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Nonparametric confidence intervals for ranked set samples

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Abstract In this work, we propose several different confidence interval methods based on ranked-set samples. First, we develop bootstrap bias-corrected and accelerated method for constructing confidence intervals based on ranked-set samples. Usually, for this method, the accelerated constant is computed by employing jackknife method. Here, we derive an analytical expression for the accelerated constant, which results in reducing the computational burden of this bias-corrected and accelerated bootstrap method. The other proposed confidence interval approaches are based on a monotone transformation along with normal approximation. We also study the asymptotic properties of the proposed methods. The performances of these methods are then compared with those of the conventional methods. Through this empirical study, it is shown that the proposed confidence intervals can be successfully applied in practice. The usefulness of the proposed methods is further illustrated by analyzing a real-life data on shrubs.

Keywords Bootstrap · Edgeworth expansion · Bias corrected and accelerated · Monotone transformations

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1 Introduction

Ranked set sampling (RSS) is an alternative method of data collection and has been known as a cost-efficient sampling procedure for many years. This approach to data collection was first proposed by McIntyre (1952) as a method to improve the precision of estimated pasture yield. Later, Takahasi and Wakimoto (1968) established a rigorous statistical foundation for the theory of RSS. For more details and real applications of RSS scheme, one may refer to Patil et al. (1999), Chen (2007), Linder et al. (2015), Samawi et al. (2017), and Al-Omari and Bouza (2014). An RSS utilizes the basic intuitive properties associated with simple random sampling (SRS). However, it involves the extra structure induced through the judgment ranking and the independence of the resulting order statistics. As a result, the procedures based on RSS lead to more efficient estimators of population parameters than those based on an SRS with the same sample size. The existing literature also includes works on hypothesis testing as well as point and interval estimation under both parametric and nonparametrics settings. See, for example, Bohn and Wolfe (1992), Chen et al. (2006), Fligner and MacEachern (2006), Frey (2007), Ozturk and Balakrishnan (2009), and the references cited therein.

The most basic version of RSS is the so-called balanced RSS. The process of generating an RSS involves drawing k^2 units at random from the target population. These items are then randomly divided into k sets of k units each. Within each set, the units are then ranked by some means other than direct measurement. For example, the ranking can be done either visually or by using a concomitant measurement that is comparatively cheaper to measure and also easier to obtain than the measurement of interest itself. Finally, one item from each set is chosen for actual quantification. To be more specific, from the first set we select the item with the smallest judgment-rank for measurement, from the second set we select the item with the second smallest judgment-rank, and so on, until the unit ranked largest is chosen from the kth set. This complete procedure, called a cycle, is repeated independently m times to obtain a Ranked set sample of size mk. Therefore, a balanced RSS of size mk requires a total of mk^2 units to be selected, but only mk of them are actually measured. Hence, a wider range of the population can be covered while greatly reducing the sampling cost. According to Takahasi and Wakimoto (1968), for easy implementation of RSS, the set size k is usually kept as small as 4 or less. However, we can obtain a large sample by increasing the cycle size m. Another option is that of unbalanced RSS. In an unbalanced RSS, $n \times k$ units are selected at random from the target population. These items are then randomly divided into n sets of k units each. Units in each set are judgment ranked without measuring the actual units. In this setting, let m_r denote the number of sets allocated to measure units having the rth judgment-rank such that $n = \sum_{r=1}^{k} m_r$. The measured observations then constitute an unbalanced RSS of size n.

Recently, several bootstrap methods have been developed based on RSS. Hui et al. (2004) proposed a bootstrap confidence interval method for the population mean based on RSS via linear regression, wherein they applied the bootstrap method to estimate the variance of the estimator of the population mean for constructing confidence intervals. In a similar vein, Modarres et al. (2006) developed many bootstrap procedures for balanced RSS and established their consistency for the sample mean. Drikvandi et al. (2006) proposed a bootstrap method to test for the symmetry of the distribution function about an unknown median based on RSS. Finally, Frey (2014) developed confidence bands for the CDF based on RSS by using the bootstrap method.

In this article, we develop three confidence interval methods for the mean parameter of RSS. First, we suggest the bias-corrected and accelerated (BC_a) confidence interval method. The BC_a was proposed by Efron (1987) for SRS and it is an improvement over the bootstrap percentile confidence interval method in terms of coverage probability (see Hall 1988). The BC_a method has been considered by several researchers in different contexts. This method requires numerical implementation of the acceleration constant that is computed by using the jackknife method. However, in the case of RSS, the numerical computation of the acceleration constant becomes intensive as set size increases. This motivates us to develop an alternative approach to reduce this computational burden. For this purpose, we derive a formula for the acceleration constant based on Edgeworth expansion which avoids the cumbersome numerical implementation. Next, two confidence interval methods are proposed based on monotone transformations. These methods transform the studentized pivot into another one based on a monotone transformation so that the resulting distribution of the transformed pivot is nearly symmetric and then we can construct confidence intervals by inverting back the transformation. Various transformations of the studentized pivot have been investigated by Johnson (1978), Hall (1992a), Zhou and Gao (2000), and Cojbasic and Loncar (2011) under SRS.

The remainder of this paper is organized as follows. Section 2 introduces a Ranked set sample. In Sects. 3 and 4, we develop BC_a method and transformation confidence interval methods, respectively. Simulation results are presented in Sect. 5. Section 6 presents a real data application. We finally conclude the article with some brief remarks in Sect. 7. Proofs are relegated to the "Appendix".

2 Ranked set sample

Let nk units be drawn randomly from a population with an unknown distribution $F(x)$. Let μ and σ^2 be the mean and variance of $F(x)$, respectively. These units are then randomly divided into n groups $G_1, \ldots, G_n$ of size k each. The rth group G_r consists of $\{X_{r,1}, X_{r,2}, \ldots, X_{r,k}\}$. Then, the units in each n subgroups are ordered on the attribute of interest by the use of some ranking process. Let m_r be the number of actual measurements on units having rank r, $r = 1, \ldots, k$, such that $n = \sum_{i=1}^{k} m_r$. Under the assumption of perfect ranking, let $X_{(r),j}$ denote the measurement on the jth unit having rank r and let the m_r resulting measurements on units with rank r be labeled as $\{X_{(r),1}, \ldots, X_{(r),m_r}\}$. Therefore, the resulting RSS of size n drawn from that underlying distribution $F(x)$ is given by $\mathcal{X}_{RSS} = \{X_{(r),j}, r = 1, \ldots, k; j = 1 \ldots, m_r\}$. When $m_r = m, r = 1, \ldots, k$, RSS leads to the balanced ranked set sample of size mk.

Let $\{X_{(r),1}, \ldots, X_{(r),m_r}\}$ be a random sample from F_r, where F_r denotes the distribution function of the rth order statistic from $F(x)$. Let μ_r denote the mean of F_r. We are then interested in constructing confidence intervals for

$$\mu = k^{-1} \sum_{r=1}^{k} \mu_r$$

based on $\mathcal{X}_{RSS}$. As stated by Dell and Clutter (1972), the above identity holds under both perfect and imperfect rankings. An unbiased estimator of μ can be obtained as

$$\bar{X}_{RSS} = \frac{1}{k} \sum_{r=1}^{k} \bar{X}_r = \frac{1}{k} \sum_{r=1}^{k} \frac{1}{m_r} \sum_{j=1}^{m_r} X_{(r),j},$$

where $\bar{X}_r$ is the sample mean based on $\{X_{(r),1}, \ldots, X_{(r),m_r}\}$. Let τ^2 be the variance of $\bar{X}_{RSS}$ given by

$$\tau^2 = \frac{1}{k^2} \sum_{r=1}^{k} \frac{\sigma_r^2}{m_r},$$

where σ_r^2 is the variance of $X_{(r),j}$. Set $S_r^2 = \frac{1}{m_r} \sum_{j=1}^{m_r} (X_{(r),j} - \bar{X}_r)^2$, a plug-in estimator for σ_r^2, so that the corresponding plug-in estimator for τ^2 becomes

$$\hat{\tau}^2 = \frac{1}{k^2} \sum_{r=1}^{k} \frac{S_r^2}{m_r}.$$

While seeking confidence intervals for μ, let t_ζ denote the ζth quantile of the distribution of the pivot

$$T_{RSS} = \frac{(\bar{X}_{RSS} - \mu)}{\hat{\tau}} = \frac{\sum_{r=1}^{k} (\bar{X}_r - \mu_r)}{\sqrt{\sum_{r=1}^{k} \frac{S_r^2}{m_r}}}$$

such that $P(T_{RSS} \leq t_\zeta) = \zeta$. For the rest of this article, we consider α as the nominal coverage probability of a confidence interval. Then, $I_0 = [\bar{X}_{RSS} - t_\alpha \hat{\tau}, \infty)$, $I_1 = (-\infty, \bar{X}_{RSS} - t_{1-\alpha} \hat{\tau}]$, and $I_2 = [\bar{X}_{RSS} - t_{(1+\alpha)/2} \hat{\tau}, \bar{X}_{RSS} - t_{1-(1+\alpha)/2} \hat{\tau}]$ are the ideal lower, upper, and two-sided confidence intervals for μ, respectively. However, these intervals are unknown since t_ζ is unknown. Usually, we estimate these by employing the normal approximation, based on the central limit theorem, to the distribution of T_{RSS}. An alternative method to the normal approximation is through bootstrap method which has become a standard tool for estimating unknown confidence intervals.

Our proposed confidence intervals depend on detailed properties of the Edgeworth expansions of the distributions of T_{RSS} and $S_{RSS} = (\bar{X}_{RSS} - \mu)/\tau$. For this purpose, we assume that the distribution of $(X_{(r)}, X_{(r)}^2)$ satisfies Cramér's continuity condition (see Hall 1992b, pp. 66–67) for each $r = 1, 2, \ldots, k$ and that $E(X^8) < \infty$. We also assume that $m_1, \ldots, m_r$ are of the same order, i.e., $\lim_{n \to \infty} (m_r/n) = \lambda_r \in (0, 1)$. These conditions are sufficient for all the results derived in this paper. We conclude this section by presenting the Edgeworth expansions of the distributions of T_{RSS} and S_{RSS}.

Theorem 2.1 *Under the above conditions, the distributions of S_{RSS} and T_{RSS} have the following Edgeworth expansions:*

$$P(S_{RSS} \leq x) = \Phi(x) + n^{-1/2} p_1(x)\phi(x) + O(n^{-1}),$$

and

$$P(T_{RSS} \leq x) = \Phi(x) + n^{-1/2} q_1(x)\phi(x) + O(n^{-1}),$$

where $\Phi(x)$ is the cdf of the standard normal distribution, and $p_1(x)$ and $q_1(x)$ are even polynomials of degree 2 having the expressions

$$p_1(x) = -\frac{1}{6}\eta_1^{-3/2}\eta_2(x^2 - 1),$$
$$q_1(x) = \frac{1}{6}\eta_1^{-3/2}\eta_2(2x^2 + 1),$$

with

$$\eta_1 = \sum_{r=1}^{k} \frac{\sigma_r^2}{\lambda_r}, \quad \eta_2 = \sum_{r=1}^{k} \frac{\gamma_r}{\lambda_r^2} \text{ and } \gamma_r = E(X_{(r)} - \mu_r)^3.$$

An excellent review on the theory of Edgeworth expansion can be found in Hall (1992a). A simpler procedure to approximate the confidence intervals I_0, I_1 and I_2 is based on the normal approximation to the distribution of T_{RSS}, using the fact that $P(T_{RSS} \leq x) \to \Phi(x)$ as $n \to \infty$. Let z_ζ be the ζth quantile of the standard normal distribution. Then, $I_{0,N} = [\bar{X}_{RSS} - \hat{\tau} z_\alpha, \infty]$, $I_{1,N} = (-\infty, \bar{X}_{RSS} - \hat{\tau} z_{1-\alpha}]$ and $I_{2,N} = [\bar{X}_{RSS} - \hat{\tau} z_{(1+\alpha)/2}, \bar{X}_{RSS} - \hat{\tau} z_{1-(1+\alpha)/2}]$ are the respective lower, upper and both-sided confidence intervals for μ based on the normal approximation. For a given α, Theorem 2.1 shows that

$$P(\mu \in I_{0,N}) = P(\mu \in I_{1,N}) = \alpha + O(n^{-1/2}) \text{ and } P(\mu \in I_{2,N}) = \alpha + O(n^{-1}).$$

Hence, $I_{0,N}$ and $I_{1,N}$ are the first-order accurate confidence intervals while $I_{2,N}$ is the second-order accurate confidence interval. In the next section, we develop BC_a confidence intervals for μ using RSS.

3 Bootstrap for ranked set samples

To facilitate the development of bootstrap confidence interval methods based on RSS, we first give a short description of the existing resampling methods, namely, BRSSR (bootstrap RSS by row), BRSS (bootstrap RSS), and MRBRSS (mixed row bootstrap RSS), for RSS. Chen et al. (2004) introduced BRSSR and Modarres et al. (2006) subsequently studied its properties. They showed that BRSSR is asymptotically consistent in estimating the distribution of the standardized sample mean under RSS. Since RSS can be viewed as k independent random samples from k different distributions, BRSSR can therefore be easily implemented by drawing bootstrap samples independently from each of the k independent random samples.

BRSS and MRBRSS were proposed by Modarres et al. (2006) for balanced RSS under perfect rankings. BRSS method draws RSS from the observed RSS to perform the bootstrap method. They established that under balanced RSS along with perfect rankings, BRSS consistently estimates the distribution of the standardized sample mean, i.e., BRSS estimator of the distribution of the standardized sample mean converges to its true distribution almost surely as $n \to \infty$. However, BRSS may not be appropriate for unbalanced RSS as it may introduce some bias (Modarres et al. 2006). The other resampling method MRBRSS is not an appealing resampling method for RSS since it does not provide a consistent estimator of the distribution of the standardized sample mean (Modarres et al. 2006). The BC_a confidence intervals constructed here are therefore based on BRSSR.

