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Copula parameter estimation by maximum-likelihood and minimum-distance estimators: a simulation study

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Abstract The purpose of this paper is to present a comprehensive Monte Carlo simulation study on the performance of minimum-distance (MD) and maximum-likelihood (ML) estimators for bivariate parametric copulas. In particular, I consider Cramér-von-Mises-, Kolmogorov-Smirnov- and L_1-variants of the CvM-statistic based on the empirical copula process, Kendall's dependence function and Rosenblatt's probability integral transform. The results presented in this paper show that regardless of the parametric form of the copula, the sample size or the location of the parameter, maximum-likelihood yields smaller estimation biases at less computational effort than any of the MD-estimators. The MD-estimators based on copula goodness-of-fit metrics, on the other hand, suffer from large biases especially when used for estimating the parameters of archimedean copulas. Moreover, the results show that the bias and efficiency of the minimum-distance estimators are strongly influenced by the location of the parameter. Conversely, the results for the maximum-likelihood estimator are relatively stable over the parameter interval of the respective parametric copula.

Keywords Copulas · Minimum-distance method · Simulation · L_1-variant · Maximum likelihood

1 Introduction

Copula models have become a major tool in statistics for modelling and analysing dependence structures between random variables due to the fact that in contrast to

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linear correlation a copula captures the complete dependence structure inherent in a set of random variables (Embrechts et al. 2002). Particularly in finance, copulas have attracted much attention in the analysis of contagion between financial markets (Rodriguez 2007; Chen and Poon 2007), the analysis of risky portfolios of stocks (Malevergne and Sornette 2003; Junker and May 2005) or the modelling of credit default (Li 2000). Copula Parameter estimation in these studies is usually performed by a fully parametric (ML), stepwise parametric (the so called inference function for margins or IFM method) or semi-parametric maximum-likelihood approach depending on the information on the marginal distributions. Kim et al. (2007) show in a recent simulation study that the semiparametric pseudo-maximum-likelihood (PML) approach in which the marginal distributions are substituted by their empirical counterparts with the copula parameters being subsequently estimated via maximum-likelihood is much better suited for parameter estimation than the fully or stepwise parametric approach.

A parallel branch of research has concentrated on deriving goodness-of-fit test statistics in a copula setting (Savu and Trede 2008; Fermanian 2005; Kole et al. 2007; Genest et al. 2009) and comparing these GoF-statistics with each other (Berg 2009). While early approaches for copula model selection were based on AIC or variants of it (Breymann et al. 2003; Chen and Fan 2006; Huard et al. 2006) recent GoF-tests are based on some comparison of the hypothesised parametric copula and Deheuvels' empirical copula which converges uniformly to the true underlying copula (Deheuvels 1978, 1981).

In contrast to the different ML-estimators, minimum-distance (MD) estimators for copulas have attracted only little attention. In one of the few papers, Biau and Wegkamp (2005) derive an upper bound for the minimum L_1-distance estimate for parametric copula densities. Tsukahara (2005) explores the empirical asymptotic behaviour of Cramér-von-Mises (CvM) and Kolmogorov-Smirnov (KS) distances between the hypothesised and empirical copula in a simulation study. He finds that the PML-estimator should be preferred to the MD-Kolmogorov-Smirnov estimator. His analysis, however, is only based on a sample size of 100 and three choices of parameters and does not include the Gaussian and Student's t copula which are of particular interest in finance. In a related paper, Mendes et al. (2007) derive weighted minimum-distance estimators based on the empirical copula process. In their simulation study they show that these MD-estimators are robust against contaminations of the data. A common feature of these studies is, however, that they all consider only MD-estimators based on the empirical copula process.

The purpose of this paper is to present a comprehensive simulation study on the performance of minimum-distance and maximum-likelihood estimators for bivariate parametric copulas. For five popular parametric copulas, classical maximum-likelihood is compared to a total of nine different minimum-distance estimators. In particular, I consider CvM-, KS- and L_1-variants of the CvM-statistic based on the empirical copula process, Kendall's dependence function and Rosenblatt's probability integral transform. While the importance of copula selection and misspecification of the copula (Durrleman et al. 2000; Ané and Kharoubi 2003) as well as goodness-of-fit tests have been intensely discussed (Fermanian 2005; Genest et al. 2009), the empirical properties of estimators for copula parameters have not been considered in detail in the literature. In this paper, I focus on assessing the estimation error caused purely by estimating the copula parameters without the further bias that is caused by any particular choice of marginal model. Consequently, no specific model has to be imposed on the marginal distributions and the random samples are used directly as arguments of the likelihood or distance functions.

This paper is closely related to the works of Tsukahara (2005) and Mendes et al. (2007), though I extend their simulation studies in a number of important ways: Firstly, the parametric copulas considered in this study include both elliptical and archimedean copulas in order to cover a broader range of dependence structures. Secondly, I compare the estimators' performance for different sample sizes in order to assess their asymptotic empirical properties. Thirdly, a close-meshed grid of parameter values is used for simulating samples from the copulas in order to assess the influence of the location of the true parameter on the estimators' bias. Forthly and most importantly, in addition to the minimum-distance estimator based on the empirical copula process which is used in Tsukahara (2005) and Genest et al. (2009), I include two further minimum-distance estimators based on Rosenblatt's transform and Kendall's dependence function in my analysis. Furthermore, not only do I consider Cramér-von-Mises- and Kolmogorov-Smirnov distances but also L_1-variants of the Cramér-von-Mises statistic (Schmid and Trede 1996) for a general description of this L_1-variant and Biau and Wegkamp (2005).

The results presented in this paper show that regardless of the parametric form of the copula, the sample size or the location of the parameter, maximum-likelihood yields smaller estimation biases at less computational effort than any of the MD-estimators. The MD-estimators based on copula goodness-of-fit metrics, on the other hand, suffer from large biases especially when used for estimating the parameters of archimedean copulas.

The remainder of this article is structured as follows. Section 2 discusses the different parametric and nonparametric copulas as well as the parameter estimation procedures. The simulation procedure is outlined in Sect. 3. In Sect. 3.2 results are presented. Concluding remarks are given in Sect. 4.

2 Preliminary copula theory

The purpose of this section is to shortly restate the basic definitions of the parametric copulas used later on and to outline the different parameter estimation techniques. In short, copulas allow for the coupling of marginal distributions to their joint distribution by estimating the marginals and the dependence structure separately.

2.1 Parametric copulas

The mathematical basis for the analysis of copulas was founded by Sklar (1959) and Hoeffding (1940). In the following, a basic definition of a copula and Sklar's theorem are described [for a more detailed description of copulas see Nelsen (2006) or Joe (1997)].

Consider a random vector $\mathbf{X} \equiv (X_1, \ldots, X_d)^t$ of dimension d with a joint cumulative distribution function (cdf) G and marginal cdfs $F_1, \ldots, F_d$. A d-dimensional *copula* is a d-variate cumulative distribution function $C : [0; 1]^d \rightarrow [0; 1]$ with uniformly distributed marginals (hereafter called d-*copula*). The central result in copula theory is Sklar's theorem which ensures the existence of a unique copula under relatively weak conditions:

Theorem 1 (Sklar) *Let G be a joint cumulative distribution function with d marginals F_i. Then there exists a d-dimensional Copula C such that for all $x \in \bar{\mathbb{R}}^d$,*

$$G(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)). \tag{1}$$

If all marginals F_i are continuous, then the Copula C is unique.

Vice versa, if a d-Copula C and d cumulative distribution fuctions F_i are given then (1) *yields a d-variate cumulative distribution function with marginals F_i.*

Prominent examples of copulas are the ones inherent in multivariate Gaussian and Student's t-distributions. The Gaussian copula is given by the cdf

$$C_d^{\Phi}(\mathbf{u}; \Sigma) = \Phi_{\Sigma}^{(d)}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)) \tag{2}$$

with $\mathbf{u} \equiv (u_1, u_2, \ldots, u_d)^t \in [0; 1]^d$. It can be obtained by applying the inversion method on a d-variate standard Gaussian distribution $\Phi^{(d)}$ with correlation matrix Σ and d univariate standard Gaussian distributions as marginals (Nelsen 2006). The Gaussian copula is tail independent for imperfectly correlated marginals (Sibuya 1960; Resnick 1987).

