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Using the cost function to generate Marshallian demand systems*

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Abstract. Limited data means that prior structure is needed when working with large demand systems. The cost function is a convenient vehicle for generating demand systems incorporating such structure. While the cost function directly yields Hicksian demand functions they will not usually have an explicit representation as Marshallian demand equations i.e. in terms of the observable variables. With fast hardware and modern software, however, this need not hinder the estimation of the (implied) Marshallian demand equations. This paper develops the formal theory for using cost functions in this context, and reports on initial trials on the operational feasibility of the method.

Key words: Demand systems, cost functions, separability, numerical methods

JEL classification: D11, D12

1. Motivation and plan of paper

Most existing consumer demand systems suffer from the following practical shortcomings:

- It is difficult to incorporate prior ideas about the structure of preferences (yet such information is almost always required because of limited sample size).
- Such systems remain regular over ranges that often are too narrow: (i) for

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realistic policy simulation in an environment of rapid income growth; (ii) for historical analysis over long periods; or (iii) for empirical analysis of demand using international comparisons data spanning countries widely separated on the development spectrum.

This paper describes the first stage in the development of a tool which aims to overcome these limitations.

The use of extraneous information on the structure of demand is inevitable in applied work at the level of detail used by large economy-wide models. This information (which must be matched to the aggregation level at which estimation is to proceed) often takes the form of statements about relative substitutability among items within different commodity groups (see, e.g., Clements and Smith, 1983). The approach taken in this paper facilitates the imposition of such constraints.

Due to the relative ease with which its structure can be related to that of the Slutsky matrix, the cost function is a natural vehicle for generating empirically oriented demand functions. Moreover, because concavity is preserved under addition and nesting of concave functions, the generation of wide classes of regular cost functions is straightforward. Differentiation of a finally chosen cost function with respect to prices yields the demand functions (Shephard's Lemma). While these equations are explicit in the (unobservable) utility level, they may lack a closed-form representation in terms of the observable variables (prices and expenditure). With modern software, this need not hinder estimation (as was pointed out by McLaren, 1991, and implemented in a related context by Rimmer and Powell, 1992 & 1996, and in a finance context by Brown, 1993). A simple one-dimensional numerical inversion allows the estimation of the parameters of the cost function via the parameters of the (implied) Marshallian demand equations.

Section 2 of the paper develops these theoretical foundations formally. In Section 3 we explore some options for the specification of the cost function. These are meant to illustrate the feasibility of the method rather than to be taken seriously in their own right. A good starting point is the familiar linear expenditure system (LES), which has explicit closed forms for the direct and indirect utility functions, and for the cost function. We show that the proposed method generates all of the well-known results in this uncomplicated exemplar. We then move to a rank three generalization of (and therefore a more flexible system than) LES in which the indirect utility function lacks an explicit closed form, illustrating both that the cost-function method is feasible for the estimation of this GLES system, and that the more general Engel specification overcomes the second shortcoming mentioned above. The final part of Section 3 outlines an alternative candidate for empirical use, namely a recursive linear expenditure system (RecLES) which embodies a principle for the hierarchical disaggregation of commodities using cost-function ideas.

A first empirical exploration of the models is reported in Section 4 where the story is kept simple by working with just four commodities, and applying simple corrections for autocorrelation if required. After identifying the time-series data used, we describe empirical estimation of LES, GLES and RecLES using the new methodology. The properties of the estimated GLES and RecLES are briefly compared with LES estimates from the same data.

Section 5 contains concluding remarks and an agenda for further empirical work.

2. Theoretical foundations

Let R^N represent N space, let Ω^N represent the nonnegative orthant, and let Ω_+^N represent the strictly positive orthant. Let $x \in \Omega^N$ represent an N vector of commodities, $p \in \Omega_+^N$ the corresponding N-vector of prices, and $c \in \Omega_+^1$ a level of expenditure. Functions conditioned on expenditure and prices will be referred to as "Marshallian", while those conditioned on utility and prices will be termed "Hicksian". The demand functions are written respectively as $X^M(c, p)$ and $X^H(u, p)$. This notation is followed below, where upper-case letters denote functions whose values are indicated by the corresponding lower-case letters.

By duality theory, preferences may be equivalently expressed by the (direct) utility function $u = U(x)$, the indirect utility function $U^M(c, p) = U(X^M(c, p))$, or the cost function $c = C(u, p) = p'X^H(u, p)$, each subject to appropriate regularity conditions (see, e.g., Blackorby, Primont and Russell 1978). If the aim is to derive a regular Marshallian demand system, then usually one would begin with a regular indirect utility function, and apply Roy's Identity; while if Hicksian demands are required, one usually would apply Shephard's Lemma to a chosen indirect utility function. Since the exogenous variables in most applied work are prices and expenditure, it is natural to search for general specifications of indirect utility functions (see e.g., Cooper and McLaren, 1992, 1996). But other approaches to the specification of regular demand systems may have merit. Since in practice any specification involving more than a few commodities will require that some structure be imposed, it is important that the system be represented in a form convenient for such imposition. Further, it is likely that such structure will be built up by simple operations such as composition of lower dimensional functions, and preservation of regularity will be simpler under some representations than others. In general, it is the cost function specification that is most attractive in this regard. Finally, one might argue that the specification should be matched with the final aim of the analysis, rather than the intermediate step of parameter estimation. In welfare analysis, for example, an empirically calibrated cost function may be more useful than the corresponding indirect utility function.