3.1 Construction of bootstrap confidence intervals

Confidence intervals for μ based on RSS can be easily constructed by extending Efron's (1979) bootstrap percentile method in the case of RSS. This method has some attractive features, such as it is invariant to monotone transformations, but it suffers from poor coverage probabilities. Efron (1987) proposed a correction to the percentile-method that reduces the coverage error while retaining the invariance property. The resulting confidence interval method is known as bias-corrected and accelerated (BC_a) bootstrap. To facilitate the construction of BC_a confidence intervals based on RSS, let $\{X^*_{(1),1}, \dots X^*_{(1),m_1}\}, \dots, \{X^*_{(k),1}, \dots X^*_{(k),m_k}\}$ denote k bootstrap samples drawn independently and randomly with replacement from the k sets, $\{X_{(1),1}, \dots X_{(1),m_1}\}, \dots, \{X_{(k),1}, \dots X_{(k),m_k}\}$, respectively. Then, the resulting bootstrap sample $\mathcal{X}^*_{RSS} = \{X^*_{(r),j}, r = 1, \dots, k;\ j = 1 \dots, m_r\}$ is known as BRSSR. Let us denote $\bar{X}^*_r = m_r^{-1} \sum_{i=1}^{m_r} X^*_{(r),i}$ and $S^{*2}_r = m_r^{-1} \sum_{i=1}^{m_r} (X^*_{(r),i} - \bar{X}^*_r)^2$. The bootstrap versions of $\bar{X}_{RSS}$ and $\hat{\tau}^2$ are then

$$\bar{X}^*_{RSS} = k^{-1} \sum_{r=1}^{k} \bar{X}^*_r \text{ and } \hat{\tau}^{*2} = k^{-2} \sum_{r=1}^{k} \frac{S^{*2}_r}{m_r}.$$

Define

$$\hat{u}_\xi = sup\{u : P(\bar{X}^*_{RSS} \leq u | \mathcal{X}_{RSS}) \leq \xi\}.$$

Then, $I_{0,BP} = [\hat{u}_{1-\alpha}, \infty)$, $I_{1,BP} = (-\infty, \hat{u}_\alpha]$ and $I_{2,BP} = [\hat{u}_{(1-\alpha)/2}, \hat{u}_{(1+\alpha)/2}]$ are the respective lower, upper and both-sided percentile-method confidence intervals for

μ. It can be shown that

$$P\{\mu \in I_{0,BP}\} = P\{\mu \in I_{1,BP}\} = \alpha + O(n^{-1/2}) \text{ and } P\{\mu \in I_{2,BP}\} = \alpha + O(n^{-1}).$$

For constructing BC_a confidence intervals for μ based on RSS, let us define

$$\hat{G}_{RSS}(x) = P(\bar{X}^*_{RSS} \le x | \mathcal{X}_{RSS}),$$

the bootstrap distribution of $\bar{X}^*_{RSS}$. Put

$$\hat{d} = \Phi^{-1}\{\hat{G}_{RSS}(\bar{X}_{RSS})\}, \tag{3.1}$$

$$l_{\hat{a}}(\alpha) = \Phi[\hat{d} + (\hat{d} + z_\alpha)\{1 - \hat{a}(\hat{d} + z_\alpha)\}^{-1}], \tag{3.2}$$

where $\hat{d}$ and $\hat{a}$ are called bias-correction and acceleration constant, respectively. We now define Efron's (1987) BC_a method confidence intervals for μ based on RSS as follows:

$$I_{0,BC_a} = [\hat{u}_{l_{\hat{a}}(1-\alpha)}, \infty),$$
$$I_{1,BC_a} = (-\infty, \hat{u}_{l_{\hat{a}}(\alpha)}]$$

and

$$I_{2,BC_a} = [\hat{u}_{l_{\hat{a}}((1-\alpha)/2)}, \hat{u}_{l_{\hat{a}}((1+\alpha)/2)}],$$

respectively. The bootstrap percentile and the bias-corrected methods can be viewed as special cases of BC_a, which can be obtained by letting $\hat{d} = \hat{a} = 0$ and $\hat{a} = 0$, respectively. In particular, the non-zero values of $\hat{d}$ and $\hat{a}$ change the quantiles used for BC_a. In practice, $\hat{d}$ is computed as

$$\hat{d} = \Phi^{-1}\left(\frac{\#\{\bar{X}^*_{RSS} \le \bar{X}_{RSS}\}}{B}\right),$$

where B is the number of bootstrap samples. The acceleration constant $\hat{a}$ can be computed using the jackknife method (for details, see Efron 1987; Efron and Tibshirani 1993), which becomes computationally burdensome as the set size in RSS increases. This computational burden involved in BC_a method for RSS can be substantially reduced by letting

$$\hat{a} = \frac{1}{6} n^{-1/2} \hat{\eta}_1^{-3/2} \hat{\eta}_2 = \frac{1}{6} n^{-1/2} \left(\sum_{r=1}^{k} \frac{\hat{\sigma}_r^2}{\lambda_r} \right)^{-3/2} \sum_{r=1}^{k} \frac{\hat{\gamma}_r}{\lambda_r^2}.$$

The above formula for $\hat{a}$ is obtained by extending Hall's (1988) finding to the case of RSS. Then, I^*_{0,BC_a}, I^*_{1,BC_a} and I^*_{2,BC_a} are versions of I_{0,BC_a}, I_{1,BC_a} and I_{2,BC_a} based on the above expression for $\hat{a}$. The following result establishes the second-order accuracy of I^*_{0,BC_a}, I^*_{1,BC_a} and I^*_{2,BC_a}.

Theorem 3.1 *Under the assumptions of Theorem 2.1,*

$$P\{\mu \in I^*_{0,BC_a}\} = P\{\mu \in I^*_{1,BC_a}\} = P\{\mu \in I^*_{2,BC_a}\} = \alpha + O(n^{-1}).$$

This result shows that BC_a method-based lower and upper confidence intervals are more accurate than those based on normal approximation and bootstrap percentile-method, which provide a coverage error of order $O(n^{-1/2})$. However, for two sided-confidence intervals, normal approximation and bootstrap percentile methods are similar to BC_a method in terms of the coverage probability, as they all result in a coverage error of order $O(n^{-1})$. It is important to note that the normal approximation confidence intervals do not respect transformation.

4 Confidence intervals based on monotone transformations

Considerable work has been done on obtaining a simple and accurate approximation to the distribution of a statistic or a pivot. Many researchers (Johnson 1978; Hall 1992a; Zhou and Gao 2000; Cojbasic and Loncar 2011) have investigated the effects of a monotone transformation on the distribution of an asymptotic pivot for simple random samples. Such a transformation is useful in reducing the effects of skewness of data on the distribution of an asymptotic pivot. That is, the distribution of the transformed pivot becomes more symmetric than that of the original asymptotic pivot. Johnson (1978) proposed a modified one-sample t test based on a quadratic transformation that is less affected by the population skewness than the conventional t test. However, Hall (1992a) noticed that this transformation has some drawbacks in that it is not monotone and fails to correct adequately for skewness. For this reason, Hall (1992a) proposed a monotone transformation based on a cubic polynomial that has a simple inverse function. In our case, Hall's (1992a) cubic transformation can be defined as

$$g_1(x) = x + n^{-1/2}\frac{1}{3}\hat{\eta}x^2 + n^{-1}\frac{1}{27}\hat{\eta}^2x^3 + n^{-1/2}\frac{1}{6}\hat{\eta},$$

where $\hat{\eta} = \hat{\eta}_1^{-3/2}\hat{\eta}_2$. Under the assumption of Theorem 2.1, it can be shown that $P\{g_1(T_{RSS}) \leq x\} = \Phi(x) + O(n^{-1})$; that is, the distribution of the transformed pivot, $g_1(T_{RSS})$, is more symmetric than that of T_{RSS}.

Let z_ζ be the ζth quantile of the standard normal distribution and define

$$I_{0,g_1} = [\bar{X}_{RSS} - g_1^{-1}(z_\alpha)\hat{\tau}, \infty),$$
$$I_{1,g_1} = (-\infty, \bar{X}_{RSS} - g_1^{-1}(z_{1-\alpha})\hat{\tau}]$$

and

$$I_{2,g_1} = [\bar{X}_{RSS} - \hat{\tau}g_1^{-1}(z_\gamma), \bar{X}_{RSS} - \hat{\tau}g_1^{-1}(z_{1-\gamma})],$$

where $\gamma = \frac{1}{2}(1 + \alpha)$. Then, I_{0,g_1}, I_{1,g_1} and I_{2,g_1} are the lower, upper and two-sided confidence intervals for μ, based on the transformation $g_1(.)$ with $g_1^{-1}(x) =$

$n^{1/2}(\frac{1}{3}\hat{\eta})^{-1}[\{1+\hat{\eta}(n^{-1/2}x-\frac{1}{6}n^{-1}\hat{\eta})\}^{1/3}-1]$. Hall (1992a) also proposed exponential-type transformation which is monotone and has a simple inverse function, and for RSS it is given by

$$g_2(x) = \left(\frac{2}{3}n^{-1/2}\hat{\eta}\right)^{-1}\left\{\exp\left(\frac{2}{3}n^{-1/2}\hat{\eta}x\right)-1\right\}+\frac{1}{6}n^{-1}\hat{\eta}.$$

Then, in this case, we have

$$I_{0,g_2} = [\bar{X}_{RSS} - \hat{\tau}g_2^{-1}(z_\alpha), \infty),$$
$$I_{1,g_2} = (-\infty, \bar{X}_{RSS} - \hat{\tau}g_2^{-1}(z_{1-\alpha})]$$

and

$$I_{2,g_2} = [\bar{X}_{RSS} - \hat{\tau}g_2^{-1}(z_\gamma), \bar{X}_{RSS} - \hat{\tau}g_2^{-1}(z_{1-\gamma})],$$

as the respective lower, upper and two-sided confidence intervals for μ, based on the transformation $g_2(.)$, with $g_2^{-1}(x) = (\frac{2}{3}n^{-1/2}\hat{\eta})^{-1}\log\{1+\frac{2}{3}n^{-1/2}\hat{\eta}(x-n^{-1}\frac{1}{6}\hat{\eta})\}$. The following result shows that the intervals I_{0,g_i}, I_{1,g_i} and I_{2,g_i} are second-order accurate, for $i = 1, 2$.

Theorem 4.1 *Under the assumptions of Theorem 2.1,*

$$P\{\mu \in I_{0,g_i}\} = P\{\mu \in I_{1,g_i}\} = P\{\mu \in I_{2,g_i}\} = \alpha + O(n^{-1}), \quad i = 1, 2.$$

Theorem 4.1 follows from the fact that $P\{g_i(T_{RSS}) \leq x\} = \Phi(x) + O(n^{-1})$, $i = 1, 2$. The above results show that the coverage error associated with the intervals I_{0,g_i} and I_{1,g_i} are of order $O(n^{-1})$ and hence these intervals are improvements over the intervals $I_{0,N}$ and $I_{1,N}$ in terms of coverage errors. Theorems 3.1 and 4.1 imply that the bias-corrected and accelerated and transformation methods are asymptotically equivalent in terms of coverage errors. But, the bias-corrected and accelerated method is transformation-respecting, while the transformation methods are easy to apply.

5 Simulation study

Simulation studies are performed to compare the proposed confidence interval methods in Sects. 3 and 4 with three conventional confidence interval methods, namely, the bootstrap percentile method, the bootstrap percentile-*t* method, and the normal approximation method. Lower, upper and two-sided confidence intervals were constructed based on each of these methods.

To facilitate the discussion of our simulation results, the lower, upper and two-sided confidence intervals are denoted by LCL, UCL and TCL, respectively. The symbols N, BP, BT, BC_a stand for normal approximation method, bootstrap percentile method, bootstrap percentile-*t* method, and bias-corrected and accelerated method. Moreover, N_{g_1} and N_{g_2} denote the normal approximation along with transformations g_1 and

g_2, respectively. We also consider the confidence interval method proposed by Ahn et al. (2014) as a competing procedure. This method uses t critical values based on the Welch-type approximation and we denote this procedure by t_{ALW}. All these different confidence interval methods are then compared with respect to their coverage probabilities, average lower confidence limits, average upper confidence limits, and average interval widths. We also compare RSS and simple random sampling (SRS) with respect to all these methods.

We now describe in detail the setting of our simulation study. The RSS were generated based on several balanced RSS and unbalanced RSS designs with different sample sizes when the set sizes were chosen to be $k = 2$, 3 and 5. Data were generated from three underlying distributions with various degrees of skewness and kurtosis: a chi-square distribution with $df = 1(\chi_1^2)$; the standard exponential distribution $(Exp(1))$, and the half normal distribution denoted by $HN(0, 1)$. The means are, respectively, 1, 1 and $\sqrt{(}2/\pi)$ for these three distributions.