Similarly as the Gaussian copula can be derived from a multivariate Gaussian distribution, the t-copula can be obtained from a (non-singular) n-dimensional Student's t-distribution $\mathcal{T}_d(\boldsymbol{\mu}; \Omega; \nu)$ with density

$$f(\mathbf{x}) = \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\pi\nu)^d|\Omega|}} \left(1 + \frac{(\mathbf{x}-\boldsymbol{\mu})'\Omega^{-1}(\mathbf{x}-\boldsymbol{\mu})}{\nu}\right)^{\frac{-\nu+d}{2}}, \tag{3}$$

ν degrees of freedom, mean vector $\boldsymbol{\mu}$ and dispersion matrix Ω [note that the dispersion matrix does not equal the covariance matrix in this case, see Demarta and McNeil (2005)]. As copulas are invariant under strictly increasing transformations of the marginals, we can obtain the t-copula from the standardised d-dimensional t-distribution $\mathcal{T}_d(\mathbf{0}; \Sigma; \nu)$ yielding

$$C_d^{\mathcal{T}}(\mathbf{u}; \nu; \Sigma) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\pi\nu)^d|\Sigma|}} \left(1 + \frac{\mathbf{x}'\Sigma^{-1}\mathbf{x}'}{\nu}\right)^{-\frac{\nu+d}{2}} dx, \tag{4}$$

with t_ν^{-1} being the inverted cdf of a standard univariate Student's t-distribution with ν degrees of freedom. The t-copula is symmetrically tail dependent and converges to the Gaussian copula for $\nu \rightarrow \infty$. This characteristic will be of particular interest in the simulation study, where the different estimation techniques' ability to distinguish between the Gaussian the Student's t-copula will be analysed.

Another symmetrically tail independent copula that will be implemented in the simulation study is the Frank copula given by

$$C_n^{\mathcal{F}}(\mathbf{u};\delta) = -\frac{1}{\delta}\log\left(1 + \frac{\prod_{i=1}^{n}(\exp(-\delta u_i) - 1)}{(\exp(-\delta) - 1)^{n-1}}\right), \tag{5}$$

with parameter $\delta \in \mathbb{R}^+$ [for some properties of the bivariate Frank copula see Genest et al. (1986)].

The aforementioned copulas exhibit tail independence (Gaussian and Frank) and symmetric tail dependence (Student's t), respectively. For the purpose of capturing different patterns of tail dependence, the Gumbel copula which is asymmetrically tail dependent (upper tail dependence and lower tail independence) shall be considered in the simulation study as well. Its cdf is given by

$$C_d^{\mathcal{G}}(\mathbf{u};\lambda) = \exp\left[-\left(\sum_{i=1}^{d} -(\log u_i)^{\lambda}\right)^{\frac{1}{\lambda}}\right], \tag{6}$$

where the parameter λ satisfies $\lambda \geq 1$.

The last parametric copula exhibiting lower tail dependence that will be considered in the simulation study is the Clayton copula [sometimes also called the Cook-Johnson or Pareto copula, see (Genest and Mackay 1986; Hutchinson and Lai 1990)]. The Clayton copula is given by

$$C_d^{\mathcal{C}}(\mathbf{u};\theta) = (u_1^{-\theta} + \cdots + u_d^{-\theta} - d + 1)^{-1/\theta}, \tag{7}$$

with $\theta \geq 0$ with the independence copula being the limiting case for $\theta \to 0$.

In the following, the five different estimators for the copula parameters are discussed.

2.2 Parameter estimation via maximum-likelihood

Parameter estimation via maximum-likelihood usually requires a parametric or nonparametric approximation of the marginal distributions of X. As the random variates used in this study are simulated directly from the respective copula, maximum-likelihood estimation of the copula parameters is based solely on the copula density [the same approach is used when data are first transformed into a pseudo-sample of the copula in canonical maximum-likelihood, see McNeil et al. (2005)].

In the following, random samples in the unit square will be indicated by upper case letters while arbitrary vectors in the unit square will be denoted with lower case letters. The ML-estimate of the parameter $\hat{\theta}_n^{ML}$ estimated from a sample $\mathbf{U} \equiv \left(U_{ij}\right)$ of size n ($i = 1, 2, \ldots, n$; $j \in \{1; 2\}$; for ease of simplicity, I only consider the bivariate case) is computed by numerically maximising

$$\mathcal{L}_{\mathbf{U}}(\theta) = \sum_{i=1}^{n} \log\ c(U_{i,1}, U_{i,2}|\theta) \tag{8}$$

with $\theta \in \Theta \subset \mathbb{R}^p$ ($p \geq 1$) being the parameter space of the respective bivariate parametric copula C, $\mathbf{U_i} \equiv (U_{i,1}, U_{i,2})^t \in [0; 1]^2$ being the i-th sample from the copula and c being the copula's density parameterised by θ given by

$$c(u_1, u_2|\theta) = \frac{\partial C(u_1, u_2|\theta)}{\partial u_1 \cdot \partial u_2},\ u_1, u_2 \in [0; 1]. \tag{9}$$

The ML-estimator thus is given by

$$\hat{\theta}_n^{ML}(\mathbf{U}) \equiv \arg\max_{\theta \in \Theta} \mathcal{L}_{\mathbf{U}}(\theta) \tag{10}$$

Note that all parametric copulas discussed so far have densities and the densities of the copulas which are given only implicitly (like e.g. the Gaussian copula) can be derived by differentiating the joint cdf with the inverse marginal cdfs as its arguments.

The ML-estimator is consistent and asymptotically normal under some regularity conditions (Genest et al. 1995).

2.3 Minimum-distance estimators

In the following, the different minimum-distance estimators (and the closely related goodness-of-fit test statistics) will be presented.

2.3.1 Minimum-distance estimators based on the empirical copula process

Most of the goodness-of-fit tests related to copulas that have been proposed recently are based on a comparison between Deheuvels' empirical and the hypothesised parametric copula. The bivariate empirical copula estimated from an i.i.d. sample of size n is defined on the lattice

$$L = \left\{ \left(\frac{i_1}{n}, \frac{i_2}{n} \right)^t \in [0; 1]^2 \middle| i_1, i_2 = 0, \ldots, n; \right\}. \tag{11}$$

as

$$C_n(\mathbf{u}) \equiv \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(U_{i,1} \leq u_1, U_{i,2} \leq u_2,), \quad \mathbf{u} \equiv (u_1, u_2)^t \in [0; 1]^2 \tag{12}$$

with $\mathbf{1}(\cdot)$ being a logical indicator function (Genest et al. 2009). Equivalently, we can construct C_n from the 2-dimensional empirical cdf $\mathbb{H}_n$ of a random vector X and its univariate marginals $\mathbb{F}_{1,n}$ and $\mathbb{F}_{2,n}$ by the use of

$$C_n(\mathbf{u}) = \mathbb{H}_n(\mathbb{F}_{1,n}^{-1}(u_1), \mathbb{F}_{2,n}^{-1}(u_2)). \tag{13}$$

with $\mathbb{H}_n$ and $\mathbb{F}_{i,n}$ ($i \in \{1; 2\}$) all being estimated from an i.i.d. sample of size n (Fermanian et al. 2004).

Analogous to a histogram approximating a cdf, the empirical copula constitutes a discontinuous approximation to the true underlying copula to which it converges uniformly (Deheuvels 1978, 1981). As pointed out by Genest et al. (2009), Deheuvels' empirical copula is arguably the most objective approximation to the true underlying copula as it is completely nonparametric. Not surprisingly, goodness-of-fit tests that are based on computing a distance between the empirical and the hypothesised copula have been shown empirically to perform well in power studies (Genest et al. 2009).

