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A dynamic approach to the Shapley value based on associated games

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Abstract. We propose a dynamic process leading to the Shapley value of TU games or any solution satisfying Inessential Game (IG) and Continuity (CONT), based on a modified version of Hamiache's notion of an associated game.

Key words: Shapley value, associated game, dynamic process.

1. Introduction

Roughly speaking there are two approaches to cooperative games. A solution concept can be given *axiomatic justification*. Alternatively, *dynamic processes* can be defined that lead the players to that solution, starting from an arbitrary Pareto-optimal payoff vector. The foundation of a dynamic theory was laid by Stearns (1968). He defined transfer schemes which, starting from an arbitrary Pareto-optimal payoff vector, produce a resulting payoff that always converges to *bargaining sets*. A continuous analogue was developed by Billera (1972). In his paper, he extended the notion of transfer sequence and, relying heavily on Stearns' methods, proved the necessary convergence results.

Hart and Mas-Colell (1989) introduced a notion of consistency and used it to axiomatize the Shapley value. Maschler and Owen (1989) successfully adopted the reduced game used by Hart and Mas-Colell to provide a dynamic process leading to the Shapley value for the class of hyperplane games. On the other hand, Hamiache (2001) introduced the notion of associated game and gave a new characterization for the Shapley value. The axioms are inessential game, associated consistency and continuity, but the central one is a consistency notion that is defined in terms of his associated game. Naturally, as the important role of reduced game played in both the characterization of, and

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dynamic process leading, to the Shapley value, one would like to find a dynamic process leading to the Shapley value that is based on associated games. The aim of this paper is to do so.

The paper is organized as follows: The definitions and notations are given in section 2. In section 3, we describe the dynamic process and prove that it converges to the Shapley value. The same process also applies to any solution satisfying *IG* and *CONT*. The discussion is in section 4.

2. Definitions and Notations

Let U be a non-empty and finite set of players. A coalition is a non-empty subset of U. A coalitional game with transferable utility (TU game) is a pair (N, v) where N is a coalition and v is a mapping such that $v : 2^N \longrightarrow \mathbb{R}$ and $v(\emptyset) = 0$. We denote by G the set of all games. The notation $S \subset T$ and $S \subseteq T$ mean that S is a proper subset of T and S is a subset of T, respectively. We use the lowercase letter to denote the number of elements of a set if no confusion arises.

Definition 1. A game (N, v) is inessential if for each coalition $S \subseteq N$,

$$v(S) = \sum_{j \in S} v(\{j\}).$$

Definition 2. A solution on G is a function σ which associates with each game $(N, v) \in G$ an element $\sigma(N, v)$ of $\mathbb{R}^N$.

Definition 3. The Shapley value f is the solution on G which associates with each game (N, v) and each player $j \in N$ the value

$$f_j(N, v) = \sum_{\substack{S \subseteq N \\ j \in S}} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus \{j\})]. \tag{1}$$

The value $f_j(N, v)$ can be interpreted as the expectation of the marginal contribution of j to coalition S, $j \in S$.

Shapley (1953) introduced the Shapley value, and several axiomatic characterizations of this value were given by Shapley (1953), Young (1985) and Hart and Mas-Colell (1989). In 2001, Hamiache presented a consistency axiom to characterize it. His axiom, called associated consistency, is defined in terms of the notion of an "associated game".

Definition 4. (Hamiache, 2001) *For each game (N, v) in G, the associated game is the game denoted $(N, v^*_{\lambda,H})$, whose characteristic function $v^*_{\lambda,H}$ is defined by*

$$v^*_{\lambda,H}(S) = \begin{cases} 0 & \text{, if } S = \emptyset \\ v(S) + \lambda \sum_{j \in N \setminus S} [v(S \cup \{j\}) - v(S) - v(\{j\})], & \text{o.w..} \end{cases} \tag{2}$$

An interpretation of this game is as follows: Let us assume, as in Myerson (1977), that a communication structure exists, that allows bilateral meetings between players. Using this device the proposed associated game is a modification of Hamiache (1999), which was justified by a double assumption, a myopic vision of the environment and a "divide and rule" behavior of coalitions. The myopia assumption is that a coalition S ignores the links between players in $N \setminus S$. As a consequence, coalition S considers itself at the center of a star-like graph, which is equivalent to saying that S considers the players in $N \setminus S$ as isolated elements. Following the "divide and rule" behavior, coalition S may believe that the appropriation of at least a part of the surplus $[v(S \cup \{j\}) - v(S) - v(\{j\})]$, generated by its cooperation with each one of the isolated players $j \in N \setminus S$, is within reach. Thus coalition S may evaluate its own new worth, $v^*_{\lambda,H}(S)$, as the sum of its worth in the original game, $v(S)$, and of a given percentage λ, $0 \leq \lambda \leq 1$, of all the possible previous surpluses.[1]

Hamiache used the following three axioms to characterize the Shapley value.

