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Uniform design with prior information of factors under weighted wrap-around L_2-discrepancy

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Abstract
Uniform design is one of the most frequently used designs of experiment, and all factors are usually regarded as equally important in the existing literature of uniform design. If some prior information of certain factors is known, the potential importance of factors should be distinguished. In this paper, by assigning different weights to factors with different importance, the weighted wrap-around L_2-discrepancy is proposed to measure the uniformity of design when some prior information of certain factors are known. The properties of weighted wrap-around L_2-discrepancy are explored. Accordingly, the weighted generalized wordlength pattern is proposed to describe the aberration of these kinds of designs. The relationship between the weighted wrap-around L_2-discrepancy and weighted generalized wordlength pattern is built, and a lower bound of weighted wrap-around L_2-discrepancy is obtained. Numerical results show that both weighted wrap-around L_2-discrepancy and weighted generalized wordlength pattern are precisely to capture the difference of importance among the columns of design.

Keywords Lower bound · Prior information of factor · Uniform design · Weighted generalized wordlength pattern · Weighted wrap-around L_2-discrepancy

1 Introduction

Uniform design has been frequently used in physical and computer experiments (see Fang et al. 2006, 2018). Uniform design scatters its experimental points uniformly throughout the design domain under some discrepancy criteria. As a measure of uniformity, discrepancy plays a key role in uniform design. Various discrepancies have been proposed by using the tool of reproducing kernel Hilbert space. The widely used discrepancies include, centered L_2-discrepancy and symmetric L_2-discrepancy (Hickernell 1998a), wrap-around L_2-discrepancy (Hickernell 1998b), discrete discrepancy

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(Hickernell and Liu 2002; Qin and Fang 2004), Lee discrepancy (Zhou et al. 2008), generalized discrete discrepancy (Chatterjee and Qin 2011) and mixture discrepancy (Zhou et al. 2013). To overcome the drawbacks in projection uniformity measured by symmetric L_2-discrepancy, the projection weighted symmetric L_2-discrepancy is proposed by He et al. (2020).

The general problem considered in design of experiment is how to select the “best" fractional factorial designs. In situations where we have little or no knowledge about the effects that are potentially important, it is appropriate to select designs using the minimum aberration criterion (Fries and Hunter 1980) or its generalization based on (generalized) wordlength pattern. The minimum aberration criterion has been frequently used in the selection of regular fractional factorial designs (Mukerjee and Wu 2006; Wu and Hamada 2009). In order to compare general factorial designs, Tang and Deng (1999) and Xu and Wu (2001) proposed the generalized minimum aberration criterion. They further justified the criterion for designs with qualitative factors under an ANOVA model. Meanwhile, the minimum generalized aberration criterion was proposed by Ma and Fang (2001) based on code theory. Motivating by the desire to unify minimum aberration and minimum β-aberration criteria, the concept of wordlength enumerator is proposed by Tang and Xu (2020) for general fractional factorial designs.

Usually, all factors are regarded as equally important in the existing literature of uniform design. In fact, not all factors are equally important in the design of experiment. If some knowledge or information indicates that certain effects are potentially important or unimportant, the potential importance of factors should be carefully distinguished.

Developing a new uniformity measure is motivated by the desire to reflect the importance of each factor that some important or unimportant factors are detected. Based on a sound statistical principle, the new concept of weighted wrap-around L_2-discrepancy is proposed in this paper, which is capable of measuring the uniformity of design with some potentially important or unimportant factors. Some properties of weighted wrap-around L_2-discrepancy are further explored. Accordingly, the weighted generalized wordlength pattern is proposed to describe the aberration of design with some potentially important or unimportant factors. The relationship between the weighted wrap-around L_2-discrepancy and weighted generalized wordlength pattern is built, and a lower bound of weighted wrap-around L_2-discrepancy is obtained. Numerical results show that both weighted wrap-around L_2-discrepancy and weighted generalized wordlength pattern perform well in capturing the difference of importance among experimental factors. We will focus our discussion on two-level designs. However, most of our arguments are quite general.

This paper is organized as follows. Some notations and preliminaries are described in Sect. 2. The weighted wrap-around L_2-discrepancy is defined in Sect. 3, and the projection property of weighted wrap-around L_2-discrepancy is also studied in this section. In Sect. 4, the weighted generalized wordlength pattern is defined, the relationship between the weighted wrap-around L_2-discrepancy and weighted generalized wordlength pattern is built, and a lower bound of weighted wrap-around L_2-discrepancy is obtained. Some numerical examples are provided in Sect. 5 to illustrate the theoretical results. Section 6 concludes this paper with some remarks and future work.

2 Notations and preliminaries

Consider a class of n runs and m factors with q levels U-type designs, denoted as $\mathcal{U}(n; q^m)$. A design d in $\mathcal{U}(n; q^m)$ can be presented as an $n \times m$ matrix with entries $0, 1, \ldots, q-1$, and each element occurs equally often in each column. Let d be a design in $\mathcal{U}(n; q^m)$, and d is regarded as a set of m columns $d = (x_1, \ldots, x_m)$, where $x_j = (x_{1j}, \ldots, x_{nj})'$ is the j-th column of d, $j = 1, \ldots, m$. Each row of d corresponds to a run and each column of d to an experimental factor in the design.

It is to be noted that any treatment combination $(x_{i1}, \ldots, x_{im})$ in design $d \in \mathcal{U}(n; q^m)$ can be mapped to $(u_{i1}, \ldots, u_{im})$, where $u_{ij} = \frac{2x_{ij}+1}{2q}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Traditionally, the uniformity of design $d \in \mathcal{U}(n; q^m)$ is measured by the wrap-around L_2-discrepancy, due to Hickernell (1998b), given by

$$[WD(d)]^2 = -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j=1}^{m}\left(\frac{3}{2} - \alpha_{il}^j\right), \tag{1}$$

where $\alpha_{il}^j = \left|u_{ij} - u_{lj}\right| - \left|u_{ij} - u_{lj}\right|^2$, $j = 1, \ldots, m; i, l = 1, \ldots, n$.

In particular, for two-level U-type designs $d \in \mathcal{U}(n; 2^m)$, another formulation of wrap-around L_2-discrepancy is presented in Fang et al. (2003) based on the row distance as follows

$$[WD(d)]^2 = -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\left(\frac{5}{4}\right)^m\sum_{i,j=1}^{n}\left(\frac{6}{5}\right)^{h_{ij}}, \tag{2}$$

where h_{ij} is the coincidence number between the i-th and j-th rows of design d. Moreover, a lower bound of $[WD(d)]^2$ for $d \in \mathcal{U}(n; 2^m)$ is obtained in their paper as follows

$$[WD(d)]^2 \geq -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{n-1}{n}\left(\frac{5}{4}\right)^m\left(\frac{6}{5}\right)^{\lambda}, \tag{3}$$

where $\lambda = \frac{m(n-2)}{2(n-1)}$.

On the other hand, the generalized minimum aberration (GMA) criterion is proposed by Xu and Wu (2001) for comparing fractional factorial designs. For any design $d \in \mathcal{U}(n; q^m)$ and $j = 0, \ldots, m$, define

$$B_j(d) = \frac{1}{n}\left|\{(a, b) | d_H(a, b) = j, a \in d, b \in d\}\right|, \tag{4}$$

where a and b are two runs of d, and $d_H(a, b)$ is the Hamming distance between a and b, namely, the number of places where they differ. The vector $(B_0(d), B_1(d), \ldots, B_m(d))$ is called as the distance distribution of design d. The generalized wordlength pattern of $d \in \mathcal{U}(n; q^m)$ is defined by $(A_0(d), A_1(d), \ldots, A_m(d))$, where

$$A_i(d) = \frac{1}{n} \sum_{j=0}^{m} P_i(j; m, q) B_j(d), \;\; i = 0, \ldots, m, \tag{5}$$

and $P_i(j; m, q) = \sum_{r=0}^{i} (-1)^r \binom{j}{r} \binom{m-j}{i-r}$ is the Krawtchouk polynomial, $\binom{s}{k} = s(s-1) \cdots (s-k+1)/k!$ and $\binom{s}{k} = 0$ for $s < k$. By the orthogonality of the Krawtchouk polynomials, it is easy to show that

$$B_j(d) = \frac{n}{q^m} \sum_{i=0}^{m} P_j(i; m, q) A_i(d), \;\; j = 0, \ldots, m. \tag{6}$$

The generalized minimum aberration criterion for selecting optimal design is to sequentially minimize $A_i(d)$ for $i = 1, \ldots, m$. One can refer to Xu and Wu (2001) for more details.

Meanwhile, Ma and Fang (2001) proposed the minimum generalized aberration (MGA) criterion based on the generalized wordlength pattern, which is similar to the ones defined in Xu and Wu (2001). In particular, the two generalized wordlength patterns in Ma and Fang (2001) and Xu and Wu (2001) are exactly the same for two-level design $d \in \mathcal{U}(n; 2^m)$. Furthermore, the relationship between wrap-around L_2-discrepancy and generalized wordlength pattern for two-level design $d \in \mathcal{U}(n; 2^m)$ is obtained in Ma and Fang (2001) as follows

$$[WD(d)]^2 = -\left(\frac{4}{3}\right)^m + \left(\frac{11}{8}\right)^m \sum_{j=0}^{m} \frac{A_j(d)}{11^j}. \tag{7}$$

3 Weighted wrap-around L_2-discrepancy

Under the assumption that each factor is equally important, the wrap-around L_2-discrepancy is commonly used to measure the uniformity of designs. When some prior information about the importance of factors is obtained before experiment, it is not appropriate to use only the wrap-around L_2-discrepancy to measure the uniformity of the design. The following uniform design measured by wrap-around L_2-discrepancy is a heuristic example, it is shown that the importance of each factor is not equally important.