The first type of minimum-distance estimators that will be considered in this work is based on the empirical process

$$\mathbb{C}_n \equiv \sqrt{n}(C_n - C_{\hat{\theta}}) \tag{14}$$

where C_n is Deheuvels' empirical copula and $C_{\hat{\theta}}$ is the hypothesised copula from a parametric family parameterised by the parameter estimate $\hat{\theta}$ obtained from the sample $\mathbf{U}$. The process $\mathbb{C}_n$ is briefly mentioned in Fermanian (2005) and used in detail by Tsukahara (2005); Mendes et al. (2007); Berg (2009) and Genest et al. (2009). Simple Cramér-von-Mises and Kolmogorov-Smirnov statistics based on $\mathbb{C}_n$ are given by

$$\rho_{emp}^{CvM} \equiv \int\limits_{[0;1]^2} \mathbb{C}_n(\mathbf{u})^2 dC_n(\mathbf{u}) \;\text{ and }\; \rho_{emp}^{KS} \equiv \sup_{\mathbf{u}\in[0;1]^2} |\mathbb{C}_n(\mathbf{u})|\,. \tag{15}$$

The empirical version of ρ_1^{CvM} is e.g. given by (Genest et al. 2009)

$$\hat{\rho}_{emp}^{CvM}(\mathbf{U};\theta) \equiv \sum_{i=1}^{n} \left\{C_n(\mathbf{U_i}) - C_{\hat{\theta}}(\mathbf{U_i})\right\}^2 \tag{16}$$

with $\mathbf{U}_i$ being the i-th sample (similarly, an empirical approximation to ρ_1^{KS} can be derived).

In addition to the usual Cramér-von-Mises statistic ρ_{emp}^{CvM} and the Kolmogorov-Smirnov statistic ρ_{emp}^{KS}, I consider the following L_1-variant of the Cramér-von-Mises statistic

$$\rho_{emp}^{L_1} \equiv \sqrt{n} \int\limits_{[0;1]^2} |\mathbb{C}_n(\mathbf{u})| dC_n(\mathbf{u}). \tag{17}$$

A general version of this statistic was proposed by Schmid and Trede (1996) for arbitrary continuous cdfs. Note that similarly to the Cramér-von-Mises and

Kolmogorov-Smirnov statistic, the L_1-variant $\rho_{emp}^{L_1}$ is a continuous functional of the empirical process $\mathbb{C}_n$. In the simulation study, I consider the empirical approximation

$$\hat{\rho}_{emp}^{L_1}(\mathbf{U};\theta) = \sum_{i=1}^{n} \left| C_n(\mathbf{U_i}) - C_{\hat{\theta}}(\mathbf{U_i}) \right|. \tag{18}$$

The minimum distance estimators are then given by

$$\hat{\theta}_n^{emp,L_1}(\mathbf{U}) \equiv \arg\min_{\theta\in\Theta} \hat{\rho}_{emp}^{L_1}(\mathbf{U};\theta), \tag{19}$$

$$\hat{\theta}_n^{emp,CvM}(\mathbf{U}) \equiv \arg\min_{\theta\in\Theta} \hat{\rho}_{emp}^{CvM}(\mathbf{U};\theta) \text{ and} \tag{20}$$

$$\hat{\theta}_n^{emp,KS}(\mathbf{U}) \equiv \arg\min_{\theta\in\Theta} \hat{\rho}_{emp}^{KS}(\mathbf{U};\theta). \tag{21}$$

The convergence of $\mathbb{C}_n$ under appropriate regularity conditions is established in Genest and Rémillard (2008).

2.3.2 Minimum-distance estimators based on Kendall's dependence function

The second type of MD-estimators used in the simulation study was proposed in Savu and Trede (2008) and Genest et al. (2006) and is based on Kendall's probability integral transform. The specific transform for an arbitrary random vector $\mathbf{X}$ with joint cdf G and margins F_i $(i \in \mathbb{N}_d)$ is given by (Genest et al. 2009)

$$\mathbf{X} \mapsto V = G(\mathbf{X}) = C(\mathcal{U}_1, \ldots, \mathcal{U}_d), \tag{22}$$

where the joint cdf of $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_d)^t$ is C and $\mathcal{U}_i = F_i(X_i)$. Let K be the cdf of the probability integral transform V. Then a nonparametric estimation of K based on the transformed sample $V_i \equiv C_n(\mathbf{U_i})$ of size n is given by (Genest and Rivest 1993)

$$K_n(\omega) \equiv \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(V_i \leq \omega), \quad \omega \in [0;1]. \tag{23}$$

If $\mathcal{U}$ is distributed as $C_{\hat{\theta}}$, a parametric estimation of K is given by the distribution $K_{\hat{\theta}}$ of the Kendall transform $C_{\hat{\theta}}(\mathcal{U})$. Goodness-of-fit tests can then be based on the empirical process

$$\mathbb{K}_n \equiv \sqrt{n}(K_n - K_{\hat{\theta}}). \tag{24}$$

The specific test statistics are given by

$$\rho_K^{L_1} \equiv \sqrt{n} \int_0^1 |\mathbb{K}_n(\omega)|\, dK_{\hat{\theta}}(\omega), \tag{25}$$

$$\rho_K^{CvM} \equiv \int_0^1 \mathbb{K}_n(\omega)^2 dK_{\hat{\theta}}(\omega) \text{ and} \tag{26}$$

$$\rho_K^{KS} \equiv \sup_{\omega \in [0;1]} |\mathbb{K}_n(\omega)|\,. \tag{27}$$

Empirical versions of these statistics are given in appendix A. The convergence of the empirical process $\mathbb{K}_n$ underlying these estimators is established in Genest et al. (2006) under appropriate regularity conditions.

2.3.3 Minimum-distance estimators based on Rosenblatt's transform

The third type of MD-estimators is based on Rosenblatt's probability integral transform proposed by Rosenblatt (1952) which transforms a set of dependent variables into a set of independent $U([0;1])$ variables, given the multivariate distribution. For a given random vector $\mathbf{X} \equiv (X_1, X_2)^t$ with marginal cdfs $F_j(x_j)$ $(j \in \{1; 2\})$ and conditional cdf $F_{2|1}$, Rosenblatt's transform of $\mathbf{X}$ is defined by (Berg 2009) $\mathcal{R}(\mathbf{X}) \equiv (\mathcal{R}_1(X_1), \mathcal{R}_2(X_2))^t$ where

$$\mathcal{R}_1(X_1) \equiv F_1(x_1), \mathcal{R}_2(X_2) \equiv F_{2|1}(x_2|x_1). \tag{28}$$

As stated in Berg (2009) Rosenblatt's transform can be used for multivariate GoF tests by applying it to a random sample assuming a parametric null hypothesis copula. As the transformed sample $\mathbf{V} \equiv \mathcal{R}(\mathbf{X})$ is i.i.d. $U([0;1])^2$, Genest et al. (2009) propose to measure the distance between the empirical copula and the independence copula at each element of the transformed matrix $\mathbf{V}$ which is dependent on the null hypothesis copula $C_{\hat{\theta}}$. A Cramér-von-Mises statistic for this approach is then given by

$$\rho_{Ros}^{CvM} \equiv n \int_{[0;1]^2} \{C_n(\mathbf{V}) - C_\perp(\mathbf{V})\}^2\, dC_n(\mathbf{V}) \tag{29}$$

with $\mathbf{V_i}$ being the i-th transformed sample from the copula. The specific test statistics are given by

$$\rho_{Ros}^{L_1} \equiv \sqrt{n} \int_{[0;1]^2} |C_n(\mathbf{V}) - C_\perp(\mathbf{V})|\, dC_n(\mathbf{V}), \tag{30}$$

$$\rho_{Ros}^{CvM} \equiv n \int_{[0;1]^2} \{C_n(\mathbf{V}) - C_\perp(\mathbf{V})\}^2\, dC_n(\mathbf{V}) \text{ and} \tag{31}$$

$$\rho_{Ros}^{KS} \equiv \sup_{\mathbf{u}\in[0;1]^2} |C_n(\mathbf{V}) - C_\perp(\mathbf{V})|\,. \tag{32}$$

An empirical version for the Cramér-von-Mises statistic is e.g. given by

$$\hat{\rho}_{Ros}^{CvM}(\mathbf{V}) = \sum_{i=1}^{n} \{C_n(\mathbf{V_i}) - C_\perp(\mathbf{V_i})\}^2\,. \tag{33}$$

Note that the distances depend indirectly on the parameter θ through Rosenblatt's transform. The asymptotic null behaviour of the underlying empirical process and the convergence of the test statistics are established in Ghoudi and Rémillard (2004) and Genest et al. (2009).