For every distribution and sample size combination, we generated 5000 simulated samples. The coverage probability for each method for each combination was then estimated by the proportion of times the method covered the true parameter of the underlying distribution. Different bootstrap confidence intervals were constructed by using 3000 bootstrap samples. The obtained simulation results are presented in Tables 1, 2, 3, 4, 5 and 6 for $k = 2, 3$. The simulation results for $k = 5$ are provided in the supplement. Based on the results in these tables, we have the following findings:

(i) The BC_a method gives most accurate coverage probabilities of LCL for χ_1^2 and $Exp(1)$ when $n \leq 20$. However, for these distributions, the coverage probabilities of LCL based on BC_a, N_{g_1} and N_{g_2} are comparable for $n > 20$. We also observe that the LCL based on BP, BT, t_{ALW} and N methods gives over-coverage in most cases for χ_1^2 and $Exp(1)$. The methods BC_a, N_{g_1} and N_{g_2} give similar coverage probabilities for the lower confidence interval for most of the cases when data are generated from $HN(0, 1)$. The coverage probabilities for the lower confidence interval corresponding to all methods, except N, are also similar for $HN(0, 1)$ when $n \geq 30$. BC_a method provides most accurate average lower limits compared to all other methods when $n \leq 20$ for χ_1^2 and $Exp(1)$. However, for $n > 20$, average lower limits are comparable for all the methods. Average lower limits of all methods corresponding to $HN(0, 1)$ are also comparable, but BC_a method provides the most accurate average lower limits for $n = 10$. Based on our simulation study, BC_a is the one we would recommend for the construction of lower confidence intervals.

(ii) For UCL, BT method gives best coverage accuracy among all the methods when $n \leq 20$. However, the coverage probability of UCL corresponding to the BT method is far removed from the nominal coverage for χ_1^2 and $Exp(1)$ when $n \leq 20$. For large $n(> 30)$, the methods BT, BC_a, t_{ALW}, N_{g_1} and N_{g_2} all give similar coverage probabilities for χ_1^2 and $Exp(1)$. For the case of $HN(0, 1)$, the methods BT, BC_a, t_{ALW}, N_{g_1} and N_{g_2} all give similar coverage probabilities for UCL when $n \geq 20$. This may be due to the fact that $HN(0, 1)$ is less skewed as compared to χ_1^2 and $Exp(1)$. The N and BP methods undercover consistently for

Table 1 Coverage probabilities of 90% confidence intervals for mean of χ_1^2 data

Methods	$(k=2, n=10), (m_1=5, m_2=5)$			$(k=2, n=10), (m_1=4, m_2=6)$			SRS, $n=10$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.938 (0.56)	0.773 (1.46)	0.798 (1.16)	0.943 (0.57)	0.768 (1.43)	0.793 (1.10)	0.942 (0.52)	0.741 (1.48)	0.771 (1.23)
BP	0.916 (0.61)	0.757 (1.42)	0.767 (1.05)	0.923 (0.62)	0.750(1.40)	0.762(0.99)	0.940 (0.53)	0.747 (1.49)	0.780 (1.23)
BT	0.937 (0.58)	0.835 (2.04)	0.889 (2.50)	0.938 (0.60)	0.852 (1.87)	0.896 (2.00)	0.928 (0.57)	0.855 (2.33)	0.892 (2.54)
t_{ALW}	0.969 (0.48)	0.788 (1.50)	0.841 (1.40)	0.960 (0.51)	0.794 (1.48)	0.841 (1.31)			
BC_a	0.899 (0.64)	0.784 (1.51)	0.780 (1.09)	0.900 (0.65)	0.779 (1.47)	0.776 (1.04)	0.907 (0.60)	0.790 (1.65)	0.799 (1.36)
N_{g_1}	0.926 (0.60)	0.796 (1.54)	0.824 (1.30)	0.927 (0.61)	0.791 (1.51)	0.824 (1.24)	0.923 (0.58)	0.802 (1.70)	0.822 (1.41)
N_{g_2}	0.921 (0.61)	0.802 (1.56)	0.816 (1.23)	0.920 (0.63)	0.796 (1.52)	0.816 (1.17)	0.922 (0.57)	0.797 (1.67)	0.825 (1.46)
	$(k=2, n=20), (m_1=10, m_2=10)$			$(k=2, n=20), (m_1=12, m_2=8)$			SRS, $n=20$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.934 (0.67)	0.806 (1.34)	0.832 (0.86)	0.941 (0.64)	0.782 (1.36)	0.809 (0.92)	0.938 (0.63)	0.797 (1.37)	0.825 (0.95)
BP	0.919 (0.69)	0.799 (1.33)	0.818 (0.81)	0.927 (0.67)	0.774 (1.34)	0.793 (0.86)	0.934 (0.64)	0.805 (1.38)	0.830 (0.94)
BT	0.921 (0.70)	0.874 (1.56)	0.893 (1.19)	0.923 (0.68)	0.851 (1.66)	0.885 (1.41)	0.925 (0.68)	0.876 (1.65)	0.888 (1.33)
t_{ALW}	0.960 (0.64)	0.826 (1.36)	0.863 (0.95)	0.953(0.60)	0.795 (1.40)	0.846 (1.05)			
BC_a	0.899 (0.73)	0.836 (1.39)	0.822 (0.86)	0.904 (0.70)	0.805 (1.41)	0.806 (0.91)	0.898 (0.70)	0.845 (1.48)	0.844 (1.03)
N_{g_1}	0.912 (0.71)	0.840 (1.42)	0.866 (0.98)	0.920 (0.68)	0.814 (1.44)	0.846 (1.06)	0.908 (0.69)	0.850 (1.51)	0.861 (1.06)
N_{g_2}	0.895 (0.73)	0.851 (1.45)	0.847 (0.96)	0.904 (0.70)	0.825 (1.48)	0.838 (1.02)	0.907 (0.69)	0.848 (1.50)	0.860 (1.09)

Table 1 continued

	$(k = 2, n = 30), (m_1 = 15, m_2 = 15)$			$(k = 2, n = 30), (m_1 = 18, m_2 = 12)$			SRS, $n = 30$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.938 (0.72)	0.814 (1.25)	0.851 (0.72)	0.943 (0.70)	0.811 (1.31)	0.838 (0.78)	0.939 (0.69)	0.812 (1.31)	0.844 (0.79)
BP	0.930 (0.74)	0.811 (1.28)	0.843 (0.69)	0.931 (0.72)	0.809 (1.30)	0.829 (0.75)	0.931 (0.70)	0.819 (1.32)	0.849 (0.79)
BT	0.919 (0.75)	0.885 (1.41)	0.896 (0.89)	0.918 (0.73)	0.875 (1.45)	0.895 (1.02)	0.909 (0.73)	0.878 (1.46)	0.894 (0.99)
t_{ALW}	0.940 (0.71)	0.825 (1.29)	0.864 (0.77)	0.949 (0.68)	0.822 (1.33)	0.866 (0.85)			
BC_a	0.903 (0.77)	0.845 (1.33)	0.850 (0.73)	0.901 (0.75)	0.836 (1.36)	0.841 (0.78)	0.898 (0.74)	0.858 (1.39)	0.859 (0.85)
N_{g1}	0.914 (0.76)	0.855 (1.35)	0.886 (0.82)	0.914 (0.74)	0.845 (1.38)	0.872 (0.90)	0.905 (0.73)	0.867 (1.41)	0.875 (0.88)
N_{g2}	0.913 (0.76)	0.852 (1.34)	0.866 (0.78)	0.914 (0.74)	0.843 (1.37)	0.854 (0.84)	0.903 (0.74)	0.863 (1.40)	0.872 (0.88)
	$(k = 2, n = 60), (m_1 = 30, m_2 = 30)$			$(k = 2, n = 30), (m_1 = 25, m_2 = 35)$			SRS, $n = 60$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.929 (0.80)	0.840 (1.20)	0.872 (0.52)	0.925 (0.81)	0.851 (1.19)	0.873 (0.49)	0.925 (0.78)	0.845 (1.23)	0.875 (0.58)
BP	0.921 (0.80)	0.842 (1.20)	0.868 (0.51)	0.918 (0.82)	0.851 (1.20)	0.869 (0.49)	0.920 (0.78)	0.849 (1.24)	0.877 (0.58)
BT	0.901 (0.82)	0.887 (1.26)	0.897 (0.58)	0.908 (0.83)	0.894 (1.24)	0.905 (0.54)	0.899 (0.80)	0.899 (1.30)	0.898 (0.66)
t_{ALW}	0.936 (0.80)	0.860 (1.21)	0.894 (0.54)	0.936 (0.80)	0.852 (1.19)	0.884 (0.50)			
BC_a	0.897 (0.82)	0.870 (1.23)	0.873 (0.53)	0.893 (0.84)	0.877 (1.22)	0.873 (0.50)	0.894 (0.80)	0.886 (1.28)	0.877 (0.61)
N_{g1}	0.903 (0.82)	0.876 (1.24)	0.905 (0.58)	0.899 (0.83)	0.883 (1.23)	0.903 (0.55)	0.897 (0.80)	0.889 (1.29)	0.888 (0.63)
N_{g2}	0.903 (0.82)	0.875 (1.24)	0.883 (0.54)	0.898 (0.83)	0.881 (1.23)	0.880 (0.52)	0.897 (0.80)	0.887 (1.28)	0.885 (0.62)

Table 2 Coverage probabilities of 90% confidence intervals for mean of χ_1^2 data

Methods:	$(k=3, n=10)$, $(m_1=3, m_2=3, m_3=4)$			$(k=3, n=10)$,$(m_1=4, m_2=3, m_3=3)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.940 (0.60)	0.760 (1.39)	0.785 (1.01)	0.933 (0.57)	0.748 (1.43)	0.756 (1.10)
BP	0.909 (0.66)	0.734 (1.34)	0.732 (0.86)	0.897 (0.64)	0.707 (1.36)	0.689 (0.86)
BT	0.942 (0.59)	0.844 (1.85)	0.902 (2.32)	0.951 (0.50)	0.841 (2.28)	0.878 (3.10)
t_{ALW}	0.955 (0.42)	0.835 (1.60)	0.862 (1.67)	0.963 (0.42)	0.806 (1.58)	0.853 (1.71)
BC_a	0.892 (0.68)	0.753 (1.39)	0.742 (0.90)	0.892 (0.66)	0.720 (1.38)	0.704 (0.94)
N_{g1}	0.927 (0.64)	0.787 (1.46)	0.816 (1.15)	0.925 (0.60)	0.761 (1.48)	0.779 (1.19)
N_{g2}	0.925 (0.64)	0.785 (1.45)	0.808 (1.07)	0.925 (0.60)	0.760 (1.48)	0.770 (1.14)
	$(k=3, n=15)$, $(m_1=5, m_2=5, m_3=5)$			$(k=3, n=15)$, $(m_1=7, m_2=3, m_3=5)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.934 (0.65)	0.775 (1.35)	0.804 (0.89)	0.932 (0.64)	0.788 (1.37)	0.810 (0.93)
BP	0.910 (0.69)	0.758(1.32)	0.767 (0.80)	0.901 (0.69)	0.766 (1.33)	0.767 (0.82)
BT	0.940 (0.67)	0.858 (1.67)	0.895 (1.55)	0.938 (0.65)	0.854 (1.70)	0.899 (1.67)
t_{ALW}	0.954 (0.61)	0.820 (1.41)	0.869 (1.09)	0.959 (0.57)	0.813 (1.41)	0.864 (1.16)
BC_a	0.893 (0.72)	0.781 (1.37)	0.773 (0.83)	0.886 (0.71)	0.784(1.38)	0.775 (0.85)
N_{g1}	0.917 (0.69)	0.806 (1.43)	0.833 (1.02)	0.917 (0.68)	0.811 (1.45)	0.833 (1.04)
N_{g2}	0.916 (0.69)	0.804 (1.42)	0.821 (0.96)	0.916 (0.68)	0.809(1.43)	0.821 (0.99)

Table 2 continued

	$(k=3, n=30), (m_1=10, m_2=10, m_3=10)$			$(k=3, n=30), (m_1=10, m_2=8, m_3=12)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.936 (0.74)	0.818 (1.25)	0.845 (0.66)	0.933 (0.75)	0.833 (1.25)	0.856 (0.63)
BP	0.917 (0.76)	0.819 (1.25)	0.829 (0.63)	0.921 (0.77)	0.828 (1.24)	0.840 (0.60)
BT	0.914 (0.77)	0.866 (1.38)	0.886 (0.83)	0.912 (0.78)	0.884 (1.34)	0.902 (0.76)
t_{ALW}	0.947 (0.73)	0.831 (1.27)	0.875 (0.71)	0.942 (0.74)	0.835 (1.25)	0.876 (0.66)
BC_a	0.891 (0.79)	0.844 (1.30)	0.840 (0.66)	0.896 (0.79)	0.853 (1.28)	0.844 (0.63)
N_{g1}	0.902 (0.76)	0.857 (1.32)	0.882 (0.76)	0.911 (0.78)	0.866 (1.30)	0.886 (0.72)
N_{g2}	0.902 (0.78)	0.855 (1.32)	0.863 (0.72)	0.909 (0.78)	0.865 (1.30)	0.866 (0.67)
	$(k=3, n=60), (m_1=20, m_2=20, m_3=20)$			$(k=3, n=60), (m_1=25, m_2=10, m_3=25)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.927 (0.81)	0.841 (1.18)	0.871 (0.48)	0.930 (0.81)	0.848 (1.18)	0.872 (0.47)
BP	0.920 (0.82)	0.839 (1.18)	0.862 (0.46)	0.922 (0.82)	0.845 (1.18)	0.862 (0.46)
BT	0.904 (0.83)	0.884 (1.24)	0.893 (0.54)	0.912 (0.83)	0.883 (1.23)	0.896 (0.52)
t_{ALW}	0.935 (0.81)	0.857 (1.20)	0.883 (0.50)	0.929 (0.81)	0.863 (1.19)	0.890 (0.49)
BC_a	0.896 (0.84)	0.865 (1.21)	0.863 (0.48)	0.896 (0.84)	0.871 (1.21)	0.864 (0.47)
N_{g1}	0.903 (0.83)	0.874 (1.22)	0.901 (0.54)	0.905 (0.84)	0.880 (1.22)	0.897 (0.53)
N_{g2}	0.902 (0.83)	0.873 (1.22)	0.877 (0.50)	0.904 (0.84)	0.879 (1.22)	0.875 (0.50)