Example 1 Consider the two-level design $d_1 = (x_1, \ldots, x_8) \in \mathcal{U}(8; 2^8)$ given below, it is a uniform design in $\mathcal{U}(8; 2^8)$ under the wrap-around L_2-discrepancy. There are eight factors in design d_1 and its factor correlation graph is given in Fig. 1.

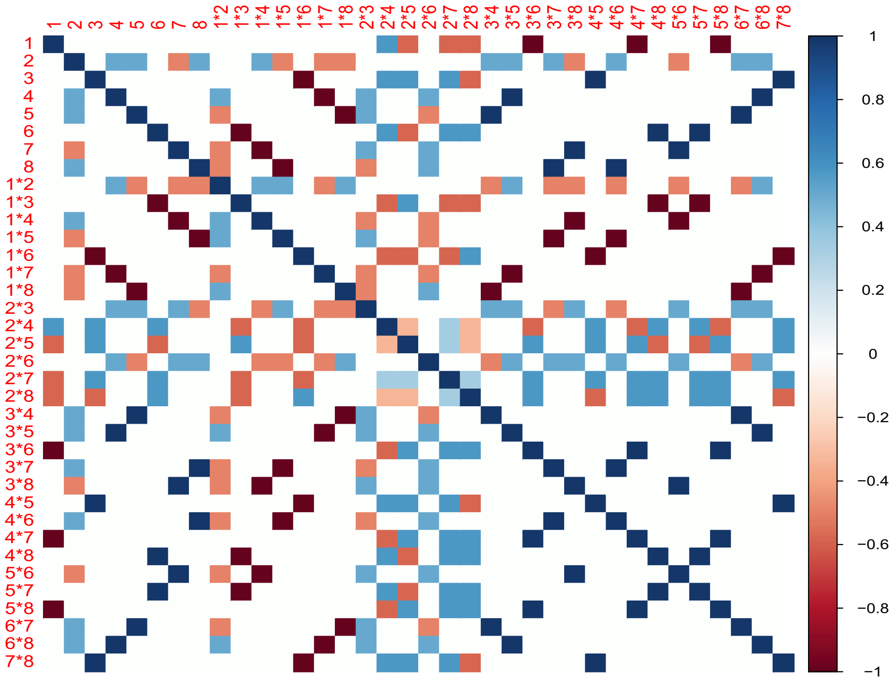

Fig. 1 Factor correlation graph of design d_1

$$d_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

In Fig. 1, the main effects of experimental factors $x_1, \ldots, x_8$ are respectively represented by 1, …, 8, and the two-factor interactions between factors a and b are denoted by $a * b$, where $a, b \in \{1, \ldots, 8\}$ and $a < b$. The correlations among all the 8 main effects and 28 two-factor interactions of design d_1 are visualized in Fig. 1. The largest absolute correlations for the two-factor interactions equal 1, and they are marked by the darkest off-diagonal cells. From Fig. 1, it is shown that there are 4 factor main effects (4, 5, 7, 8) and 12 two-factor interactions $(1 * 4, 1 * 5, 1 * 7, 1 * 8, 3 * 4, 3 * 5, 3 * 7, 3 * 8, 4 * 6, 5 * 6, 6 * 7, 6 * 8)$ that have significant correlations with factor x_2, and the degree of correlation of factor x_2 with the main effects and two-factor interactions is significantly higher than the other 7 factors. Therefore, Fig. 1 provides some prior information about importance of factors in design d_1, and the importance of factor x_2 in design d_1 is obviously weaker than the rest factors from the perspective of aberration. On the other hand, there is not factor main effect and there are only 7 two-factor interactions that have significant correlations with factors x_1, x_3, x_6. Thus, the importance of factors x_1, x_3, x_6 in design d_1 are obviously stronger than the rest factors.

To reflect the importance of each factor, the factors could be weighted such that the importance of each factor is represented by the value of its weight. Inspired by Example 1, a factor-weighted approach is proposed in this section, which can overcome the lack of sensitivity of factor importance in the uniform designs measured by wrap-around L_2-discrepancy. A new concept of weighted wrap-around L_2-discrepancy is defined as follows. The resulted uniform designs measured by weighted wrap-around L_2-discrepancy are able to distinguish the importance of each factor without changing the overall uniformity, which is specifically reflected in the degree of aberration among factor effects.

Definition 1 Let $d \in \mathcal{U}(n; q^m)$, $w = (w_1, \ldots, w_m)$ be the weight vector of design d, where $0 \leq w_j < 6$ is the weight of the j-th factor, $j = 1, \ldots, m$. The weighted wrap-around L_2-discrepancy of design d with weight vector w is defined as

$$[WWD(d)]^2 = -\left(\frac{4}{3}\right)^m + \frac{1}{n^2} \sum_{i,l=1}^{n} \prod_{j=1}^{m} \left(\frac{3}{2} - w_j \alpha_{il}^j\right), \tag{8}$$

where α_{il}^j's are defined in (1).

Remark 1 If there is some prior information about the experimental factors obtained before the experiment, the weights of these factors could be taken value larger or smaller than 1, and the weights of rest factors without prior information are assigned as 1. In Definition 1, the more important of the j-th factor, the larger its weight w_j takes, $j = 1, \ldots, m$. In order to guarantee $\frac{3}{2} - w_j \alpha_{il}^j > 0$, the weight w_j of the j-th factor should be less than 6. The tighter upper bound on w_j also depends on the number of weighted factors, which will be discussed in more details later. In particular, when all the w_j's are equal to 1, the weighted wrap-around L_2-discrepancy is just the wrap-around L_2-discrepancy defined in (1).

The relationship between the weighted wrap-around L_2-discrepancies of design $d \in \mathcal{U}(n; q^m)$ and its $(m-1)$-dimension projection design is given in Theorem 1 as follows. In addition, it will be helpful to rapidly search uniform designs measured by weighted wrap-around L_2-discrepancy.

Theorem 1 *Let $d \in \mathcal{U}(n; q^m)$, $d_{(-k)}$ be any $(m-1)$-dimension projection design of d obtained by deleting the k-th column of d, $k = 1, \ldots, m$, $w = (w_1, \ldots, w_m)$ is the weight vector of design d. Then the weighted wrap-around L_2-discrepancy of design d with weight vector w can be rewritten as*

$$\begin{aligned}[WWD(d)]^2 = &\frac{3}{2}[WWD(d_{(-k)})]^2 + \frac{1}{6}\left(\frac{4}{3}\right)^{m-1} \\ &- \frac{1}{n^2} \sum_{i,l=1}^{n} w_k \alpha_{il}^k \prod_{j=1, j\neq k}^{m} \left(\frac{3}{2} - w_j \alpha_{il}^j\right). \end{aligned} \tag{9}$$

Proof Without loss of generality, we only consider the case of $(m-1)$-dimension projection design $d_{(-m)}$ of design d obtained by deleting the last column of d, since the order of columns in design d is interchangeable. Thus, from the definition of $[WWD(d)]^2$ in (8)

$$
\begin{aligned}
&[WWD(d)]^2\\
&=-\left(\frac{4}{3}\right)^m+\frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j=1}^{m}\left(\frac{3}{2}-w_j\alpha_{il}^j\right)\\
&=-\left(\frac{4}{3}\right)^m+\frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j=1}^{m-1}\left(\frac{3}{2}-w_j\alpha_{il}^j\right)\left(\frac{3}{2}-w_m\alpha_{il}^m\right)\\
&=-\left(\frac{4}{3}\right)^m+\frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j=1}^{m-1}\frac{3}{2}\left(\frac{3}{2}-w_j\alpha_{il}^j\right)\\
&\quad+\frac{1}{n^2}\sum_{i,l=1}^{n}\left(-w_m\alpha_{il}^m\right)\prod_{j=1}^{m-1}\left(\frac{3}{2}-w_j\alpha_{il}^j\right)\\
&=-\left(\frac{4}{3}\right)^m+\frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j=1}^{m-1}\frac{3}{2}\left(\frac{3}{2}-w_j\alpha_{il}^j\right)-\frac{3}{2}\left(\frac{4}{3}\right)^{m-1}+\frac{3}{2}\left(\frac{4}{3}\right)^{m-1}\\
&\quad-\frac{1}{n^2}\sum_{i,l=1}^{n}w_m\alpha_{il}^m\prod_{j=1}^{m-1}\left(\frac{3}{2}-w_j\alpha_{il}^j\right)\\
&=\frac{3}{2}[WWD(d_{(-m)})]^2+\frac{1}{6}\left(\frac{4}{3}\right)^{m-1}-\frac{1}{n^2}\sum_{i,l=1}^{n}w_m\alpha_{il}^m\prod_{j=1}^{m-1}\left(\frac{3}{2}-w_j\alpha_{il}^j\right),
\end{aligned}
$$

which completes the proof. □

When all the w_j's in the weight vector w are equal to 1, the relationship between uniform design d and $(m-1)$-dimension projection $d_{(-k)}$ of design d is established in Theorem 1. Following the line of Theorem 1, $[WWD(d_{(-k)})]^2$ can also be expressed as the weighted wrap-around L_2-discrepancy of the $(m-2)$-dimension projection of design $d_{(-k)}$, and so on. If there are k elements of w_j's in the weight vector w are equal to 1, $k=1,\ldots,m-2$, that is, there are k factors (e.g. $\{x_{i_1},\ldots,x_{i_k}\}$) of design d with weight 1, the k-dimension projection design could be taken as the subdesign $d_k=(x_{i_1},\ldots,x_{i_k})$, and the weighted wrap-around L_2-discrepancy $[WWD(d)]^2$ of design d could be expressed as the wrap-around L_2-discrepancy $[WD(d_k)]^2$ of the k-dimension projection design d_k as the following theorem.