3 Simulation study

A large-scale simulation study was conducted to compare the classical ML-estimator to the nine minimum-distance estimators described in the previous section and to assess the bias and efficiency of the three approaches. The aim of this simulation study is to compute and compare the bias, mean squared error (MSE) and relative efficiency of the three approaches. This goal is achieved by comparing the true parameter with the parameters estimated with the above mentioned strategies under the premise that the parametric form of the copula is chosen correctly.

3.1 Design of the simulation study

In the following, the design of the simulation study is outlined in detail. After the description of the simulation steps, the different choices of parameters are given.

For each bivariate parametric copula family parameterised by the (true) parameter θ repeat the following steps K times where K is some large integer:

(1) Simulate a sample ${}^t(u_1, \ldots, u_n)$ of size n from the copula C_θ.
(2) Compute the parameter estimates with the maximum-likelihood and each of the nine minimum-distance estimators under the premise that the parametric form of the copula is known.
(3) Compare the parameter estimates $\hat{\theta}$ with the true parameter vector θ by computing the bias and MSE for all estimators $\hat{\theta}_l$ with $l \in \{1; \ldots; 10\}$ and the efficiency r of each MD-estimator $\hat{\theta}_{l'}^{MD}$ with $l' \in \{1; \ldots; 9\}$ relative to the ML-estimator $\hat{\theta}^{ML}$ given by
   (a) $BIAS(\hat{\theta}_l) \equiv \eta - E(\hat{\theta}_l)$
   (b) $MSE(\hat{\theta}_l) \equiv E(\theta - \hat{\theta}_l)^2$
   (c) $r \equiv \frac{\sqrt{MSE(\hat{\theta}_{l'}^{MD})}}{\sqrt{MSE(\hat{\theta}^{ML})}}$

In the simulation study, the number of simulated samples K was chosen to be 1000. The procedure outlined above was repeated for different sample sizes n with

$n \in \{30, 40, 50, 100, 300, 500\}$ to assess the improvement in the bias and efficiency of the estimators with increasing sample size. Furthermore, the simulation procedure was repeated for a large set of parameters to analyse the influence of the location of the parameter on the performance of the estimators. To limit the computational complexity, only bivariate copulas were considered as the true copula. The parameters of the true copula were

– $\rho \in \{-0.9; -0.8; \cdots ; 0.8; 0.9\}$ for the Gaussian copula
– $\rho \in \{-0.9; -0.8; \cdots ; 0.8; 0.9\}$ for the Student's t copula with the degrees of freedom being ν $\{3; 5; 10\}$
– $\lambda \in \{1.5; 2.0; \cdots ; 9.5; 10\}$ for the Gumbel copula and
– $\theta, \delta \in \{0.5; 1.0; \cdots ; 9.5; 10\}$ for the Frank and Clayton copula.

The choice of n and K are comparable to similar studies like Nikoloulopoulos and Karlis (2008) and Kim et al. (2007). The inclusion of the (for practical problems in finance more important) Gaussian and Student's t copula as well as the more closely meshed coverages of the parameter spaces, however, are distinctive features of this study making it more comprehensive than the aforementioned studies.

All computations were performed in R version 2.6.0 on the HPC Compute Cluster of the *RWTH Aachen University* using the procedure *optimise*. For all estimates, the found optima were polished by the additional use of the function *optim*. The parameters ρ and ν of the Student's t copula were estimated in alternating steps until convergence of the optimising algorithm.

3.2 Results

The results of the simulation study are presented in Figs. 1, 2, 3, 4, 5, 6, 7, 8 and 9. The figures present the bias, MSE and the efficiency relative to the ML-estimator of the respective copula for three different sample sizes. As the results for the remaining three samples sizes ($n = 40$, $n = 50$ and $n = 300$) were in line with our expectation that the increase in sample size will improve the parameter estimation, the corresponding results were omitted from the figures for brevity. The results for the Student's t copula with 5 degrees of freedom were omitted as well as the results dit not differ from those for the two other Student's t copulas. Finally, the results concerning the time needed for computing the parameters are only summarised; the detailed results as well as the *R* workspaces are available from the author upon request.

3.2.1 Which estimator performs best?

Surprisingly, the results given in Figs. 1, 2, 3, 4, 5, 6, 7, 8 and 9 show an unequivocal picture. The ML-estimator $\hat{\theta}_n^{ML}$ yields the best results regardless of sample size, parametric copula and parameter location. Even for the smallest sample size of $n = 30$, maximum-likelihood estimation produces estimates whose bias is smaller by a factor of 10 to 100 than the bias of the best minimum-distance estimator. The same results hold true for the ML-estimator's efficiency and needed computation time.

Among the minimum-distance estimators, the results are more ambiguous. While the estimates with the lowest bias among MD-estimators for the elliptical copulas are

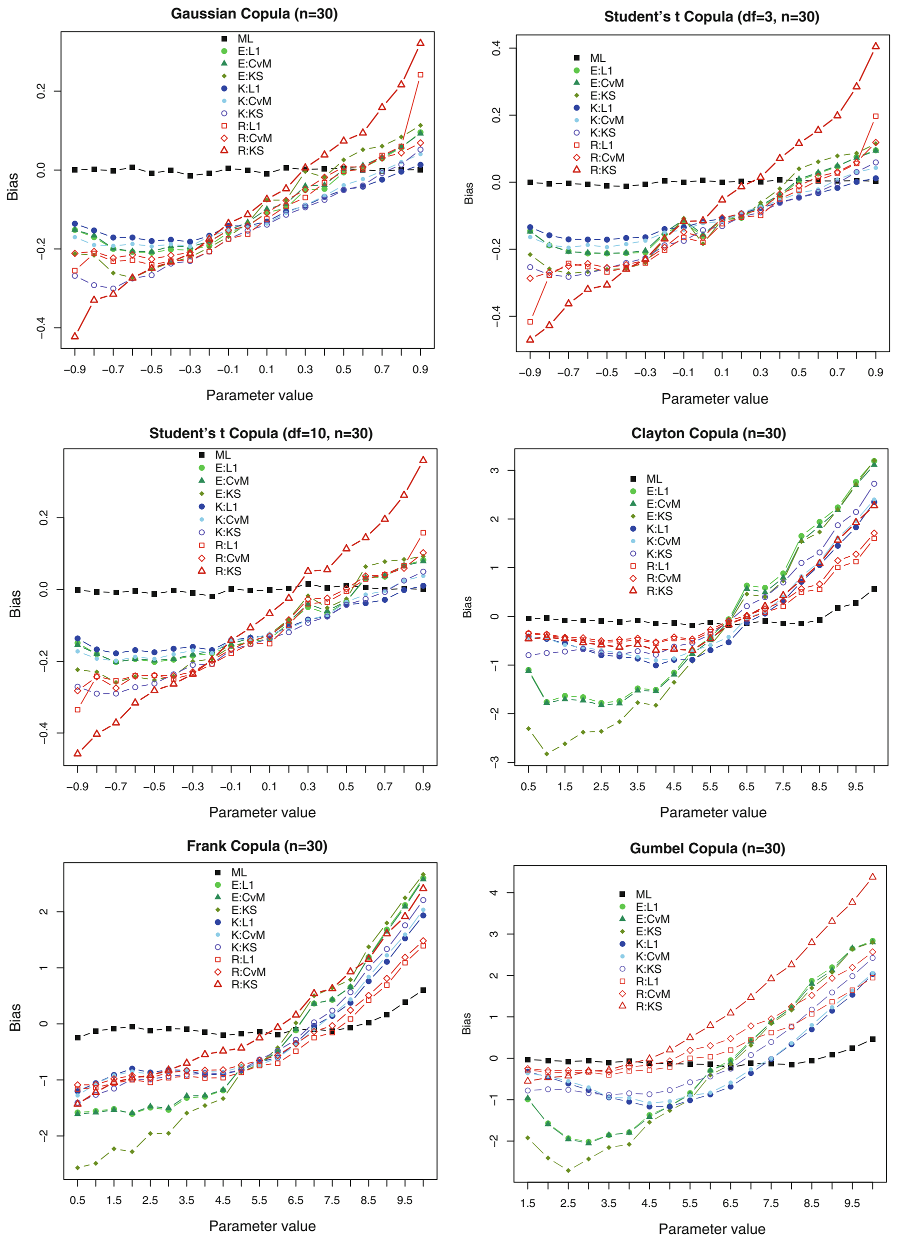

Fig. 1 Mean estimation bias for the maximum-likelihood (ML) and the minimum-distance estimators based on the empirical copula process (E:L1, E:CvM and E:KS), on Kendall's transform (K:L1, K:CvM and K:KS) and Rosenblatt's transform (R:L1, R:CvM and R:KS) for different parametric copulas and a sample size of $n = 30$ depending on the location of the true parameter under the hypothesis that the parametric copula family has been correctly specified