The starting point for this paper will be the cost function, defined by

$$\begin{aligned} c &= C(u, p) \\ &= \min_x \{p'x : U(x) \geq u, x \in \Omega^N\} \\ &= p'X^H(u, p). \end{aligned} \tag{1}$$

The cost function satisfies the set of regularity conditions RC (see, e.g., Blackorby, Primont and Russell 1978 p. 24):

RC1: $C : R \times \Omega^N \to \Omega^1$
RC2: C is continuous
RC3: C is a positive linearly homogeneous function of p
RC4: C is nondecreasing in p
RC5: C is increasing in u
RC6: C is concave in p.

By Shephard's Lemma the Hicksian demand equations can be derived from the cost function by differentiation with respect to prices.

There are at least two ways by which general cost functions may be constructed. The first is the standard method by which duality is normally exploited, in which a known functional form is generalized in a way which preserves the regularity properties. Because of the form of the regularity properties RC, it might be expected that the cost function would be more easily generalized than other functions. For example, it is known that increasing concave functions of concave functions are concave, whereas an analogous result is not necessarily true for quasi-convex functions, as would be required for generalizing indirect utility functions. In section 3.2, the cost function corresponding to the linear expenditure system is generalized in this way. A second way in which a functional form may be generated is by the imposition of separable structure, and the successive composition of cost functions. This procedure will be used in section 3.3 to construct general but regular cost functions.

Consider now the possibility of using a cost function to specify preferences, but with the use of data to estimate Marshallian demand equations. Take as given a cost function $C(u, p)$ satisfying RC. By Shephard's Lemma the ith Hicksian demand equation is[1]

$$X_i^H = C_{p_i}(u, p). \tag{2}$$

Now, if the explicit functional form of the corresponding indirect utility function were available, then these Hicksian demands could be "Marshallianized" by replacing the unobservable u by

$$u \equiv U^M(c, p) \tag{3}$$

to give

$$X_i^M(c, p) = X_i^H(U^M(c, p), p) = C_{p_i}(U^M(c, p), p). \tag{4}$$

Indeed, this was exactly the procedure followed by Deaton and Muellbauer (1980) in deriving the Almost Ideal Demand System, whereby they first specified the PIGLOG cost function, then derived the Hicksian demand system, and finally inverted the cost function to give the implied indirect utility function that was used to eliminate the unobservable u. If such an explicit inversion is available, then this procedure is of course equivalent to beginning with the corresponding indirect utility function, and there is no gain in generality in starting with a cost function. But the class of preferences for which there exists explicit solutions for both the cost function and the indirect utility function is quite limited.

Of interest for this paper is the class of cost functions for which such explicit inversion is not available. Thus, for a given functional form for the cost function with parameters π

$$C(u, p; \pi), \tag{5}$$

[1] In general, subscripts other than indices denote partial differentiation.

the Marshallian demand equations can be expressed implicitly by the set of equations

$$\begin{aligned} X_i^H &= C_{p_i}(u, p; \pi), \quad i = 1, \ldots N \\ c &= C(u, p; \pi). \end{aligned} \tag{6}$$

Given regularity condition RC5, then it is always possible numerically to invert the cost function to express u as a function of c and p. At each iterative step of the maximization of the likelihood function, there is a given set of parameter values. For these parameter values, (5) can be numerically inverted to recover the value of utility u consistent with given values of expenditure c and prices p, and this value of utility can then be used to eliminate the unknown value of utility from the Hicksian demand equations. This requires one inversion for each set of parameter values at each set of sample points, but is the same solution for all N demand equations, and thus the order of numerical complexity does not increase with the number of commodities. A similar procedure will be implemented for demand equations expressed as shares. This will be illustrated in the next section using the LES as an example.

3. Cost function specification – some options

3.1. The LES as a starting point and illustration

To illustrate the derivation of demand equations and of price and expenditure elasticities in general, the methods will first be applied to the LES without resort to the analytical inversion of the cost function that is available in this case. The general structure of the cost function generating the linear expenditure system can be written as

$$C = uP1 + P2 \tag{7}$$

where $P1$ and $P2$ are two functions of prices satisfying those properties of RC that relate to prices, and regularity requires that $u > 0$, or equivalently that $c > P2$. The particular form of the LES results if these functions are specified as

$$P1(p) = \prod_{i=1}^{N} p_i^{\beta_i} \quad \left(\sum_{i=1}^{N} \beta_i = 1 \right) \tag{8}$$

and

$$P2(p) = \sum_{i=1}^{N} \gamma_i p_i. \tag{9}$$

Applying Shephard's Lemma gives Hicksian (compensated) demand functions as

$$X_i^H = \frac{\beta_i u P1}{p_i} + \gamma_i \tag{10}$$

or in share form,

$$W_i^H = \frac{\beta_i uP1 + p_i\gamma_i}{C(u,p)}. \tag{11}$$

Elimination of u from (10) by the inversion of (7) leads immediately to the familiar Marshallian demands. It is transparent that, for any given values of the parameters, β and γ, and for any given data on c and p, the numerical inversion of (7) to give u in terms of c and p, and its substitution in (10), would give the same result as analytical inversion.