Theorem 2 *Let $d=(x_1,\ldots,x_m)\in\mathcal{U}(n;q^m)$. Suppose there are k factors $\{x_{i_1},\ldots,x_{i_k}\}$ of design d with weight 1, and the other $(m-k)$ factors $\{x_{j_1},\ldots,x_{j_{m-k}}\}$ of design d with weight $w_{j_l}\neq 1$, $k=1,\ldots,m-2$ and $l=1,\ldots,m-k$. Let $d_k=(x_{i_1},\ldots,x_{i_k})$ be the k-dimension projection design of d. Then the weighted* *wrap-around L_2-discrepancy of design d with weight vector w can be rewritten as*

$$[WWD(d)]^2 = \left(\frac{3}{2}\right)^{m-k} [WD(d_k)]^2 + \left(\frac{4}{3}\right)^k \left(\frac{3}{2}\right)^{m-k} - \left(\frac{4}{3}\right)^m$$
$$-\frac{1}{n^2}\sum_{t=0}^{m-k-2}\sum_{i,l=1}^{n}\left(\frac{3}{2}\right)^t w_{j_{t+1}}\alpha_{il}^{j_{t+1}} \prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)\prod_{g=t+2}^{m-k}\left(\frac{3}{2}-w_{j_g}\alpha_{il}^{j_g}\right)$$
$$-\frac{1}{n^2}\sum_{i,l=1}^{n}\left(\frac{3}{2}\right)^{m-k-1} w_{j_{m-k}}\alpha_{il}^{j_{m-k}} \prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right). \quad (10)$$

Proof According to Definition 1,

$$
\begin{aligned}
&[WWD(d)]^2\\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j=1}^{m}\left(\frac{3}{2}-w_j\alpha_{il}^j\right)\\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)\prod_{g=1}^{m-k}\left(\frac{3}{2}-w_{j_g}\alpha_{il}^{j_g}\right)\\
&= -\left(\frac{4}{3}\right)^m + \frac{3}{2}\frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)\prod_{g=2}^{m-k}\left(\frac{3}{2}-w_{j_g}\alpha_{il}^{j_g}\right)\\
&\quad -\frac{1}{n^2}\sum_{i,l=1}^{n} w_{j_1}\alpha_{il}^{j_1}\prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)\prod_{g=2}^{m-k}\left(\frac{3}{2}-w_{j_g}\alpha_{il}^{j_g}\right)\\
&= -\left(\frac{4}{3}\right)^m + \left(\frac{3}{2}\right)^2\frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)\prod_{g=3}^{m-k}\left(\frac{3}{2}-w_{j_g}\alpha_{il}^{j_g}\right)\\
&\quad -\frac{1}{n^2}\sum_{i,l=1}^{n} w_{j_2}\alpha_{il}^{j_2}\prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)\prod_{g=3}^{m-k}\left(\frac{3}{2}-w_{j_g}\alpha_{il}^{j_g}\right)\\
&\quad -\frac{1}{n^2}\sum_{i,l=1}^{n} w_{j_1}\alpha_{il}^{j_1}\prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)\prod_{g=2}^{m-k}\left(\frac{3}{2}-w_{j_g}\alpha_{il}^{j_g}\right)\\
&= -\left(\frac{4}{3}\right)^m + \left(\frac{3}{2}\right)^{m-k}\frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)\\
&\quad -\frac{1}{n^2}\sum_{t=0}^{m-k-2}\left(\frac{3}{2}\right)^t\sum_{i,l=1}^{n} w_{j_{t+1}}\alpha_{il}^{j_{t+1}}\prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)\prod_{g=t+2}^{m-k}\left(\frac{3}{2}-w_{j_g}\alpha_{il}^{j_g}\right)\\
&\quad -\frac{1}{n^2}\left(\frac{3}{2}\right)^{m-k-1}\sum_{i,l=1}^{n} w_{j_{m-k}}\alpha_{il}^{j_{m-k}}\prod_{j\in\{i_1,\ldots,i_k\}}\left(\frac{3}{2}-\alpha_{il}^j\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{3}{2}\right)^{m-k} \left[-\left(\frac{4}{3}\right)^{m-k} + \frac{1}{n^2} \sum_{i,l=1}^{n} \prod_{j \in \{i_1,\ldots,i_k\}} \left(\frac{3}{2} - \alpha_{il}^{j}\right) \right] \\
&\quad + \left(\frac{4}{3}\right)^{k} \left(\frac{3}{2}\right)^{m-k} - \left(\frac{4}{3}\right)^{m} \\
&\quad - \frac{1}{n^2} \sum_{t=0}^{m-k-1} \left(\frac{3}{2}\right)^{t} \sum_{i,l=1}^{n} w_{j_{t+1}} \alpha_{il}^{j_{t+1}} \prod_{j \in \{i_1,\ldots,i_k\}} \left(\frac{3}{2} - \alpha_{il}^{j}\right) \prod_{g=t+2}^{m-k} \left(\frac{3}{2} - w_{j_g} \alpha_{il}^{j_g}\right) \\
&\quad - \frac{1}{n^2} \left(\frac{3}{2}\right)^{m-k-1} \sum_{i,l=1}^{n} w_{j_{m-k}} \alpha_{il}^{j_{m-k}} \prod_{j \in \{i_1,\ldots,i_k\}} \left(\frac{3}{2} - \alpha_{il}^{j}\right) \\
&= \left(\frac{3}{2}\right)^{m-k} [WD(d_k)]^2 + \left(\frac{4}{3}\right)^{k} \left(\frac{3}{2}\right)^{m-k} - \left(\frac{4}{3}\right)^{m} \\
&\quad - \frac{1}{n^2} \sum_{t=0}^{m-k-1} \sum_{i,l=1}^{n} \left(\frac{3}{2}\right)^{t} w_{j_{t+1}} \alpha_{il}^{j_{t+1}} \prod_{j \in \{i_1,\ldots,i_k\}} \left(\frac{3}{2} - \alpha_{il}^{j}\right) \prod_{g=t+2}^{m-k} \left(\frac{3}{2} - w_{j_g} \alpha_{il}^{j_g}\right) \\
&\quad - \frac{1}{n^2} \left(\frac{3}{2}\right)^{m-k-1} \sum_{i,l=1}^{n} w_{j_{m-k}} \alpha_{il}^{j_{m-k}} \prod_{j \in \{i_1,\ldots,i_k\}} \left(\frac{3}{2} - \alpha_{il}^{j}\right),
\end{aligned}
$$

which completes the proof. □

Remark 2 When $k = m$, $[WWD(d)]^2 = [WD(d)]^2$, that is, the weighted wrap-around L_2-discrepancy is exactly as the wrap-around L_2-discrepancy. When $k = m-1$, it is a special case of Theorem 1, and the relationship between the weighted wrap-around L_2-discrepancy $[WWD(d)]^2$ of design d and the wrap-around L_2-discrepancy $[WD(d_k)]^2$ of d_k can be directly obtained by Theorem 1.

4 One factor has prior information in two-level designs

Before carrying out an experiment, weighting the factors is considered only when there is some prior information of specific factors obtained or in some special circumstances, which reflects the difference between factors with prior information or special factors and other factors.

Suppose that there are m_2 factors of d existing prior information and the rest m_1 ($=m - m_2$) factors without prior information. For convenience, all these kinds of two-level design d are denoted by $\mathcal{U}(n; 2^{m_1} \cdot 2^{m_2})$. First, the distance distribution of design $d \in \mathcal{U}(n; 2^{m_1} \cdot 2^{m_2})$ when some factors of d existing prior information are introduced. For any run a of design $d \in \mathcal{U}(n; 2^{m_1} \cdot 2^{m_2})$, it can be split into two parts, for example, $a = (a_1, a_2)$, where a_1 is the part with the first m_1 elements of a and a_2 is the part with the last m_2 elements of a. For $i = 0, 1, \ldots, m_1$ and $j = 0, 1, \ldots, m_2$, define

$$
B_{ij}(d) = \frac{1}{n} |\{(a, b)| d_H(a_1, b_1) = i, d_H(a_2, b_2) = j,
$$

$$a = (a_1, a_2), b = (b_1, b_2), a, b \in d\}|$$

as the distance distribution of design $d \in \mathcal{U}(n; 2^{m_1} \cdot 2^{m_2})$. Furthermore, following the line of Chatterjee et al. (2005), the MacWilliams transforms of the distance distribution $\{B_{ij}(d)\}$ of design $d \in \mathcal{U}(n; 2^{m_1} \cdot 2^{m_2})$ are

$$A_{gh}(d) = \frac{1}{n} \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} P_g(i; m_1, 2) P_h(j; m_2, 2) B_{ij}(d), \tag{11}$$

where $g = 0, 1, \ldots, m_1$; $h = 0, 1, \ldots, m_2$.