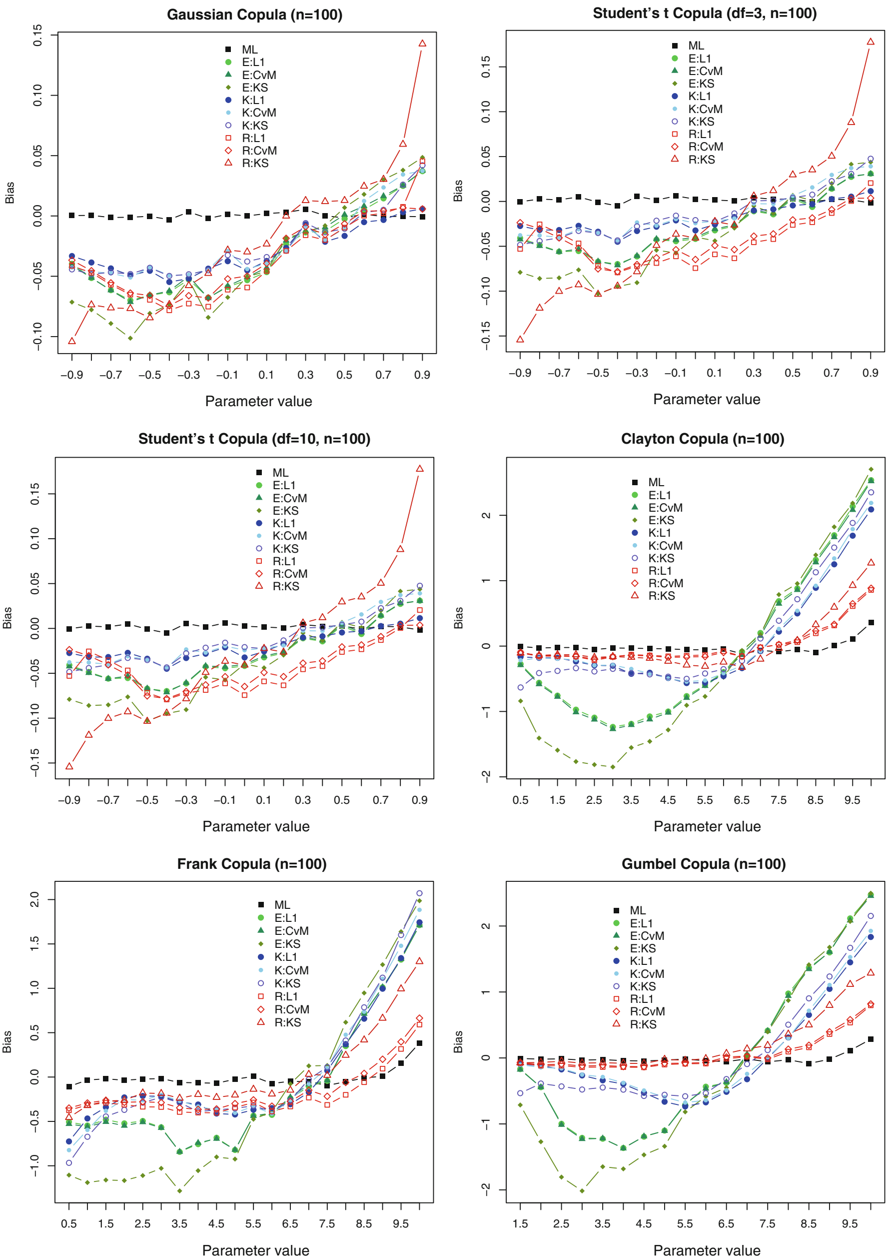

Fig. 2 Mean estimation bias for the maximum-likelihood (ML) and the minimum-distance estimators based on the empirical copula process (E:L1, E:CvM and E:KS), on Kendall's transform (K:L1, K:CvM and K:KS) and Rosenblatt's transform (R:L1, R:CvM and R:KS) for different parametric copulas and a sample size of $n = 100$ depending on the location of the true parameter under the hypothesis that the parametric copula family has been correctly specified

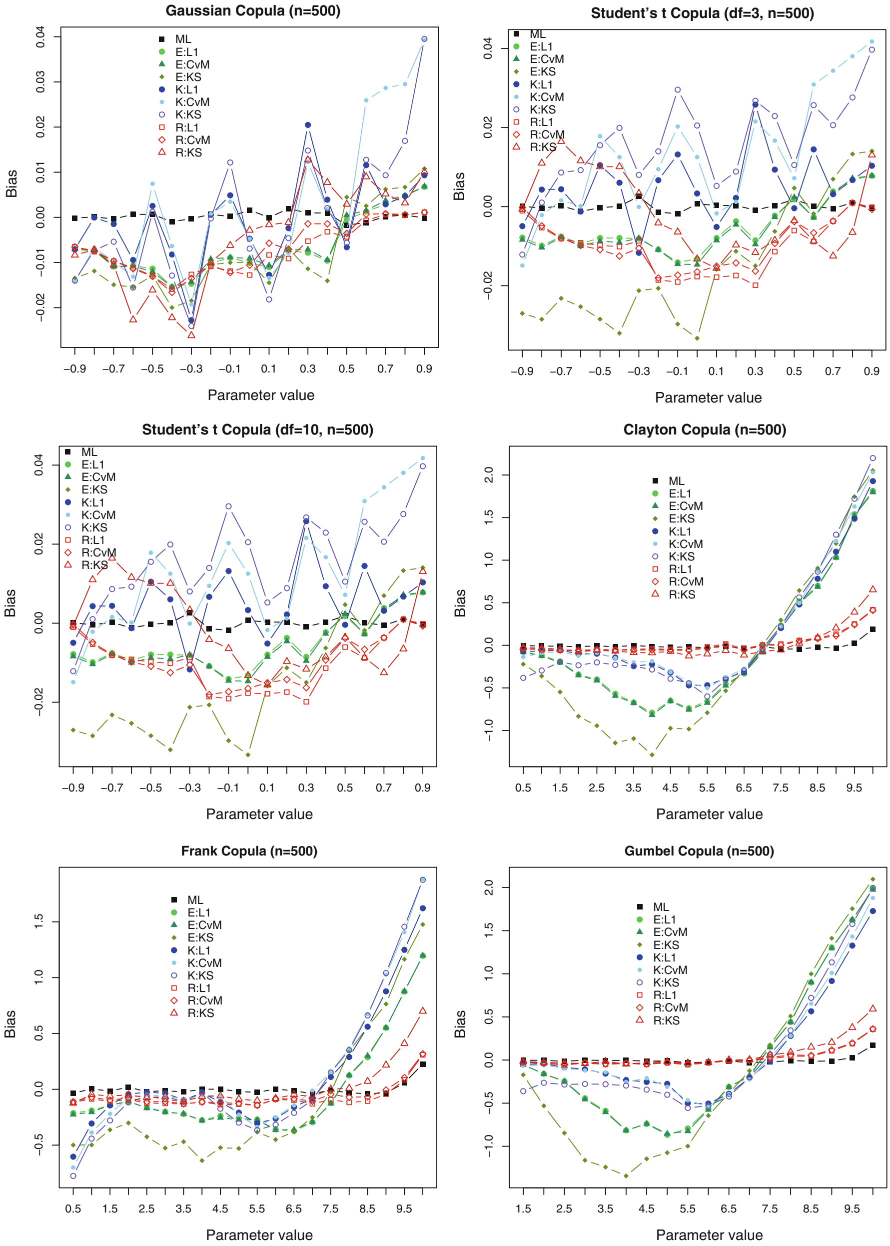

Fig. 3 Mean estimation bias for the maximum-likelihood (ML) and the minimum-distance estimators based on the empirical copula process (E:L1, E:CvM and E:KS), on Kendall's transform (K:L1, K:CvM and K:KS) and Rosenblatt's transform (R:L1, R:CvM and R:KS) for different parametric copulas and a sample size of $n = 500$ depending on the location of the true parameter under the hypothesis that the parametric copula family has been correctly specified