Table 3 Coverage probabilities of 90% confidence intervals for mean of $Exp(1)$ data

Methods	$(k=2, n=10), (m_1=5, m_2=5)$			$(k=2, n=10), (m_1=4, m_2=6)$			SRS, $n=10$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.932 (0.68)	0.796 (1.32)	0.815 (0.82)	0.926 (0.69)	0.802 (1.31)	0.820 (0.79)	0.928(0.64)	0.780 (1.35)	0.807 (0.91)
BP	0.910 (0.71)	0.779 (1.29)	0.784 (0.74)	0.907 (0.72)	0.787 (1.28)	0.784 (0.71)	0.924 (0.65)	0.787 (1.36)	0.810 (0.91)
BT	0.932 (0.68)	0.859 (1.56)	0.897 (1.35)	0.927 (0.70)	0.863 (1.51)	0.899 (1.18)	0.923 (0.66)	0.871 (1.68)	0.892 (1.40)
t_{ALW}	0.950 (0.63)	0.810 (1.36)	0.859 (1.00)	0.955 (0.65)	0.825 (1.33)	0.876 (0.97)			
BC_a	0.894 (0.73)	0.797 (1.33)	0.789 (0.76)	0.891 (0.74)	0.804 (1.32)	0.788 (0.73)	0.903 (0.69)	0.823 (1.44)	0.818 (0.97)
N_{g1}	0.922 (0.70)	0.816 (1.36)	0.836 (0.89)	0.914 (0.71)	0.820 (1.35)	0.839 (0.86)	0.916 (0.67)	0.832 (1.47)	0.845 (1.03)
N_{g2}	0.922 (0.70)	0.815 (1.35)	0.827 (0.85)	0.914 (0.71)	0.819 (1.34)	0.829 (0.82)	0.916 (0.67)	0.829 (1.45)	0.842 (1.03)
	$(k=2, n=20), (m_1=10, m_2=10)$			$(k=2, n=20), (m_1=12, m_2=8)$			SRS, $n=20$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.925 (0.764)	0.828 (1.24)	0.855 (0.61)	0.931 (0.75)	0.821 (1.25)	0.842 (0.65)	0.934 (0.73)	0.830 (1.27)	0.854 (0.69)
BP	0.911 (0.78)	0.818 (1.23)	0.837 (0.56)	0.917 (0.77)	0.814 (1.24)	0.823 (0.61)	0.929 (0.74)	0.836 (1.27)	0.855 (0.69)
BT	0.914 (0.79)	0.882 (1.33)	0.893 (0.74)	0.909 (0.77)	0.866 (1.39)	0.883 (0.86)	0.913 (0.76)	0.885 (1.39)	0.902 (0.85)
t_{ALW}	0.933 (0.75)	0.832 (1.25)	0.867 (0.66)	0.941 (0.78)	0.849 (1.22)	0.884 (0.59)			
BC_a	0.892 (0.80)	0.841 (1.26)	0.838 (0.59)	0.899 (0.79)	0.833 (1.28)	0.829 (0.63)	0.900 (0.77)	0.865 (1.33)	0.867 (0.72)
N_{g1}	0.907 (0.79)	0.853 (1.28)	0.880 (0.67)	0.916 (0.77)	0.843 (1.29)	0.863 (0.72)	0.910 (0.76)	0.871 (1.34)	0.880 (0.76)
N_{g2}	0.907 (0.79)	0.852 (1.27)	0.864 (0.63)	0.916 (0.77)	0.843 (1.29)	0.850 (0.68)	0.910 (0.76)	0.869 (1.33)	0.877 (0.75)

Table 3 continued

	$(k=2, n=30), (m_1=15, m_2=15)$			$(k=2, n=30), (m_1=18, m_2=12)$			SRS, $n=30$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.921 (0.81)	0.844 (1.20)	0.867 (0.50)	0.933 (0.79)	0.830 (1.21)	0.857 (0.54)	0.931 (0.78)	0.843 (1.23)	0.866 (0.57)
BP	0.909 (0.82)	0.840 (1.19)	0.856 (0.48)	0.921 (0.80)	0.826 (1.21)	0.844 (0.52)	0.922 (0.78)	0.847 (1.23)	0.870 (0.57)
BT	0.919 (0.82)	0.897 (1.26)	0.905 (0.58)	0.919 (0.81)	0.877 (1.28)	0.890 (0.64)	0.907 (0.80)	0.892 (1.30)	0.898 (0.66)
t_{ALW}	0.945 (0.80)	0.858 (1.21)	0.891 (0.53)	0.933 (0.78)	0.838 (1.22)	0.877 (0.58)			
BC_a	0.899 (0.83)	0.861 (1.22)	0.856 (0.50)	0.902 (0.82)	0.849 (1.23)	0.852 (0.54)	0.898 (0.81)	0.877 (1.27)	0.873 (0.59)
N_{g1}	0.898 (0.83)	0.869 (1.23)	0.890 (0.55)	0.912 (0.81)	0.857 (1.25)	0.884 (0.60)	0.904 (0.80)	0.881 (1.25)	0.884 (0.62)
N_{g2}	0.896 (0.83)	0.868 (1.23)	0.871 (0.52)	0.912 (0.81)	0.856 (1.24)	0.869 (0.56)	0.903 (0.80)	0.880 (1.27)	0.883 (0.61)
	$(k=2, n=60), (m_1=30, m_2=30)$			$(k=2, n=60), (m_1=25, m_2=35)$			SRS, $n=60$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.912 (0.86)	0.872 (1.15)	0.879 (0.36)	0.922 (0.86)	0.856 (1.13)	0.878 (0.34)	0.922 (0.84)	0.861 (1.16)	0.881 (0.41)
BP	0.904 (0.87)	0.871 (1.15)	0.872 (0.36)	0.917 (0.87)	0.86 (1.13)	0.872 (0.34)	0.919 (0.84)	0.864 (1.16)	0.879 (0.41)
BT	0.898 (0.87)	0.893 (1.17)	0.894 (0.39)	0.898 (0.88)	0.891 (1.16)	0.899 (0.37)	0.902 (0.85)	0.892 (1.19)	0.897 (0.44)
t_{ALW}	0.919 (0.86)	0.860 (1.14)	0.884 (0.37)	0.920 (0.86)	0.870 (1.13)	0.894 (0.35)			
BC_a	0.903 (0.88)	0.888 (1.16)	0.873 (0.36)	0.899 (0.88)	0.872 (1.15)	0.877 (0.34)	0.895 (0.86)	0.885 (1.19)	0.883 (0.42)
N_{g1}	0.892 (0.88)	0.892 (1.17)	0.899 (0.39)	0.903 (0.88)	0.876 (1.15)	0.900 (0.37)	0.900 (0.85)	0.887 (1.19)	0.889 (0.43)
N_{g2}	0.891 (0.88)	0.891 (1.16)	0.882 (0.37)	0.903 (0.88)	0.875 (1.15)	0.883 (0.35)	0.899 (0.85)	0.887 (1.19)	0.888 (0.43)

Table 4 Coverage probabilities of 90% confidence intervals for mean of $Exp(1)$ data

Methods	$(k=3, n=10), (m_1=3, m_2=3, m_3=4)$			$(k=3, n=10), (m_1=4, m_2=3, m_3=3)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.926 (0.72)	0.813 (1.28)	0.823 (0.71)	0.924 (0.70)	0.775 (1.29)	0.786 (0.76)
BP	0.894 (0.76)	0.781 (1.24)	0.773 (0.60)	0.884 (0.75)	0.739 (1.24)	0.714 (0.61)
BT	0.939 (0.70)	0.871 (1.48)	0.905 (1.27)	0.937 (0.63)	0.862 (1.64)	0.893 (1.56)
t_{ALW}	0.945 (0.67)	0.845 (1.33)	0.888 (0.92)	0.948 (0.62)	0.842 (1.39)	0.877 (1.12)
BC_a	0.883 (0.78)	0.797 (1.26)	0.776 (0.62)	0.880 (0.78)	0.751 (1.26)	0.723 (0.64)
N_{g1}	0.916 (0.74)	0.828 (1.31)	0.841 (0.78)	0.916 (0.71)	0.787 (1.32)	0.803 (0.81)
N_{g2}	0.915 (0.74)	0.828 (1.31)	0.835 (0.74)	0.915 (0.72)	0.787 (1.32)	0.797 (0.78)
	$(k=3, n=15), (m_1=5, m_2=5, m_3=5)$			$(k=3, n=15), (m_1=7, m_2=3, m_3=5)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.933 (0.76)	0.816 (1.24)	0.841 (0.62)	0.921 (0.75)	0.806 (1.25)	0.819 (0.65)
BP	0.923 (0.78)	0.800 (1.22)	0.800 (0.56)	0.894 (0.78)	0.779 (1.22)	0.776 (0.56)
BT	0.928 (0.76)	0.876 (1.38)	0.903 (0.88)	0.930 (0.74)	0.868 (1.41)	0.900 (0.99)
t_{ALW}	0.942 (0.73)	0.830 (1.27)	0.876 (0.73)	0.943 (0.71)	0.854 (1.31)	0.893 (0.82)
BC_a	0.882 (0.80)	0.813 (1.25)	0.809 (0.57)	0.886 (0.79)	0.794 (1.25)	0.779 (0.58)
N_{g1}	0.906 (0.78)	0.842 (1.28)	0.860 (0.69)	0.913 (0.76)	0.821 (1.29)	0.842 (0.71)
N_{g2}	0.906 (0.78)	0.840 (1.28)	0.847 (0.65)	0.913 (0.77)	0.820 (1.28)	0.833 (0.67)

Table 4 continued

	$(k=3, n=30), (m_1=10, m_2=10, m_3=10)$			$(k=3, n=30), (m_1=10, m_2=8, m_3=12)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.923 (0.82)	0.853 (1.18)	0.878 (0.46)	0.930 (0.83)	0.858 (1.17)	0.886 (0.44)
BP	0.918 (0.84)	0.845 (1.17)	0.861 (0.43)	0.914 (0.84)	0.851 (1.17)	0.867 (0.42)
BT	0.922 (0.83)	0.875 (1.22)	0.902 (0.52)	0.905 (0.84)	0.885 (1.22)	0.896 (0.50)
t_{ALW}	0.924 (0.82)	0.857 (1.19)	0.878 (0.47)	0.932 (0.82)	0.863 (1.18)	0.889 (0.46)
BC_a	0.891 (0.85)	0.865(1.20)	0.863 (0.44)	0.894 (0.85)	0.871 (1.188)	0.868 (0.43)
N_{g1}	0.905 (0.84)	0.877 (1.21)	0.900 (0.50)	0.904 (0.85)	0.884 (1.20)	0.905 (0.48)
N_{g2}	0.904 (0.84)	0.876 (1.21)	0.884 (0.47)	0.904 (0.85)	0.883 (1.20)	0.884 (0.46)
	$(k=3, n=60), (m_1=20, m_2=20, m_3=20)$			$(k=3, n=60), (m_1=25, m_2=10, m_3=25)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.925 (0.87)	0.864 (1.13)	0.885 (0.33)	0.914 (0.87)	0.866 (1.13)	0.883 (0.33)
BP	0.917 (0.88)	0.860 (1.13)	0.879 (0.32)	0.904 (0.88)	0.862 (1.13)	0.871 (0.32)
BT	0.906 (0.88)	0.893 (1.15)	0.903 (0.35)	0.904 (0.88)	0.895 (1.15)	0.900 (0.36)
t_{ALW}	0.917 (0.87)	0.880 (1.13)	0.897 (0.34)	0.929 (0.86)	0.859 (1.13)	0.892 (0.34)
BC_a	0.898 (0.89)	0.880 (1.13)	0.879 (0.32)	0.896 (0.89)	0.882 (1.14)	0.872 (0.33)
N_{g1}	0.903 (0.88)	0.887 (1.14)	0.903 (0.35)	0.895 (0.88)	0.889 (1.15)	0.906 (0.36)
N_{g2}	0.903 (0.88)	0.886 (1.14)	0.890 (0.33)	0.894 (0.88)	0.888 (1.15)	0.890 (0.34)

Table 5 Coverage probabilities of 90% confidence intervals for mean of $HN(0, 1)$

