varepsilon'}(x(T)).$$

Hence, we obtain

$$\begin{aligned}\rho_0 &= \int_{t_0}^{T} L(s, x(s), \alpha(v(\cdot))(s), v(s))\mathrm{d}s + \rho(T) \\ &\geq \int_{t_0}^{T} L(s, x(s), \alpha(v(\cdot))(s), v(s))\mathrm{d}s + \Psi_{\varepsilon'}(x(T)) \\ &\geq J_{\varepsilon'}(t_0, x_0, \alpha(v(\cdot)), v(\cdot)) \geq \inf_{v(\cdot)\in\mathcal{V}(t_0,z_0)} J_{\varepsilon'}(t_0, x_0, \alpha(v(\cdot)), v(\cdot)). \end{aligned} \tag{14}$$

Recalling that (14) holds true for any $\alpha(\cdot) \in S_U(t_0, x_0)$, we get

$$\rho_0 \geq \sup_{\alpha(\cdot)\in S_U(t_0,x_0)} \inf_{v(\cdot)\in\mathcal{V}(t_0,z_0)} J_{\varepsilon'}(t_0, x_0, \alpha(v(\cdot)), v(\cdot))$$

Since the right hand side of the inequality above is bounded by ρ_0 and increases as $\varepsilon' \searrow 0^+$, we can pass to the limit:

$$\rho_0 \geq \lim_{\varepsilon'\to 0} \sup_{\alpha(\cdot)\in S_U(t_0,x_0)} \inf_{v(\cdot)\in\mathcal{V}(t_0,z_0)} J_{\varepsilon'}(t_0, x_0, \alpha(v(\cdot)), v(\cdot)) = v^\sharp(t_0, x_0).$$

This implies $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K}) \subset \mathcal{E}pi(v^\sharp)$.

Conversely, let us consider $\rho_0 > v^\sharp(t_0, x_0)$. Then for any $\varepsilon > 0$ we have $\rho_0 > \sup_{\alpha(\cdot)\in S_U(t_0,x_0)} \inf_{v(\cdot)\in\mathcal{V}(t_0,z_0)} J_\varepsilon(t_0, x_0, \alpha(v(\cdot)), v(\cdot))$ and, so, for any strategy $\alpha \in S_U(t_0, x_0)$

$$\rho_0 > \inf_{v(\cdot)\in\mathcal{V}(t_0,z_0)} J_\varepsilon(t_0, x_0; \alpha(v(\cdot)), v(\cdot)).$$

Now we take a control $v(\cdot) \in \mathcal{V}(t_0, z_0)$ such that $\rho_0 \geq J_\varepsilon(t_0, x_0; \alpha(v(\cdot)), v(\cdot))$. The ρ-component of the trajectory $X[X_0, \alpha(v(\cdot))(\cdot), v(\cdot)](\cdot)$ satisfies the condition $\rho(T) = \rho_0 - \int_{t_0}^{T} L(s, x(s), \alpha(v(\cdot))(s), v(s))\mathrm{d}s$. Hence, we obtain $\rho(T) \geq J_\varepsilon(t_0, x_0; \alpha(v(\cdot)), v(\cdot)) - \int_{t_0}^{T} L(s, x(s), \alpha(v(\cdot))(s), v(s))\mathrm{d}s = \Psi_\varepsilon(x(T))$ and, finally, $X(\tau) \in \mathcal{E} + \varepsilon B$ for a time $\tau \leq T$. □

3 Approximation by constrained trajectories

An important problem in order to get suitable estimations on constrained trajectories, is to obtain a kind of Filippov Theorem with constraints. Namely a result which allows to approach – in a suitable sense – a given trajectory of the dynamics by a constrained trajectory. Note that similar results exist in the literature (cf. Arisawa and Lions 1996; Frankowska and Rampazzo 2000; Loreti and Tessitore 1994) but in the present paper we need a construction of the constrained trajectory in a nonanticipative way. This result also contains, as a particular case the correct proof of Lemma 4.4 in Cardaliaguet et al. (2001), and can be seen as a Corrigendum of this proof. The present section is concerned with the proof of the following

Proposition 3.1 *Assume that conditions* (4) *are satisfied. For any* $R > 0$ *there exist* $C_0 = C_0(R) > 0$ *such that for any initial time* $t_0 \in [0, T]$, *for any* $y_0, y_1 \in K_U$ *with* $|y_0|, |y_1| \le R$, *there is a nonanticipative strategy* $\sigma : \mathcal{U}(t_0, y_0) \longrightarrow \mathcal{U}(t_0, y_1)$ *with the following property: for any* $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$ *and for any* $t \in [t_0, T]$ *we have*

$$
\begin{aligned}
&|y_0(t) - y_1(t)| + \int_{t_0}^{t} |g(y_0(s), u_0(s)) - g(y_1(s), \sigma(u_0(\cdot))(s))|\mathrm{d}s \\
&\quad \le C_0|y_0 - y_1|e^{C_0(t-t_0)}
\end{aligned}
\tag{15}
$$

where we have set for simplicity $y_0(t) = y[t_0, y_0; u_0(\cdot)](t)$ *and* $y_1(t) = y[t_0, y_1; \sigma(u_0(\cdot))](t)$.

In particular if g is affine with respect to the control u, namely

$$
g(y, u) = g_1(y)u + g_2(y)
$$

where $g_1(y)$ *is an invertible matrix with Lipschitz continuous inverse, then we have*

$$
|y_0(t) - y_1(t)| + \int_{t_0}^{t} |u_0(s) - \sigma(u_0(\cdot))(s)|\mathrm{d}s \le C_1|y_0 - y_1|\mathrm{e}^{C_1(t-t_0)}. \tag{16}
$$

for some constant $C_1 = C_1(R) > 0$.

The rest of this section is devoted to the proof of Proposition 3.1. For this, let us fix an admissible control $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$, and let us set $y_0(\cdot) = y[t_0, y_0; u_0(\cdot)](\cdot)$, take a new starting point $\bar{y} \in K_U$. We wish to build in a nonanticipative way a control $\bar{u}(\cdot)$ satisfying:

$$
\begin{aligned}
&\int_{t_0}^{t} |g(y[t_0, y_0; u_0(\cdot)](s), u_0(s)) - g(y[t_0, \bar{y}; \bar{u}](s), \bar{u}(s))\mathrm{d}s \\
&\quad + |y[t_0, y_0; u_0(\cdot)](t) - y[t_0, \bar{y}; \bar{u}](t)| \le C_0|y_0 - \bar{y}|e^{C_0(t-t_0)}.
\end{aligned}
\tag{17}
$$

To this end, we consider the system:

$$
\begin{cases}
y'(t) = \pi_{G(y(t)) \cap T_{K_U}(y(t))}\big(g(y(t), u_0(t))\big) \\
y(t_0) = \bar{y} \in K_U,
\end{cases}
\tag{18}
$$

where $G(y) := g(y, U)$ and $\pi_{G(y(t))\cap T_{K_U}(y(t))}\big(g(y(t), u_0(t))\big)$ denotes the projection of $g(y(t), u_0(t))$ onto $G(y(t)) \cap T_{K_U}(y(t))$. Notice that $\pi_{G(y)\cap T_{K_U}(y)}\big(g(y, u_0(t))\big) = g(y, u_0(t))$ whenever y belongs to the interior of K_U or if y is on the boundary of K_U and

$$\langle g(y(t), u_0(t)), \nabla\phi_U(y(t))\rangle \leq 0.$$

Let us also underline that, since the set $G(y) \cap T_{K_U}(y)$ is convex, the projection onto $G(y) \cap T_{K_U}(y)$ is unique. We denote it by $g(y, \bar{u}(y, u))$ and we note that the control $\bar{u}(y, u) \in U$ is not necessarily unique. Our goal is to show that $\bar{u}(y, u)$ is a suitable feedback, which enables us to build the control $\bar{u}$ in a nonanticipative way. First we show that there is a solution to (18).

Lemma 3.2 *System* (18) *admits at least one solution.*

Proof of Lemma 3.2 We claim that the set of solutions of system (18) is the same as the set of solutions of the following system

$$\begin{cases} y'(t) \in \tilde{G}(t, y(t)), \quad y(t) \in K_U \\ y(t_0) = \bar{y} \in K_U, \end{cases} \tag{19}$$

where

$$\tilde{G}(t, y) := \begin{cases} g(y, u_0(t)) & \text{if } y \in \mathrm{Int}(K_U) \\ \overline{\mathrm{co}}\,\{g(y, u_0(t)); g(y, \bar{u}(y, u_0(t)))\} & \text{if } y \in \partial K_U \end{cases}$$

Before proving the claim, let us note that, since the set-valued function $\tilde{G}$ is clearly Lebesgue–Borel measurable in (t, y) and upper semicontinuous with respect to y, by the measurable Viability Theorem of Frankowska et al. (1995), we obtain that system (19) and, so, according to the claim, also system (18), has a solution for any starting point $\bar{y} \in K_U$ at any initial time t_0.