The typical elements of the Slutsky substitution matrix are the partial derivatives of the i th Hicksian demand with respect to the j th price, and (with a slight abuse of notation) are given as:

$$X_{ii}^H = -\frac{\beta_i(1-\beta_i)}{p_i^2}uP1 \tag{12}$$

$$X_{ij}^H = \frac{\beta_i\beta_j}{p_ip_j}uP1 \tag{13}$$

which clearly illustrate the known properties of the LES that diagonal terms are negative, and all off-diagonal terms are positive. These allow calculation of the compensated price elasticities as:

$$E_{ij}^H = \frac{X_{ij}^H p_j}{X_i^H}. \tag{14}$$

The Marshallian demands can be expressed as

$$X_i^M(c,p) = X_i^H(U^M(c,p),p) = \frac{\beta_i}{p_i}[U^M(c,p)]P1 + \gamma_i; \tag{15}$$

differentiating with respect to expenditure gives

$$\begin{aligned} X_{ic}^M &= \frac{\partial X_i^M}{\partial c} = \frac{\partial X_i^H}{\partial u}\frac{\partial U^M}{\partial c} \\ &= \frac{\partial X_i^H}{\partial u}\left[\frac{\partial C}{\partial u}\right]^{-1} \end{aligned} \tag{16}$$

which, when evaluated using (7)–(10), directly gives

$$X_{ic}^M = \beta_i/p_i. \tag{17}$$

Because of the linearity in u of both the cost function and the Hicksian demand function the above expression does not contain u. In the more general models considered below, this is no longer the case, and in the final step of the derivation any dependence on u must then be eliminated by the inversion of the cost function. Using this method, the expenditure elasticities in general are:

$$E_{ic}^{M} = \frac{X_{ic}^{M} c}{X_{i}^{M}} \tag{18}$$

which, for LES, gives the familiar result:

$$E_{ic}^{M} = \frac{\beta_i c}{p_i \gamma_i + \beta_i (c - P2)}. \tag{19}$$

3.2. GLES (a generalized LES)

The LES is known to have a number of restrictions over and above those implied by demand theory: constant marginal budget shares; the fact that all pairs of goods must be gross complements but Hicksian substitutes; the impossibility of inferior goods; and the implication of direct additivity that all cross Allen-Uzawa partial substitution elasticities are proportional to the product of the corresponding Engel elasticities. A first generalization that does not have these restrictions arises from the extension of (7)

$$C = uP1 + P2, \tag{20}$$

where, for this particular generalization, the price indices have the familiar LES structure:

$$P1 = \prod_{i=1}^{N} p_i^{B_i} \tag{21}$$

$$P2 = \sum_{i=1}^{N} \gamma_i p_i. \tag{22}$$

but the constant marginal budget shares, β_i, from (8), are replaced by utility-varying coefficients

$$B_i = \frac{\alpha_i + \beta_i u^*}{1 + u^*} \quad \text{where} \quad \sum \alpha_i = \sum \beta_i = 1 \tag{23}$$

and where $u^* = u/u_0$ is the normalization of utility which takes the value of unity in the same base period as the price indexes p_i. Note that, from the definitions (21) and (23), it is possible to solve (20) explicitly for the value of u in the base period, u_0, solely in terms of parameters and the base period value of c. Provided $u > 0$, the structure (20) maintains all of the regularity properties in prices of the cost function corresponding to the LES.

The Hicksian demands are given by:

$$X_i^H = \frac{B_i}{p_i} uP1 + \gamma_i \tag{24}$$

or in share form,

$$W_i^H = \frac{B_i uP1 + p_i\gamma_i}{C(u,p)}. \tag{25}$$

To convert these to Marshallian demands, the unobservable u is replaced by the inversion of (20), noting that $P1$ and B_i are now also functions of u.

Compensated price elasticities of demand require the terms of the Slutsky matrix, which are:

$$X_{ii}^H = -\frac{B_i(1-B_i)}{p_i^2} uP1 \tag{26}$$

$$X_{ij}^H = \frac{B_i B_j}{p_i p_j} uP1. \tag{27}$$

Expenditure elasticities require the derivatives of demand with respect to expenditure, which are derived by Marshallianizing and differentiating (24) with respect to the argument c:

$$X_{ic}^M = \frac{B_i}{p_i} + \frac{U^M}{p_i} \frac{u_0(\beta_i - \alpha_i)}{(u_0 + U^M)^2 + U^M u_0 \log \prod_{j=1}^{N} p_j^{\beta_j - \alpha_j}} \tag{28}$$

where $U^M \equiv U^M(c,p)$ is the inversion of (20).