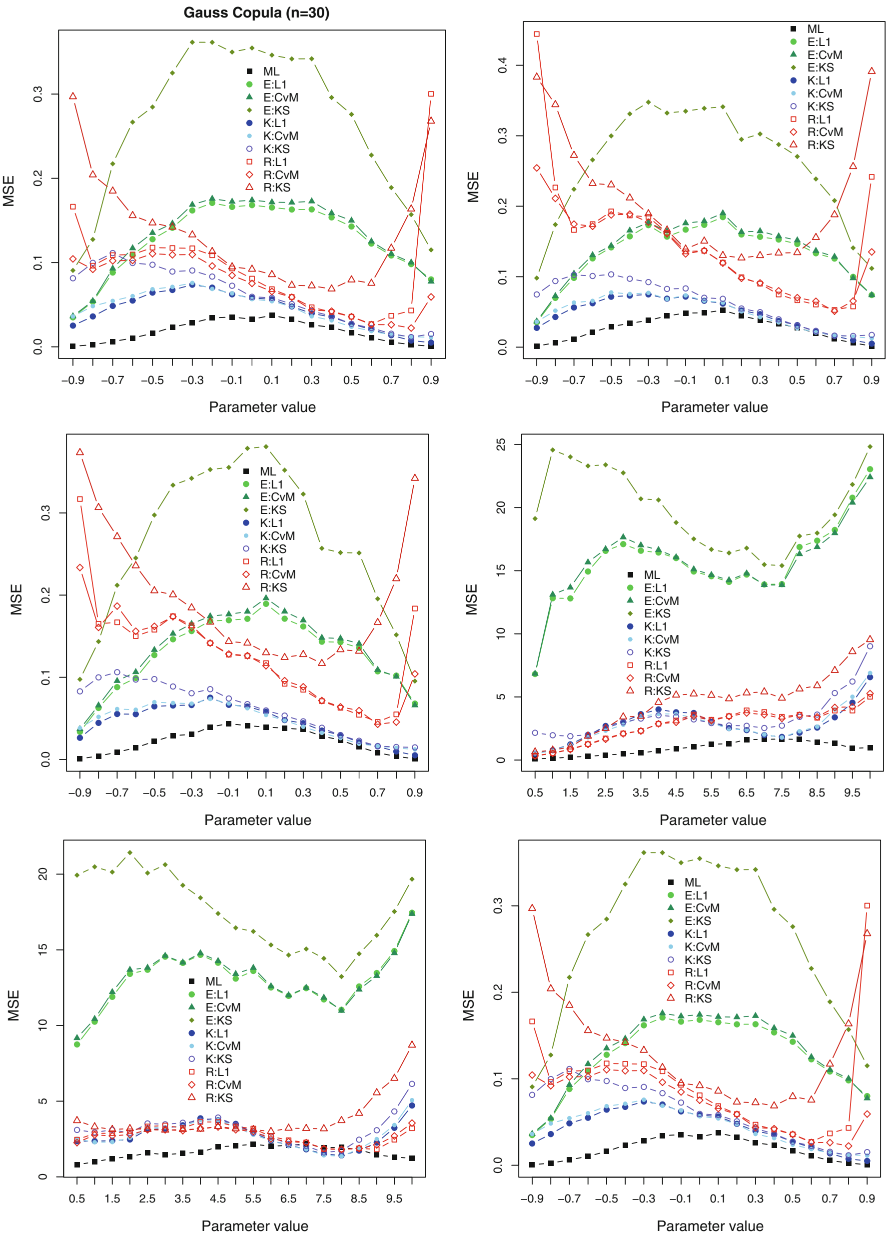

Fig. 4 Mean squared error (MSE) for the maximum-likelihood (ML) and the minimum-distance estimators based on the empirical copula process (E:L1, E:CvM and E:KS), on Kendall's transform (K:L1, K:CvM and K:KS) and Rosenblatt's transform (R:L1, R:CvM and R:KS) for different parametric copulas and a sample size of $n = 30$ depending on the location of the true parameter under the hypothesis that the parametric copula family has been correctly specified

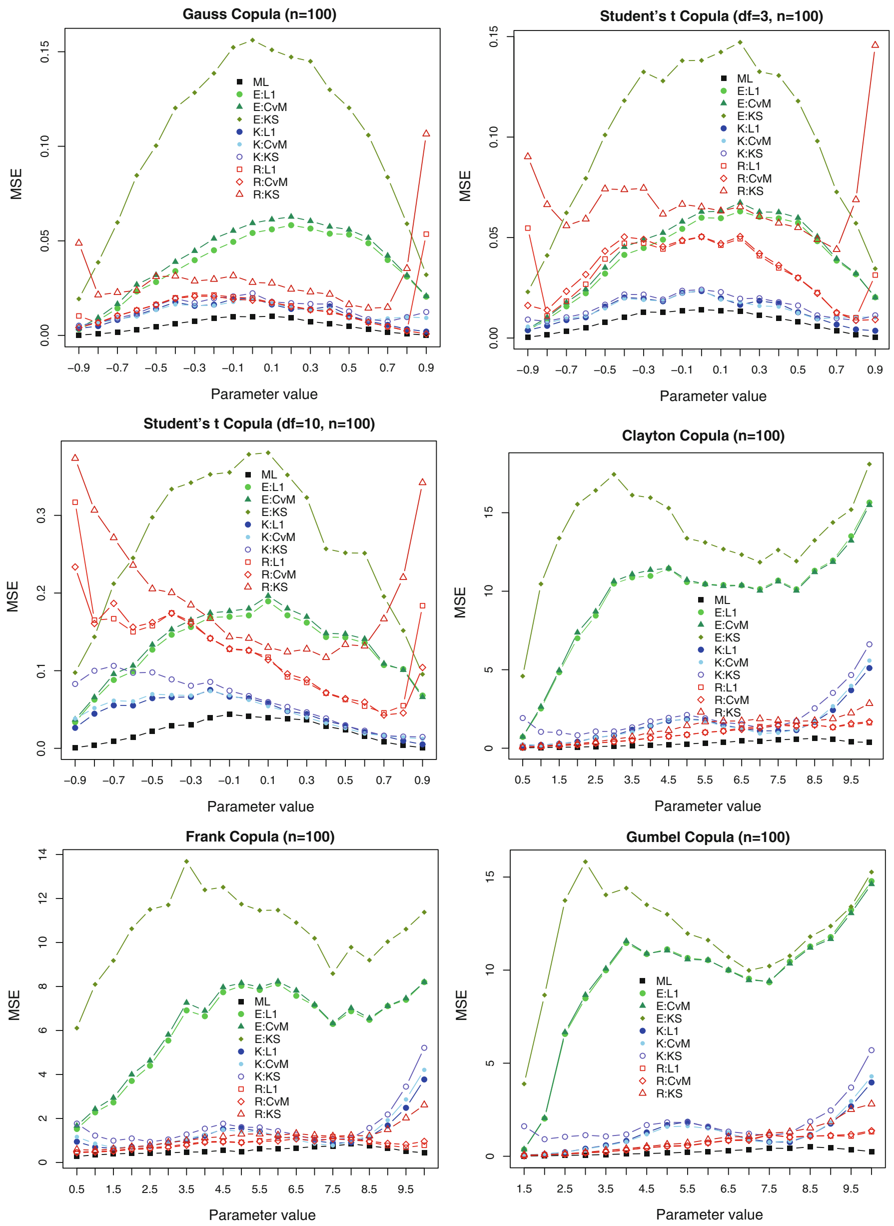

Fig. 5 Mean squared error (MSE) for the maximum-likelihood (ML) and the minimum-distance estimators based on the empirical copula process (E:L1, E:CvM and E:KS), on Kendall's transform (K:L1, K:CvM and K:KS) and Rosenblatt's transform (R:L1, R:CvM and R:KS) for different parametric copulas and a sample size of $n = 100$ depending on the location of the true parameter under the hypothesis that the parametric copula family has been correctly specified