We now prove the claim. Since

$$\pi_{G(y)\cap T_{K_U}(y)}\big(g(y, u_0(t))\big) \subset \tilde{G}(t, y)$$

then any solution of (18) is also a solution of (19). Conversely, suppose that $y(\cdot)$ is a solution of (19) and consider the set

$$\mathcal{D} := \Big\{t \mid \exists y'(t) \text{ with } y'(t) \notin \pi_{G(y(t))\cap T_{K_U}(y(t))}\big(g(y(t), u_0(t))\big)\Big\}$$

We have $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ where $\mathcal{D}_1 = \{t \in \mathcal{D} \mid y(t) \in \partial K_U\}$ and $\mathcal{D}_2 = \{t \in \mathcal{D} \mid y(t) \in \mathrm{Int}(K_U)\}$. The measure of $\mathcal{D}_2$ is zero because

$$\tilde{G}(t, y)\Big|_{\mathrm{Int}(K_U)} \equiv \pi_{G(y)\cap T_{K_U}(y)}\big(g(y, u_0(t))\big).$$

On the other hand, the measure of $\mathcal{D}_1$ is also zero; indeed, for almost every t such that $y(t) \in \partial K_U$ the derivative $y'(t)$ exists, and, since t is a local maximum for the function $s \mapsto \phi_U(y(s))$, the derivative of $\phi_U(y(s))$ with respect to s at time t vanishes:

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\phi_U(y(s))\bigg|_{s=t} = \langle y'(t), \nabla\phi_U(y(t))\rangle.$$

If $\langle g(y(t), u_0(t)), \nabla\phi_U(y(t))\rangle > 0$ then we obtain $y'(t) = g(y(t), \bar{u}(y(t), u_0(t)))$; otherwise $g(y(t), \bar{u}(y(t), u_0(t))) = g(y(t), u_0(t))$. In any case we get

$$y'(t) \in \pi_{G(y)\cap T_{K_U}(y)}\big(g(y(t), u_0(t))\big). \qquad \square$$

Next Lemma allows to compare $g(y, u)$ and $g(y, \bar{u}(y, u))$:

Lemma 3.3 *Under assumption* (4)*, for any* $R > 0$*, there is a constant* $C > 0$ *such that for any* $y \in \partial K_U$ *with* $|y| \leq R$ *and any* $u \in U$*, we have*

$$|g(y, \bar{u}(y, u)) - g(y, u)| \leq C\,(\langle g(y, u), \nabla\phi_U(y)\rangle)_+, \tag{20}$$

where $(x)_+ = \max\{x, 0\}$.

Proof of Lemma 3.3 From (4-vi), we can choose $\eta > 0$ such that:

$$\sup_{y\in\partial K_U,\ |y|\leq R}\ \inf_{u\in U}\langle g(y, u), \nabla\phi_U(y)\rangle < -\eta < 0.$$

Fix $y \in \partial K_U$ and consider $u_1 \in U$ such that

$$\langle g(y, u_1), \nabla\phi_U(y)\rangle < -\eta < 0.$$

Let us set

$$\lambda = \frac{(\langle g(y, u), \nabla\phi_U(y)\rangle)_+}{\eta + (\langle g(y, u), \nabla\phi_U(y)\rangle)_+}.$$

Note that $\lambda \in [0, 1]$. From the convexity of $g(y, U)$, we can find some $u_\lambda \in U$ such that

$$g(y, u_\lambda) = (1 - \lambda)g(y, u) + \lambda g(y, u_1).$$

Then

$$\langle g(y, u_\lambda), \nabla\phi_U(y)\rangle \leq -\eta\lambda + (1 - \lambda)\,(\langle g(y, u), \nabla\phi_U(y)\rangle)_+ = 0.$$

Hence, observing that $\lambda \leq \frac{1}{\eta} (\langle g(y,u), \nabla\phi_U(y)\rangle)_+$, we obtain

$$|g(y,\bar{u}(y,u)) - g(y,u)| \leq |g(y,u_\lambda) - g(y,u)| \leq \lambda |g(y,u_1) - g(y,u)| \\ \leq \frac{M}{\eta} (\langle g(y,u), \nabla\phi_U(y)\rangle)_+ ,$$

for some constant $M = M(R)$. □

If $\bar{y}(\cdot)$ is a solution of (18), by the measurable selection theorem there exists an admissible control $\bar{u}(\cdot)$ such that

$$\begin{cases} \bar{y}'(t) = g(\bar{y}(t), \bar{u}(t)) = \pi_{G(\bar{y}(t)) \cap T_{K_U}(\bar{y}(t))}\big(g(\bar{y}(t), u_0(t))\big) \\ y(t_0) = \bar{y} \in K_U, \end{cases} \tag{21}$$

Lemma 3.4 *Assume that conditions (4) hold. For any positive constant R there exists a positive $\tilde{C} = \tilde{C}(R)$ such that for any y_0, $\bar{y} \in K_U$ with $|y_0|$, $|\bar{y}| \leq R$ and for any admissible control $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$, the admissible control $\bar{u}(\cdot) \in \mathcal{U}(t_0, \bar{y})$ is such that for all $t \in [t_0, T]$*

$$\int_{t_0}^{t} |g(\bar{y}(s), \bar{u}(s)) - g(y_0(s), u_0(s))| \mathrm{d}s \leq \tilde{C} \left(|\bar{y} - y_0| + \int_{t_0}^{t} |\bar{y}(s) - y_0(s)| \mathrm{d}s \right), \tag{22}$$

where $\bar{y}(s) := y[t_0, \bar{y}; \bar{u}(\cdot)](s)$ and $y_0(s) := y[t_0, y_0; u_0(\cdot)](s)$.

Proof Recall that $(\bar{u}(\cdot), \bar{y}(\cdot))$ denotes the couple control-trajectory which satisfies system (21).

In order to fix the ideas, let us assume that $\bar{y} \in \mathrm{Int}(K_U)$. The case in which $\bar{y}$ belongs to the boundary of K_U can be treated similarly. Let us define the following set:

$$O := \{s \in (t_0, t) \mid \bar{y}(s) \in \mathrm{Int}(K_U)\} = \{s \in (t_0, t) \mid \phi_U(\bar{y}(s)) < 0\}.$$

The set O is open in $[t_0, t]$ and it is an enumerable union of open disjoint intervals, I_n,

$$O = \bigcup_{n \in \mathbb{N}} I_n.$$

For any $\varepsilon > 0$ we can choose a finite number of these intervals, say I_i for $i = 1, \ldots, k$, such that

$$\left| O \Big\backslash \bigcup_{i=1}^{k} I_i \right| \leq \varepsilon$$

where by $|E|$ we denote the Lebesgue measure of a subset $E \subset R$. Let us call $O_k := \mathrm{Int}\big(\overline{\bigcup_{i=1}^{k} I_i}\big)$; notice that $O_k = \bigcup_{j=0}^{h} J_j$, where J_j are open intervals: $J_j =]t_{2j}, t_{2j+1}[$ with $t_{2j+1} \le t_{2j+2}$. Observe that

$$\left| O_k \vartriangle \left(\bigcup_{i=1}^{k} I_i \right) \right| = 0$$

and that $\phi_U(\bar{y}(t_{2j})) = \phi_U(\bar{y}(t_{2j+1})) = 0$ for any j. Moreover we have

$$O_k^c = [t_0, t] \Big\backslash O_k = [t_0, t] \setminus \bigcup_{j=0}^{h} J_j = \bigcup_{j=0}^{h} [t_{2j+1}, t_{2j+2}],$$

where $t_{2h+2} = t$.

We claim that there is a constant C, independent of the control u_0 and of the initial positions y_0 and $\bar{y}$, such that for almost every $s \in O^c$ we have

$$|g(\bar{y}(s), \bar{u}(s)) - g(\bar{y}(s), u_0(s))| \le C \langle g(\bar{y}(s), u_0(s)), \nabla \phi_U(\bar{y}(s)) \rangle. \tag{23}$$

For this we apply Lemma 3.3, for the constant $\tilde{R}$ such that any solution starting from a point $y \in RB \cap K_U$ remains in $\tilde{R}B$ on the time interval $[0, T]$. We have now to explain how to remove the "plus" in the inequality (20) of Lemma 3.3. Let E be the set where the derivative of $\bar{y}(\cdot)$ exists. For any $s \in E \cap O^c$, we obtain

$$0 = \left. \frac{\mathrm{d}}{\mathrm{d}\tau} \phi_U(\bar{y}(\tau)) \right|_{\tau = s} = \langle \bar{y}'(s), \nabla \phi_U(\bar{y}(s)) \rangle \tag{24}$$

because s is a local maximum for $\tau \longmapsto \phi_U(\bar{y}(\tau))$. Since $|E \cap O^c| = |O^c|$, by (24) we obtain that for almost every $s \in O^c$ either $\bar{u}(s) = u_0(s)$ and, hence,

$$\langle g(\bar{y}(s), u_0(s)), \nabla \phi_U(\bar{y}(s)) \rangle = \langle g(\bar{y}(s), \bar{u}(s)), \nabla \phi_U(\bar{y}(s)) \rangle = 0,$$

or $\bar{u}(s) \neq u_0(s)$ and, so, $\bar{u}(s) = \bar{u}(\bar{y}(s), u_0(s))$ and

$$\langle g(\bar{y}(s), u_0(s)), \nabla \phi_U(\bar{y}(s)) \rangle > \langle g(\bar{y}(s), \bar{u}(s)), \nabla \phi_U(\bar{y}(s)) \rangle = 0.$$

Thanks to Lemma 3.3 we have (23).