Axiom 1. Inessential Game (IG): For each inessential game (N, v),

$$\sigma_j(N, v) = v(\{j\}) \quad \forall j \in N.$$

Axiom 2. Associated Consistency (AC): For each game (N, v),

$$\sigma_j(N, v) = \sigma_j(N, v^*_{\lambda,H}) \quad \forall j \in N.$$

Axiom 3. Continuity (CONT): For each convergent sequence $\{(N, v_k)\}_{k=1}^{\infty}$ the limit of which is game $(N, \tilde{v})$, *we have*

$$\lim_{k\to\infty} \sigma_j(N, v_k) = \sigma_j(N, \tilde{v}) \quad \forall j \in N.$$

Theorem 1. *(Hamiache, 2001) The Shapley value is the unique solution on G satisfying IG, AC for $0 < \lambda < \frac{2}{n}$, and CONT.*

3. A Dynamic Approach to the Shapley Value

Let (N, v) be a game in G. The *preimputation set* of (N, v) is $X(N, v) = \{x \in \mathbb{R}^N : \sum_{j\in N} x_j = v(N)\}$. In this section, we give a dynamic process that leads the players to the Shapley value, starting from an arbitrary point x in $X(N, v)$.

Let (N, v) be a game. For each $\lambda \in \mathbb{R}$ and each preimputation $x \in X(N, v)$, we define the associated game $(N, v^*_{\lambda,x})$ as well as the m-fold associated game $(N, v^{m*}_{\lambda,x})$ inductively by $v^{(m+1)*}_{\lambda,x} = (v^{m*}_{\lambda,x})^*_{\lambda,x}$, where $v^{0*}_{\lambda,x} = v$ and $v^{1*}_{\lambda,x} = v^*_{\lambda,x}$ itself is given by $v^*_{\lambda,x}(\emptyset) = 0$ and

$$v^*_{\lambda,x}(S) = v(S) + \lambda \sum_{j\in N\setminus S} [v(S \cup \{j\}) - v(S) - x_j], \forall S \subseteq N, S \neq \emptyset, i.e., \tag{3}$$

$$v^*_{\lambda,x}(S) = [1 - (n - s)\lambda]v(S) + \lambda \sum_{j\in N\setminus S} v(S \cup \{j\}) - \lambda \sum_{j\in N\setminus S} x_j. \tag{4}$$

[1]We may imagine that there is an arbitrator to determine the value of λ.

Note that Hamiache's associated game is obtained by replacing the payoff x_j by the individual worth $v(\{j\}), j \in N$.

In view of the alternative description (4) of the associated game, we arrive at the following general representation of the m-fold associated game $(N, v^{m*}_{\lambda,x})$. For each coalition $S \subseteq N$, there exist certain coefficients $a^s_m(t)$ and b_m such that

$$v^{m*}_{\lambda,x}(S) = \sum_{\substack{T \subseteq N \\ S \subseteq T}} a^s_m(t) v(T) + \sum_{k \in N \setminus S} b_m x_k. \tag{5}$$

The following two lemmata concern the determination of the $(m+1)$-fold associated game $(N, v^{(m+1)*}_{\lambda,x})$. We can simplify the notation of the coefficients as well in equation (5) that they do not depend on coalitions nor players, but merely on sizes of coalitions.