This model generalizes and nests the LES which is obtained when $\alpha_i \equiv \beta_i, \forall i$. The generalization is motivated by the AIDADS model of Rimmer and Powell (1996), who derive an expenditure system with an underlying cost function similar to (20) to (23) corresponding to their implicitly additive direct utility function. This generalization is of particular interest because it generates regular demand equations with Engel rank equal to 3. While the generalization from rank 2 to rank 3 does not necessarily have implications for the regularity of the cost function, it does mean that the dimension of the function space spanned by the Engel curves has been increased from two to three (see Lewbel (1991), p. 711), allowing far more flexible modelling of Engel responses. On the basis of nonparametric estimation of Engel responses in US and UK household survey data, Lewbel (1991) argues that rank 3 is required to capture the stylized facts. This conclusion is supported by Rimmer and Powell's (1994) nonparametric estimates from Australian household data. This generalization thus goes a long way towards removing the empirically unacceptable restrictions of the LES. The estimation of this example will be illustrated in Section 4, and will be referred to as GLES.

3.3. RecLES (a recursive LES)

A second way in which a functional form may be generated is by the imposition of separable structure. Following Blackorby, Primont and Russell, define the commodity index set I as

$$I = \{1, 2, \ldots, N\}$$

and partition it as

$$\hat{I} = \{I^1, I^2, \ldots, I^m\} \quad m \leq N.$$

Then the price vector can be similarly partitioned as

$$p = (p^1, p^2, \ldots, p^m).$$

If C is separable in $\hat{I}$ then C has the structure

$$C(u, p) = \bar{C}(u, C^1(u, p^1), \ldots, C^r(u, p^r), \ldots, C^m(u, p^m)). \tag{29}$$

Blackorby, Primont and Russell refer to this structure on preferences as felicitous decentralizability: in order to allocate expenditure optimally within group r, the consumer need only know the aggregate level of expenditure on all goods in group r, the prices in group r, and the level of utility or "real income" u. If the cost function C satisfies RC and is separable, then it is "almost" necessary that the C^r and $\bar{C}$ satisfy RC, the only property in question being the strict monotonicity of $\bar{C}$ in u (see Blackorby, Primont and Russell pp. 70–76 for a discussion of this point). Nevertheless, the use in (29) of functions C^r and $\bar{C}$ that satisfy RC is a sufficient condition to generate a valid cost function, and hence provides an attractive means of construction of regular cost functions from more basic regular generating functions. Because the C^r satisfy RC, it is tempting to think of them either as group expenditure functions, or as group price indices. Neither of these interpretations is strictly correct. The C^r are not group cost functions because total expenditure is not the sum of these C^r (unless the aggregator function is a sum), and they are not simple price indices because of their dependence on utility. However, as elaborated on below (and congruent with Shephard's Lemma), the C^r do behave somewhat like group cost functions, in that the share of commodity i expenditure in group r expenditure is given by

$$\frac{p_i C^r_{p_i}}{C^r}, \tag{30}$$

and also operate somewhat like group price indices, in that the total value of expenditure on group r commodities is given by

$$\bar{C}_{C^r} C^r. \tag{31}$$

The structure in (29) can be made more flexible by noting that, since conditional on the level of utility, each of the C^r acts like a price index, the separable structure can be defined recursively, using at each stage a regular cost function as the functional form for the generic function.

The interpretation of a function C as a cost function depends on the identity of its arguments: below we define *quasi-cost functions* C^r structured like C, preserving the linear homogeneity in their arguments that are themselves quasi-cost functions, but where the latter are functions of only a subset of prices (namely, those relevant to the level at which these quasi-cost functions occur). Of itself this would not invalidate the interpretation of these C^r as sub-cost functions specific to the level at which they occur; it is the appearance of the global utility level u as an argument of these functions which makes such an interpretation inappropriate.

To introduce a notation to handle this recursive definition, subscripts will be reserved to refer to individual commodities, while superscripts (other than H and M) will be used to refer to groups of commodities, with the number of successive superscripts indicating the order of successive levels of grouping. Hence define

$$C(u,p) = \bar{C}\left(u, \frac{C^1(u,p^1)}{C^1(0)}, \ldots, \frac{C^r(u,p^r)}{C^r(0)}, \ldots, \frac{C^m(u,p^m)}{C^m(0)}\right) \tag{32}$$

where the notation $C^r(0)$ indicates evaluation at the base period values of the arguments, and at the next level define

$$C^r(u,p^r) = \bar{C}^r\left(u, \frac{C^{r1}(u,p^{r1})}{C^{r1}(0)}, \ldots, \frac{C^{rm^r}(u,p^{rm^r})}{C^{rm^r}(0)}\right), \quad r = 1, \ldots, m. \tag{33}$$