Now, by using (23), we obtain

$$\int_{t_0}^{t} |g(\bar{y}(s), \bar{u}(s)) - g(y_0(s), u_0(s))| \mathrm{d}s$$

$$\le \int_{O^c} |g(\bar{y}(s), \bar{u}(s)) - g(\bar{y}(s), u_0(s))| \mathrm{d}s + \int_{t_0}^{t} |g(\bar{y}(s), u_0(s)) - g(y_0(s), u_0(s))| \mathrm{d}s$$

$$
\begin{aligned}
&\leq C \int\limits_{O^c} \langle g(\bar{y}(s), u_0(s)), \nabla\phi_U(\bar{y}(s))\rangle \mathrm{d}s + M \int\limits_{t_0}^{t} |\bar{y}(s) - y_0(s)| \mathrm{d}s \\
&= C \left[\int\limits_{O_k^c} \langle g(\bar{y}(s), u_0(s)), \nabla\phi_U(\bar{y}(s))\rangle \mathrm{d}s - \int\limits_{O_k^c \setminus O^c} \langle g(\bar{y}(s), u_0(s)), \nabla\phi_U(\bar{y}(s))\rangle \mathrm{d}s \right] \\
&\quad + M \int\limits_{t_0}^{t} |\bar{y}(s) - y_0(s)| \mathrm{d}s \\
&\leq C \left[\int\limits_{O_k^c} \langle g(\bar{y}(s), u_0(s)), \nabla\phi_U(\bar{y}(s))\rangle \mathrm{d}s + \varepsilon \| \langle g(\bar{y}, u_0), \nabla\phi_U(\bar{y})\rangle \|_\infty \right] \\
&\quad + M \int\limits_{t_0}^{t} |\bar{y}(s) - y_0(s)| \mathrm{d}s \left(\text{because} \quad \left|O_k^c \setminus O^c\right| \leq \varepsilon\right) \\
&= C \left[\sum_{j=0}^{h} \int\limits_{t_{2j+1}}^{t_{2j+2}} \langle g(\bar{y}(s), u_0(s)), \nabla\phi_U(\bar{y}(s))\rangle \mathrm{d}s + \varepsilon \bar{M} \right] + M \int\limits_{t_0}^{t} |\bar{y}(s) - y_0(s)| \mathrm{d}s \\
&\leq C \Bigg[\sum_{j=0}^{h} \int\limits_{t_{2j+1}}^{t_{2j+2}} \langle g(y_0(s), u_0(s)), \nabla\phi_U(y_0(s))\rangle \mathrm{d}s \\
&\quad + K_0 \sum_{j=0}^{h} \int\limits_{t_{2j+1}}^{t_{2j+2}} |\bar{y}(s) - y_0(s)| \mathrm{d}s + \varepsilon \bar{M} \Bigg] + M \int\limits_{t_0}^{t} |\bar{y}(s) - y_0(s)| \mathrm{d}s,
\end{aligned}
$$

where K_0 and $\bar{M}$ are suitable constants depending on R.

Observe that

$$
\begin{aligned}
&\sum_{j=0}^{h} \int\limits_{t_{2j+1}}^{t_{2j+2}} \langle g(y_0(s), u_0(s)), \nabla\phi_U(y_0(s))\rangle \mathrm{d}s = \sum_{j=0}^{h} \left(\phi_U(y_0(t_{2j+2})) - \phi_U(y_0(t_{2j+1}))\right) \\
&= \phi_U(y_0(t_{2h+2})) - \phi_U(y_0(t_1)) - \sum_{j=1}^{h} \left(\phi_U(y_0(t_{2j+1})) - \phi_U(y_0(t_{2j}))\right) \\
&= \phi_U(y_0(t_{2h+2})) - \phi_U(y_0(t_1)) - \sum_{j=1}^{h} \int\limits_{t_{2j}}^{t_{2j+1}} \langle g(y_0(s), u_0(s)), \nabla\phi_U(y_0(s))\rangle \mathrm{d}s \\
&\leq -\phi_U(y_0(t_1)) - \sum_{j=1}^{h} \int\limits_{t_{2j}}^{t_{2j+1}} \langle g(\bar{y}(s), u_0(s)), \nabla\phi_U(\bar{y}(s))\rangle \mathrm{d}s
\end{aligned}
$$

$$+K_0 \sum_{j=1}^{h} \int_{t_{2j}}^{t_{2j+1}} |\bar{y}(s) - y_0(s)|\mathrm{d}s$$

$$= -\phi_U(y_0(t_1)) - \sum_{j=1}^{h} \left(\phi_U(\bar{y}(t_{2j+1})) - \phi_U(\bar{y}(t_{2j}))\right) + K_0 \sum_{j=1}^{h} \int_{t_{2j}}^{t_{2j+1}} |\bar{y}(s) - y_0(s)|\mathrm{d}s$$

$$= -\phi_U(y_0(t_1)) + K_0 \sum_{j=1}^{h} \int_{t_{2j}}^{t_{2j+1}} |\bar{y}(s) - y_0(s)|\mathrm{d}s,$$

because the points $\bar{y}(t_{2j})$ and $\bar{y}(t_{2j+1})$ for each $j = 1, \dots, h$ belong to the boundary. Moreover, we have:

$$-\phi_U(y_0(t_1)) \le -\phi_U(\bar{y}(t_1)) + K_0|\bar{y}(t_1) - y_0(t_1)|$$
$$\le K_0 \left[|\bar{y} - y_0| + \int_{t_0}^{t_1} |\bar{y}(s) - y_0(s)|\mathrm{d}s \right]$$

because $-\phi_U(\bar{y}(t_1)) = 0$ and by applying Gronwall Lemma. Then, finally, we obtain:

$$\int_{t_0}^{t} |g(\bar{y}(s), \bar{u}(s)) - g(y_0(s), u_0(s))|\mathrm{d}s$$

$$\le C \Bigg\{ K_0 \left[|\bar{y} - y_0| + \int_{t_0}^{t_1} |\bar{y}(s) - y_0(s)|\mathrm{d}s \right]$$

$$+ K_0 \sum_{j=1}^{h} \int_{t_{2j}}^{t_{2j+1}} |\bar{y}(s) - y_0(s)|\mathrm{d}s + K_0 \sum_{j=0}^{h} \int_{t_{2j+1}}^{t_{2j+2}} |\bar{y}(s) - y_0(s)|\mathrm{d}s + \varepsilon \bar{M} \Bigg\}$$

$$+M \int_{t_0}^{t} |\bar{y}(s) - y_0(s)|\mathrm{d}s \le C \left\{ K_1 \left[|\bar{y} - y_0| + \int_{t_0}^{t} |\bar{y}(s) - y_0(s)|\mathrm{d}s \right] + \varepsilon \bar{M} \right\},$$

for some constant $K_1 > 0$. This gives (22) because ε is arbitrary. □

Proof of Proposition 3.1 For any admissible control $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$, we claim that it is possible to construct (in a nonanticipative way) an admissible control $u_1(\cdot) \in \mathcal{U}(t_0, y_1)$ such that $\forall\, t \in [t_0, T]$

$$|y_1(t)-y_0(t)|+\int_{t_0}^{t}|g(y_1(s),u_1(s))-g(y_0(s),u_0(s))|\mathrm{d}s \le C_0|y_1-y_0|\mathrm{e}^{C_0(t-t_0)} \quad (25)$$

where $y_0(t) = y[t_0, y_0; u_0(\cdot)](t)$ and $y_1(t) = y[t_0, y_1; u_1(\cdot)](t)$. Indeed, let $(u_1(\cdot), y_1(\cdot))$ the couple control-trajectory which satisfies system (21) with the starting point $y_1 = \bar{y}$. We get

$$\begin{aligned}|y_1(t) - y_0(t)| &\le |y_1 - y_0| + \int_{t_0}^{t} |g(y_1(s), u_1(s)) - g(y_0(s), u_0(s))|\mathrm{d}s \\ &\le (1 + \tilde{C})|y_1 - y_0| + \tilde{C}\int_{t_0}^{t} |y_1(s) - y_0(s)|\mathrm{d}s, \qquad (26)\end{aligned}$$

invoking Lemma 3.4. Thus

$$\begin{aligned}&|y_1(t) - y_0(t)| + \int_{t_0}^{t} |g(y_1(s), u_1(s)) - g(y_0(s), u_0(s)|\mathrm{d}s \\ &\le C_0|y_1 - y_0| + C_0\int_{t_0}^{t} |y_1(s) - y_0(s)|\mathrm{d}s\end{aligned}$$

for some positive constant C_0 and, thanks to the Gronwall's Lemma, we obtain (25).