Lemma 1. *Concerning the representation (5) of the m-fold associated game* $(N, v^{m*}_{\lambda,x})$, *the coefficients* $a^s_m(t)$ *and* b_m *satisfy the following recursive relationships:*

$$b_{m+1} = [1-\lambda] b_m - \lambda \text{ where } b_0 = 0 \tag{6}$$

$$a^s_{m+1}(s) = [1-(n-s)\lambda] a^s_m(s) \text{ where } a^s_0(s) = 1 \tag{7}$$

$$a^s_{m+1}(t) = [1-(n-s)\lambda] a^s_m(t) + (t-s)\lambda a^{s+1}_m(t) \text{ for each } s < t \leq n. \tag{8}$$

Proof: Let $S \subseteq N$. On the one hand, by applying formula (5) to $m+1$, we have

$$v^{(m+1)*}_{\lambda,x}(S) = \sum_{\substack{T \subseteq N \\ S \subseteq T}} a^s_{m+1}(t) v(T) + \sum_{k \in N \setminus S} b_{m+1} x_k.$$

On the other hand, some combinatorial calculations yield the following chain of equalities:

$$\begin{aligned}
v^{(m+1)*}_{\lambda,x}(S) &= (v^{m*}_{\lambda,x})^*_{\lambda,x}(S) \\
&= [1-(n-s)\lambda](v^{m*}_{\lambda,x})(S) + \lambda \sum_{j \in N \setminus S} (v^{m*}_{\lambda,x}(S \cup \{j\}) - \lambda \sum_{j \in N \setminus S} x_j \text{ (by (4))} \\
&= [1-(n-s)\lambda] \left[\sum_{\substack{T \subseteq N \\ S \subseteq T}} a^s_m(t) v(T) + \sum_{k \in N \setminus S} b_m x_k \right] \text{ (by (5))} \\
&\quad + \lambda \sum_{j \in N \setminus S} \left[\sum_{\substack{T \subseteq N \\ S \cup \{j\} \subseteq T}} a^{s+1}_m(t) v(T) + \sum_{k \in N \setminus (S \cup \{j\})} b_m x_k \right] - \lambda \sum_{k \in N \setminus S} x_k \\
&= [1-(n-s)\lambda] a^s_m(s) v(S) \\
&\quad + [1-(n-s)\lambda] \sum_{\substack{T \subseteq N \\ S \subset T}} a^s_m(t) v(T) + \lambda \sum_{j \in N \setminus S} \left[\sum_{\substack{T \subseteq N \\ S \cup \{j\} \subseteq T}} a^{s+1}_m(t) v(T) \right]
\end{aligned}$$

$$+[1-(n-s)\lambda]\sum_{k\in N\setminus S} b_m x_k + \lambda \sum_{j\in N\setminus S}\sum_{k\in N\setminus (S\cup\{j\})} b_m x_k - \lambda \sum_{k\in N\setminus S} x_k$$

$$= [1-(n-s)\lambda] a_m^s(s) v(S)$$

$$+ \sum_{\substack{T\subseteq N\\ S\subset T}} \Big[[1-(n-s)\lambda] a_m^s(t) + (t-s)\lambda a_m^{s+1}(t)\Big] v(T)$$

$$+ \sum_{k\in N\setminus S} \Big[[1-(n-s)\lambda] b_m + [(n-s-1)\lambda] b_m - \lambda\Big] x_k.$$

The comparison of the coefficients obtained by following the two approaches yields the three recursive relationships (6)–(8). ■

Lemma 2. *Concerning the representation (5) of the m-fold associated game* $(N, v_{\lambda,x}^{m*})$, *the coefficients* $a_m^s(t)$ *and* b_m *satisfy the following recursive relationships:*

$$b_{m+1} = b_m - \lambda \sum_{t=s}^{n-1} \binom{n-1-s}{t-s} a_m^s(t) \quad \text{where } b_0 = 0 \tag{9}$$

$$a_{m+1}^s(s) = [1-(n-s)\lambda] a_m^s(s) \quad \text{where } a_0^s(s) = 1 \tag{10}$$

$$a_{m+1}^s(t) = [1-(n-t)\lambda] a_m^s(t) + (t-s)\lambda a_m^s(t-1) \text{ for each } s < t \leq n. \tag{11}$$

Proof: Let $S \subseteq N$. On the one hand, by applying formula (5) to $m+1$, we have

$$v_{\lambda,x}^{(m+1)*}(S) = \sum_{\substack{T\subseteq N\\ S\subseteq T}} a_{m+1}^s(t) v(T) + \sum_{k\in N\setminus S} b_{m+1} x_k.$$