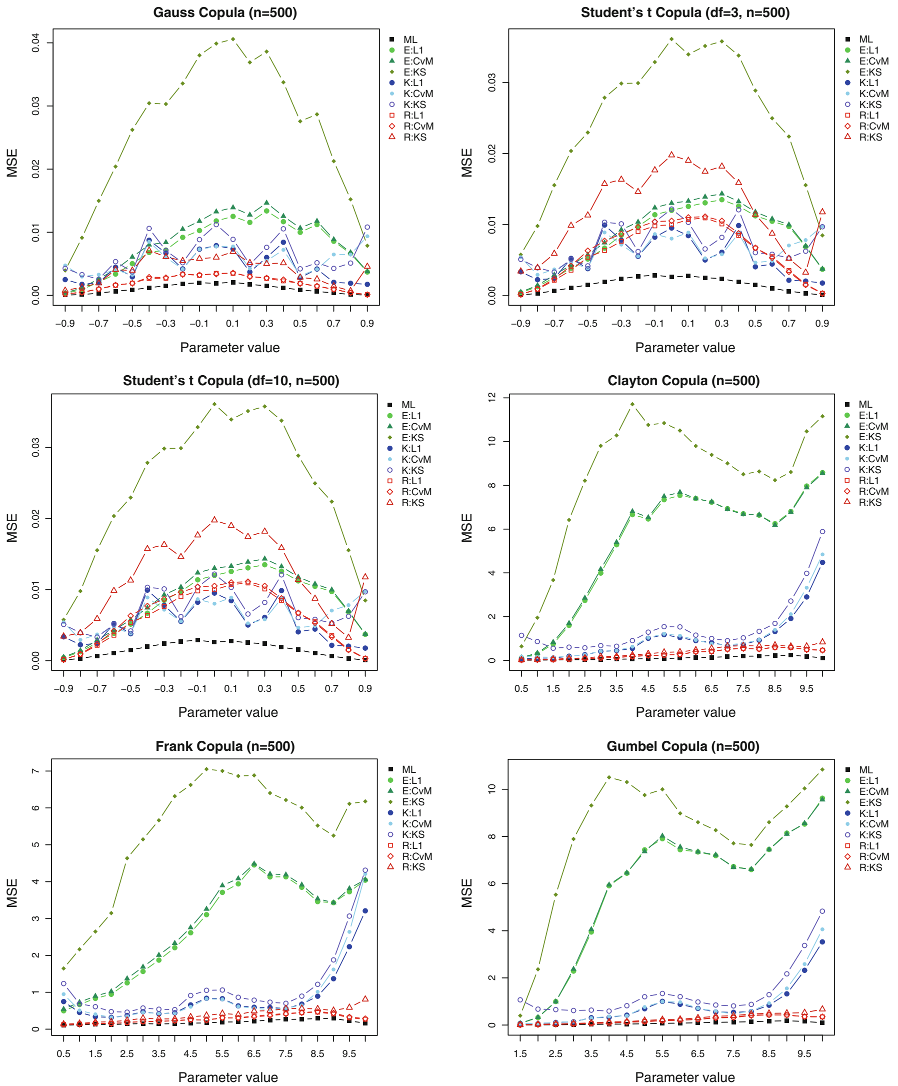

Fig. 6 Mean squared error (MSE) for the maximum-likelihood (ML) and the minimum-distance estimators based on the empirical copula process (E:L1, E:CvM and E:KS), on Kendall's transform (K:L1, K:CvM and K:KS) and Rosenblatt's transform (R:L1, R:CvM and R:KS) for different parametric copulas and a sample size of $n = 500$ depending on the location of the true parameter under the hypothesis that the parametric copula family has been correctly specified

produced by the estimators $\hat{\theta}_n^{K,\ldots}$ based on Kendall's dependence function, the estimates with the lowest bias for the archimedean copulas are given by the estimators $\hat{\theta}_n^{Ros,\ldots}$ based on Rosenblatt's transform. In most configurations, the three estimators $\hat{\theta}_n^{emp,\ldots}$ based on the empirical copula process yield the worst bias and MSE results among the MD-estimators. This finding is particularly interesting as the corresponding

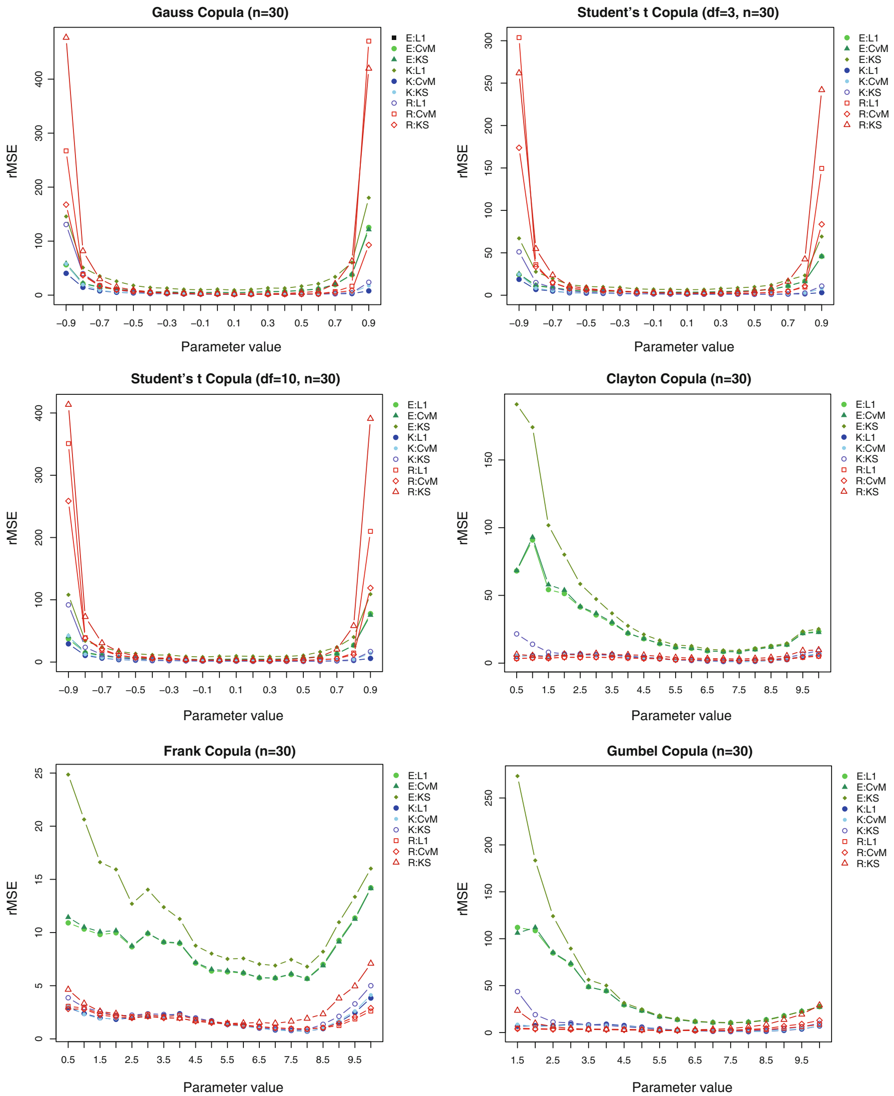

Fig. 7 Mean squared error for the minimum-distance estimators based on the empirical copula process (E:L1, E:CvM and E:KS), on Kendall's transform (K:L1, K:CvM and K:KS) and Rosenblatt's transform (R:L1, R:CvM and R:KS) relative to the ML-estimator (rMSE) for different parametric copulas and a sample size of $n = 30$ depending on the location of the true parameter under the hypothesis that the parametric copula family has been correctly specified

goodness-of-fit-tests based on the empirical copula process were found to perform well in recent studies (Berg 2009; Genest et al. 2009). Moreover, with the exception of the Student copula, the MD-estimators based on the empirical copula require considerably more computation time than the estimators based on Rosenblatt's transform, though both groups yield comparable estimation biases. Thus, there is no evident reason why