In most cases, the factors with known prior information are rarely detected. In this section, the one factor weighted wrap-around L_2-discrepancy $[WWD(d)]^2$ of two-level design $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$, that is, there is only one factor that has prior information, which will be explored in detail. Without loss of generality, the special factor with prior information is arranged in the last column of design d. For simplicity, the one factor weighted wrap-around L_2-discrepancy of two-level design $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$ is denoted by $OWWD_2(d)$.

Fang et al. (2003) expressed the wrap-around L_2-discrepancy of two-level design $d \in \mathcal{U}(n; 2^m)$ as the coincidence numbers between rows of d, which is reviewed in (2). Similarly, the one factor weighted wrap-around L_2-discrepancy $OWWD_2$ of two-level design $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$ can also be expressed as the coincidence numbers between rows of projection design of d as follows.

Theorem 3 *Let $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$, $d_{(-m)}$ be the projection design of d by deleting the last column of d, $w = (\mathbf{1}_{m-1}, w_m)$ be the weight vector of design d, where $\mathbf{1}_{m-1}$ is the $1 \times (m-1)$ vector with all elements unity, $0 \le w_m < 6$ and $w_m \neq 1$. The one factor weighted wrap-around L_2-discrepancy $OWWD_2$ of design d can be expressed as*

$$[OWWD_2(d)]^2 = -\left(\frac{4}{3}\right)^m + \frac{3}{2n^2}\left(\frac{5}{4}\right)^{m-1} \sum_{i,l=1}^{n} \left(\frac{6}{5}\right)^{h'_{il}} - \frac{w_m}{4n^2}\left(\frac{5}{4}\right)^{m-1} \sum_{i,l=1}^{n} \delta(u_{im} - u_{lm}) \left(\frac{6}{5}\right)^{h'_{il}}, \tag{12}$$

where

$$\delta(x) = \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases} \tag{13}$$

and h'_{il}'s are the coincidence numbers between the i-th and l-th rows of design $d_{(-m)}$.

Proof According to Theorem 1,

$$[OWWD_2(d)]^2 = \frac{3}{2}[WWD(d_{(-m)})]^2 + \frac{1}{6}\left(\frac{4}{3}\right)^{m-1}$$

$$-\frac{1}{n^2}\sum_{i,l=1}^{n} w_m\alpha_{il}^m \prod_{j=1}^{m-1}\left(\frac{3}{2}-w_j\alpha_{il}^j\right).$$

Since the first $(m-1)$ elements in the weight vector w are all equal to 1, the weighted wrap-around L_2-discrepancy $[WWD(d_{(-m)})]^2$ of $d_{(-m)}$ is just the wrap-around L_2-discrepancy $[WD(d_{(-m)})]^2$ of $d_{(-m)}$, that is, $[WWD(d_{(-m)})]^2 = [WD(d_{(-m)})]^2$. When $q = 2$, u_{ij} can only take $\frac{1}{4}$ or $\frac{3}{4}$, thus $\frac{3}{2}-\alpha_{il}^j$ can only take $\frac{3}{2}$ or $\frac{5}{4}$. Therefore, through expressing $[WD(d_{(-m)})]^2$ as the form of (2)

$$\begin{aligned}
&[OWWD_2(d)]^2\\
&=\frac{3}{2}[WD(d_{(-m)})]^2+\frac{1}{6}\left(\frac{4}{3}\right)^{m-1}-\frac{1}{n^2}\sum_{i,l=1}^{n} w_m\alpha_{il}^m\prod_{j=1}^{m-1}\left(\frac{3}{2}-w_j\alpha_{il}^j\right)\\
&=\frac{3}{2}\left[-\left(\frac{4}{3}\right)^{m-1}+\frac{1}{n^2}\left(\frac{5}{4}\right)^{m-1}\sum_{i,l=1}^{n}\left(\frac{6}{5}\right)^{h'_{il}}\right]+\frac{1}{6}\left(\frac{4}{3}\right)^{m-1}\\
&-\frac{1}{n^2}\sum_{i,l=1}^{n} w_m\alpha_{il}^m\prod_{j=1}^{m-1}\left(\frac{3}{2}-w_j\alpha_{il}^j\right)\\
&=-\left(\frac{4}{3}\right)^{m}+\frac{3}{2n^2}\left(\frac{5}{4}\right)^{m-1}\sum_{i,l=1}^{n}\left(\frac{6}{5}\right)^{h'_{il}}-\frac{w_m}{n^2}\sum_{i,l=1}^{n}\alpha_{il}^m\left(\frac{3}{2}\right)^{h'_{il}}\left(\frac{5}{4}\right)^{m-1-h'_{il}}\\
&=-\left(\frac{4}{3}\right)^{m}+\frac{3}{2n^2}\left(\frac{5}{4}\right)^{m-1}\sum_{i,l=1}^{n}\left(\frac{6}{5}\right)^{h'_{il}}\\
&-\frac{w_m}{4n^2}\left(\frac{5}{4}\right)^{m-1}\sum_{i,l=1}^{n}\delta(u_{im}-u_{lm})\left(\frac{6}{5}\right)^{h'_{il}},
\end{aligned}$$

which completes the proof. □

Based on the expression of $[WD(d)]^2$ in (2), a lower bound of wrap-around L_2-discrepancy $[WD(d)]^2$ of design d is obtained in Fang et al. (2003), which is given in (3). Following the line of Fang et al. (2003), a lower bound of the one factor weighted wrap-around L_2-discrepancy $[OWWD_2(d)]^2$ of two-level design $d \in \mathcal{U}(n; 2^{m-1}\cdot 2^1)$ is provided in the following theorem.

Theorem 4 *Let $d \in \mathcal{U}(n; 2^{m-1}\cdot 2^1)$, $w = (\mathbf{1}_{m-1}, w_m)$ be the weight vector of design d, where $0 \le w_m < 6$ and $w_m \ne 1$. Then $[OWWD_2(d)]^2 \ge LB_1$, where*

$$LB_1 = -\left(\frac{4}{3}\right)^m+\frac{1}{n}\left(\frac{3}{2}\right)^m+\frac{n-1}{n}\left(\frac{5}{4}\right)^{m-\gamma}\left(\frac{3}{2}-\frac{w_m}{4}\right)^{\gamma}\left(\frac{6}{5}\right)^{\lambda}, \quad (14)$$

where $\gamma = 1-\frac{n-2}{2(n-1)}$, $\lambda = \frac{m(n-2)}{2(n-1)}$. When $w_m = 1$, the lower bound LB_1 of $OWWD_2$ in (14) is the same as the ones in (3), which is obtained by Fang et al. (2003).

Proof Let $t_{il}^{m} = \delta(u_{im} - u_{lm})$ be the number of different element between the i-th and l-th rows in the m-th column of design d. According to Definition 1,

$$[OWWD_2(d)]^2 = -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j=1}^{m}\left(\frac{3}{2} - w_j\alpha_{il}^{j}\right)$$
$$= -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\sum_{i,l=1}^{n}\left(\frac{3}{2}\right)^{h_{il}}\left(\frac{5}{4}\right)^{m-h_{il}-t_{il}^m}\left(\frac{3}{2} - \frac{w_m}{4}\right)^{t_{il}^m}$$
$$= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n}\left(\frac{3}{2}\right)^{h_{il}}\left(\frac{5}{4}\right)^{m-h_{il}-t_{il}^m}\left(\frac{3}{2} - \frac{w_m}{4}\right)^{t_{il}^m}$$
$$= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n}\left(\frac{6}{5}\right)^{h_{il}}\left(\frac{5}{4}\right)^{m-t_{il}^m}\left(\frac{3}{2} - \frac{w_m}{4}\right)^{t_{il}^m}.$$

Based on Jenssen's inequality,

$$\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n}\left(\frac{6}{5}\right)^{h_{il}} \geq \left(\frac{6}{5}\right)^{\lambda},$$

thus,

$$[OWWD_2(d)]^2$$
$$\geq -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{1}{n^2}\left(\frac{5}{4}\right)^{m-\gamma}\left(\frac{3}{2} - \frac{w_m}{4}\right)^{\gamma}\left(\frac{6}{5}\right)^{\lambda} n(n-1)$$
$$= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{n-1}{n}\left(\frac{5}{4}\right)^{m-\gamma}\left(\frac{3}{2} - \frac{w_m}{4}\right)^{\gamma}\left(\frac{6}{5}\right)^{\lambda},$$

which completes the proof. □

By adding nonnegativity constraint on the lower bound LB_1 of $[OWWD_2(d)]^2$ obtained in Theorem 4, an upper bound of w_m in the one factor weighted wrap-around L_2-discrepancy $OWWD_2$ is provided as follows

$$w_m \leq 6 - 4\left\{\frac{n}{n-1}\left(\frac{4}{5}\right)^{m-\gamma}\left(\frac{5}{6}\right)^{\lambda}\left[\left(\frac{4}{3}\right)^m - \frac{1}{n}\left(\frac{3}{2}\right)^m\right]\right\}^{\frac{1}{\gamma}} \triangleq U_w. \quad (15)$$

The following lemma will be helpful in establishing another lower bound of the one factor weighted wrap-around L_2-discrepancy $OWWD_2$.