Methods	$(k = 2, n = 10), (m_1 = 5, m_2 = 5)$			$(k = 2, n = 10), (m_1 = 4, m_2 = 6)$			SRS, $n = 10$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.913 (0.60)	0.834 (0.99)	0.843 (0.50)	0.915(0.60)	0.837(0.99)	0.844 (0.49)	0.911 (0.57)	0.827 (1.02)	0.834 (0.57)
BP	0.889 (0.62)	0.818 (0.97)	0.802 (0.44)	0.884 (0.63)	0.813 (0.97)	0.808 (0.44)	0.910 (0.58)	0.834 (1.03)	0.841 (0.57)
BT	0.915 (0.60)	0.892 (1.07)	0.911 (0.69)	0.920 (0.58)	0.888 (1.08)	0.914 (0.73)	0.920 (0.57)	0.894 (1.12)	0.915 (0.76)
t_{ALW}	0.934 (0.58)	0.877 (1.02)	0.894 (0.60)	0.933 (0.58)	0.880 (1.01)	0.902 (0.58)			
BC_a	0.890 (0.63)	0.831 (0.98)	0.812 (0.46)	0.893 (0.63)	0.830 (0.98)	0.814 (0.45)	0.898 (0.59)	0.859 (1.05)	0.852 (0.59)
N_{g1}	0.908 (0.61)	0.852 (1.00)	0.857 (0.52)	0.913 (0.61)	0.845 (1.00)	0.856 (0.51)	0.909 (0.58)	0.867 (1.06)	0.875 (0.63)
N_{g2}	0.908 (0.61)	0.852 (1.00)	0.852 (0.51)	0.913 (0.61)	0.847 (1.00)	0.852 (0.50)	0.908 (0.58)	0.864 (1.06)	0.872 (0.62)
	$(k = 2, n = 20),(m_1 = 10, m_2 = 10)$			$(k = 2, n = 20), (m_1 = 12, m_2 = 8)$			SRS, $n = 20$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.908 (0.65)	0.857 (0.94)	0.866 (0.36)	0.909 (0.65)	0.867 (0.95)	0.870 (0.38)	0.910 (0.63)	0.863 (0.96)	0.869 (0.42)
BP	0.887 (0.66)	0.849 (0.93)	0.849 (0.34)	0.889 (0.67)	0.859 (0.94)	0.852 (0.36)	0.910 (0.64)	0.869 (0.97)	0.869 (0.42)
BT	0.909 (0.66)	0.899 (0.97)	0.907 (0.41)	0.909 (0.66)	0.897 (0.985)	0.909 (0.45)	0.907 (0.64)	0.904 (1.00)	0.904 (0.48)
t_{ALW}	0.929 (0.65)	0.873 (0.95)	0.905 (0.39)	0.919 (0.67)	0.873 (0.926)	0.890 (0.34)			
BC_a	0.896 (0.67)	0.863 (0.94)	0.854 (0.35)	0.893 (0.67)	0.871 (0.95)	0.853 (0.36)	0.897 (0.65)	0.889 (0.98)	0.873 (0.43)
N_{g1}	0.898(0.66)	0.873 (0.95)	0.881 (0.38)	0.900 (0.66)	0.882 (0.96)	0.882 (0.40)	0.903 (0.64)	0.893 (0.99)	0.885 (0.45)
N_{g2}	0.898 (0.66)	0.874 (0.95)	0.873 (0.38)	0.898 (0.66)	0.882 (0.96)	0.874 (0.39)	0.903 (0.64)	0.893 (0.99)	0.884 (0.44)

Table 5 continued

	$(k=2, n=30), (m_1=15, m_2=15)$			$(k=2, n=30), (m_1=18, m_2=12)$			SRS, $n=30$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.915 (0.68)	0.874 (0.92)	0.879 (0.30)	0.915(0.68)	0.876 (0.91)	0.891 (0.29)	0.914 (0.66)	0.856 (0.93)	0.868 (0.35)
BP	0.906 (0.68)	0.868 (0.91)	0.870 (0.30)	0.906 (0.69)	0.869 (0.908)	0.879 (0.28)	0.912 (0.66)	0.861 (0.94)	0.871 (0.35)
BT	0.904 (0.69)	0.903 (0.93)	0.906 (0.32)	0.911 (0.68)	0.895 (0.94)	0.905 (0.34)	0.907 (0.67)	0.893 (0.95)	0.898 (0.38)
t_{ALW}	0.918 (0.68)	0.885 (0.92)	0.901 (0.31)	0.922 (0.67)	0.867 (0.92)	0.883 (0.33)			
BC_a	0.895 (0.69)	0.881 (0.92)	0.876 (0.29)	0.896 (0.69)	0.881 (0.92)	0.884 (0.28)	0.899 (0.67)	0.879 (0.95)	0.879 (0.36)
N_{g1}	0.903 (0.69)	0.888 (0.92)	0.897 (0.31)	0.903 (0.69)	0.889 (0.92)	0.905 (0.30)	0.904 (0.67)	0.882 (0.95)	0.885 (0.36)
N_{g2}	0.903 (0.69)	0.887 (0.92)	0.889 (0.30)	0.903 (0.69)	0.889 (0.92)	0.895 (0.29)	0.904 (0.67)	0.881 (0.95)	0.883 (0.36)
	$(k=2, n=60), (m_1=30, m_2=30)$			$(k=2, n=60), (m_1=25, m_2=35)$			SRS, $n=60$		
	LCL	UCL	TCL	LCL	UCL	TCL	LCL	UCL	TCL
N	0.911 (0.71)	0.878 (0.88)	0.895 (0.21)	0.903 (0.72)	0.882 (0.88)	0.888 (0.21)	0.905 (0.70)	0.871 (0.90)	0.875 (0.25)
BP	0.906 (0.72)	0.877 (0.88)	0.889 (0.21)	0.898 (0.72)	0.879 (0.88)	0.884 (0.20)	0.903 (0.70)	0.873 (0.90)	0.875 (0.25)
BT	0.908 (0.72)	0.901 (0.89)	0.904 (0.22)	0.896 (0.72)	0.895 (0.89)	0.894 (0.21)	0.897 (0.70)	0.893 (0.91)	0.894 (0.26)
t_{ALW}	0.913 (0.71)	0.899 (0.882)	0.911 (0.22)	0.915 (0.72)	0.886 (0.880)	0.897 (0.21)			
BC_a	0.896 (0.72)	0.886 (0.88)	0.896 (0.21)	0.894 (0.72)	0.889 (0.88)	0.886 (0.20)	0.890 (0.71)	0.886 (0.90)	0.883 (0.25)
N_{g1}	0.900 (0.72)	0.890 (0.88)	0.908 (0.22)	0.893 (0.72)	0.895 (0.88)	0.900 (0.22)	0.895 (0.70)	0.890 (0.90)	0.888 (0.26)
N_{g2}	0.900 (0.72)	0.889 (0.88)	0.900 (0.21)	0.893 (0.72)	0.895 (0.88)	0.892 (0.21)	0.895 (0.70)	0.889 (0.90)	0.887 (0.26)

Table 6 Coverage probabilities of 90% confidence intervals for mean of $HN(0, 1)$ data

Methods	$(k = 3, n = 10), (m_1 = 3, m_2 = 3, m_3 = 4)$			$(k = 3, n = 10), (m_1 = 4, m_2 = 3, m_3 = 3)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.913 (0.63)	0.851 (0.97)	0.849 (0.43)	0.909 (0.62)	0.835 (0.97)	0.828 (0.45)
BP	0.867 (0.66)	0.816 (0.94)	0.784 (0.36)	0.863 (0.65)	0.802 (0.94)	0.76 (0.37)
BT	0.926 (0.61)	0.892 (1.03)	0.918 (0.63)	0.926 (0.58)	0.898 (1.06)	0.915 (0.72)
t_{ALW}	0.933 (0.60)	0.878 (1.00)	0.904 (0.54)	0.922 (0.58)	0.877 (1.02)	0.907 (0.62)
BC_a	0.892 (0.66)	0.826 (0.95)	0.788 (0.37)	0.894 (0.65)	0.811 (0.95)	0.761 (0.38)
N_{g1}	0.899 (0.64)	0.859 (0.98)	0.858 (0.45)	0.896 (0.63)	0.844 (0.98)	0.835 (0.47)
N_{g2}	0.898 (0.64)	0.858 (0.98)	0.852 (0.44)	0.895 (0.63)	0.844 (0.98)	0.830 (0.46)
	$(k = 3, n = 15), (m_1 = 5, m_2 = 5, m_3 = 5)$			$(k = 3, n = 15), (m_1 = 7, m_2 = 3, m_3 = 5)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.919 (0.65)	0.855 (0.94)	0.858 (0.37)	0.910 (0.65)	0.846 (0.95)	0.847 (0.39)
BP	0.886 (0.67)	0.834 (0.93)	0.822 (0.33)	0.877 (0.66)	0.819 (0.93)	0.798 (0.34)
BT	0.914 (0.65)	0.888 (0.97)	0.905 (0.45)	0.912 (0.64)	0.893 (1.00)	0.910 (0.51)
t_{ALW}	0.926 (0.64)	0.886 (0.95)	0.904 (0.42)	0.923 (0.62)	0.878 (0.97)	0.897 (0.47)
BC_a	0.890 (0.67)	0.848 (0.93)	0.825 (0.33)	0.891 (0.67)	0.830 (0.94)	0.804 (0.34)
N_{g1}	0.902 (0.66)	0.870 (0.95)	0.869 (0.38)	0.900 (0.65)	0.857 (0.96)	0.856 (0.40)
N_{g2}	0.902 (0.66)	0.870 (0.95)	0.861 (0.37)	0.900 (0.65)	0.857 (0.96)	0.854 (0.39)

Table 6 continued

	$(k=3, n=30), (m_1=10, m_2=10, m_3=10)$			$(k=3, n=30), (m_1=10, m_2=8, m_3=12)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.908 (0.70)	0.884 (0.90)	0.885 (0.26)	0.911 (0.70)	0.881 (0.90)	0.884 (0.26)
BP	0.895 (0.70)	0.874 (0.90)	0.871 (0.25)	0.898 (0.70)	0.870 (0.90)	0.866 (0.25)
BT	0.907 (0.70)	0.900 (0.91)	0.911 (0.28)	0.908 (0.70)	0.899 (0.91)	0.904 (0.28)
t_{ALW}	0.920 (0.69)	0.866 (0.90)	0.890 (0.28)	0.919 (0.69)	0.883 (0.90)	0.898 (0.27)
BC_a	0.895 (0.71)	0.885 (0.90)	0.870 (0.25)	0.897 (0.71)	0.883 (0.90)	0.871 (0.25)
N_{g1}	0.895 (0.70)	0.896 (0.91)	0.895 (0.27)	0.900 (0.70)	0.894 (0.91)	0.894 (0.27)
N_{g2}	0.899 (0.70)	0.896 (0.91)	0.889 (0.27)	0.900 (0.70)	0.894 (0.91)	0.887 (0.26)
	$(k=3, n=60), (m_1=20, m_2=20, m_3=20)$			$(k=3, n=60), (m_1=25, m_2=10, m_3=25)$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.915 (0.73)	0.881 (0.87)	0.897 (0.19)	0.909 (0.72)	0.880 (0.87)	0.885 (0.19)
BP	0.909 (0.73)	0.875 (0.87)	0.889 (0.18)	0.902 (0.72)	0.873 (0.87)	0.872 (0.19)
BT	0.900 (0.73)	0.893 (0.88)	0.889 (0.19)	0.908 (0.72)	0.895 (0.88)	0.900 (0.21)
t_{ALW}	0.925 (0.72)	0.890 (0.87)	0.903 (0.19)	0.913 (0.72)	0.887 (0.88)	0.893 (0.21)
BC_a	0.901 (0.73)	0.888 (0.87)	0.891 (0.18)	0.895 (0.73)	0.885 (0.88)	0.878 (0.19)
N_{g1}	0.909 (0.73)	0.894 (0.88)	0.907 (0.19)	0.899 (0.72)	0.893 (0.88)	0.896 (0.21)
N_{g2}	0.909 (0.73)	0.894 (0.87)	0.899 (0.19)	0.899 (0.72)	0.892 (0.88)	0.888 (0.20)

all sample sizes. Also, when n is small, t_{ALW} method produces better coverage probabilities compared to BC_a, N_{g_1} and N_{g_2} methods for UCL. Overall, BT method appears to be the best in terms of coverage accuracy for UCL when $n \leq 30$.

(iii) For two-sided confidence intervals, BT method performs best in terms of coverage accuracy with very wide confidence intervals for small to moderate sample sizes. For large sample sizes ($n = 60$), all methods provides very similar coverage probabilities. But, N_{g_1} and t_{ALW} give better coverage probabilities for TCL than N, BP, BC_a and N_{g_2} methods for small to moderate sample sizes. In summary, we find BT method to give best coverage for TCL and produces wider confidence intervals for small to moderate samples sizes. For large sample sizes, N_{g_1}, N_{g_2} and t_{ALW} all give comparable coverage as BT in addition to being simple to compute and also requiring less computational effort.

(iv) The simulation results show that the SRS scheme produces better coverage probabilities for the methods N, BP, BT, BC_a, N_{g_1} and N_{g_2} than the RSS when $n \leq 20$. However, both sampling methods give similar coverages for all methods when $n \geq 30$. The most important observation is that for each distribution and sample size combination that we have considered, all confidence interval methods give more accurate average lower and upper limits, and shorter average interval widths for LCL, UCL and TCL under the RSS scheme as compared to the SRS scheme.

(v) Simulation results (see the supplement) for set size $k = 5$ provide similar conclusions as for set sizes $k = 2, 3$.

5.1 Imperfect ranking

Throughout this work, we have assumed the RSS scheme under perfect ranking. However, in practice, it may involve ranking errors. Therefore, it will be of natural interest to evaluate the performance of all these interval estimation methods under imperfect ranking. Several nonparametric tests have been proposed by Li and Balakrishnan (2008) to test the assumption of perfect ranking in RSS. In order to simulate imperfect RSS samples, we employ the model proposed by Dell and Clutter (1972). Consider the model $Y_i = X_i + \epsilon_i$, where ϵ_i represents the error involved in the judgment ranking. This model can be implemented by generating n independent realizations X_i from a given distribution $F(x)$ and n independent normally distributed random errors with zero mean and variance σ^2. We compute Y_i's and order them. Let $Y_{(r)}$ denote the measurement corresponding to the true rth order statistic. Then, the corresponding X value represents the rth judgment order statistic, $X_{[r]}$. The variance component attached to the error term, σ^2, controls the degree of judgment error. In particular, the correlation coefficient between Y and X can be expressed as $\rho = \phi/\sqrt{\phi^2 + \sigma^2}$, where ϕ^2 is the variance of X. It is quite evident that the RSS scheme involves perfect ranking when ϵ has a degenerate distribution. We consider ρ to be 0.5 and 0.75 and the values of σ^2 are selected accordingly. The obtained simulation results under imperfect ranking are presented in Tables 7, 8 and 9. For brevity, simulation results below highlights our findings based on a set size of $k = 2$ with the balanced design. All these different confidence interval methods appeared to be robust under imperfect ranking. That is, we have similar conclusions as reported under the perfect ranking scenario. We also observe that, given a sample size, as the correlation increases these methods result in a shorter interval.