Finally, one can check that the set-valued map $\Sigma : \mathcal{U}(t_0, y_0) \rightsquigarrow \mathcal{U}(t_0, y_1)$ defined by:

$$\Sigma(u_0(\cdot)) := \{u(\cdot) \in \mathcal{U}(t_0, y_1)|(u(\cdot), y(\cdot)) \text{ solves } (21)\}$$

is nonexpansive with nonempty (*)-closed values (cf. Cardaliaguet and Plaskacz 2000). Hence, by Plaskacz Lemma (see Lemma 2.7 of Cardaliaguet and Plaskacz 2000) there exists a nonanticipative selection σ of Σ: namely, $\sigma(u_0(\cdot)) \in \Sigma(u_0(\cdot))$ for any $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$.

The proof of (16) is a direct consequence of the assumptions and of (15). □

We now apply the result just proved to system (12).

Corollary 3.5 *Under the assumptions* (4, 5, 6)*, for any terminal time* $T > 0$*, for any* $R > 0$*, there is a constant* C_2 *such that, for any* $X_0, X_1 \in [0, T] \times K_U \times K_V \times R$, $|X_0| \le R$, $|X_1| \le R$*, for any nonanticipative strategy* $\alpha \in S_U(t_0, x_0)$*, there is a nonanticipative strategy* $\alpha' \in S_U(t_1, x_1)$ *and a nonanticipative strategy* $\tau : \mathcal{V}(t_1, z_1) \to \mathcal{V}(t_0, z_0)$*, such that:* $\forall\, t \in [t_1, T]$ *and* $\forall\, v \in \mathcal{V}(t_1, z_1)$

$$\left|X[X_1, \alpha'(v), v](t) - X[X_0, \alpha \circ \tau(v), \tau(v)](t - t_1 + t_0)\right| \le C_2|X_1 - X_0|.$$

Remark 3.6

(1) The strategy $\tau : \mathcal{V}(t_1, z_1) \to \mathcal{V}(t_0, z_0)$ is nonanticipative in the following sense: if two control v_1, v_2 of $\mathcal{V}(t_1, z_1)$ coincide on the time interval $[t_1, t_1 + \theta]$ for some $\theta > 0$, then the images $\tau(v_1)$ and $\tau(v_2)$ coincide on $[t_0, t_0 + \theta]$.
(2) Of course a symmetric result holds true for a nonanticipative strategy $\beta \in S_V(t_0, x_0)$.
(3) If $z_0 = z_1$ and $t_0 = t_1$, then we can take $\tau(v) = v$ for any $v \in \mathcal{V}(t_1, z_1)$.

Proof of Corollary 3.5 To fix the ideas, we suppose to be under condition **2** of (6). Set $T' = T + |t_1 - t_0|$. From Proposition 3.1 [see estimate (16)] there is some constant $C_1 > 0$ and some nonanticipative strategy $\sigma_1 : \mathcal{V}(t_0, z_1) \to \mathcal{V}(t_0, z_0)$ such that: $\forall\, v \in \mathcal{V}(t_0, z_1), \forall\, t \in [t_0, T']$,

$$|z[t_0, z_1, v](t) - z[t_0, z_0, \sigma_1(v)](t)| + \int_{t_0}^{t} |v(s) - \sigma_1(v)(s)|\mathrm{d}s \leq C_1 \mathrm{e}^{C_1(t-t_0)}|z_0 - z_1|. \tag{27}$$

Let us define τ_1 and τ_2 as the shifts $\tau_1(\phi)(s) = \phi(s - t_0 + t_1)$ and $\tau_2 = \tau_1^{-1}$. We also set $\tau = \sigma_1 \circ \tau_1$. Then we get: $\forall\, v \in \mathcal{V}(t_1, z_1), \forall\, t \in [t_1, T]$,

$$\begin{aligned}&|z[t_1, z_1, v](t) - z[t_0, z_0, \tau(v)](t + t_0 - t_1)| \\ &\quad + \int_{t_1}^{t} |v(s) - \tau_2 \circ \sigma_1 \circ \tau_1(v)(s)|\mathrm{d}s \leq C_1 e^{C_1(t-t_1)}|z_0 - z_1|,\end{aligned} \tag{28}$$

by applying inequality (27) at time $t + t_0 - t_1$ to the control $\tau_1(v)$. In the same way, there is a nonanticipative strategy $\sigma_2 : \mathcal{U}(t_0, y_0) \to \mathcal{U}(t_0, y_1)$ such that: $\forall\, u \in \mathcal{U}(t_0, y_0), \forall\, t \in [t_0, T']$,

$$\begin{aligned}&|y[t_0, y_1, \sigma_2(u)](t) - y[t_0, y_0, u](t)| + \int_{t_0}^{t} |\sigma_2(u)(s) - u(s)|\mathrm{d}s \\ &\quad \leq C_1 \mathrm{e}^{C_1(t-t_0)}|y_0 - y_1|.\end{aligned}$$

Hence we have: $\forall\, u \in \mathcal{U}(t_0, y_0), \forall\, t \in [t_1, T]$,

$$\begin{aligned}&|y[t_1, y_1, \tau_2 \circ \sigma_2(u)](t) - y[t_0, y_0, u](t - t_1 + t_0)| \\ &\quad + \int_{t_1}^{t} |\tau_2 \circ \sigma_2(u)(s) - \tau_2(u)(s)|\mathrm{d}s \leq C_1 e^{C_1(t-t_1)}|y_0 - y_1|.\end{aligned} \tag{29}$$

Let us set $\alpha' = \tau_2 \circ \sigma_2 \circ \alpha \circ \sigma_1 \circ \tau_1$. Then we have: $\forall v \in \mathcal{V}(t_1, z_1)$, $\forall t \in [t_1, T]$,

$$\begin{aligned}
&\left|X[X_1, \alpha'(v), v](t) - X[X_0, \alpha \circ \tau(v), \tau(v)](t - t_1 + t_0)\right| \\
&\quad \leq \left|y[t_1, y_1, \alpha'(v)](t) - y[t_0, y_0, \alpha \circ \tau(v)](t - t_1 + t_0)\right| \\
&\qquad + |z[t_1, z_1, v](t) - z[t_0, z_0, \tau(v)](t - t_1 + t_0)| \\
&\qquad + \left|\rho_1 + \int_{t_1}^{t} L(s, x_1(s), \alpha'(v)(s), v(s)]ds \right. \\
&\qquad\qquad \left. -\rho_0 - \int_{t_0}^{t-t_1+t_0} L(s, x_0(s), \alpha \circ \tau(v)(s), \tau(v)(s))\mathrm{d}s\right|
\end{aligned}$$

(where $x_0(\cdot) = (y_0(\cdot), z_0(\cdot))$ and $x_1(\cdot) = (y_1(\cdot), z_1(\cdot))$ are the x-component of $X[X_0, \alpha \circ \tau(v), \tau(v)]$ and $X[X_1, \alpha'(v), v]$ respectively)

$$\begin{aligned}
&\leq |y_1(t) - y_0(t - t_1 + t_0)| + |z_1(t) - z_0(t - t_1 + t_0)| + |\rho_1 - \rho_0| \\
&\quad + \int_{t_1}^{t} \left|L(s, x_1(s), \alpha'(v)(s), v(s)) - \right. \\
&\qquad\quad L(s - t_1 + t_0, x_0(s - t_1 + t_0), \tau_2 \circ \alpha \circ \tau(v)(s), \tau_2 \circ \tau(v)(s))|\, \mathrm{d}s \\
&\leq |y_1(t) - y_0(t - t_1 + t_0)| + |z_1(t) - z_0(t - t_1 + t_0)| + |\rho_1 - \rho_0| \\
&\quad + M \int_{t_1}^{t} \big(|t_1 - t_0| + |x_1(s) - x_0(s + t_1 - t_0)| + |\alpha'(v)(s) - \tau_2 \circ \alpha \circ \tau(v)(s)| \\
&\quad + |v(s) - \tau_2 \circ \tau(v)(s))|\big)\, \mathrm{d}s \\
&\leq \bar{C}_1 \mathrm{e}^{C_1(t-t_1)} |X_0 - X_1|
\end{aligned}$$

for some constant $\bar{C}_1 \geq C_1$, thanks to (28) and (29) applied to $u := \alpha \circ \tau(v)$. □

4 Proof of Proposition 2.3

Throughout this section we consider the dynamic (12). It is worth pointing out that this new dynamic do satisfy assumptions (4) (when K_U is replaced by $R^+ \times K_U \times R$). Note also that under the conditions **1-** or **2-** of (6) the Hamiltonian $\widetilde{H}$ (defined by (11)) satisfies the Isaacs condition

$$\inf_{u \in U(y)} \sup_{v \in V(y)} \langle \tilde{f}, p \rangle = \sup_{v \in V(y)} \inf_{u \in U(y)} \langle \tilde{f}, p \rangle, \quad \forall\, y, p$$

The following result expresses a property of discriminating domains in terms of trajectories. Since this is a rather direct adaptation of results of Cardaliaguet (1996), we omit its proof.