Proceed recursively with successive levels of nesting until remaining sets of prices are singletons, at which point the dependence on utility is ignored and the quasi-cost function is defined as that (normalized) price. To avoid the need to use completely general notation, equation (33) will henceforth be used as an archetype to illustrate general results. If at each stage in this recursive structure the functional form for the aggregator $\bar{C}$ is of the LES form (7)–(9), then the resulting specification will be referred to as RecLES (Recursive LES). Thus for (33) this form is

$$\begin{aligned}
\bar{C}^r &= uP1\left(\frac{C^{r1}(u,p^{r1})}{C^{r1}(0)}, \ldots, \frac{C^{rm^r}(u,p^{rm^r})}{C^{rm^r}(0)}\right) \\
&\quad + P2\left(\frac{C^{r1}(u,p^{r1})}{C^{r1}(0)}, \ldots, \frac{C^{rm^r}(u,p^{rm^r})}{C^{rm^r}(0)}\right) \\
&= u\prod_{i=1}^{m^r}\left(\frac{C^{ri}(u,p^{ri})}{C^{ri}(0)}\right)^{\beta^{ri}} + \sum_{i=1}^{m^r}\gamma^{ri}\left(\frac{C^{ri}(u,p^{ri})}{C^{ri}(0)}\right).
\end{aligned} \tag{34}$$

Via Shephard's Lemma, logarithmic differentiation of $C(u,p)$ with respect to prices produces an equation system for the shares W_i^H $(i = 1,2,3,\ldots,N)$ of the N commodities in total expenditure. While the LES parentage of the recursive structure ensures that there will still be some (possibly quite complicated) relationship between substitution and Engel elasticities, the unacceptable strict proportionality among them no longer applies. The above recursive structure also allows a recursive definition of shares. Define S^{rs} as the share of expenditure on the second level group s in expenditure on the first level group r. (Note that the superscript H has been omitted from S for simplicity). The usefulness of the quasi-cost functions, and their similarity to both cost functions and price indices, is as follows:

Proposition. *Logarithmic differentiation of a quasi-cost function at level r with respect to lower-order quasi-costs produces a subsystem of equations for the sub-shares of commodity groups in the level of expenditure on all the commodities identified by the quasi-cost function at level r. Shares of individual commodities in total expenditure can be constructed by successive compounding of these sub-shares.*

Take, for example, the logarithmic differentiation of C^r with respect to C^{rs}. In this case the proposition states that $S^{rs} = \partial \log C^r / \partial \log C^{rs}$.

Proof of Proposition: The proposition is proved using Shephard's Lemma, the chain rule and the homogeneity of degree 1 in its arguments of each of the aggregator functions.

To avoid the notational complications of a completely general recursive system, we will now show how this notation applies to the empirical example estimated in the next section. This example uses just four commodities, with the commodities arranged in three levels as follows:

Top level	The commodities $\{1, 2, 3\}$ being taken as group 1, and commodity 4 as the remaining group
Middle level	The commodities $\{1, 2\}$ being taken as group 1, and commodity 3 as the remaining group
Bottom level	Commodity 1 as group 1 and commodity 2 as group 2

Although this hierarchical allocation structure is recursive, the solution of the equation system generated by it is simultaneous. This is because the utility level u appears at all levels of the structure. Thus, *conditional upon the level of utility* being enjoyed, the allocation of expenditure between good 4 and the rest at the top level depends only on the price of good 4 relative to a suitable price index of the remaining commodities. Likewise, at the middle level, and again conditional on the level of utility, the allocation of expenditure net of that devoted to commodity 4 depends only on the price of commodity 3 relative to a suitable index of the prices of commodities lower in the allocation scheme (namely commodities 1 and 2). These considerations identify the type of structure depicted in (32) as an analogue in the cost function space of the utility tree used to depict separable preferences. More specifically, because the allocation at each level of our scheme is binary, it is the analogue of a utility tree in which bifurcations, but not higher-order branching, occur at each node.

For this example the full structure and notation is:

$$C(u,p) = \bar{C}\left(u, \frac{C^1(u,p^1)}{C^1(0)}, \frac{C^2(u,p^2)}{C^2(0)}\right) \tag{35}$$

$$C^1(u,p) = \bar{C}^1\left(u, \frac{C^{11}(u,p^{11})}{C^{11}(0)}, \frac{C^{12}(u,p^{12})}{C^{12}(0)}\right) \tag{36}$$

$$C^2(u,p^2) = \bar{C}^2(u,p_4) = p_4 \tag{37}$$

$$C^{11}(u, p^{11}) = \bar{C}^{11}\left(u, \frac{C^{111}(u, p^{111})}{C^{111}(0)}, \frac{C^{112}(u, p^{112})}{C^{112}(0)}\right) \tag{38}$$

$$C^{12}(u, p^{12}) = \bar{C}^{12}(u, p_3) = p_3 \tag{39}$$

$$C^{111}(u, p^{111}) = \bar{C}^{111}(u, p_1) = p_1 \tag{40}$$

$$C^{112}(u, p^{112}) = \bar{C}^{112}(u, p_2) = p_2. \tag{41}$$

The notation for parameters corresponding to the LES specifications for these functions at each of the three levels is as follows:

$$\beta^1, \beta^2, \gamma^1, \gamma^2; \beta^{11}, \beta^{12}, \gamma^{11}, \gamma^{12}; \beta^{111}, \beta^{112}, \gamma^{111}, \gamma^{112}. \tag{42}$$

To compose the system of equations for (Hicksian) shares W_i^H of individual commodities in aggregate expenditure we cascade the relevant sub-shares multiplicatively. Thus from the top level we find,

$$W_4^H = S^2. \tag{43}$$

From the top and middle levels we compose

$$W_3^H = S^1 S^{12}. \tag{44}$$

Finally, using all three levels we compose the final two shares:

$$W_1^H = S^1 S^{11} S^{111} \tag{45}$$

and

$$W_2^H = S^1 S^{11} S^{112}. \tag{46}$$

The detailed formulae required to carry out the computations reported in the next Section can be found in the Appendix.