Lemma 1 (Hu et al. 2020) *Suppose* $\sum_{i=1}^{n} x_i = c_1$ *and* $\sum_{i=1}^{n} y_i = c_2$*, where* x_i *and* y_i *are nonnegative real numbers. Let* $z_i = ax_i + by_i$ *for* $i = 1, \ldots, n$*, and* $c = ac_1 + bc_2$*,* *where $a > 0, b > 0$. Denote $z_{(1)}, \ldots, z_{(l)}$ as the ordered arrangements of the distinct possible values of $z_1, \ldots, z_n$, where $1 \le l \le n$. Then for any positive integer t*

$$\sum_{i=1}^{n} z_i^t \ge p z_{(k)}^t + q z_{(k+1)}^t,$$

where k is the largest integer such that $z_{(k)} \le c/n \le z_{(k+1)}$, p and q are nonnegative real numbers such that $p + q = n$ and $p z_{(k)} + q z_{(k+1)} = c$.

A new lower bound of the one factor weighted wrap-around L_2-discrepancy $OWWD_2$ can be obtained from Lemma 1, which is given in the following theorem.

Theorem 5 *Let $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$, $w = (\mathbf{1}_{m-1}, w_m)$ be the weight vector of design d, where $w_m \in [0, U_w]$ and $w_m \neq 1$. Define $z_{il} = a h'_{il} + b h_{il}^m$ for $i, l = 1, \ldots, n$ and $i \neq l$, where $a = \ln(\frac{6}{5})$, $b = \ln(\frac{6}{6-w_m})$. Denote $z_{(1)}, \ldots, z_{(s)}$ the ordered arrangements of the distinct possible values of $\{z_{il}\}$, where $1 \le s \le n(n-1)$, Then $[OWWD_2(d)]^2 \ge LB_2$, where*

$$LB_2 = -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{6-w_m}{4n^2}\left(\frac{5}{4}\right)^{m-1}[p\mathrm{e}^{z_{(k)}} + q\mathrm{e}^{z_{(k+1)}}], \qquad (16)$$

k is the largest integer such that $z_{(k)} \le \frac{c}{n(n-1)} < z_{(k+1)}$, p and q are nonnegative real numbers such that $p + q = n(n-1)$ and $p z_{(k)} + q z_{(k+1)} = c = \frac{1}{2} n(n-2)(m-1)a + \frac{1}{2} n(n-2) b$.

Proof Let $h_{il}^m = 1 - t_{il}^m$, where t_{il}^m is defined in proof for Theorem 4. According to Definition 1,

$$\begin{aligned}
[OWWD_2(d)]^2 &= -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j=1}^{m}\left(\frac{3}{2} - w_j \alpha_{il}^j\right) \\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{6-w_m}{4n^2}\left(\frac{5}{4}\right)^{m-1}\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n}\left(\frac{6}{5}\right)^{h'_{il}}\left(\frac{6}{6-w_m}\right)^{h_{il}^m} \\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{6-w_m}{4n^2}\left(\frac{5}{4}\right)^{m-1}\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n}\mathrm{e}^{a h'_{il} + b h_{il}^m} \\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{6-w_m}{4n^2}\left(\frac{5}{4}\right)^{m-1}\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n}\sum_{t=0}^{\infty}\frac{z_{il}^t}{t!} \\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{6-w_m}{4n^2}\left(\frac{5}{4}\right)^{m-1}\sum_{t=0}^{\infty}\frac{1}{t!}\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n} z_{il}^t.
\end{aligned}$$

It is easy to verify that $\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n} h'_{il} = c_1 = \frac{1}{2}n(n-2)(m-1)$ and $\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n} h_{il}^{m} = c_2 = \frac{1}{2}n(n-2)$. From Lemma 1,

$$\sum_{i=1}^{n}\sum_{l(\neq i)=1}^{n} z_{il}^{t} \geq p z_{(k)}^{t} + q z_{(k+1)}^{t}.$$

Thus,

$$\begin{aligned}
&[OWWD_2(d)]^2 \\
&\geq -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{6-w_m}{4n^2}\left(\frac{5}{4}\right)^{m-1}\sum_{t=0}^{\infty}\frac{1}{t!}\left[p z_{(k)}^{t} + q z_{(k+1)}^{t}\right] \\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m + \frac{6-w_m}{4n^2}\left(\frac{5}{4}\right)^{m-1}\left[p e^{z_{(k)}} + q e^{z_{(k+1)}}\right],
\end{aligned}$$

which completes the proof. □

From Theorems 4 and 5, an improved lower bound of one factor weighted wrap-around L_2-discrepancy $OWWD_2$ can be obtained in the following.

Theorem 6 *Let $d \in \mathcal{U}(n; 2^{m-1}\cdot 2^1)$, $w = (\mathbf{1}_{m-1}, w_m)$ be the weight vector of design d, where $w_m \in [0, U_w]$ and $w_m \neq 1$. Then $[OWWD_2(d)]^2 \geq \max\{LB_1, LB_2\} \triangleq LB$, where LB_1 and LB_2 are the lower bounds of $OWWD_2$ obtained in Theorems 4 and 5, respectively.*

The following theorem provides the relationship between $[OWWD_2(d)]^2$ and the distance distribution $\{B_{ij}(d)\}$ of design $d \in \mathcal{U}(n; 2^{m-1}\cdot 2^1)$.

Theorem 7 *Let $d \in \mathcal{U}(n; 2^{m-1}\cdot 2^1)$, $w = (\mathbf{1}_{m-1}, w_m)$ be the weight vector of design d, where $w_m \in [0, U_w]$ and $w_m \neq 1$. Then $[OWWD_2(d)]^2$ can be expressed as*

$$\begin{aligned}
[OWWD_2(d)]^2 &= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m\sum_{i=0}^{m-1}\sum_{j=0}^{1}\left(\frac{5}{6}\right)^i\left(\frac{6-w_m}{6}\right)^j B_{ij}(d) \\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m\sum_{i=0}^{m-1}\left(\frac{5}{6}\right)^i B_{i0}(d) + \frac{6-w_m}{6n}\left(\frac{3}{2}\right)^m\sum_{i=0}^{m-1}\left(\frac{5}{6}\right)^i B_{i1}(d).
\end{aligned}$$

Proof According to Definition 1 and the definition of B_{ij}'s, $[OWWD_2(d)]^2$ can be rewritten as

$$\begin{aligned}
&[OWWD_2(d)]^2 \\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\sum_{i,l=1}^{n}\prod_{j=1}^{m}\left(\frac{3}{2} - w_j\alpha_{il}^{j}\right) \\
&= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m\sum_{i=0}^{m-1}\sum_{j=0}^{1}\left(\frac{3}{2}\right)^{m-1-i}\left(\frac{5}{4}\right)^i\left(\frac{3}{2}\right)^{1-j}\left(\frac{6-w_m}{4}\right)^j B_{ij}(d)
\end{aligned}$$

$$= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m \sum_{i=0}^{m-1}\sum_{j=0}^{1}\left(\frac{5}{6}\right)^i\left(\frac{6-w_m}{6}\right)^j B_{ij}(d)$$

$$= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m \sum_{i=0}^{m-1}\left(\frac{5}{6}\right)^i B_{i0}(d)$$

$$+\frac{1}{n}\left(\frac{3}{2}\right)^m\left(\frac{6-w_m}{6}\right)\sum_{i=0}^{m-1}\left(\frac{5}{6}\right)^i B_{i1}(d),$$

which completes the proof. □

For design $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$, the relationship between $[OWWD_2(d)]^2$ and the MacWilliams transforms $\{A_{gh}(d)\}$ of the distance distribution $\{B_{ij}(d)\}$ is built in the following theorem.