Proposition 4.1 *Assume that the conditions* (4), (5) *and* (6) *are satisfied. Then a closed set* $D \subset [0,T] \times K_U \times K_V \times R$ *is a discriminating domain for* $\widetilde{H}$ *if and only if for any* $X_0 = (t_0, x_0, \rho_0) \in D$ *there is some nonanticipative strategy* $\beta \in S_V(t_0, x_0)$ *such that, for any control* $u(\cdot) \in \mathcal{U}(t_0, y_0)$*, the trajectory* $X[X_0; u(\cdot), \beta(u(\cdot))](s)$ *remains in D until it reaches* $\mathcal{E}$*.*

Proposition 4.2 *Assume that the conditions* (4), (5) *and* (6) *are satisfied. A closed set* $D \subset [0,T] \times K_U \times K_V \times R$ *is a discriminating domain for* $\widetilde{H}$ *if and only if for any* $X_0 = (t_0, x_0, \rho_0) \in D$*, for any nonanticipative strategy* $\alpha \in S_U(t_0, x_0)$*, for any* $\varepsilon > 0$*, there is some control* $v(\cdot) \in \mathcal{V}(t_0, z_0)$ *such that the trajectory* $X[X_0; \alpha(v(\cdot)), v(\cdot)](s)$ *remains in* $D + \varepsilon B$ *on* $[t_0, T]$*, at least as long as it does not reach* $\mathcal{E} + \varepsilon B$*: namely, there is a time* $T' \in [t_0, T]$ *such that*

$$X[X_0; \alpha(v(\cdot)), v(\cdot)](s) \in D + \varepsilon B \quad \forall s \in [t_0, T']$$

with either $T' = T$*, or* $X[X_0; \alpha(v(\cdot)), v(\cdot)](T') \in \mathcal{E} + \varepsilon B$*.*

Proof The condition is sufficient. We claim that for any $X_0 \in D$, for any admissible strategy $\alpha \in S_U(t_0, x_0)$ and for any $\varepsilon > 0$, there is a control $v \in \mathcal{V}(t_0, z_0)$ and a time $T' \in [t_0, T]$ such that

$$d_D(X[X_0, \alpha(v), v)](t)) \leq \varepsilon \quad \forall t \in [0, T'] \tag{30}$$

with either $T' = T$, or $X[X_0; \alpha(v(\cdot)), v(\cdot)](T') \in \mathcal{E} + \varepsilon B$.

Let $X_0 \in D$, $\alpha \in S_U(t_0, x_0)$ and $\varepsilon > 0$. Since D is a discriminating domain, Proposition 4.1 states that there is some nonanticipative strategy $\beta_0 \in S_V(t_0, x_0)$ such that, for any control $u \in \mathcal{U}(t_0, y_0)$, the solution $X[X_0, u, \beta_0(u)]$ remains in D as long as it does not reach the evasion set $\mathcal{E}$.

Let us also fix $\bar{v} \in \mathcal{V}(t_0, z_0)$ and let us set $z_1 = z[t_0, z_0, \bar{v}](\tau)$, for some $\tau > 0$ to be defined later. Note for later use that $|z_0 - z_1| \leq M_g \tau$, where M_g is a bound on g. We set $X_1 = (t_0, y_0, z_1, \rho_0)$. By Corollary 3.5 and Remark 3.6-3), (for any $\beta \in S_V(t_0, x_0)$) there is a nonanticipative strategy $\beta_1 \in S_V(t_1, x_1)$ and a constant $C_2 > 0$ such that, for any $u \in \mathcal{U}(t_0, y_0)$ (recall that $(t_0, y_0) = (t_1, y_1)$), $\forall t \in [t_0, T]$,

$$|X[X_1, u, \beta_1(u)](t) - X[X_0, u, \beta(u)](t)| \leq C_2 |z_1 - z_0| \leq C_2 M_g \tau.$$

We then define $\beta \in S_V(t_0, x_0)$ by setting

$$\beta(u)(t) = \begin{cases} \bar{v}(t) & \text{if } t \in [t_0, t_0 + \tau) \\ \beta_1(u)(t - \tau) & \text{if } t \geq t_0 + \tau \end{cases}$$

We note that β is a nonanticipative strategy with delay, i.e.,

$$\text{if } u_1 = u_2 \text{ on } [t_0, t] \text{ then } \beta(u_1) = \beta(u_2) \text{ on } [t_0, t + \tau].$$

The next step of the proof consists in comparing $X[X_0, u, \beta(u)]$ and $X[X_0, u, \beta_0(u)]$ for any control $u \in \mathcal{U}(t_0, y_0)$. Let us recall that

$$X[X_0, u, \beta(u)](t) = (t, y[y_0, u](t), z[t_0, z_0, \beta(u)](t), \rho[X_0, u, \beta(u)](t))$$

and

$$X[X_0, u, \beta_0(u)](t) = (t, y[y_0, u](t), z[t_0, z_0, \beta_0(u)](t), \rho[X_0, u, \beta_0(u)](t)).$$

We note that the y-component of $X[X_0, u, \beta(u)]$ and $X[X_0, u, \beta_0(u)]$ is the same. It remains to compare the z and ρ components. We first compare $z_1(\cdot) := z[t_0, z_0, \beta(u)](\cdot)$ and $z_0(\cdot) := z[t_0, z_0, \beta_0(u)](\cdot)$. For any $t \geq t_0 + \tau$, we have $z_1(t) = z[t_0, z_1, \beta_1(u)](t - \tau)$ and so

$$\begin{aligned} |z_1(t) - z_0(t)| &\leq |z[t_0, z_1, \beta_1(u)](t - \tau) - z_0(t - \tau)| + |z_0(t - \tau) - z_0(t)| \\ &\leq C_2 M_g \tau + M_g \tau. \end{aligned}$$

Next we compare $\rho_1(\cdot) := \rho[X_0, u, \beta(u)](\cdot)$ and $\rho_0(\cdot) := \rho[X_0, u, \beta_0(u)](\cdot)$. To fix the ideas we work in the case **2** of assumption (6). For any $t \geq t_0$, we have

$$\rho_1(t) = \rho_0 - \int_{t_0}^{t} L(s, y_0(s), z_1(s), u(s), \beta(u)(s))\mathrm{d}s.$$

So, setting $x_0(\cdot) = (y[t_0, y_0, u](\cdot), z_0(\cdot))$, we get

$$\left|\rho_1(t) - \rho_0 + \int_{t_0}^{t} L(s, y_0(s), z_0(s), u(s), \beta(u)(s))\mathrm{d}s\right| \leq M \int_{t_0}^{t} |z_0(s) - z_1(s)|\mathrm{d}s \leq C\tau$$

for some constant C. Then, from the structure condition (6-condition 2), we have

$$\begin{aligned} |\rho_0(t) - \rho_1(t)| &\leq \left|\rho_0(t) - \rho_0 + \int_{t_0}^{t} L(s, x_0(s), u(s), \beta(u)(s))\mathrm{d}s\right| + C\tau \\ &\leq \left|\int_{t_0}^{t} (L_2(s, x_0(s))\beta_0(u)(s) - L_2(s, x_0(s))\beta(u)(s))\, \mathrm{d}s\right| + C\tau \\ &\leq \left|\int_{t_0}^{t_0+\tau} L_2(s, x_0(s))\beta(u)(s)\mathrm{d}s\right| + \left|\int_{t-\tau}^{t} L_2(s, x_0(s))\beta_0(u)(s)\mathrm{d}s\right| \end{aligned}$$

$$+\left|\int_{t_0}^{t-\tau}(L_2(s,x_0(s))\beta_0(u)(s)-L_2(s+\tau,x_0(s+\tau))\beta_1(u)(s))\mathrm{d}s\right|+C\tau$$
$$\leq C'\tau$$

for some new constant $C' \geq C$ which does not depend on u and β_0.

From now on, we choose τ such that $C'\tau \leq \varepsilon$. We have proved that, for any $t \in [t_0, T]$,

$$|X[X_0, u, \beta(u)](t) - X[X_0, u, \beta_0(u)](t)| \leq C'\tau \leq \varepsilon.$$

Since $X[X_0, u, \beta_0(u)]$ remains in D until it reaches the target $\mathcal{E}$ at some time T', we have

$$d_D(X[X_0, u, \beta(u)](t)) \leq |X[X_0, u, \beta(u)](t) - X[X_0, u, \beta_0(u)](t)| \leq \varepsilon$$

for all $t \in [0, T']$ and

$$d_{\mathcal{E}}(X[X_0, u, \beta(u)](T')) \leq |X[X_0, u, \beta(u)](T') - X[X_0, u, \beta_0(u)](T')| \leq \varepsilon.$$

Hence $X[X_0, u, \beta(u)]$ remains in $D + \varepsilon B$ until it reaches the target $\mathcal{E} + \varepsilon B$.