On the other hand, some combinatorial calculations yield the following chain of equalities:

$$v_{\lambda,x}^{(m+1)*}(S) = (v_{\lambda,x}^*)_{\lambda,x}^{m*}(S)$$

$$= \sum_{\substack{T\subseteq N\\ S\subseteq T}} a_m^s(t) v_{\lambda,x}^*(T) + \sum_{k\in N\setminus S} b_m x_k \text{ (by (5))}$$

$$= \sum_{\substack{T\subseteq N\\ S\subseteq T}} a_m^s(t) \left[[1-(n-t)\lambda] v(T) + \lambda \sum_{j\in N\setminus T} v(T\cup\{j\}) - \lambda \sum_{j\in N\setminus T} x_j\right]$$

$$+ \sum_{k\in N\setminus S} b_m x_k$$

$$= \sum_{\substack{T\subseteq N\\ S\subseteq T}} [1-(n-t)\lambda] a_m^s(t) v(T) + \lambda \sum_{\substack{T\subseteq N\\ S\subseteq T}} a_m^s(t) \sum_{j\in N\setminus T} v(T\cup\{j\})$$

$$- \lambda \sum_{\substack{T\subseteq N\\ S\subseteq T}} a_m^s(t) \sum_{j\in N\setminus T} x_j + \sum_{k\in N\setminus S} b_m x_k$$

$$= \sum_{\substack{T\subseteq N \\ S\subseteq T}}[1-(n-t)\lambda]a_m^s(t)v(T) + \lambda\sum_{\substack{T\subseteq N \\ S\subset T}}(t-s)a_m^s(t-1)v(T)$$

$$-\lambda\sum_{k\in N\setminus S}\left[\sum_{t=s}^{n-1}\binom{n-1-s}{t-s}a_m^s(t)\right]x_k + \sum_{k\in N\setminus S} b_m x_k$$

$$= [1-(n-s)\lambda]a_m^s(s)v(S)$$

$$+\sum_{\substack{T\subseteq N \\ S\subseteq T}}\Big[[1-(n-t)\lambda]a_m^s(t) + (t-s)\lambda a_m^s(t-1)\Big]v(T)$$

$$+\sum_{k\in N\setminus S}\left[b_m - \lambda\sum_{t=s}^{n-1}\binom{n-1-s}{t-s}a_m^s(t)\right]x_k.$$

The comparison of the coefficients obtained by following the two approaches yields the three recursive relationships (9)–(11). ■

It is important to notice that (11) agrees with (8) if and only if the so-called triangle equality holds, namely $a_m^s(t) + a_m^s(t-1) = a_m^{s+1}(t)$ for each $1 \le s < t$. By (10), it follows immediately that $a_m^s(s) = [1-(n-s)\lambda]^m$ for each $s \ge 1$. Now we claim that these triangle equalities determine the formula (12) mentioned below concerning the term $a_m^s(t)$, $t \ge s$, provided that the special term $a_m^s(s)$ is known for each $s \ge 1$, where $a_m^s(s) = [1-(n-s)\lambda]^m$.

For instance, these triangle equalities imply both $a_m^s(s+1) = a_m^{s+1}(s+1) - a_m^s(s)$ as well as $a_m^s(s+2) = a_m^{s+1}(s+2) - a_m^s(s+1) = a_m^{s+2}(s+2) - 2a_m^{s+1}(s+1) + a_m^s(s)$. Formally, we have the following lemma:

Lemma 3.

$$a_m^s(s+k) = \sum_{d=0}^{k}(-1)^{k-d}\binom{k}{d}a_m^{s+d}(s+d) \text{ for each } 0 \le k \le n-s, \text{ each } m \ge 1. \tag{12}$$

Proof: The proof proceeds by induction on the number k, $k \ge 0$. The case $k=0$ reduces to the trivial equality $a_m^s(s) = a_m^s(s)$, whereas the case $k=1$ reduces to the triangle equality $a_m^s(s+1) = a_m^{s+1}(s+1) - a_m^s(s)$. The case $k>1$, the triangle equality and the induction hypothesis yield the following:

$$a_m^s(s+k) = a_m^{s+1}(s+k) - a_m^s(s+k-1)$$

$$= \sum_{d=0}^{k-1}(-1)^{k-1-d}\binom{k-1}{d}a_m^{s+1+d}(s+1+d)$$

$$-\sum_{d=0}^{k-1}(-1)^{k-1-d}\binom{k-1}{d}a_m^{s+d}(s+d)$$

$$= a_m^{s+k}(s+k) + \sum_{e=1}^{k-1}(-1)^{k-e}\binom{k-1}{e-1}a_m^{s+e}(s+e)$$

$$+\sum_{e=0}^{k-1}(-1)^{k-e}\binom{k-1}{e}a_m^{s+e}(s+e)$$

$$= a_m^{s+k}(s+k) + (-1)^k a_m^s(s) + \sum_{e=1}^{k-1}(-1)^{k-e}\binom{k}{e} a_m^{s+e}(s+e),$$

where the latter equality is due to the fact that $\binom{k-1}{e} + \binom{k-1}{e-1} = \binom{k}{e}$ *for each* $1 \le e \le k-1$. ■

Lemma 4.

(i) The solution of the recursive formula (6) $b_{m+1} = [(1-\lambda)]b_m - \lambda$ *is given by* $b_m = (1-\lambda)^m - 1$ *for each* $m \ge 1$, *where* $b_0 = 0$.

(ii) The solution of the recursive formula (12) $a_m^s(s+k) = \sum_{d=0}^{k}(-1)^{k-d}\binom{k}{d} a_m^{s+d}(s+d)$ *is given by*

$$a_m^s(s+k) = \sum_{d=0}^{k}(-1)^{k-d}\binom{k}{d}[1-\lambda(n-s-d)]^m \tag{13}$$

for each $0 \le k \le n-s$, *each* $m \ge 1$, *where* $a_0^s(s) = 1$.

Proof: To prove (i), a recursive calculation yields

$$\begin{aligned} b_m &= (1-\lambda)b_{m-1} - \lambda \\ &= (1-\lambda)[(1-\lambda)b_{m-2} - \lambda] - \lambda \\ &= (1-\lambda)^2 b_{m-2} - \lambda(1-\lambda) - \lambda \\ &= \cdots \\ &= (1-\lambda)^m b_0 - \sum_{p=0}^{m-1}\lambda(1-\lambda)^p \\ &= -\lambda\sum_{p=0}^{m-1}(1-\lambda)^p \\ &= -\lambda\cdot\frac{1-(1-\lambda)^m}{1-(1-\lambda)} \\ &= -(1-(1-\lambda)^m) \\ &= (1-\lambda)^m - 1. \end{aligned}$$

To prove (ii), a recursive calculation yields

$$\begin{aligned} a_m^s(s+k) &= \sum_{d=0}^{k}(-1)^{k-d}\binom{k}{d} a_m^{s+d}(s+d) \\ &= \sum_{d=0}^{k}(-1)^{k-d}\binom{k}{d}[1-(n-s-d)\lambda]a_{m-1}^{s+d}(s+d) \text{ (by (10))} \\ &= \sum_{d=0}^{k}(-1)^{k-d}\binom{k}{d}[1-(n-s-d)\lambda]^2 a_{m-2}^{s+d}(s+d) \text{ (by (10))} \\ &= \cdots \\ &= \sum_{d=0}^{k}(-1)^{k-d}\binom{k}{d}[1-\lambda(n-s-d)]^m. \end{aligned}$$

■

Lemma 5. *Concerning the representation (5) of the m–fold associated game* $(N, v^{m*}_{\lambda,x})$, *the coefficients* $a^s_m(t)$ *and* b_m *satisfy the following convergence results:*

(i) For each $0 < \lambda \leq 1$, *the sequence* $\{b_m\}_{m=0}^{\infty} = \{(1-\lambda)^m - 1\}_{m=0}^{\infty}$ *converges to* -1.
(ii) Let $0 < \lambda < \frac{2}{n-1}$. *For each* $s \leq t < n$, *the sequence* $\{a^s_m(t)\}_{m=0}^{\infty}$ *converges to* 0, *while the sequence* $\{a^s_m(n)\}_{m=0}^{\infty}$ *converges to 1 for each* $1 \leq s \leq n$.
(iii) Let $0 < \lambda < \frac{2}{n-1}$, *the sequence of m-fold associated games* $\{(N, v^{m*}_{\lambda,x})\}_{m=0}^{\infty}$ *converges to the limit game* $(N, \tilde{v}_x)$ *given by* $\tilde{v}_x(s) = \sum_{j \in S} x_j$ *for each* $S \subseteq N$, $S \neq \emptyset$.