Theorem 8 *Let $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$, $w = (\mathbf{1}_{m-1}, w_m)$ be the weight vector of design d, where $w_m \in [0, U_w]$ and $w_m \neq 1$, $\tilde{w}_m = \frac{12-w_m}{11}$. Then $[OWWD_2(d)]^2$ can be expressed as*

$$[OWWD_2(d)]^2 = -\left(\frac{4}{3}\right)^m + \frac{12-w_m}{11}\left(\frac{11}{8}\right)^m \sum_{g=0}^{m-1}\sum_{h=0}^{1}\left(\frac{w_m}{12-w_m}\right)^h \frac{A_{gh}(d)}{11^g}$$

$$= -\left(\frac{4}{3}\right)^m + \left(\frac{11}{8}\right)^m\left[\tilde{w}_m \sum_{g=0}^{m-1}\frac{A_{g0}(d)}{11^g} + w_m \sum_{g=0}^{m-1}\frac{A_{g1}(d)}{11^{g+1}}\right].$$

Proof Appling the orthogonality of Krawtchouk polynomials to $\{A_{gh}(d)\}$ defined in (11), the expression of $[OWWD_2(d)]^2$ in Theorem 7 can be rewritten as

$$[OWWD_2(d)]^2$$

$$= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m \sum_{i=0}^{m-1}\sum_{j=0}^{1}\left(\frac{5}{6}\right)^i\left(\frac{6-w_m}{6}\right)^j B_{ij}(d)$$

$$= -\left(\frac{4}{3}\right)^m + \frac{1}{n}\left(\frac{3}{2}\right)^m \sum_{i=0}^{m-1}\sum_{j=0}^{1}\left(\frac{5}{6}\right)^i\left(\frac{6-w_m}{6}\right)^j$$

$$\times\left[\frac{n}{2^m}\sum_{g=0}^{m-1}\sum_{h=0}^{1} A_{gh}(d)P_i(g; m-1, 2)P_j(h; 1, 2)\right]$$

$$= -\left(\frac{4}{3}\right)^m + \left(\frac{3}{4}\right)^m \sum_{i,g=0}^{m-1}\left(\frac{5}{6}\right)^i P_i(g; m-1, 2)$$

$$\times \sum_{j,h=0}^{1} P_j(h; 1, 2)\left(\frac{6-w_m}{6}\right)^j A_{gh}(d)$$

$$= -\left(\frac{4}{3}\right)^m + \frac{12-w_m}{6}\left(\frac{3}{4}\right)^m\left(\frac{11}{6}\right)^{m-1}\sum_{g=0}^{m-1}\sum_{h=0}^{1}\left(\frac{w_m}{12-w_m}\right)^h\frac{A_{gh}(d)}{11^g}$$
$$= -\left(\frac{4}{3}\right)^m + \frac{12-w_m}{11}\left(\frac{11}{8}\right)^m\sum_{g=0}^{m-1}\sum_{h=0}^{1}\left(\frac{w_m}{12-w_m}\right)^h\frac{A_{gh}(d)}{11^g}$$
$$= -\left(\frac{4}{3}\right)^m + \frac{12-w_m}{11}\left(\frac{11}{8}\right)^m\sum_{g=0}^{m-1}\frac{A_{g0}(d)}{11^g} + \frac{w_m}{11}\left(\frac{11}{8}\right)^m\sum_{g=0}^{m-1}\frac{A_{g1}(d)}{11^g}$$
$$= -\left(\frac{4}{3}\right)^m + \left(\frac{11}{8}\right)^m\left[\tilde{w}_m\sum_{g=0}^{m-1}\frac{A_{g0}(d)}{11^g} + w_m\sum_{g=0}^{m-1}\frac{A_{g1}(d)}{11^{g+1}}\right],$$

which completes the proof. □

In order to define the weighted generalized wordlength pattern of design $d \in \mathcal{U}(n; 2^{m-1}\cdot 2^1)$, the expression of $[OWWD(d)]^2$ in Theorem 8 can be rewritten as

$$[OWWD_2(d)]^2 = -\left(\frac{4}{3}\right)^m + \left(\frac{11}{8}\right)^m$$
$$\times\left[\frac{\tilde{w}_m A_{00}(d)}{11^0} + \tilde{w}_m\sum_{g=1}^{m-1}\frac{A_{g0}(d)}{11^g} + w_m\sum_{g=1}^{m-1}\frac{A_{(g-1)1}(d)}{11^{(g-1)+1}} + \frac{w_m A_{(m-1)1}(d)}{11^m}\right]$$
$$= -\left(\frac{4}{3}\right)^m + \left(\frac{11}{8}\right)^m$$
$$\times\left[\frac{\tilde{w}_m A_{00}(d)}{11^0} + \sum_{g=1}^{m-1}\frac{\tilde{w}_m A_{g0}(d) + w_m A_{(g-1)1}(d)}{11^g} + \frac{w_m A_{(m-1)1}(d)}{11^m}\right]$$
$$= -\left(\frac{4}{3}\right)^m + \left(\frac{11}{8}\right)^m\sum_{g=0}^{m}\frac{WA_g(d)}{11^g}, \tag{17}$$

where

$$WA_g(d) = \begin{cases} \tilde{w}_m A_{00}(d), & g = 0, \\ \tilde{w}_m A_{g0}(d) + w_m A_{(g-1)1}(d), & g = 1, \ldots, m-1, \\ w_m A_{(m-1)1}(d), & g = m, \end{cases} \tag{18}$$
$$= \delta(m-g)\tilde{w}_m A_{g0}(d) + \delta(g) w_m A_{(g-1)1}(d),\ g = 0, 1, \ldots, m, \tag{19}$$

and $\tilde{w}_m$ is defined in Theorem 8, $\delta(\cdot)$ is the delta function defined in (13).

From the definition of $\tilde{w}_m$ in Theorem 8, it is easy to see that

$$\begin{cases} \tilde{w}_m > 1, & \text{when } 0 \le w_m < 1, \\ \tilde{w}_m = 1, & \text{when } w_m = 1, \\ \tilde{w}_m < 1, & \text{when } 1 < w_m \le U_w. \end{cases}$$

Thus, from the definition of $WA_g(d)$'s in (18) or (19), when $0 \leq w_m < 1$ leads to $\tilde{w}_m > 1$, it means that $WA_g(d)$ is mainly decided by $A_{g0}(d)$ and $A_{(g-1)1}(d)$ plays a relatively minor role in $WA_g(d)$ for unimportant factor x_m. Conversely, when $1 < w_m \leq U_w$ leads to $\tilde{w}_m < 1$, it means that $WA_g(d)$ is mainly decided by $A_{(g-1)1}(d)$ and $A_{g0}(d)$ plays a relatively minor role in $WA_g(d)$ for important factor x_m. In particular, when $w_m = 1$ leads to $\tilde{w}_m = 1$, it means that $A_{g0}(d)$ and $A_{(g-1)1}(d)$ play the same role in $WA_g(d)$ for without prior information factor x_m. In brief, the aberration among factors in design $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$ is precisely captured by $WA_g(d)$, $g = 1, \ldots, m$.

On the other hand, comparing the equations (7) and (17), the only difference is that $A_g(d)$ is substituted by $WA_g(d)$ in turn, $g = 0, 1, \ldots, m$. Based on the $WA_g(d)$'s defined in (18) or (19), the weighted generalized wordlength pattern of design $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$ is formally defined as follows.

Definition 2 Let $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$, $w = (\mathbf{1}_{m-1}, w_m)$ be the weight vector of design d, where $w_m \in [0, U_w]$ and $w_m \neq 1$. The one factor weighted generalized wordlength pattern (OWGWP) of design d is defined as $WA(d) = (WA_0(d), WA_1(d), \ldots, WA_m(d))$, where $WA_g(d)$ is defined in (18) or (19).

When $w_m = 1$, the weighted generalized wordlength pattern of design $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$ in Definition 2 is just the same as the generalized wordlength pattern of design $d \in \mathcal{U}(n; 2^m)$ defined in (5). From the perspective of aberration, the optimality of designs in $\mathcal{U}(n; 2^{m-1} \cdot 2^1)$ can be measured by sequentially minimizing the weighted generalized wordlength pattern defined in Definition 2, and it is formally defined as follows.

Definition 3 Let d_1, d_2 be two designs in $\mathcal{U}(n; 2^{m-1} \cdot 2^1)$, if there exists some $k \in \{1, \ldots, m\}$ such that $WA_k(d_1) < WA_k(d_2)$ and $WA_j(d_1) = WA_j(d_2)$, $j = 0, 1, \ldots, k-1$, then design d_1 is said to have less weighted generalized aberration than design d_2. A design $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$ has minimum weighted generalized aberration if no other design in $\mathcal{U}(n; 2^{m-1} \cdot 2^1)$ has less weighted generalized aberration than it.

From the relationship between $[OWWD_2(d)]^2$ and $\{WA_g(d)\}$ given in (17), the leading factor of $WA_g(d)$ in $[OWWD_2(d)]^2$ decrease exponentially with g, thus design $d \in \mathcal{U}(n; 2^{m-1} \cdot 2^1)$ which has less weighted generalized aberration, in the sense of having small $WA_g(d)$ for small values of g, behaves well in terms of uniformity measured by $[OWWD_2(d)]^2$.

5 Illustrative examples

In this section, we give some examples to show our theoretical results.

Example 1 (Continued) Consider the design $d_1 \in \mathcal{U}(8; 2^8)$ in Example 1 again. Assuming that only one of the experimental factors has prior information, and the special factor is arranged in rotation to each column of d_1, the corresponding weight is respectively assigned to 0.5 for unimportant factor and 1.5 for important factor. All the numerical results are listed in Table 1.