Since α is nonanticipative and β is nonanticipative with delay, it can be easily proved that there is a unique pair $(u, v) \in \mathcal{U}(t_0, y_0) \times \mathcal{V}(t_0, z_0)$ such that

$$\alpha(v) = u \quad \text{and} \quad \beta(u) = v.$$

Then, for this control v we have

$$X[X_0, \alpha(v), v] = X[X_0, u, \beta(u)]$$

which shows that for this choice of v, the trajectory $X[X_0, \alpha(v), v]$ remains in $D + \varepsilon B$ until it reaches the target $\mathcal{E} + \varepsilon B$. This completes the proof of the sufficiency part of the Proposition.

Now we show that the condition is necessary. We suppose to be under condition **1-** of (6) for simplicity of notations (case **2-** of (6) is quite similar). Assume that D is not discriminating for $\widetilde{H}$. Let $X_0 = (t_0, x_0, \rho_0) \in \partial D \setminus \mathcal{E}$ be such that $\exists p \in NP_D(X_0)$ and $\exists \gamma > 0$ with the property:

$$\widetilde{H}(X_0, p) \geq 2\gamma.$$

There exists $\bar{u} \in U(y_0)$ such that

$$p_t + \inf_V \left\{\langle h(z_0, v), p_z\rangle - p_\rho L(t, x_0)\right\} + \langle g(y_0, \bar{u}), p_z\rangle \geq \gamma.$$

The set valued map $y \mapsto g(y, U(y))$ is lower semicontinuous with convex compact values, so by Michael's selection Theorem [see for instance Aubin and Frankowska (1990)] there exists $\tilde{u} : K_U \longrightarrow U$ such that

- $y \mapsto g(y, \tilde{u}(y))$ is continuous
- $g(y_0, \tilde{u}(y_0)) = g(y_0, \bar{u})$.

Hence, there exists a neighborhood of (t_0, x_0), say $I_{(t_0,x_0)}$, such that $\forall (t, x) \in I_{(t_0,x_0)}$

$$\inf_V \left\{ \langle \tilde{f}(X, \tilde{u}(y), v), p \rangle \right\} \geq \frac{\gamma}{2}.$$

So, $\forall\, v(\cdot) \in \mathcal{V}(t_0, z_0)$, there exists a control $u(\cdot) = \tilde{u}(y(\cdot))$ such that for all $s \in (0, \tau)$, there exists $\tau > 0$ such that:

$$\begin{aligned} &|X[X_0, u(\cdot), v(\cdot)](s) - X_0 - p|^2 \\ &\quad \leq |p|^2 - 2 \int_{t_0}^{t_0+s} \langle \tilde{f}(X(\sigma), \tilde{u}(y(\sigma)), v(\sigma)), p\rangle \mathrm{d}\sigma + Cs^2 \leq |p|^2 - \gamma s + Cs^2. \end{aligned}$$

Hence, we obtain

$$\mathrm{d}_D(X[X_0, u(\cdot), v(\cdot)](s)) \geq |p|^2 - [|p|^2 - \gamma s + Cs^2]^{\frac{1}{2}}$$

which is positive $\forall s \in (0, \tau)$ for $\tau > 0$ sufficiently small. Therefore, there exists a nonanticipative strategy is $\alpha : \mathcal{V}(t_0, z_0) \longrightarrow \mathcal{U}(t_0, y_0)$, defined by the constant map $\alpha(v(\cdot)) = u(\cdot)$ *for any* $v(\cdot) \in \mathcal{V}(t_0, z_0)$, such that the trajectory $X[X_0, \alpha(v(\cdot)), v(\cdot)](s)$ *for all* $v(\cdot) \in \mathcal{V}(t_0, z_0)$ leaves $D + \varepsilon B$ before the time τ for $\varepsilon > 0$ suitably small. □

Recall now the notation $\mathcal{K} \leq [0, T] \times \mathbb{R}^n \times R$. The next step toward the proof of Proposition 2.3 is the characterization of the set

$$\begin{aligned} \mathcal{L} := \Big\{ & X_0 \in \mathcal{K} \,|\, \forall \alpha \in S_U(t_0, x_0),\ \forall \varepsilon > 0, \exists v(\cdot) \in \mathcal{V}(t_0, z_0),\ \exists T' \in [t_0, T] \quad \text{s.t.} \\ & X[X_0, \alpha(v(\cdot)), v(\cdot)](T') \ \in \mathcal{E} + \varepsilon B \Big\} \end{aligned}$$

Lemma 4.3 *Under the assumptions* (4, 5, 6)*, suppose that* $X_0 \in \mathcal{K} \setminus \mathcal{L}$*. Then there is* $\eta_T > 0$*,* $\varepsilon > 0$ *and, for any* $X_1 \in \{X_0 + \eta_T B\}$*, a nonanticipative strategy* $\alpha \in S_V(t_1, x_1)$ *such that, for any control* $v(\cdot) \in \mathcal{V}(t_1, z_1)$*, the trajectory* $X[X_1, \alpha(v(\cdot)), v(\cdot)](\cdot)$ *never reaches* $\mathcal{E} + \varepsilon B$ *before* T*.*

Proof By the very definition of $\mathcal{L}$, there exists a strategy $\alpha_0 \in S_U(t_0, x_0)$ and a real number $\varepsilon_0 > 0$ such that for any $v_0(\cdot) \in \mathcal{V}(t_0, z_0)$, the solution $X[X_0, \alpha_0(v_0(\cdot)), v_0(\cdot)]$ never reaches $\mathcal{E} + \varepsilon_0 B$ between t_0 and T.

Let us set $\eta_T := \varepsilon_0/(2(M_0 + C_2))$, where C_2 is the constant which appears in Corollary 3.5 for the time $T + 1$, $R = |X_0| + 1$ and M_0 is a bound on the dynamics on a sufficiently large ball. From Corollary 3.5, for any $X_1 \in X_0 + \eta_T B$, there is a nonanticipative strategy $\alpha \in S_U(t_1, x_1)$ and a nonanticipative strategy $\tau : \mathcal{V}(t_1, z_1) \to \mathcal{V}(t_0, z_0)$, such that: $\forall v \in \mathcal{V}(t_1, z_1)$, $\forall t \in [t_1, T + 1]$,

$$|X[X_1, \alpha(v), v](t) - X[X_0, \alpha_0(\tau(v)), \tau(v)](t - t_1 + t_0)| \leq C_2|X_1 - X_0|.$$

Let us fix $v \in \mathcal{V}(t_1, z_1)$ and let us set $X_1 := X[X_1, \alpha(v), v]$, $X_0 := X[X_0, \alpha_0(\tau(v)), \tau(v)]$ and $T_1 = \min\{T, T - t_1 + t_0\}$. Then, for any $t \in [t_1, T_1]$,

$$\begin{aligned} d_{\mathcal{E}}(X_1(t)) &\geq d_{\mathcal{E}}(X_0(t - t_1 + t_0)) - |X_0(t - t_1 + t_0) - X_1(t)| \\ &\geq \varepsilon_0 - C_2|X_1 - X_0| \geq \varepsilon_0/2 \end{aligned}$$

since $d_{\mathcal{E}}(X_0(t - t_1 + t_0)) \geq \varepsilon$ on $[t_1, T_1]$. In conclusion, there is a nonanticipative strategy $\alpha \in S_U(t_1, x_1)$ such that, for any $v \in \mathcal{V}(t_1, z_1)$, the solution $X[X_1, \alpha(v), v](t)$ never reaches $\mathcal{E} + \varepsilon B$ on $[t_0, T]$, where $\varepsilon := \varepsilon_0/2$. □

We now complete the proof of Proposition 2.3.

- Part (ii) of Proposition 2.3 can be easily deduced from the previous results. We refer the reader to (Cardaliaguet et al. 2001) for the complete proof, recalling only its scheme: Proposition 4.2 implies $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K}) \subset \mathcal{L}$. On the other hand, $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K}) \supset \mathcal{L}$ is given by Lemma 4.3. So (ii) is proved.
- Let us prove (i) of Proposition 2.3. From Proposition 4.2, we obtain that $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ is contained in the set of points $X_0 = (t_0, x_0, \rho_0) \in [0, T] \times K_U \times K_V \times R$, for which there is some $\beta \in S_V(t_0, x_0)$, such that, for any control $u(\cdot) \in \mathcal{U}(t_0, y_0)$, the solution $X[X_0; u, \beta(u)]$ remains in $\mathcal{K}$ as long as it does not reach $\mathcal{E}$.

Let us prove the converse inclusion. Consider $X_0 \in \mathcal{K}\backslash\mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$. For any β: $\mathcal{U}(t_0, y_0) \mapsto \mathcal{V}(t_0, z_0)$, we want to construct $u(\cdot) \in \mathcal{U}(t_0, y_0)$ such that $X[X_0; u(\cdot), \beta(u(\cdot))]$ leaves $\mathcal{K}$ before reaching $\mathcal{E}$. For doing this, we adopt a method described in Cardaliaguet et al. (2001), Osipov (1971). We claim that $\bigcap_n K_n = \mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ where K_n is defined as follows:

$$\begin{cases} K_0 = \mathcal{K} \\ K_{n+1} := \left\{ X_0 \in K_n \mid \begin{array}{c} \forall u(\cdot) \in \mathcal{U}(t_0, y_0),\ \exists v(\cdot) \in \mathcal{V}(t_0, z_0),\ \exists \tau \in [0, +\infty] \\ \text{s.t. } X[X_0, u(\cdot), v(\cdot)](\tau) \in \mathcal{E} \quad \text{if } \tau < +\infty, \quad \text{and} \\ \forall t \in [0, \tau),\ X[X_0, u(\cdot), v(\cdot)](t) \in K_n \end{array} \right\} \end{cases}.$$

Note that $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K}) \subset K_n$ for any n. One can also easily check that $\bigcap_n K_n$ is a discriminating domain. Hence $\bigcap_n K_n \subset \mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ which proves our claim.