6 Illustrative example

In this section, we use a data from a study involving 46 shrubs. The dataset was first reported in Muttlak and McDonald (1990). First three transect lines were laid out across the area and all shrubs intersecting each transect were sampled. Finally, the size of each shrub was measured. This technique is good for sampling a very large area relatively quickly. These data were further used by Ghosh and Tiwari (2004) to construct an RSS. The original sample was broken into 15 groups, each containing 3 shrubs (leaving one out). The 3 shrubs in the first group were ranked based on their sizes, and the shortest of all was included in the sample. This process was repeated 5 times, which resulted in 5 replicates. For the next 5 groups, the ones with the second smallest size were included in the sample. Finally, from each of the remaining five groups, the largest shrubs were chosen. This process resulted in a balanced RSS with set size 3 and cycle size 5. The data so obtained are presented in Ghosh and Tiwari (2004). Figure 1 presents the density plot of shrub sizes and it shows that the sample has a bimodal skewed distribution. The resulting confidence intervals for the mean shrub size are presented in Table 10.

It can be seen that BP method gives largest lower limit for the 90% lower confidence interval for mean size of shrubs. The lower limit corresponding to the BC_a method is close to that of BP method. Method N is similar to the methods N_{g_1} and N_{g_2}, but their lower limits are smaller than those of BP and BC_a methods. BT method gives the smallest lower limit. Based on our simulation study, BC_a method should be chosen for the lower confidence interval for mean size of shrubs.

For the 90% upper confidence limit, BP method produces smallest upper limit which is close to that of BC_a method. N_{g_1} and N_{g_2} methods give same upper limits for mean size of shrubs and their upper limits are bigger than those of BP and BC_a methods, but close to the upper limit based on N method. BT method gives largest upper limits for mean size of shrubs, but very similar to that of t_{ALW} method. The results of the simulation study suggest that the BT method often has the best coverage error for upper confidence intervals. Therefore, BT method should be chosen for the upper confidence interval for mean size of shrubs.

For the 90% two-sided confidence interval, BC_a and BP methods have shortest interval width compared to other methods. BT method produces the widest 90% two-sided confidence interval for mean size of shrubs. Our simulation study shows BT method to have best coverage probabilities for small to moderate sample sizes, but it produces widest two-sided confidence intervals. t_{ALW} gives the next best coverage for two-sided confidence intervals. Hence, t_{ALW} or BT method should be chosen for the two-sided confidence interval for mean size of shrubs.

Table 7 Coverage probabilities of 90% confidence intervals for mean of χ_1^2 data under imperfect ranking

Methods	$(k=2, n=10)$, $\rho=0.5$			$(k=2, n=10)$, $\rho=0.75$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.946 (0.51)	0.763 (1.50)	0.786 (1.27)	0.935 (0.52)	0.765 (1.47)	0.790 (1.22)
BP	0.925 (0.57)	0.746 (1.46)	0.762 (1.14)	0.911 (0.58)	0.749 (1.43)	0.764 (1.09)
BT	0.938 (0.55)	0.852 (2.19)	0.897 (2.59)	0.942 (0.55)	0.853 (2.05)	0.896 (2.50)
BC_a	0.907 (0.61)	0.772 (1.55)	0.775 (1.19)	0.898 (0.62)	0.777 (1.51)	0.773 (1.13)
N_{g1}	0.933 (0.56)	0.787 (1.59)	0.819 (1.42)	0.922 (0.57)	0.794 (1.55)	0.822 (1.35)
N_{g2}	0.933 (0.56)	0.786 (1.58)	0.811 (1.33)	0.922 (0.57)	0.792 (1.55)	0.812 (1.27)
	$(k=2, n=30)$, $\rho=0.5$			$(k=2, n=30)$, $\rho=0.75$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.934 (0.69)	0.817 (1.30)	0.845 (0.78)	0.938 (0.70)	0.822 (1.29)	0.857 (0.76)
BP	0.922 (0.71)	0.817 (1.30)	0.838 (0.75)	0.928 (0.72)	0.820 (1.29)	0.847 (0.73)
BT	0.917 (0.73)	0.879 (1.43)	0.895 (0.95)	0.916 (0.74)	0.885 (1.43)	0.901 (0.94)
BC_a	0.895 (0.75)	0.850 (1.36)	0.852 (0.79)	0.903 (0.75)	0.852 (1.34)	0.857 (0.77)
N_{g1}	0.907 (0.73)	0.859 (1.38)	0.886 (0.90)	0.912 (0.74)	0.861 (1.36)	0.896 (0.87)
N_{g2}	0.906 (0.73)	0.857 (1.37)	0.865 (0.83)	0.911 (0.74)	0.859 (1.36)	0.872 (0.81)
	$(k=2, n=60)$, $\rho=0.5$			$(k=2, n=60)$, $\rho=0.75$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.938 (0.77)	0.848 (1.22)	0.875 (0.57)	0.933 (0.79)	0.855 (1.22)	0.879 (0.55)
BP	0.933 (0.78)	0.848 (1.22)	0.871 (0.56)	0.924 (0.79)	0.854 (1.22)	0.878 (0.54)
BT	0.902 (0.80)	0.892 (1.28)	0.897 (0.63)	0.898 (0.81)	0.892 (1.28)	0.900 (0.61)
BC_a	0.908 (0.81)	0.877 (1.25)	0.882 (0.57)	0.900 (0.82)	0.881 (1.25)	0.877 (0.56)
N_{g1}	0.914 (0.80)	0.884 (1.26)	0.910 (0.64)	0.906 (0.81)	0.888 (1.26)	0.908 (0.62)
N_{g2}	0.914 (0.80)	0.883 (1.26)	0.890 (0.59)	0.905 (0.81)	0.887 (1.25)	0.887 (0.58)

Table 8 Coverage probabilities of 90% confidence intervals for mean of $Exp(1)$ data under imperfect ranking

Methods	$(k = 2, n = 10)$, $\rho = 0.5$			$(k = 2, n = 10)$, $\rho = 0.75$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.931 (0.64)	0.805 (1.37)	0.826 (0.93)	0.925 (0.66)	0.811 (1.36)	0.825 (0.90)
BP	0.906 (0.68)	0.790 (1.34)	0.792 (0.83)	0.901 (0.70)	0.793 (1.33)	0.789 (0.81)
BT	0.927 (0.65)	0.867 (1.63)	0.898 (1.45)	0.935 (0.66)	0.869 (1.61)	0.907 (1.45)
BC_a	0.890 (0.71)	0.809 (1.38)	0.800 (0.85)	0.889 (0.72)	0.813 (1.37)	0.798 (0.83)
N_{g_1}	0.920 (0.67)	0.823 (1.41)	0.847 (1.00)	0.916 (0.68)	0.829 (1.40)	0.844 (0.97)
N_{g_2}	0.920 (0.67)	0.822 (1.41)	0.839 (0.95)	0.916 (0.68)	0.829 (1.40)	0.836 (0.93)
	$(k = 2, n = 30)$, $\rho = 0.5$			$(k = 2, n = 30)$, $\rho = 0.75$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.932 (0.78)	0.850 (1.22)	0.876 (0.56)	0.922 (0.79)	0.856 (1.22)	0.874 (0.54)
BP	0.910 (0.85)	0.863 (1.16)	0.875 (0.40)	0.916 (0.85)	0.865 (1.15)	0.874 (0.38)
BT	0.904 (0.80)	0.888 (1.28)	0.898 (0.63)	0.901 (0.81)	0.889 (1.27)	0.890 (0.61)
BC_a	0.889 (0.86)	0.884 (1.17)	0.878 (0.40)	0.897 (0.86)	0.881 (1.17)	0.878 (0.39)
N_{g_1}	0.896 (0.86)	0.888 (1.18)	0.905 (0.44)	0.904 (0.86)	0.886 (1.17)	0.902 (0.42)
N_{g_2}	0.896 (0.86)	0.888 (1.18)	0.884 (0.41)	0.903 (0.86)	0.885 (1.17)	0.886 (0.40)
	$(k = 2, n = 60)$, $\rho = 0.5$			$(k = 2, n = 60)$, $\rho = 0.75$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.917 (0.84)	0.862 (1.16)	0.879 (0.40)	0.922 (0.85)	0.865 (1.15)	0.880 (0.39)
BP	0.916 (0.85)	0.866 (1.16)	0.874 (0.41)	0.927 (0.84)	0.864 (1.16)	0.886 (0.41)
BT	0.905 (0.86)	0.903 (1.19)	0.905 (0.43)	0.902 (0.86)	0.900 (1.18)	0.901 (0.41)
BC_a	0.917 (0.84)	0.887 (1.18)	0.892 (0.43)	0.928 (0.84)	0.883 (1.18)	0.902 (0.44)
N_{g_1}	0.900 (0.6)	0.893 (1.19)	0.906 (0.45)	0.911 (0.85)	0.889 (1.18)	0.914 (0.46)
N_{g_2}	0.900 (0.86)	0.892 (1.19)	0.884 (0.43)	0.910 (0.85)	0.888 (1.18)	0.895 (0.430)

Table 9 Coverage probabilities of 90% confidence intervals for mean of $HN(0, 1)$ data under imperfect ranking

Methods	$(k = 2, n = 10)$, $\rho = 0.5$			$(k = 2, n = 10)$, $\rho = 0.75$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.917 (0.57)	0.844 (1.02)	0.846 (0.58)	0.909 (0.59)	0.856 (1.02)	0.859 (0.55)
BP	0.897 (0.59)	0.827 (1.00)	0.814 (0.52)	0.883 (0.61)	0.834 (1.00)	0.825 (0.49)
BT	0.927 (0.57)	0.889 (1.11)	0.916 (0.77)	0.918 (0.58)	0.898 (1.09)	0.920 (0.74)
BC_a	0.890 (0.60)	0.840 (1.02)	0.822 (0.53)	0.877 (0.62)	0.849 (1.01)	0.834 (0.50)
N_{g_1}	0.912 (0.58)	0.856 (1.04)	0.861 (0.61)	0.903 (0.60)	0.869 (1.03)	0.870 (0.57)
N_{g_2}	0.912 (0.58)	0.856 (1.04)	0.858 (0.59)	0.903 (0.60)	0.869 (1.03)	0.869 (0.56)
	$(k = 2, n = 30)$, $\rho = 0.5$			$(k = 2, n = 30)$, $\rho = 0.75$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.914 (0.66)	0.875 (0.93)	0.886 (0.34)	0.903 (0.67)	0.873 (0.93)	0.878 (0.33)
BP	0.905 (0.67)	0.870 (0.93)	0.874 (0.33)	0.893 (0.68)	0.867 (0.92)	0.868 (0.31)
BT	0.901 (0.67)	0.903 (0.95)	0.905 (0.37)	0.908 (0.67)	0.896 (0.94)	0.902 (0.35)
BC_a	0.897 (0.68)	0.887 (0.94)	0.878 (0.34)	0.885 (0.68)	0.880 (0.93)	0.872 (0.32)
N_{g_1}	0.905 (0.67)	0.894 (0.94)	0.899 (0.36)	0.891 (0.68)	0.887 (0.93)	0.893 (0.34)
N_{g_2}	0.905 (0.67)	0.893 (0.94)	0.890 (0.35)	0.891 (0.68)	0.887 (0.93)	0.885 (0.33)
	$(k = 2, n = 60)$, $\rho = 0.5$			$(k = 2, n = 60)$, $\rho = 0.75$		
	LCL	UCL	TCL	LCL	UCL	TCL
N	0.910 (0.70)	0.883 (0.89)	0.892 (0.24)	0.906 (0.71)	0.884 (0.89)	0.887 (0.23)
BP	0.905 (0.70)	0.883 (0.89)	0.887 (0.24)	0.900 (0.71)	0.882 (0.89)	0.881 (0.23)
BT	0.901 (0.71)	0.901 (0.90)	0.897 (0.25)	0.899 (0.71)	0.905 (0.89)	0.905 (0.24)
BC_a	0.898 (0.71)	0.892 (0.90)	0.888 (0.24)	0.893 (0.71)	0.890 (0.89)	0.882 (0.23)
N_{g_1}	0.902 (0.71)	0.895 (0.90)	0.905 (0.25)	0.896 (0.71)	0.895 (0.89)	0.899 (0.24)
N_{g_2}	0.902 (0.71)	0.895 (0.90)	0.895 (0.25)	0.896 (0.71)	0.895 (0.89)	0.890 (0.23)

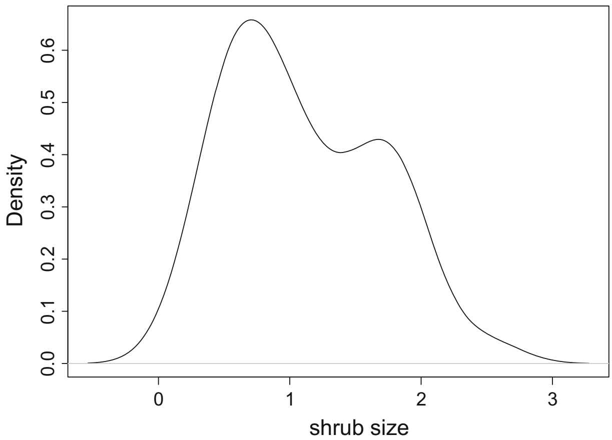

Fig. 1 Estimated density function for shrub sizes

Table 10 Summary of 90% LCL, UCL and TCL for mean shrub size

Methods	LCL	UCL	TCL	Interval length
N	$[0.891, \infty)$	$(-\infty, 1.268]$	(0.838, 1.321)	0.483
BP	$[0.906, \infty)$	$(-\infty, 1.235]$	(0.858, 1.291)	0.433
BT	$[0.847, \infty)$	$(-\infty, 1.301]$	(0.606, 1.387)	0.781
t_{ALW}	$[0.865, \infty)$	$(-\infty, 1.293]$	(0.788, 1.371)	0.583
BC_a	$[0.900, \infty)$	$(-\infty, 1.239]$	(0.851, 1.282)	0.432
N_{g_1}	$[0.884, \infty)$	$(-\infty, 1.261]$	(0.838, 1.311)	0.473
N_{g_2}	$[0.884, \infty)$	$(-\infty, 1.261]$	(0.826, 1.311)	0.485