Table 1 Numerical results of design d_1

Weight vector w	$[OWWD(d_1)]^2$	LB	$WA(d_1)$
(1, 1, 1, 1, 1, 1, 1, 1)	2.999591	2.999591	(0, 1, 10, 11, 4, 3, 2, 0)
(1, 1, 1, 1, 1, 1, 1, 0.5)	3.553198	3.514575	(0, 0.9, 8.4, 8.5, 2.8, 1.9, 1.1, 0)
(1, 1, 1, 1, 1, 1, 0.5, 1)	3.553198	3.514575	(0, 0.9, 8.4, 8.5, 2.8, 1.9, 1.1, 0)
(1, 1, 1, 1, 1, 0.5, 1, 1)	3.566073	3.514575	(0, 1.0, 8.3, 8.2, 3.1, 2.0, 1.0, 0)
(1, 1, 1, 1, 0.5, 1, 1, 1)	3.553198	3.514575	(0, 0.9, 8.4, 8.5, 2.8, 1.9, 1.1, 0)
(1, 1, 1, 0.5, 1, 1, 1, 1)	3.553198	3.514575	(0, 0.9, 8.4, 8.5, 2.8, 1.9, 1.1, 0)
(1, 1, 0.5, 1, 1, 1, 1, 1)	3.566073	3.514575	(0, 1.0, 8.3, 8.2, 3.1, 2.0, 1.0, 0)
(1, 0.5, 1, 1, 1, 1, 1, 1)	3.514575	3.514575	(0, 0.5, 8.8, 9.3, 2.0, 1.5, 1.5, 0)
(0.5, 1, 1, 1, 1, 1, 1, 1)	3.566073	3.514575	(0, 1.0, 8.3, 8.2, 3.1, 2.0, 1.0, 0)
(1, 1, 1, 1, 1, 1, 1, 1.5)	2.445983	2.395101	(0, 1.1, 11.6, 13.2, 5.2, 4.1, 2.9, 0)
(1, 1, 1, 1, 1, 1, 1.5, 1)	2.445983	2.395101	(0, 1.1, 11.6, 13.2, 5.2, 4.1, 2.9, 0)
(1, 1, 1, 1, 1, 1.5, 1, 1)	2.433108	2.395101	(0, 1.0, 11.7, 13.8, 4.9, 4.0, 3.0, 0)
(1, 1, 1, 1, 1.5, 1, 1, 1)	2.445983	2.395101	(0, 1.1, 11.6, 13.2, 5.2, 4.1, 2.9, 0)
(1, 1, 1, 1.5, 1, 1, 1, 1)	2.445983	2.395101	(0, 1.1, 11.6, 13.2, 5.2, 4.1, 2.9, 0)
(1, 1, 1.5, 1, 1, 1, 1, 1)	2.433108	2.395101	(0, 1.0, 11.7, 13.8, 4.9, 4.0, 3.0, 0)
(1, 1.5, 1, 1, 1, 1, 1, 1)	2.484606	2.395101	(0, 1.5, 11.2, 12.7, 6.0, 4.5, 2.5, 0)
(1.5, 1, 1, 1, 1, 1, 1, 1)	2.433108	2.395101	(0, 1.0, 11.7, 13.8, 4.9, 4.0, 3.0, 0)

From Table 1, if the special factor existing some prior information indicates that it is an unimportant factor, and its weight is assigned to 0.5, the corresponding $[OWWD(d_1)]^2$ is minimum and the OWGWP is sequentially minimum when the special factor is arranged to the 2nd column of d_1, when the special factor is arranged to one of columns 1, 3, 6 of d_1, the corresponding $[OWWD(d_1)]^2$ and OWGWP are among the worst performing weighted designs. On the other hand, if the special factor existing some prior information indicates that it is an important factor, and its weight is assigned to 1.5, the corresponding $[OWWD(d_1)]^2$ is minimum and the OWGWP is sequentially minimum when the special factor is arranged to one of columns 1, 3, 6 of d_1, when the special factor is arranged to the 2nd column of d_1, the corresponding $[OWWD(d_1)]^2$ and OWGWP are among the worst performing weighted designs.

In fact, the conclusions obtained above are independent of the choice of weight. The curves of $[OWWD(d_1)]^2$ of design d_1 related to w_k are given in Fig. 2, where w_k is the weight of the k-th factor of design d_1, $k = 1, \ldots, 8$. It is shown that $[OWWD(d_1)]^2$ is a simple linear function of w_k for given design d_1, and all the lines of $[OWWD(d_1)]^2$ intersect at $w_k = 1$. Especially for $k = 2$, when $0 \leq w_k < 1$, the line of $[OWWD(d_1)]^2$ is under the other lines of $[OWWD(d_1)]^2$ for $k \neq 2$; and when $1 < w_k \leq U_w$ ($U_w = 3.346295$), the line of $[OWWD(d_1)]^2$ is upon the other lines of $[OWWD(d_1)]^2$ for $k \neq 2$. It means that the 2nd factor is suitable to arrange unimportant factor and is not suitable to arrange important factor. Similarly, factors 1, 3, 6 are suitable to arrange important factors and are not suitable to arrange unimportant factors. All of these are in accord with the numerical results in Table 1 and the factor correlation graph of design d_1 in Fig. 1.

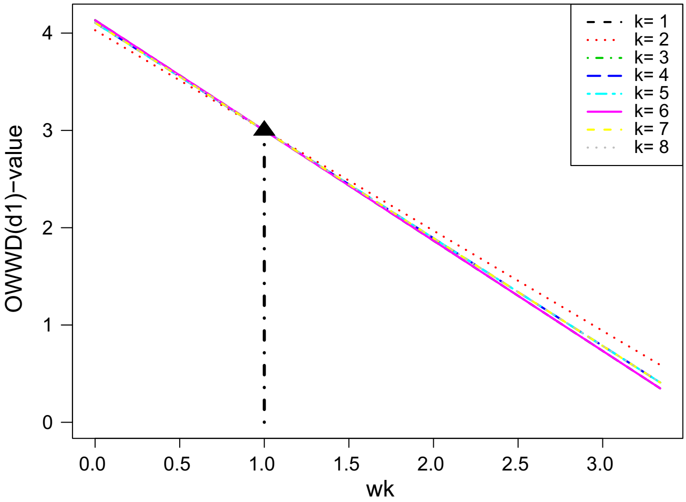

Fig. 2 The curves of $[OWWD(d_1)]^2$ of design d_1 related to w_k for $k = 1, \ldots, 8$

Example 2 Consider an experiment with nine two-level factors and eight runs, uniform design $d_2 = (x_1, \ldots, x_9) \in \mathcal{U}(8; 2^9)$ based on wrap-around L_2-discrepancy is used for the experiment. The factor correlation diagram of design d_2 is shown in Fig. 3.

$$d_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

In Fig. 3, the main effects of experimental factors $x_1, \ldots, x_9$ are respectively represented by 1, …, 9, and the two-factor interactions between factors a and b are denoted by $a * b$, where $a, b \in \{1, \ldots, 9\}$ and $a < b$. The correlations among all the 9 main effects and 36 two-factor interactions of design d_2 are visualized in Fig. 3. The largest absolute correlations for the two-factor interactions equal 1, and they are marked by the darkest off-diagonal cells. From Fig. 3, it is shown that there are 4 factor main effects (4, 5, 7, 8) and 16 two-factor interactions ($1*4$, $1*5$, $1*7$, $1*8$, $3*6$, $3*7$, $3*8$, $3*9$, $4*6$, $4*9$, $5*6$, $5*9$, $6*7$, $6*8$, $7*9$, $8*9$) that have significant correlations with factor x_2, and there are 4 factor main effects (4, 5, 6, 9) and 16 two-factor interactions ($1*4$, $1*5$, $1*6$, $1*9$, $2*6$, $2*7$, $2*8$, $2*9$, $4*7$, $4*8$, $5*7$, $5*8$, $6*7$, $6*8$, $7*9$, $8*9$) that have significant correlations with factor x_3. The degree of correlation of factors x_2 and x_3 with the main effects and two-factor interactions is significantly higher than the other 7 factors. Therefore, Fig. 3 provides some prior information about importance of factors in design d_2, and the importance of factors x_2 and x_3 in design d_2 are

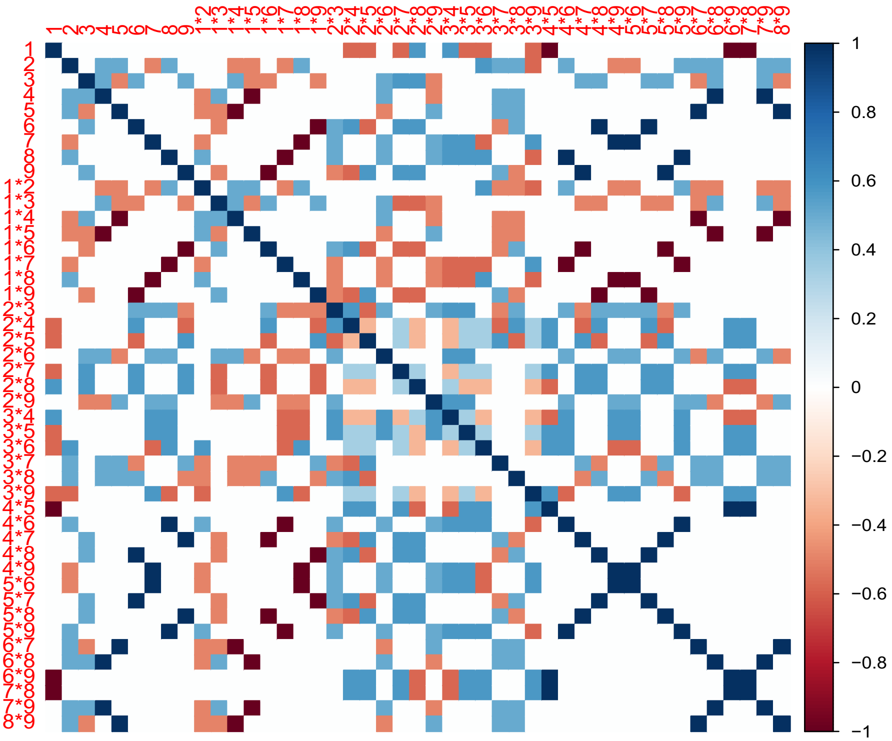

Fig. 3 Factor correlation graph of design d_2

obviously weaker than the rest factors from the perspective of aberration. On the other hand, there is not factor main effect and there are 11 two-factor interactions that have significant correlations with factor x_1. Thus, the importance of factor x_1 in design d_2 is obviously stronger than the rest factors.