4. Illustrative estimation of demand systems

4.1. The data

For illustrative purposes, the models are estimated with annual Australian time series data on four categories of private final consumption expenditure covering the period 1959/60 to 1995/96. The four categories are: Food; Alcohol and Tobacco; Clothing; and Other (including expenditure on household durables, purchase and operation of motor vehicles, electricity, etc. but excluding expenditure on dwelling rent). Current and constant price data are matched, and the price series is the implicit price deflator obtained by dividing the current price series by the corresponding constant price series. A base year of 1979/80 was selected for the price indexes. Population was used to construct per capita consumption.

4.2. *Estimation*

The computation of the maximum-likelihood estimates reported below was feasible because the GAUSS language used to program the estimators handles implicit representation of functional relationships well. All three models were estimated in share form, with one equation deleted. As is common with demand systems of this form, autocorrelation is clearly a problem. Some experimentation was carried out with various autocorrelation schemes such as the scalar autocorrelation parameter specification, the order N parameterization of the autocovariance matrix proposed by Moschini and Moro (1994), and a complete $(N-1)^2$ specification. Such results add little to the illustration of the use of the cost function approach, and are hence not all presented below, but are available in a separate paper (McLaren, Rossiter and Powell (1996)). The results reported below have been chosen solely to illustrate the feasibility of the approach, and are based on a scalar autocorrelation correction calculated using the FIML algorithm of Beach and MacKinnon (1979). The R^2 values reported in Table 1 are constructed as the squares of the correlations between the actual shares and the values of the shares predicted using the explanatory variables and the estimated parameter values. If in forming the predicted values, the serial correlation structure of the errors is ignored, then the values reported for R^2 are labelled "Static", while if this structure of the errors is exploited, the values are referred to as "Dynamic". The Durbin-Watson statistics reported relate to the properties of the implied innovations.

4.3. *Estimation results*

Comparative results for the three specifications are presented in Table 1. Engel elasticities and compensated (i.e., Hicksian) price elasticities of demand for all three demand systems are presented in Table 2. The results for LES were derived under the implicit estimation scheme to confirm their equivalence to estimates derived in the standard way, and are presented as a basis for comparison with the new models.

For GLES the main point to make about these estimates is the substantial improvement in the log-likelihood, indicating that the restrictions required in moving from GLES to LES are not supported by the data on the basis of a χ^2 test. This improved fit also manifests itself in the substantially lower value (in the metric of distance from unity) of the scalar autocorrelation coefficient ρ, the corresponding improved fit of the static share equations, and higher Durbin-Watson statistics. Thus the generalization provided by moving from LES to GLES goes a substantial way towards explaining variation that is usually attributable to autocorrelation in LES. For RecLES on the other hand, the enhanced generality over LES has contributed somewhat less to improved fit, although this could well be a function of the particular nesting structure chosen from the large number of possibilities. The elasticities reported in Table 2 show that the price and expenditure elasticities for all three models are broadly consistent. This is hardly surprising given the use of the same data set,

Table 1. Estimation results

	Models					
	LES		GLES		RecLES	
			α_1	0.0841	β^1	0.0331*
			α_2	0.1728**	β^2	#0.9669**
			α_3	0.1982**	β^{11}	0.8849**
			α_4	#0.5449**	β^{12}	#0.1151**
	β_1	0.1162**	β_1	0.1469**	β^{111}	0.8914**
	β_2	0.0338**	β_2	−0.0280*	β^{112}	#0.1086**
	β_3	0.0352**	β_3	−0.0483*	γ^1	1101.6**
	β_4	#0.8149**	β_4	#0.9295**	γ^2	−9149.0**
	γ_1	401.3**	γ_1	501.9**	γ^{11}	−2105.3
	γ_2	179.1**	γ_2	151.4**	γ^{12}	981.3*
	γ_3	177.2**	γ_3	152.4*	γ^{111}	−2448.8
	γ_4	−685.1	γ_4	454.2	γ^{112}	1910.7**
	ρ	0.9885**	ρ	0.9041**	ρ	0.9742**
Log-likelihood	**537.29**		**542.36**		**541.33**	
R^2 *("Static")*						
I. Food	0.5044		0.9377		0.8029	
II. Alcohol/Tobacco	0.5712		0.9479		0.8035	
III. Clothing	0.5051		0.9424		0.7857	
IV. Other	0.8100		0.9781		0.9230	
R^2 *("Dynamic")*						
I. Food	0.9886		0.9886		0.9900	
II. Alcohol/Tobacco	0.9902		0.9905		0.9900	
III. Clothing	0.9886		0.9895		0.9891	
IV. Other	0.9956		0.9960		0.9961	
Durbin-Watson Statistics						
I. Food	1.0389		1.2586		1.0464	
II. Alcohol/Tobacco	0.9878		1.2664		0.9680	
III. Clothing	1.2512		1.5497		1.3391	
IV. Other	1.0959		1.1835		1.1124	