7 Discussion and concluding remarks

We have developed bias-corrected and accelerated method along with transformation methods, N_{g_1} and N_{g_2}, for constructing confidence intervals for the population mean based on Ranked set samples. We have studied asymptotic properties of these methods and have shown that they are second-order accurate. These methods are asymptotically equivalent to bootstrap percentile-t in terms of coverage errors. From the simulation studies carried out, it is evident that for a right skewed distribution, the bias-corrected method gives best coverage probability in the case of lower confidence intervals, whereas bootstrap percentile-t method results in smallest average lower limits among all methods. On the other hand, when the population distribution is right skewed, the bootstrap percentile-t method gives best finite sample coverage for the upper and two-sided intervals. This behavior can be substantiated by the empirical results suggesting largest average upper limits and the widest two-sided confidence intervals associated with bootstrap percentile-t method. Performances of all methods improve as sample size increases, as one would expect. Hence, for large sample sizes, bias-corrected and accelerated, bootstrap percentile-t, t_{ALW}, N_{g_1} and N_{g_2} methods all give similar coverage probabilities for all three intervals. However, t_{ALW}, N_{g_1} and N_{g_2} require less computing in terms of bootstrap resampling. Though our proposed confidence interval methods are developed under the assumptions of perfect ranking, our simulation studies show that they are robust even in the presence of judgment error. In this research, we did not consider iterated bootstrap method. Even though this method is known to be effective in reducing coverage errors for SRS, it becomes computationally demanding as set sizes in RSS scheme increases. A reduction in the computational burden of the iterated bootstrap method in RSS may be possible by applying some analytical approximations to the nominal level. Work is currently under progress on this problem and we hope to report these findings in a future paper.

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Appendix A: Proofs

Proof of Theorem 2.1: Let us define $Y_{(r),i} = \frac{X_{(r),i}-\mu_r}{\sigma_r}$, for $r = 1, 2, \ldots, k$. Then, T_{RSS} can be expressed as

$$T_{RSS} = \frac{n\sum_{r=1}^{k}\sigma_r\bar{Y}_r}{\sqrt{\sum_{r=1}^{k}\sigma_r^2\frac{s_{r,Y}^2}{\lambda_{r,n}}}},$$

where $s_{r,Y}^2 = m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i} - \bar{Y}_r)^2$. We can express

$$\sum_{r=1}^{k}\sigma_r^2\frac{s_{r,Y}^2}{\lambda_{r,n}} = a_n\left[1 + \sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2 - 1) - \sum_{r=1}^{k}a_{r,n}\bar{Y}_r^2\right], \tag{7.1}$$

where $a_{r,n} = \frac{\lambda_{r,n}^{-1}\sigma_r^2}{a_n}$ and $a_n = \sum_{r=1}^{k}\lambda_{r,n}^{-1}\sigma_r^2$. Using (7.1), T_{RSS} can be expressed as

$$T_{RSS} = a_n^{-1/2}T_1, \tag{7.2}$$

where

$$\begin{aligned} T_1 = \sqrt{n}\sum_{r=1}^{k}\sigma_r\bar{Y}_r\Bigg[1 &- \frac{1}{2}\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2 - 1) + \frac{1}{2}\sum_{r=1}^{k}a_{r,n}\bar{Y}_r^2 \\ &+ \frac{3}{8}\left\{\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2 - 1)\right\}^2 \\ &+ \frac{3}{4}\left\{\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2 - 1)\right\}\sum_{r=1}^{k}a_{r,n}\bar{Y}_r^2\Bigg] + O_p(n^{-2}). \end{aligned} \tag{7.3}$$

In order to obtain the Edgeworth expansion of T_{RSS} given in Theorem 2.1, we first need to derive asymptotic expansions for the first three cumulants of T_{RSS}, which are given in the following lemma. □

Lemma 7.1 *Under the assumptions of Theorem 2.1, we have:*

(1) $E(T_{RSS}) = -\frac{1}{2}n^{-1/2}\left(\sum_{r=1}^{k}\frac{\sigma_r^2}{\lambda_r}\right)^{-3/2}\sum_{r=1}^{k}\frac{\gamma_r}{\lambda_{r,n}^2} + O(n^{-3/2})$,

(2) $E(T_{RSS}^2) = 1 + 2\left(\sum_{r=1}^{k}\frac{\sigma_r^2}{\lambda_r}\right)^{-3}\left[\sum_{r=1}^{k}\frac{\gamma_r}{\lambda_{r,n}^2}\right]^2 + \left(\sum_{r=1}^{k}\frac{\sigma_r^2}{\lambda_r}\right)^{-2}\left[\sum_{r=1}^{k}\frac{3\sigma_r^4}{\lambda_{r,n}^3} + \sum_{r=1}^{k}\sum_{r'>r}\frac{\sigma_r^2\sigma_{r'}^2}{\lambda_r^2\lambda_{r'}^2}(\lambda_r+\lambda_{r'})\right] + O(n^{-2})$,

(3) $E(T_{RSS}^3) = -\frac{7}{2}n^{-1/2}\left(\sum_{r=1}^{k}\frac{\sigma_r^2}{\lambda_r}\right)^{-3/2}\sum_{r=1}^{k}\frac{\gamma_r}{\lambda_{r,n}^2} + O(n^{-3/2})$.

Proof of (1) Note that $\lambda_{r,n} \to \lambda_r$, and so $\lambda_{r,n}^{-1} = O(1)$ and $a_{r,n} = O(1)$. Now, $a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2 - 1) = a_{r,n}m_r^{-1/2}O_p(1) = a_{r,n}(\lambda_{r,n}^{-1})^{1/2}O_p(n^{-1/2}) = O_p(n^{-1/2})$, and similarly we have $\bar{Y}_r^2 = O_p(n^{-1})$. These facts imply that

$$\left\{\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2 - 1)\right\}\sum_{r=1}^{k}a_{r,n}\bar{Y}_r^2 = O_p(n^{-3/2}).$$

From (7.3), we have

$$\begin{aligned}E(T_1) = &-\frac{1}{2}E\left[\sqrt{n}\sum_{r=1}^{k}\sigma_r\bar{Y}_r\left\{\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2-1)\right\}\right]\\ &+\frac{1}{2}E\left[\sqrt{n}\sum_{r=1}^{k}\sigma_r\bar{Y}_r\sum_{r=1}^{k}a_{r,n}\bar{Y}_r^2\right]\\ &+\frac{3}{8}E\left[\sqrt{n}\sum_{r=1}^{k}\sigma_r\bar{Y}_r\left\{\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2-1)\right\}^2\right] + O(n^{-3/2}). \quad (7.4)\end{aligned}$$

Since $E(Y_{(r'),i}(Y_{(r),i}^2 - 1)) = 0$ for $r \neq r'$, we have

$$\begin{aligned}E\left[\sqrt{n}\sum_{r=1}^{k}\sigma_r\bar{Y}_r\left\{\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2-1)\right\}\right] &= \sqrt{n}\sum_{r=1}^{k}\frac{\sigma_r a_{r,n}}{m_r^2}\sum_{i=1}^{m_r}E(Y_{i,r}^3)\\ &= n^{-1/2}a_n^{-1}\sum_{r=1}^{k}\frac{\gamma_r}{\lambda_{r,n}^2}, \quad (7.5)\end{aligned}$$

where the last part follows from the fact that $a_n^{-1} = (\frac{\sigma_r^2}{\lambda_{r,n}})^{-1} a_{r,n}$ and $E\{Y_{(r),i}^3\} = \sigma_r^{-3}\gamma_r$. Using arguments similar to those above, we have

$$E\left[\sqrt{n}\sum_{r=1}^{k}\sigma_r\bar{Y}_r\sum_{r=1}^{k}a_{r,n}\bar{Y}_r^2\right] = \sqrt{n}\sum_{r=1}^{k}a_{r,n}\sigma_r E(\bar{Y}_r^3) = n^{-3/2}\sum_{r=1}^{k}\frac{a_{r,n}\sigma_r}{\lambda_{r,n}}E(Y_r^3)$$
$$= O(n^{-3/2}), \tag{7.6}$$

since $\lambda_{r,n}^{-1} = O(1)$ and $a_{r,n} = O(1)$. For notational simplification, set $U_{(r),i} = Y_{(r),i}^2 - 1$. Then,

$$E\left[\sqrt{n}\sum_{r=1}^{k}\sigma_r\bar{Y}_r\left\{\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2-1)\right\}^2\right] = \sqrt{n}E\left[\sum_{i,j,l=1}^{k}\sigma_i a_{j,n}a_{l,n}\bar{Y}_i\bar{U}_j\bar{U}_l\right]$$
$$= \sqrt{n}E\left[\sum_{r=1}^{k}\sigma_r a_{r,n}^2\bar{Y}_r\bar{U}_r^2\right], \tag{7.7}$$

where the last part follows from the fact that for other choices of (i, j, l), $E(\bar{Y}_i\bar{U}_j\bar{U}_l) = 0$. Further, we have

$$E\left[\sum_{r=1}^{k}\sigma_r a_{r,n}^2\bar{Y}_r\bar{U}_r^2\right] = \sum_{r=1}^{k}\sigma_r a_{r,n}^2\frac{1}{m_r^2}E\left(Y_{(r)}(Y_{(r)}-1)^2\right) = O(n^{-2}).$$

Now, from (7.7), we have

$$E\left[\sqrt{n}\sum_{r=1}^{k}\sigma_r\bar{Y}_r\left\{\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2-1)\right\}^2\right] = O(n^{-3/2}). \tag{7.8}$$

Hence, Eqs. (7.4)–(7.6) and (7.8) imply that

$$E(T_{RSS}) = a_n^{-1/2} E(T_1) = -\frac{n^{-1/2}}{2}\left(\sum_{r=1}^{k} \frac{\sigma_r^2}{\lambda_{r,n}}\right)^{-3/2} \sum_{r=1}^{k} \frac{\gamma_r}{\lambda_{r,n}^2} + O(n^{-3/2}).$$

□

Proof of (3): From (7.3), we have

$$\begin{aligned} E(T_1^3) &= n^{3/2} E\left[\left(\sum_{r=1}^{k} \sigma_r \bar{Y}_r\right)^3 \left[1 - \frac{3}{2}\sum_{r=1}^{k} a_{r,n} m_r^{-1} \sum_{i=1}^{m_r} (Y_{(r),i}^2 - 1) - \frac{3}{2}\sum_{r=1}^{k} a_{r,n} \bar{Y}_r^2 \right.\right. \\ &\quad \left.\left. - \frac{9}{8}\left\{\sum_{r=1}^{k} a_{r,n} m_r^{-1} \sum_{i=1}^{m_r} (Y_{(r),i}^2 - 1)\right\}^2\right] + O(n^{-3/2}) \qquad (7.9) \\ &= n^{3/2}\left[A - \frac{3}{2}B - \frac{3}{2}C - \frac{9}{8}D\right] + O(n^{-3/2}), \end{aligned}$$

where

$$A = E\left(\sum_{r=1}^{k} \sigma_r \bar{Y}_r\right)^3, \quad B = E\left\{\left(\sum_{r=1}^{k} \sigma_r \bar{Y}_r\right)^3 \sum_{r=1}^{k} a_{r,n} m_r^{-1} \sum_{i=1}^{m_r} (Y_{(r),i}^2 - 1)\right\}, \quad C = E\left\{\left(\sum_{r=1}^{k} \sigma_r \bar{Y}_r\right)^3 \sum_{r=1}^{k} a_{r,n} \bar{Y}_r^2\right\}$$

and

$$D = E\left[\left(\sum_{r=1}^{k} \sigma_r \bar{Y}_r\right)^3 \left\{\sum_{r=1}^{k} a_{r,n} m_r^{-1} \sum_{i=1}^{m_r} (Y_{(r),i}^2 - 1)\right\}^2\right].$$

Now,

$$\begin{aligned} A &= E\left(\sum_{r=1}^{k} \sigma_r^3 \bar{Y}_r^3 + 3\sum_{r=1}^{k} \sigma_r \bar{Y}_r \sum_{r' \neq r = 1}^{k} \sigma_{r'}^2 \bar{Y}_{r'}^2 + \sum_{r \neq r' \neq r'' = 1}^{k} \sigma_r \sigma_{r'} \sigma_{r''} \bar{Y}_r \bar{Y}_{r'} \bar{Y}_{r''}\right) \\ &= \sum_{r=1}^{k} \sigma_r^3 m_r^{-2} E(Y_{(r)}^3) = n^{-2} \sum_{r=1}^{k} \frac{\gamma_r}{\lambda_{r,n}^2}. \qquad (7.10) \end{aligned}$$