Assuming that only one of the experimental factors has prior information, and the special factor is arranged in rotation to each column of d_2, the corresponding weight is respectively assigned to 0.5 for unimportant factor and 1.5 for important factor. All the numerical results are listed in Table 2.

From Table 2, if the special factor existing some prior information indicates that it is an unimportant factor, and its weight is assigned to 0.5, the corresponding $[OWWD(d_2)]^2$ is minimum and the OWGWP is sequentially minimum when the special factor is arranged to one of columns 2, 3 of d_2, when the special factor is arranged to the 1st column of d_2, the corresponding $[OWWD(d_2)]^2$ and OWGWP are among the worst performing weighted designs. On the other hand, if the special factor existing some prior information indicates that it is an important factor, and its weight is assigned to 1.5, the corresponding $[OWWD(d_2)]^2$ is minimum and the OWGWP is sequentially minimum when the special factor is arranged to the 1st column of d_2, when the special factor is arranged to one of columns 2, 3 of d_2, the corresponding $[OWWD(d_2)]^2$ and OWGWP are among the worst performing weighted designs.

Table 2 Numerical results of design d_2

Weight vector w	$[OWWD(d_2)]^2$	LB	$WA(d_2)$
(1, 1, 1, 1, 1, 1, 1, 1, 1)	4.747966	4.683593	(0, 2, 14, 18, 12, 10, 6, 1, 0)
(1, 1, 1, 1, 1, 1, 1, 1, 0.5)	5.507568	5.408512	(0, 2.0, 11.9, 14.3, 9.3, 6.8, 3.5, 0.6, 0)
(1, 1, 1, 1, 1, 1, 1, 0.5, 1)	5.507568	5.408512	(0, 2.0, 11.9, 14.3, 9.3, 6.8, 3.5, 0.6, 0)
(1, 1, 1, 1, 1, 1, 0.5, 1, 1)	5.507568	5.408512	(0, 2.0, 11.9, 14.3, 9.3, 6.8, 3.5, 0.6, 0)
(1, 1, 1, 1, 1, 0.5, 1, 1, 1)	5.507568	5.408512	(0, 2.0, 11.9, 14.3, 9.3, 6.8, 3.5, 0.6, 0)
(1, 1, 1, 1, 0.5, 1, 1, 1, 1)	5.491475	5.408512	(0, 1.8, 12.2, 14.5, 8.7, 6.9, 3.8, 0.5, 0)
(1, 1, 1, 0.5, 1, 1, 1, 1, 1)	5.491475	5.408512	(0, 1.8, 12.2, 14.5, 8.7, 6.9, 3.8, 0.5, 0)
(1, 1, 0.5, 1, 1, 1, 1, 1, 1)	5.456069	5.408512	(0, 1.5, 12.5, 15.0, 8.2, 6.6, 4.1, 0.5, 0)
(1, 0.5, 1, 1, 1, 1, 1, 1, 1)	5.456069	5.408512	(0, 1.5, 12.5, 15.0, 8.2, 6.6, 4.1, 0.5, 0)
(0.5,1 , 1, 1, 1, 1, 1, 1, 1)	5.526880	5.408512	(0, 2.1, 11.9, 13.9, 9.3, 7.2, 3.5, 0.5, 0)
(1, 1, 1, 1, 1, 1, 1, 1, 1.5)	3.988365	3.898798	(0, 2.0, 16.1, 21.7, 14.7, 13.2, 8.5, 1.4, 0)
(1, 1, 1, 1, 1, 1, 1, 1.5, 1)	3.988365	3.898798	(0, 2.0, 16.1, 21.7, 14.7, 13.2, 8.5, 1.4, 0)
(1, 1, 1, 1, 1, 1, 1.5, 1, 1)	3.988365	3.898798	(0, 2.0, 16.1, 21.7, 14.7, 13.2, 8.5, 1.4, 0)
(1, 1, 1, 1, 1, 1.5, 1, 1, 1)	3.988365	3.898798	(0, 2.0, 16.1, 21.7, 14.7, 13.2, 8.5, 1.4, 0)
(1, 1, 1, 1, 1.5, 1, 1, 1, 1)	4.004458	3.898798	(0, 2.2, 15.8, 21.5, 15.3, 13.1, 8.2, 1.5, 0)
(1, 1, 1, 1.5, 1, 1, 1, 1, 1)	4.004458	3.898798	(0, 2.2, 15.8, 21.5, 15.3, 13.1, 8.2, 1.5, 0)
(1, 1, 1.5, 1, 1, 1, 1, 1, 1)	4.039863	3.898798	(0, 2.5, 15.5, 21.0, 15.8, 13.4, 7.9, 1.5, 0)
(1, 1.5, 1, 1, 1, 1, 1, 1, 1)	4.039863	3.898798	(0, 2.5, 15.5, 21.0, 15.8, 13.4, 7.9, 1.5, 0)
(1.5, 1, 1, 1, 1, 1, 1, 1, 1)	3.969053	3.898798	(0, 1.9, 16.1, 22.1, 14.7, 12.8, 8.5, 1.5, 0)

As a matter of fact, the conclusions obtained above are independent of the choice of weight. The curves of $[OWWD(d_2)]^2$ of design d_2 related to w_k are given in Fig. 4, where w_k is the weight of the k-th factor of design d_2, $k = 1, \ldots, 9$. It is shown that $[OWWD(d_2)]^2$ is a simple linear function of w_k for given design d_2, and all the lines of $[OWWD(d_2)]^2$ intersect at $w_k = 1$. Especially for $k \in \{2, 3\}$, when $0 \leq w_k < 1$, the line of $[OWWD(d_2)]^2$ is under the other lines of $[OWWD(d_2)]^2$ for $k \in \{1, 4, 5, 6, 7, 8, 9\}$; and when $1 < w_k \leq U_w$ ($U_w = 3.67039$), the line of $[OWWD(d_2)]^2$ is upon the other lines of $[OWWD(d_2)]^2$ for $k \in \{1, 4, 5, 6, 7, 8, 9\}$. It means that the factors 2, 3 are suitable to arrange unimportant factors and are not suitable to arrange important factors. Similarly, the 1st factor is suitable to arrange important factor and is not suitable to arrange unimportant factor. All of these are in accord with the numerical results in Table 2 and the factor correlation graph of design d_2 in Fig. 3.

6 Concluding remarks

When some potentially important or unimportant factors are detected before experiment, it becomes very important to distinguish the potential importance of factors. In this paper, the new concept of weighted wrap-around L_2-discrepancy is defined to measure the uniformity of design with some potentially important or unimportant

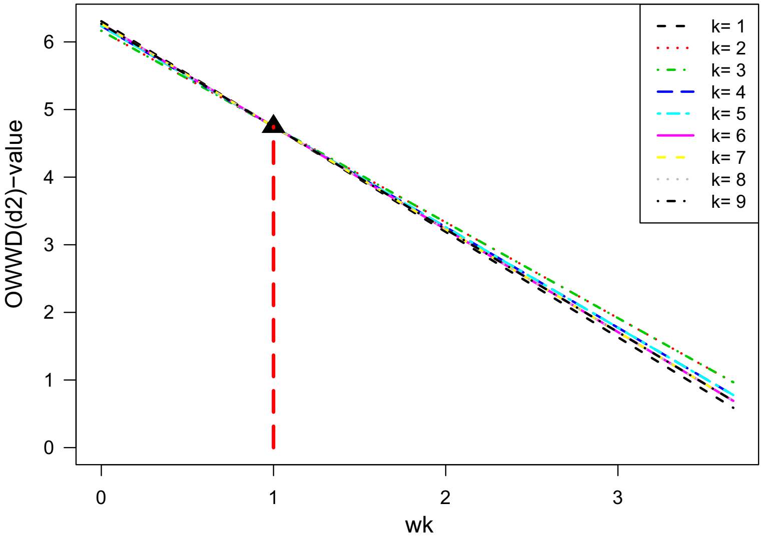

Fig. 4 The curves of $[OWWD(d_2)]^2$ of design d_2 related to w_k for $k = 1, \ldots, 9$

factors. To capture the aberration of design with some potentially important or unimportant factors, the weighted generalized wordlength pattern is accordingly proposed. The relationship between the weighted wrap-around L_2-discrepancy and weighted generalized wordlength pattern is built, and the projection properties of weighted wrap-around L_2-discrepancy is discussed. Besides, a lower bound of weighted wrap-around L_2-discrepancy is obtained, which can be as a benchmark to measure the uniformity of design with some potentially important or unimportant factors. We showed that the difference of importance among the columns of design can precisely be captured by both weighted wrap-around L_2-discrepancy and weighted generalized wordlength pattern. Furthermore, the conclusions in this paper are independent of the choice of weight. Both weighted wrap-around L_2-discrepancy and weighted generalized wordlength pattern are recommended to detect the potentially important or unimportant columns of design or reflect the difference importance among factors in practice.

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