Proof: (i) Clearly, for each $0 < \lambda \leq 1$, the sequence $\{b_m\}_{m=0}^{\infty}$ converges to -1.
(ii) Let $\lambda > 0$ and fix a coalition size $s \geq 1$. In view of formula (13) concerning $a^s_m(s+k)$, $0 \leq k \leq n-s$, it suffices to study the behavior of the fundamental expression $1-(n-s-d)\lambda$ for each $0 \leq d \leq k$. On the one hand, we have $1-(n-s-d)\lambda \geq 1-(n-1)\lambda > -1$, provided $0 < \lambda < \frac{2}{n-1}$. On the other hand, it always holds $1-(n-s-d)\lambda \leq 1$, whereas the equality holds if and only if $n-s-d=0$ (provided $0 \leq d \leq k$). That is, the equality holds if and only if $k = n-s$ occurs, in which case the sum expression (13) incorporates a unitary term.
(iii) Let $S \subseteq N$, $S \neq \emptyset$. For sufficiently large m, it follows from the representation (5) of the m–fold associated game $(N, v^{m*}_{\lambda,x})$ and the convergence results of (i)–(ii), that the approximation

$$v^{m*}_{\lambda,x}(S) = a^s_m(n) \cdot v(N) + \sum_{j \in N \setminus S} b_m \cdot x_j = v(N) - \sum_{j \in N \setminus S} x_j = \sum_{j \in S} x_j \text{ holds.}$$

■

Now, we introduce a dynamic process converging to the Shapley value, $f(N, v)$. We define a dynamic sequence $\{x^m\}_{m=0}^{\infty}$ with $x^0 = x$ and

$$x^m = x^{m-1} + [f(N, v^{(m-1)*}_{\lambda,x}) - f(N, v^{m*}_{\lambda,x})], \quad \textit{for } m \geq 1.$$

This dynamic sequence is like a reappraised process that leads the players to the Shapley value, $f(N, v)$, starting from an arbitrary point $x \in X(N, v)$. We may imagine that there is an arbitrator and that each player in N obeys the suggestion of the arbitrator. The arbitrator uses a fair rule leading the players to a reasonable allocation. In the m-th step of this dynamic process, there is a reference game $(N, v^{m*}_{\lambda,x})$. The player thinks that he plays the game $(N, v^{m*}_{\lambda,x})$ and payoff is allocated according to the allocation rule f. Since the total payoff is fixed and if player j gets a greater payoff in the game $(N, v^{m*}_{\lambda,x})$ than in the game $(N, v^{(m-1)*}_{\lambda,x})$, (i.e. $f_j(N, v^{m*}_{\lambda,x}) > f_j(N, v^{(m-1)*}_{\lambda,x})$), the arbitrator will suggest a reduction in his payoff equal to $[f_j(N, v^{m*}_{\lambda,x}) - f_j(N, v^{(m-1)*}_{\lambda,x})]$ in the new allocation x^m. On the other hand, if $f_j(N, v^{m*}_{\lambda,x}) < f_j(N, v^{(m-1)*}_{\lambda,x})$, the arbitrator will suggest the amount of $[f_j(N, v^{(m-1)*}_{\lambda,x}) - f_j(N, v^{m*}_{\lambda,x})]$ in the new allocation x^m. When $f(N, v^{m*}_{\lambda,x}) = f(N, v^{(m-1)*}_{\lambda,x})$, the dynamic process stops. In Theorem 2 , we show that the sequence $\{x^m\}_{m=0}^{\infty}$ converges to the Shapley value, $f(N, v)$.

Theorem 2. *Let* $(N, v) \in G$, *then for each* x *in* $X(N, v)$, *the dynamic sequence* $\{x^m\}_{m=0}^{\infty}$ *with* $x^0 = x$ *and*

$$x^m = x^{m-1} + [f(N, v_{\lambda,x}^{(m-1)*}) - f(N, v_{\lambda,x}^{m*})] \quad , \; m \geq 1$$

converges to the Shapley value, $f(N, v)$ *provided that* $0 < \lambda < \frac{2}{n-1}$.