Now let us fix some $\beta \in S_V(t_0, x_0)$. Since $X_0 \in \mathcal{K}\backslash\mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$, there exists some $i_0 > 0$ with $X_0 \notin K_{i_0+1}$. Thus there exists $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$ such that for any $v(\cdot) \in \mathcal{V}(t_0, z_0)$, the solution $X[X_0, u_0, v]$ leaves K_{i_0} before reaching $\mathcal{E}$. In particular, this is the case for $v = \beta(u_0)$. Let τ_0 be such that $X[X_0, u_0, v](\tau_0) \notin K_{i_0}$

and $X[X_0, u_0, \beta(u_0)] \notin \mathcal{E}$ on $[t_0, \tau_0]$. Then we can apply the same procedure for $X_1 := X[X_0, u_0, \beta(u_0)](\tau_0)$ which does not belong to K_{i_0}.

This leads to construct recursively a control $u \in \mathcal{U}(t_0, y_0)$ and a increasing sequence of times $(\tau_i)_{i=0,\dots,i_0+1}$ such that

$$X[X_0, u, \beta(u)](\tau_i) \notin K_{i_0-i}, \quad X[X_0, u, \beta(u)](s) \notin \mathcal{E} \quad \forall s \in [t_0, \tau_{i_0+1}].$$

In particular, $X[X_0, u, \beta(u)](\tau_{i_0+1}) \notin K_0 = \mathcal{K}$.

This proves that for any nonanticipative strategy $\beta \in S_V(t_0, x_0)$, there is a control $u \in \mathcal{U}(t_0, y_0)$ such that the solution $X[X_0, u, \beta(u)]$ leaves $\mathcal{K}$ before reaching $\mathcal{E}$. □

5 The Hamilton–Jacobi formulation of the problem

In this Section our aim is to characterize the value function studied before as the unique viscosity solution to the Hamilton–Jacobi–Isaacs equation:

$$\begin{cases} -\partial_t W(t,x) + H\Big(t, x, \partial_x W(t,x)\Big) = 0 & \text{on } (0,T) \times K_U \times K_V \\ W(T,x) = \Psi(x) & \text{on } K_U \times K_V, \end{cases} \tag{31}$$

where the Hamiltonian function, H, is given by:

$$H(t,x,p) := \max_{v \in V(z)} \min_{u \in U(y)} \{-\langle f(x,u,v), p\rangle - L\},$$

and where L is the running cost function. Since the function H is not continuous in general, in order to consider the notion of solution in viscosity sense we have to use the upper and lower semicontinuous envelope of H, denoted by H^* and H_*, respectively [see for instance Barles (1994) or Bardi and Capuzzo-Dolcetta (1997)]. We remind that by definition these functions are:

$$H^*(t,x,p) := \limsup_{(t',x',p') \to (t,x,p)} H(t', x', p')$$

and

$$H_*(t,x,p) := \liminf_{(t',x',p') \to (t,x,p)} H(t', x', p').$$

Remark 5.1 Suppose that assumption (6) holds true. In this case, since the set-valued maps $y \rightsquigarrow U(y)$ and $z \rightsquigarrow V(z)$ are lower semicontinuous and the functions f and L are Lipschitz continuous, it is straightforward to see that

$$H^*(t,x,p) = \max_{v \in V} \min_{u \in U(y)} \{-\langle f(x,u,v), p\rangle - L\}$$

and

$$H_*(t,x,p) = \max_{v\in V(z)} \min_{u\in U}\{-\langle f(x,u,v),p\rangle - L\}.$$

Definition 5.2 (Viscosity solution) *A viscosity super-solution for the Hamilton–Jacobi–Isaacs equation* (31) *is a lower semicontinuous function* $w : [0,T) \times K_U \times K_V \longrightarrow R$ *with the following property: for any test function* $\varphi \in \mathcal{C}^1$ *such that* $w - \varphi$ *has a local minimum at* (t_0,x_0) *then*

$$-\partial_t\varphi(t_0,x_0) + H^*(t_0,x_0,\partial_x\varphi(t_0,x_0)) \ \geq 0.$$

An upper semicontinuous function $w : [0,T) \times K_U \times K_V \longrightarrow R$ *is called viscosity subsolution of* (31) *if* $\forall\varphi \in \mathcal{C}^1$ *such that* $w - \varphi$ *has a local maximum at* (t_0,x_0) *then*

$$-\partial_t\varphi(t_0,x_0) + H_*(t_0,x_0,\partial_x\varphi(t_0,x_0)) \ \leq 0.$$

We say that a continuous function is a viscosity solution of (31) *if it is both supersolution and subsolution of* (31) *at the same time.*

5.1 Characterization of supersolution

In this section we show that the value function is the smallest supersolution of the Hamilton–Jacobi–Isaacs equation (31). For this we give an equivalent formulation of the supersolution, which involves the notion of proximal normal.

Define the set $Q := [0,T) \times K_U \times K_V \subset R^{n+1}$. Let $w : Q \longrightarrow R$ be a lower semicontinuous function. Then any proximal normal to the epigraph of w at the point $(q,w(q))$ belongs to $R^{n+1} \times R$. If we take an element $(p_q,p_\rho) \in NP_{\mathcal{E}pi(w)}(q,w(q))$ with $p_q \in R^{n+1}$ and $p_\rho \in R$, it is easy to check that $p_\rho \leq 0$.

Theorem 5.3 (See Theorem 7.2 in Cardaliaguet et al. (1999)) *Assume that the Hamiltonian* $\mathcal{H} : Q \times R \times R^{n+1} \longrightarrow R$ *is a lower semicontinuous and that* $w : Q \longrightarrow R \cup \{\infty\}$ *is an extended lower semicontinuous map. Then* w *is a viscosity super-solution to*

$$\mathcal{H}(q,w(q),Dw(q)) = 0$$

if and only if w *satisfies*

$$\forall\, q \in Q,\ \forall\, (p_q,p_\rho) \in NP_{\mathcal{E}pi(w)}(q,w(q)),\ p_\rho \neq 0 \quad \text{then} \quad \mathcal{H}\left(q,w(q),\frac{p_q}{p_\rho}\right) \geq 0.$$

Besides this theorem, we need an another result due to Rockafellar, in order to explain what happens for the proximal normals of the form $(p_q,0) \in NP_{\mathcal{E}pi(w)}(q,w(q))$ and $(p_q,p_\rho) \in NP_{\mathcal{E}pi(w)}(q,\rho)$ with $\rho > w(q)$ [see for instance Lemma 7.3 and Lemma 7.4 of Cardaliaguet et al. (1999)].

Lemma 5.4 *Let $w : Q \longrightarrow R \cup \{\infty\}$ be a lower semicontinuous map and $\bar{q}$ a point of the domain of w,* $\mathrm{Dom}(w)$.

(i) *If $(p_q, 0) \in NP_{\mathcal{E}pi(w)}(\bar{q}, w(\bar{q}))$, then there exists a sequence $q^n \in \mathrm{Dom}(w)$ and $(p_q^n, p_\rho^n) \in NP_{\mathcal{E}pi(w)}(q^n, w(q^n))$ such that*

$$q^n \to \bar{q},\ w(q^n) \to w(\bar{q}),\ p_\rho^n < 0 \quad \text{and} \quad \frac{(p_q^n, p_\rho^n)}{|(p_q^n, p_\rho^n)|} \to \frac{(p_q, 0)}{|p_q|}.$$

(ii) *If $(p_q, p_\rho) \in NP_{\mathcal{E}pi(w)}(\bar{q}, \rho)$ with $\rho > w(\bar{q})$, then we have $p_\rho = 0$ and $(p_q, 0) \in NP_{\mathcal{E}pi(w)}(\bar{q}, w(\bar{q}))$.*

Theorem 5.5 *Under the assumptions* (4, 5, 6)*, the function $\vartheta := v^\flat = v^\sharp$ is the smallest lower semicontinuous super-solution to* (31).