Let us recall $U_{(r),i} = Y_{(r),i}^2 - 1$, and so $\bar{U}_r = m_r^{-1} \sum_{i=1}^{m_r} (Y_{(r),i}^2 - 1)$. Now,

$$\begin{aligned} B &= E\left\{\left(\sum_{r=1}^{k} \sigma_r \bar{Y}_r\right)^3 \sum_{r=1}^{k} a_{r,n} \bar{U}_r\right\} \\ &= E\left(\sum_{r=1}^{k} \sigma_r^3 a_{r,n} \bar{Y}_r^3 \bar{U}_r\right) + 3\sum_{r=1}^{k} \sigma_r a_{r,n} E\left(\bar{Y}_r \bar{U}_r\right) \sum_{r' \neq r = 1}^{k} \frac{\sigma_{r'}^2}{m_{r'}}. \qquad (7.11) \end{aligned}$$

Again, after some algebraic manipulations, we obtain

$$
\begin{aligned}
E\left(\bar{Y}_r^3\bar{U}_r\right) &= m_r^{-4}\sum_{i=j=l=q=1} E\left\{Y_{(r),i}Y_{(r),j}Y_{(r),l}(Y_{(r),q}^2-1)\right\}\\
&= m_r^{-3}E(Y_{(r)}^5)+3\frac{(m_r-1)}{m_r^3}E(Y_{(r)}^3)\\
&= 3n^{-2}\frac{\sigma_r^{-3}\gamma_r}{\lambda_{r,n}^2}+O(n^{-3}) \qquad (7.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{r=1}^{k}\sigma_r a_{r,n}E\left(\bar{Y}_r\bar{U}_r\right)\sum_{r'\neq r=1}^{k}\frac{\sigma_{r'}^2}{m_{r'}} &= \sum_{r=1}^{k}\sigma_r a_{r,n}\frac{\sigma_r^{-3}\gamma_r}{m_r}\sum_{r'\neq r=1}^{k}\frac{\sigma_{r'}^2}{m_{r'}}\\
&= n^{-2}\sum_{r=1}^{k}a_{r,n}\frac{\sigma_r^{-2}\gamma_r}{\lambda_{r,n}}\sum_{r'\neq r=1}^{k}\frac{\sigma_{r'}^2}{\lambda_{r',n}}. \qquad (7.13)
\end{aligned}
$$

From Eqs. (7.11)–(7.13), we obtain

$$
\begin{aligned}
B &= 3n^{-2}\left(\sum_{r=1}^{k}a_{r,n}\frac{\gamma_r}{\lambda_{r,n}^2}+\sum_{r=1}^{k}a_{r,n}\frac{\sigma_r^{-2}\gamma_r}{\lambda_{r,n}^2}\sum_{r'\neq r=1}^{k}\frac{\sigma_{r'}^2}{\lambda_{r'}}\right)+O(n^{-3}) = 3n^{-2}\sum_{r=1}^{k}\frac{\gamma_r}{\lambda_{r,n}^2}\\
&\quad +O(n^{-3}), \qquad (7.14)
\end{aligned}
$$

where the last part follows from the fact that $\sum_{r=1}^{k}a_{r,n}=1$. Similarly, it can be shown that $C=O(n^{-3})$ and $D=O(n^{-3})$. Hence, Eqs. (7.9), (7.10) and (7.14) imply

$$
E(T_{RSS}^3)=a_n^{-3/2}E(T_1^3)=-n^{-1/2}\frac{7}{2}\left(\sum_{r=1}^{k}\frac{\sigma_r^2}{\lambda_{r,n}}\right)^{-3/2}\sum_{r=1}^{k}\frac{\gamma_r}{\lambda_{r,n}^2}+O(n^{-3/2}).
$$

□

Proof of (2): Now, from (7.3), we have

$$
\begin{aligned}
E(T_1^2) &= E\left[n\left(\sum_{r=1}^{k}\sigma_r\bar{Y}_r\right)^2\left\{1-\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2-1)\right\}\right]+O(n^{-1})\\
&= nE\left[\left\{\sum_{r=1}^{k}\sigma_r^2\bar{Y}_r^2+2\sum_{r'>r=1}^{k}\sigma_r\sigma_{r'}\bar{Y}_r\bar{Y}_{r'}\right\}\left\{1-\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2-1)\right\}\right]\\
&\quad +O(n^{-1}). \qquad (7.15)
\end{aligned}
$$

It is easy to prove that

$$E\left(\sum_{r'>r=1}^{k}\sigma_r\sigma_{r'}\bar{Y}_r\bar{Y}_{r'}\right)=0 \quad \text{and} \quad E\left[\left\{\sum_{r'>r=1}^{k}\sigma_r\sigma_{r'}\bar{Y}_r\bar{Y}_{r'}\right\}\right.$$
$$\left.\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2-1)\right]=0.$$

From (7.15) and after performing some tedious algebraic manipulations, we obtain

$$\begin{aligned}E(T_1^2)&=\left[nE\left(\sum_{r=1}^{k}\sigma_r^2\bar{Y}_r^2\right)-nE\left\{\sum_{r=1}^{k}\sigma_r^2\bar{Y}_r^2\right\}\left\{\sum_{r=1}^{k}a_{r,n}m_r^{-1}\sum_{i=1}^{m_r}(Y_{(r),i}^2-1)\right\}\right]\\&\quad+O(n^{-1})\\&=a_n-n^{-1}\sum_{r=1}^{k}\frac{\sigma_r^2a_{r,n}}{\lambda_r^2}(\kappa_{r,Y}+2)+O(n^{-1})=a_n+O(n^{-1}),\end{aligned}\tag{7.16}$$

where $\kappa_{r,Y}=E(Y_r^4)$. Equations (7.2) and (7.16) imply that $E(T_{RSS}^2)=1+O(n^{-1})$. Now, Lemma 7.1 together with some algebraic calculations, give the following asymptotic expansions for the first three cumulants, $\kappa_1(T_{RSS})$, $\kappa_2(T_{RSS})$ and $\kappa_3(T_{RSS})$, of T_{RSS}:

$$\kappa_1(T_{RSS})=-\frac{1}{2}n^{-1/2}\left(\sum_{r=1}^{k}\frac{\sigma_r^2}{\lambda_r}\right)^{-3/2}\sum_{r=1}^{k}\frac{\gamma_r}{\lambda_{r,n}^2}+O(n^{-3/2}),$$
$$\kappa_2(T_{RSS})=E(T^2)-E(T)^2=1+O(n^{-1})$$

and

$$\kappa_3(T_{RSS})=-\frac{7}{2}n^{-1/2}\left(\sum_{r=1}^{k}\frac{\sigma_r^2}{\lambda_r}\right)^{-3/2}\sum_{r=1}^{k}\frac{\gamma_r}{\lambda_{r,n}^2}+O(n^{-3/2}).$$

The characteristic function of T_{RSS} is

$$E(e^{iT_{RSS}})=\exp\left\{it\kappa_1(T)+\frac{1}{2}(it)^2\kappa_2(T)+\frac{1}{6}(it)^3\kappa_3(T)\right\}\tag{7.17}$$

and so substituting the above asymptotic formulae for the cumulants in (7.17) and expanding the right-hand side as

$$e^{-\frac{t^2}{2}}\left\{1+n^{-1/2}p_1(it)+O(n^{-1})\right\}$$

for polynomial p_1 and inverting the Fourier transformation (for details, see Section 2.3 of Hall 1992b), we get the Edgeworth expansion of $P(T_{RSS} \leq x)$ as given in Theorem 2.1.

The derivation of the Edgeworth expansion of $P(S_{RSS} \leq x)$ is straightforward, hence omitted. □

Proof of Theorem 3.1: To facilitate the proof of Theorem 3.1, let $S^*_{RSS} = (\bar{X}^*_{RSS} - \bar{X}_{RSS})/\hat{\tau}$ be the bootstrap version of S_{RSS}. Analogous to Theorem 2.1, we have

$$
\begin{aligned}
P\left\{S^*_{RSS} \leq x|\mathcal{X}_{RSS}\right\} &= \Phi(x) + n^{-1/2}\hat{p}_1(x)\phi(x) + O_p(n^{-1}),\\
P\left\{T^*_{RSS} \leq x|\mathcal{X}_{RSS}\right\} &= \Phi(x) + n^{-1/2}\hat{q}_1(x)\phi(x) + O_p(n^{-1}),
\end{aligned} \tag{7.18}
$$

respectively, where $\hat{p}_1(x)$ and $\hat{q}_1(x)$ are as given in Theorem 2.1 except that population moments are now replaced by sample moments. Let $t_\xi, s_\xi, \hat{t}_\xi$ and $\hat{s}_\xi$ be the ξth quantiles of $P(T_{RSS} \leq x)$, $P(S_{RSS} \leq x)$, $P(T^*_{RSS} \leq x|\mathcal{X}_{RSS})$ and $P(S^*_{RSS} \leq x|\mathcal{X}_{RSS})$, respectively. Then, the Cornish–Fisher expansions of these quantiles are given by

$$
\begin{aligned}
t_\xi &= z_\xi + n^{-1/2}q_{11}(z_\xi) + O(n^{-1}),\\
s_\xi &= z_\xi + n^{-1/2}p_{11}(z_\xi) + O(n^{-1}),\\
\hat{t}_\xi &= z_\xi + n^{-1/2}\hat{q}_{11}(z_\xi) + O_p(n^{-1}),\\
\hat{s}_\xi &= z_\xi + n^{-1/2}\hat{p}_{11}(z_\xi) + O_p(n^{-1});
\end{aligned} \tag{7.19}
$$

see the review of Cornish–Fisher expansion in Hall (1992b). The quantities $\hat{q}_{11}(x)$ and $\hat{p}_{11}(x)$ are the sample versions of $q_{11}(x) = -q_1(x)$ and $p_{11}(x) = -p_1(x)$, respectively. We now provide the proof for $P(\mu \in I_{1,BC_a})$. Equation (7.18) implies that

$$
\begin{aligned}
\hat{G}_{RSS}(\bar{X}_{RSS}) &= P\left\{S^*_{RSS} \leq 0|\mathcal{X}_{RSS}\right\}\\
&= \Phi(0) + n^{-1/2}\hat{p}_1(0)\phi(0) + O_p(n^{-1}).
\end{aligned} \tag{7.20}
$$

Therefore, from Eqs. (3.1) and (7.20), we have

$$
\begin{aligned}
\hat{d} &= \Phi^{-1}\left\{\Phi(0) + n^{-1/2}\hat{p}_1(0)\phi(0) + O_p(n^{-1})\right\}\\
&= n^{-1/2}\hat{p}_1(0) + O_p(n^{-1}),
\end{aligned} \tag{7.21}
$$

with the last line following from Taylor series expansion. Equation (7.19) implies that

$$
\begin{aligned}
\hat{u}_{l_{\hat{a}}}(\alpha) = \bar{X}_{RSS} + \hat{s}_{l_{\hat{a}}(\alpha)}\hat{\tau} = \bar{X}_{RSS} + \hat{\tau}\left\{z_{l_{\hat{a}}(\alpha)} + n^{-1/2}\hat{p}_{11}(z_{l_{\hat{a}}(\alpha)}) + n^{-1}\hat{p}_{21}(z_{l_{\hat{a}}(\alpha)})\right\}\\
+O_p(n^{-3/2}).
\end{aligned} \tag{7.22}
$$

Equations (3.2) and (7.20) imply that

$$z_{l_{\hat{a}}(\alpha)}2\hat{d} + z_\alpha + \hat{a}(\hat{d}^2 + 2\hat{d}z_\alpha + z_\alpha^2) + O_p(n^{-1}) = z_\alpha + n^{-1/2}(2\hat{p}_1(0) + \hat{c}z_\alpha^2) + O_p(n^{-1}), \tag{7.23}$$

where $\hat{a} = n^{-1/2}\hat{c}$ and $\hat{c} = \hat{\eta}_1^{-3/2}\hat{\eta}_2$. From (7.21) and (7.22), upon using Taylor series expansion, we get

$$\begin{aligned}\hat{u}_{l_{\hat{a}}}(\alpha) &= \bar{X}_{RSS} + \hat{\tau}\left[z_\alpha + n^{-1/2}\{2\hat{p}_1(0) + \hat{c}z_\alpha^2 + \hat{p}_{11}(z_\alpha)\}\right] + O_p(n^{-1}) \\ &= \bar{X}_{RSS} + \hat{\tau}[z_\alpha + n^{-1/2}\hat{q}_1(z_\alpha)] + O_p(n^{-1}),\end{aligned} \tag{7.24}$$

since $2\hat{p}_1(0) + \hat{c}z_\alpha^2 + \hat{p}_{11}(z_\alpha) = \hat{q}_1(z_\alpha)$. So, from (7.24), we have

$$\begin{aligned}P(\mu \in I^*_{1,BC_a}) &= P\{\hat{\tau}^{-1}(\bar{X}_{RSS} - \mu) + O_p(n^{-1}) \geq -z_\alpha - n^{-1/2}q_1(z_\alpha)\} \\ &= P\{\hat{\tau}^{-1}(\bar{X}_{RSS} - \mu) \geq z_{1-\alpha} - n^{-1/2}q_1(z_{1-\alpha})\} + O(n^{-1}),\end{aligned}$$

where the last line is obtained by the delta method (see Section 2.7, Hall 1992b) and by using the fact that $-z_\alpha = z_{1-\alpha}$. From Theorem 2.1, we then get

$$\begin{aligned}P(\mu \in I^*_{1,BC_a}) &= 1 - \left[P\left\{\hat{\tau}^{-1}(\bar{X}_{RSS} - \mu) \leq z_{1-\alpha} - n^{-1/2}q_1(z_{1-\alpha})\right\} + O(n^{-1})\right] \\ &= 1 - 1 + \alpha + O(n^{-1}) \\ &= \alpha + O(n^{-1}).\end{aligned}$$

In a similar manner, we can show that $P(\mu \in I^*_{0,BC_a}) = P(\mu \in I^*_{2,BC_a}) = \alpha + O(n^{-1})$. □

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