Proof: Given $x \in X(N, v)$, and the dynamic sequence

$$x^m = x^{m-1} + [f(N, v_{\lambda,x}^{(m-1)*}) - f(N, v_{\lambda,x}^{m*})] \quad , \; m \geq 1.$$

Using recursion, we have

$$\begin{aligned} x^m =& x^{m-1} + [f(N, v_{\lambda,x}^{(m-1)*}) - f(N, v_{\lambda,x}^{m*})] \\ =& x^{m-2} + [f(N, v_{\lambda,x}^{(m-2)*}) - f(N, v_{\lambda,x}^{m*})] \\ =& x^{m-3} + [f(N, v_{\lambda,x}^{(m-3)*}) - f(N, v_{\lambda,x}^{m*})] \\ =& \cdots \\ =& x^0 + [f(N, v_{\lambda,x}^{0*}) - f(N, v_{\lambda,x}^{m*})] \\ =& x + [f(N, v) - f(N, v_{\lambda,x}^{m*})]. \end{aligned}$$

By (iii) of Lemma 5, and f satisfies IG and CONT, $\lim_{m\to\infty} f(N, v_{\lambda,x}^{m*}) = f(N, \tilde{v}_x) = x$. Hence,

$$\begin{aligned} \lim_{m\to\infty} x^m = & \lim_{m\to\infty} \{x + [f(N, v) - f(N, v_{\lambda,x}^{m*})]\} \\ =& x + [f(N, v) - f(N, \tilde{v}_x)] \\ =& x + [f(N, v) - x] \\ =& f(N, v). \end{aligned}$$

■

Note that the convergence of the sequence $\{x^m\}_{m=0}^{\infty}$ of payoff vectors towards the Shapley value $f(N, v)$ is equivalent to the convergence of the sequence $\{f(N, v_{\lambda,x}^{m*})\}_{m=0}^{\infty}$ of the solutions towards the initial preimputation x. The latter convergence applies as soon as the Shapley value satisfies IG and CONT. Hence for each solution σ satisfying IG and CONT, we obtain immediately the following result.

Corollary 1. *Let* $(N, v) \in G$, *then for each* x *in* $X(N, v)$, *the dynamic sequence* $\{x^m\}_{m=0}^{\infty}$ *with* $x^0 = x$ *and*

$$x^m = x^{m-1} + [\sigma(N, v_{\lambda,x}^{(m-1)*}) - \sigma(N, v_{\lambda,x}^{m*})] \quad , \; m \geq 1$$

converges to $\sigma(N, v)$ *provided that* $0 < \lambda < \frac{2}{n-1}$ *and the solution* σ *satisfies IG and CONT.*

4. Discussion

In this article, we have founded a dynamic approach to the Shapley value or any solution satisfying IG and CONT, based on a modified version of Hamiache's notion of an associated game. Although the associated consistency property is a superfluous tool in the dynamic approach to solutions, it is still important for axiomatizations of the solution.

Moulin (1985) characterized the *equal allocation of nonseparable cost* (EANS) by essentially the same set of axioms as Hart and Mas-Colell (1989) used to characterize the Shapley value. The only difference is in the reduced games they used to define consistency. The different definitions help understand the difference among these solution concepts. One would like to know whether there exist different definitions of the associated game to help understand the difference among these solution concepts. We will modify the definition of Hamiache's associated game to do so. The modified version of an associated game is as follows:

Definition 5. Let (N, v) be a game in G and $\sigma(N, v)$ be a solution on G, the associated game $(N, v^*_{\lambda,\sigma})$ with respect to σ is defined by

$$v^*_{\lambda,\sigma}(S) = \begin{cases} 0 & \text{, if } S = \emptyset \\ v(S) + \lambda \sum_{j \in N \setminus S} [v(S \cup \{j\}) - v(S) - \sigma_j(N, v)] & \text{, o.w. .} \end{cases} \tag{14}$$

Note that Hamiache's associated game arises by replacing the payoff $\sigma_j(N, v)$ by the individual worth $v(\{j\}), j \in N$.

We first introduce the definition of *EANS* and two axioms AC and PO.