Proof of Theorem 5.5 By Theorem 2.2, the epigraph of ϑ is the discriminating kernel of $\mathcal{K}$ for the Hamiltonian function $\widetilde{H}$, $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$. So, for any $q = (t, x) \notin \mathcal{E}$ and for any $(p_q, p_\rho) = (p_t, p_x, p_\rho) \in NP_{\mathcal{E}pi(\vartheta)}\big((t, x), \vartheta(t, x)\big)$, by the definition of $\widetilde{H}$ and proximal normal, we get

$$p_\rho < 0 \implies \widetilde{H}(t, x, \rho; p_t, p_x, p_\rho) \le 0$$

that is:

$$-\frac{p_t}{|p_\rho|} + H(t, x, \frac{p_x}{|p_\rho|}) \ge 0.$$

So applying Theorem 5.3 with $\mathcal{H}(t, x; p_t; p_x) = p_t + H_*(t, x, p_x)$, we have that ϑ is a lower semicontinuous super-solution of the Hamilton–Jacobi–Isaacs equation (31).

On the other hand, we claim that the epigraph $\mathcal{E}pi(w)$ of any lower semi-continuous viscosity super-solution of (31) w is a discriminating domain for the Hamiltonian $\widetilde{H}$. Since $\mathcal{E}pi(w)$ is contained in $\mathcal{K}$, the discriminating domain of $\mathcal{E}pi(w)$ is contained in $\mathrm{Disc}_{\widetilde{H}}(\mathcal{K})$ which is precisely equal to the epigraph of ϑ.

Indeed, choose a point (t, x) with $t \neq T$ and take any $(p_t, p_x, p_\rho) \in NP_{\mathcal{E}pi(w)}(t, x, w(t, x))$. If $p_\rho < 0$, just applying Theorem 5.3 we obtain that

$$p_\rho < 0 \implies -\frac{p_t}{|p_\rho|} + H(t, x, \frac{p_x}{|p_\rho|}) \ge 0$$

and, so,

$$\widetilde{H}\left(t, x, \rho; p_t, p_x, p_\rho\right) \le 0.$$

If $p_\rho = 0$, then thanks to (i) of Lemma 5.4, it is possible to find sequences

$$(t^n, x^n) \quad \text{and} \quad (p_t^n, p_x^n, p_\rho^n) \in NP_{\mathcal{E}pi(w)}(t^n, x^n, w(t^n, x^n))$$

such that

$$(t^n,x^n) \to (t,x),\ w(t^n,x^n) \to w(t,x),\ p^n_\rho < 0 \quad \text{and} \quad \frac{(p^n_t,p^n_x,p^n_\rho)}{|(p^n_t,p^n_x,p^n_\rho)|} \to \frac{(p_t,p_x,0)}{|(p_t,p_x)|}.$$

Since $\widetilde{H}(t^n,x^n,\rho^n;p^n_t,p^n_x,p^n_\rho) \le 0$, $\widetilde{H}$ is lower semicontinuous and positively homogeneous, we conclude that $\widetilde{H}(t,x,\rho;p_t,p_x,0) \le 0$. If $(p_t,p_x,p_\rho) \in \mathrm{NP}_{\mathcal{E}pi(\vartheta)}((t,x),\rho)$ for some $\rho > w(t,x)$, then (ii) of Lemma 5.4 states that $p_\rho = 0$ and $(p_t,p_x,0) \in \mathrm{NP}_{\mathcal{E}pi(w)}(t,x,w(t,x))$. Hence, we obtain $\widetilde{H}(t,x,\rho;p_t,p_x,0) \le 0$ □

5.2 Lipschitz continuity of the value function

The aim of this part is to prove the following result:

Proposition 5.6 *Under the assumptions (4, 5, 6), if the final cost $\Psi = \Psi(x)$ is locally Lipschitz continuous, then the value function $\vartheta := v^\flat = v^\sharp$ is also locally Lipschitz continuous.*

Proof Since the dynamics have a linear growth, and that we work locally in space, we can assume that $\tilde{f}$ defined by (10) and Ψ are bounded by some constant M_0, and that Ψ is Lipschitz continuous with Lipschitz constant M_0.

Let $(t_0,x_0),(t_1,x_1) \in [0,T] \times K_U \times K_V$ and $\varepsilon > 0$. From the definition of $v^\flat$, there is a nonanticipative strategy β such that

$$v^\flat(t_0,x_0) + \varepsilon \ge \sup_{u\in\mathcal{U}(t_0,y_0)} J(t_0,x_0,u,\beta(u)).$$

Let us set $\rho_0 = \rho_1 = 0$, $X_0 = (t_0,x_0,\rho_0)$ and $X_1 = (t_1,x_1,\rho_1)$. From Corollary 3.5, there is a nonanticipative strategy $\beta' \in S_V(t_1,x_1)$ and a nonanticipative strategy $\tau : \mathcal{U}(t_1,y_1) \to \mathcal{U}(t_0,y_0)$, such that: $\forall\, t \in [t_1, T+1]$,

$$\left|X[X_1,u,\beta'(u)](t) - X[X_0,\tau(u),\beta(\tau(u))](t-t_1+t_0)\right| \le C_2|X_1 - X_0|.$$

Let us denote by $x_0(\cdot)$ and $x_1(\cdot)$, and by $\rho_0(\cdot)$ and $\rho_1(\cdot)$ the $x-$ and $\rho-$components of $X[X_0,\tau(u),\beta(\tau(u))]$ and $X[X_1,u,\beta'(u)]$, respectively. Then

$$\rho_0(t) = -\int_{t_0}^{t} L(s,x_0(s),\tau(u)(s),\beta(\tau(u))(s))\mathrm{d}s \qquad \forall t \ge t_0$$

and

$$\rho_1(t) = -\int_{t_1}^{t} L(s,x_1(s),u(s),\beta'(u)(s))\mathrm{d}s \quad \forall\, t \ge t_1.$$

From the choice of β, we have that

$$\Psi(x_0(T)) - \rho_0(T) = J(t_0, x_0, \tau(u), \beta(\tau(u)) \leq v^\flat(t_0, x_0) + \varepsilon.$$

Hence

$$\begin{aligned} J(t_1, x_1, u, \beta'(u)) &= \Psi(x_1(T)) - \rho_1(T) \\ &\leq \Psi(x_0(T)) - \rho_0(T) + M_0|x_0(T) - x_0(T - t_1 + t_0)| \\ &\quad + M_0|x_0(T - t_1 + t_0) - x_1(T)| \\ &\quad + |\rho_0(T) - \rho_0(T - t_1 + t_0)| + |\rho_0(T - t_1 + t_0) - \rho_1(T)| \\ &\leq v^\flat(t_0, x_0) + \varepsilon + 2M_0^2|t_1 - t_0| + M_0 C_2|X_1 - X_0|. \end{aligned}$$

Taking in the above inequality the supremum over $u \in \mathcal{U}(t_1, y_1)$, then the infimum over $\beta' \in S_V(t_1, x_1)$ and letting $\varepsilon \to 0$ finally gives:

$$v^\flat(t_1, x_1) \leq v^\flat(t_0, x_0) + C(|t_1 - t_0| + |x_1 - x_0|)$$

for some constant C. □

5.3 Uniqueness of continuous viscosity solution

The last result of this paper amounts to characterize the value function as unique viscosity solution of the Hamilton–Jacobi–Isaacs equation (31). For this we assume from now on that the data satisfy the assumptions (4, 5, 6) and that the final cost Ψ is locally Lipschitz continuous. Under these assumptions we have proved in the previous subsection that the value function ϑ is Lipschitz continuous.

Let us start by recalling a well-known result:

Lemma 5.7 *An upper semicontinuous function* $w : [0, T) \times K_U \times K_V \longrightarrow R$ *is a viscosity subsolution of* (31) *if and only if* $-w$ *is a viscosity supersolution of*

$$-\partial_t W(t, x) - H\Big(t, x, -\partial_x W(t, x)\Big) = 0 \tag{32}$$

Let us also recall that in our case the Hamiltonian function is given by

$$H(t, x, p) := \max_{v \in V(z)} \min_{u \in U(y)} \{-\langle f(x, u, v), p\rangle - L\},$$

by Remark 5.1 we deduce that

$$(-H)^*(t, x, -p) = \min_{v \in V(z)} \max_{u \in U} \{-\langle f(x, u, v), p\rangle + L\}.$$

Lemma 5.8 *The function ϑ is the largest viscosity subsolution of* (31).

Proof First observe that

$$-\vartheta(t_0,x_0) = \sup_{\beta\in S_V(t_0,x_0)} \inf_{u(\cdot)\in\mathcal{U}(t_0,y_0)} -J(t_0,x_0;u(\cdot),\beta(u(\cdot)))$$

and recall that $-\Psi$ is continuous by the assumptions. We just apply the result given by Theorem 5.5 to the function $-\vartheta$ with the Hamiltonian $\bar{H}(t,x,p) = (-H)^*(t,x,-p)$. Finally, Lemma 5.7 allows us to conclude. □

We conclude with the following uniqueness result.

Proposition 5.9 (Uniqueness) *The Hamilton–Jacobi–Isaacs equation* (31) *admits a unique continuous viscosity solution, which is given by the value function ϑ.*

Proof We have to show that if w is a continuous viscosity solution of (31) then $w = \vartheta$. Indeed, by Theorem 5.5 we get that $w \geq \vartheta$. On the other hand, thanks to Lemma 5.8 $-w \geq -\vartheta$ and, so, $w = \vartheta$. □

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