Estimated from adding-up condition.
* Significant at the 10% level
** Significant at the 1% level

and the common parentage of all three models. With this proviso, the amount of variation across models in Table 2, particularly the low expenditure elasticities for GLES for Alcohol and Tobacco, and for Clothing, demonstrates the enhanced flexibility of these models. In particular, it can be seen from the numerical results that the fixed relationship between the substitution and income elasticities that is reflected in the LES results is no longer present in either of the two more general models. The potential regularity violations caused by the negative parameter estimates is, however, a disappointing aspect of all three models, although it is interesting to note that for LES and RecLES it is the γ parameters, while for GLES it is the β parameters that violate regularity regions. While constrained estimation would be a simple option for these models, it is of course possible that the problem may be due to the level of aggregation of the data, and more investigation along these lines would be justified in searching for a regular yet parsimonious representation of the data.

Table 2. Estimated Engel and compensated price elasticities of demand for the three systems*

	Compensated Price Elasticities of Demand LES	GLES	RecLES
I. Food, *with respect to price of:*			
Food	**−0.4999**	**−0.3843**	**−0.3886**
Alcohol/Tobacco	0.0191	0.0288	0.0417
Clothing	0.0199	0.0293	0.0896
Other	0.4609	0.3262	0.2572
II. Alcohol/Tobacco, *with respect to price of:*			
Food	0.0513	0.0673	0.1073
Alcohol/Tobacco	**−0.4266**	**−0.5355**	**−0.4542**
Clothing	0.0155	0.0386	0.0896
Other	0.3598	0.4295	0.2572
III. Clothing, *with respect to price of:*			
Food	0.0571	0.0719	0.2487
Alcohol/Tobacco	0.0166	0.0405	0.0967
Clothing	**−0.4743**	**−0.5710**	**−0.6026**
Other	0.4006	0.4586	0.2572
IV. Other, *with respect to price of:*			
Food	0.1432	0.0990	0.0773
Alcohol/Tobacco	0.0416	0.0558	0.0301
Clothing	0.0433	0.0568	0.0279
Other	**−0.2282**	**−0.2116**	**−0.1353**
Elasticities of Demand with Respect to Total Expenditure			
Food	0.5737	0.6824	0.6223
Alcohol/Tobacco	0.4479	0.1949	0.3669
Clothing	0.4987	0.0763	0.2786
Other	1.2506	1.3153	1.2657

* Evaluated at values of the exogenous variables for 1984–85

5. Concluding remarks and perspective for further work

The results reported above demonstrate the feasibility of the cost-function approach to the specification and estimation of systems of demand equations. In particular, it is not necessary to have closed functional forms for Marshallian demand curves, nor for direct or indirect utility functions. We should, however, warn readers that the results reported here are for illustrative purposes only.

Data is the key to future work. For the methods described here to pay off in a policy context, we will need to work at a much greater level of disaggregation. It is here that the composition of cost functions to reflect insights (from whatever source) on the detailed structure of preferences will have its practical pay-off. Disaggregation of the time-series data to the level of about twenty commodities may be feasible.

A challenging goal is to develop routine procedures to implement the above research agenda. For example, if the structure of demand within some commodity group or sub-group is expected to have substitution elasticities following a certain pattern, the facility for composing the cost function should be able to accept this information readily and incorporate it routinely into a demand specification.

Appendix

This appendix contains the detailed formulae for the particular RecLES specification defined by equations (35) to (42). Using the notation of Section 3.1, the Hicksian (compensated) demand functions, in share form, can be efficiently defined recursively in terms of shares S at sub-levels. These shares are derived below.

As a shorthand, define price index 1 when evaluated for subgroup rs by $P1^{rs}$ (with $P1$ indicating evaluation at the top level) and define the normalized value of the quasi-cost of subgroup rs as

$$\tilde{C}^{rs} = C^{rs}/C^{rs}(0).$$

Then

$$S^1 = [\beta^1 uP1 + \gamma^1 \tilde{C}^1]/C \tag{47}$$

$$S^2 = [\beta^2 uP1 + \gamma^2 p_4]/C \tag{49}$$

$$S^{11} = [\beta^{11} uP1^1 + \gamma^{11} \tilde{C}^{11}]/C^1 \tag{50}$$

$$S^{12} = [\beta^{12} uP1^1 + \gamma^{12} p_3]/C^1 \tag{51}$$

$$S^{111} = [\beta^{111} uP1^{11} + \gamma^{111} p_1]/C^{11} \tag{52}$$

$$S^{112} = [\beta^{112} uP1^{11} + \gamma^{112} p_2]/C^{11}. \tag{53}$$

These shares, together with the functional form of the cost function, define the Marshallian share equations that were estimated.