Definition 6. The *EANS* is a solution on G which associates with each game (N, v) and each player j in N the value,

$$\begin{aligned} \phi_j(N, v) =& v(N) - v(N \setminus \{j\}) \\ &+ \frac{1}{n}\left[v(N) - \sum_{k \in N}[v(N) - v(N \setminus \{k\})]\right] \quad (15) \\ =& \frac{1}{n}\left[v(N) + \sum_{k \in N} v(N \setminus \{k\})\right] - v(N \setminus \{j\}). \quad (16) \end{aligned}$$

The term $v(N) - v(N \setminus \{j\})$ and the term $v(N) - \sum_{k \in N}[v(N) - v(N \setminus \{k\})]$ are the "separable cost" and the "nonseparable cost", respectively. One can think of many methods to allocate the nonseparable cost. The *EANS* proposes that all players should share it equally.

Axiom 4. Associated Consistency (AC): For each game (N, v) and its associated game $(N, v^*_{\lambda,\sigma})$ with respect to σ,

$$\sigma_j(N, v) = \sigma_j(N, v^*_{\lambda,\sigma}) \quad \forall j \in N.$$

Axiom 5. Pareto Optimality (PO): For each game (N, v),

$$\sum_{j \in N} \sigma_j(N, v) = v(N).$$

We are ready to show that the *EANS* satisfies IG, CONT, PO and AC.

Lemma 6. *The EANS satisfies IG, CONT, PO, and AC.*

Proof: It is easy to verify that the *EANS* satisfies the IG, CONT and PO. We will show that it satisfies AC. By the definition of the associated game, $v^*_{\lambda,\phi} = v + \lambda w$ where the game (N, w) is defined as follows:

$$w(S) = \sum_{\substack{j \\ j \in N \setminus S}} [v(S \cup \{j\}) - v(S) - \phi_j(N, v)] \text{ for } S \subseteq N,\ S \neq \emptyset.$$

Since $\phi(N, v^*_{\lambda,\phi}) = \phi(N, v) + \lambda\phi(N, w)$, we only need to prove that

$$\phi_j(N, w) = 0 \text{ for each } j \in N.$$

Thus, for each $j \in N$, by equation (16)

$$\begin{aligned}
\phi_j(N, w) &= \frac{1}{n}\left[w(N) + \sum_{k \in N} w(N \setminus \{k\})\right] - w(N \setminus \{j\}) \\
&= \frac{1}{n}\sum_{k \in N}[v(N) - v(N \setminus \{k\}) - \phi_k(N, v)] \\
&\quad - [v(N) - v(N \setminus \{j\}) - \phi_j(N, v)] \\
&= v(N) - \frac{1}{n}\sum_{k \in N} v(N \setminus \{k\}) - \frac{1}{n}\sum_{k \in N} \phi_k(N, v) \\
&\quad - v(N) + v(N \setminus \{j\}) + \phi_j(N, v) \\
&= -\frac{1}{n}\sum_{k \in N} v(N \setminus \{k\}) + v(N \setminus \{j\}) \\
&\quad - \frac{1}{n}\sum_{k \in N}\left\{\frac{1}{n}\left[v(N) + \sum_{i \in N} v(N \setminus \{i\})\right] - v(N \setminus \{k\})\right\} \\
&\quad + \frac{1}{n}\left[v(N) + \sum_{k \in N} v(N \setminus \{k\})\right] - v(N \setminus \{j\}) \\
&= -\frac{1}{n^2}\sum_{k \in N} v(N) + \frac{1}{n} v(N) \\
&\quad - \frac{1}{n^2}\sum_{k \in N}\sum_{i \in N} v(N \setminus \{i\}) + \frac{1}{n}\sum_{k \in N} v(N \setminus \{k\}) \\
&= 0.
\end{aligned}$$

■

We tried to characterize the *EANS* by means of the following axioms: IG, CONT, PO, and AC. However, there are at least two solutions satisfying these axioms. One is the *EANS*, and the other is the solution φ, where

$$\varphi_j(N, v) = \begin{cases} v(N) - v(N \setminus \{j\}) \\ \quad + \{v(N) - \sum_{k \in N}[v(N) - v(N \setminus \{k\})]\} & \text{, if j} = 1 \\ v(N) - v(N \setminus \{j\}) & \text{, if j} \neq 1, \end{cases}$$

where $N = \{1, 2, \cdots, n\}$.

It is easy to verify that φ satisfies IG, CONT, PO, and AC, so we omit it. We propose to characterize the *EANS* in terms of IG, CONT, PO, AC and some other axiom(s) in a subsequent paper. Note that Hwang (2005) also characterized the *EANS* in terms of IG, CONT, PO, and another slightly adapted notion of associated consistency.

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