Some idea of the generality allowed by this recursive structure can be gained by considering substitution possibilities. The terms in the Slutsky substitution matrix are as follows:

$$X_{11}^H = -[G(S^{11}S^{111})^2 + G^1 S^1 (S^{111})^2 + G^{11} S^1 S^{11}] \frac{C}{(p_1)^2} \tag{54}$$

$$X_{22}^H = -[G(S^{11}S^{112})^2 + G^1 S^1 (S^{112})^2 + G^{11} S^1 S^{11}] \frac{C}{(p_2)^2} \tag{55}$$

$$X_{33}^H = -[G(S^{12})^2 + G^1 S^1] \frac{C}{(p_3)^2} \tag{56}$$

$$X_{44}^H = -G \frac{C}{(p_4)^2} \tag{57}$$

$$X_{12}^H = X_{21}^H = -[G(S^{11})^2 S^{111} S^{112} + G^1 S^1 S^{111} S^{112} - G^{11} S^1 S^{11}] \frac{C}{p_1 p_2} \tag{58}$$

$$X_{13}^H = X_{31}^H = -[G S^{11} S^{12} S^{111} - G^1 S^1 S^{111}] \frac{C}{p_1 p_3} \tag{59}$$

$$X_{23}^H = X_{32}^H = -[G^1 S^{11} S^{12} S^{112} - G^2 S^1 S^{112}] \frac{C}{p_2 p_3} \tag{60}$$

$$X_{14}^H = X_{41}^H = G S^{11} S^{111} \frac{C}{p_1 p_4} \tag{61}$$

$$X_{24}^H = X_{42}^H = G S^{11} S^{112} \frac{C}{p_2 p_4} \tag{62}$$

$$X_{34}^H = X_{43}^H = G S^{12} \frac{C}{p_3 p_4} \tag{63}$$

where

$$G = \frac{\beta^1 \beta^2 u P1}{C} \tag{64}$$

$$G^1 = \frac{\beta^{11} \beta^{12} u P1^1}{C^1} \tag{65}$$

$$G^{11} = \frac{\beta^{111} \beta^{112} u P1^{11}}{C^{11}}. \tag{66}$$

Using the same method as in the previous examples, the derivatives of the Marshallian demands with respect to expenditure are:

$$X_{1c}^M = \left[\frac{\partial(S^1 C)}{\partial U} S^{11} S^{111} + S^1 \frac{\partial S^{11}}{\partial U} S^{111} C + S^1 S^{11} \frac{\partial S^{111}}{\partial U} C\right] \Big/ \left[p_1 \left\{\frac{\partial C}{\partial U}\right\}\right] \tag{67}$$

$$X_{2c}^M = \left[\frac{\partial(S^1 C)}{\partial U} S^{11} S^{112} + S^1 \frac{\partial S^{11}}{\partial U} S^{112} C + S^1 S^{11} \frac{\partial S^{112}}{\partial U} C\right] \Big/ \left[p_2 \left\{\frac{\partial C}{\partial U}\right\}\right] \tag{68}$$

$$X_{3c}^M = \left[\frac{\partial(S^1 C)}{\partial U} S^{12} + S^1 \frac{\partial S^{12}}{\partial U} C\right] \Big/ \left[p_3 \left\{\frac{\partial C}{\partial U}\right\}\right] \tag{69}$$

$$X_{4c}^M = \left[\frac{\partial(S^2 C)}{\partial U}\right] \Big/ \left[p_4 \left\{\frac{\partial C}{\partial U}\right\}\right] \tag{70}$$

where

$$\frac{\partial(S^1 C)}{\partial U} = \beta^1 P1 + (S^1 - G) \frac{C}{C^1} \left\{\frac{\partial C^1}{\partial U}\right\} \tag{71}$$

$$\frac{\partial(S^2 C)}{\partial U} = \beta^2 P1 + G \frac{C}{C^1} \left\{\frac{\partial C^1}{\partial U}\right\} \tag{72}$$

$$\frac{\partial S^{11}}{\partial U} = \frac{(\beta^{11} - S^{11})}{C^1} P1^1 + \frac{(S^{11} - G^1 - (S^{11})^2)}{C^2} P1^{11} \tag{73}$$

$$\frac{\partial S^{12}}{\partial U} = \frac{(\beta^{12} - S^{12})}{C^1} P1^1 + \frac{(G^1 - S^{11}S^{12})}{C^{11}} P1^{11} \tag{74}$$

$$\frac{\partial S^{111}}{\partial U} = \frac{(\beta^{111} - S^{111})}{C^{11}} P1^{11} \tag{75}$$

$$\frac{\partial S^{112}}{\partial U} = \frac{(\beta^{112} - S^{112})}{C^{11}} P1^{11} \tag{76}$$

and

$$\frac{\partial C}{\partial U} = P1 + S^1 \frac{C}{C^1} P1^1 + S^1 S^{11} \frac{C}{C^{11}} P1^{11} \tag{77}$$

$$\frac{\partial C^1}{\partial U} = P1^1 + S^{11} \frac{C^1}{C^{11}} P1^{11} \tag{78}$$

and where, for all terms above that are functions of utility, this level of utility is replaced by the inversion of the cost function.

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