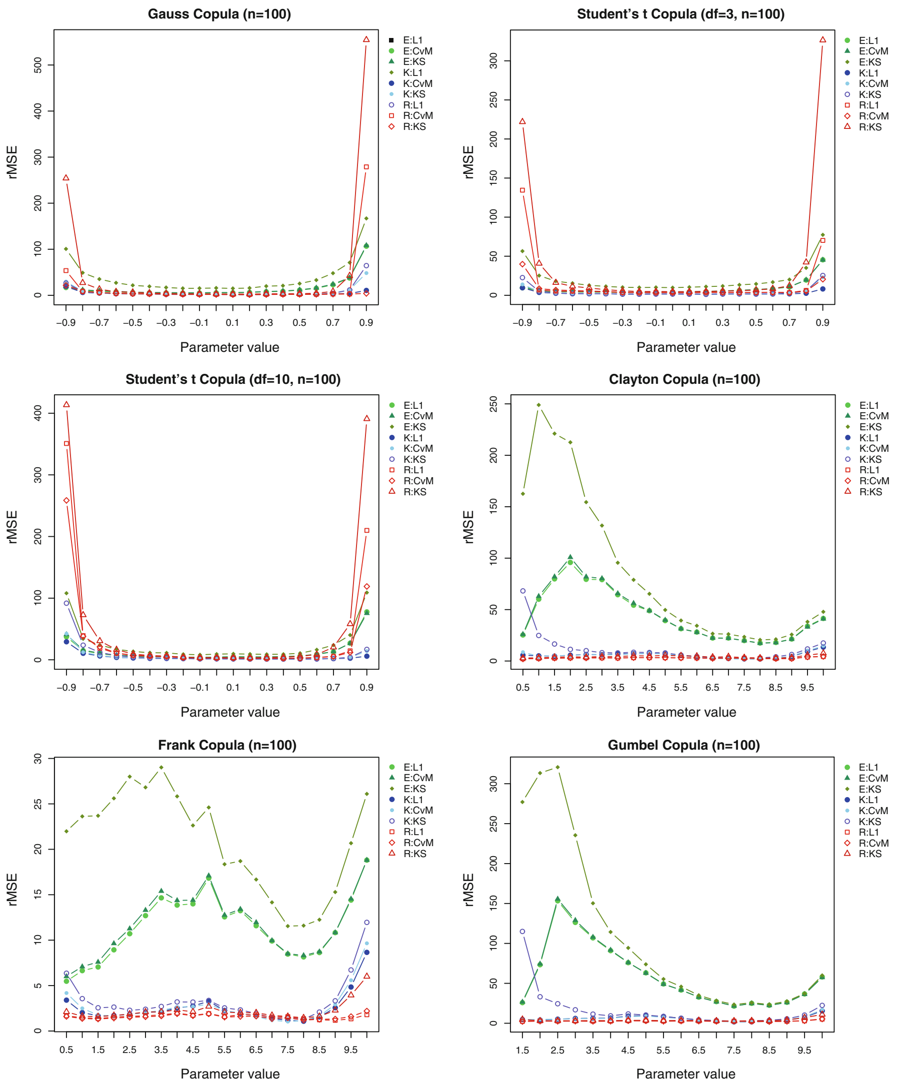

Fig. 8 Mean squared error for the minimum-distance estimators based on the empirical copula process (E:L1, E:CvM and E:KS), on Kendall's transform (K:L1, K:CvM and K:KS) and Rosenblatt's transform (R:L1, R:CvM and R:KS) relative to the ML-estimator (rMSE) for different parametric copulas and a sample size of $n = 30$ depending on the location of the true parameter under the hypothesis that the parametric copula family has been correctly specified

one would be inclined to use an MD-estimator based on the empirical copula process.

In addition to these results, biases seem to be considerably higher for archimedean copulas than for elliptical copulas. Also, especially the MD-estimators seem to suffer from large biases for smaller samples sizes yielding extremely inaccurate estimates (cf. e.g. the results for the Frank copula given in Figs. 1 and 2).

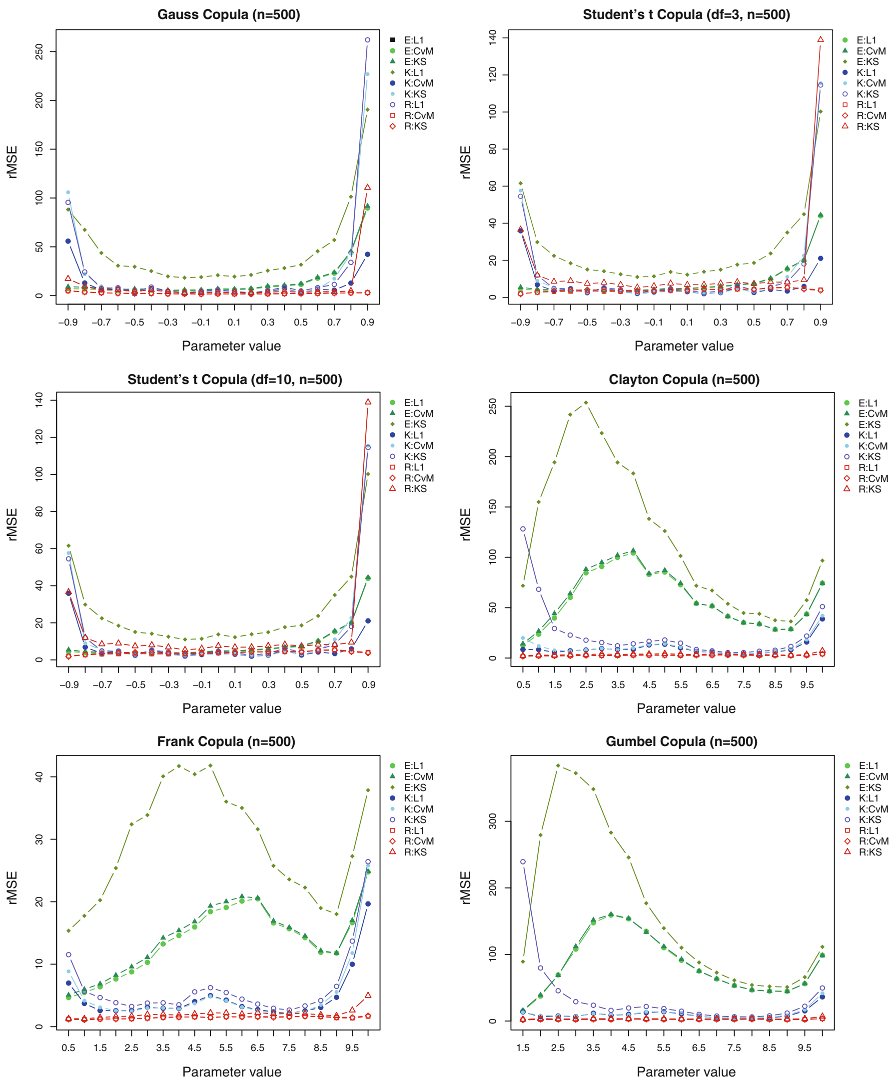

Fig. 9 Mean squared error for the minimum-distance estimators based on the empirical copula process (E:L1, E:CvM and E:KS), on Kendall's transform (K:L1, K:CvM and K:KS) and Rosenblatt's transform (R:L1, R:CvM and R:KS) relative to the ML-estimator (rMSE) for different parametric copulas and a sample size of $n = 500$ depending on the location of the true parameter under the hypothesis that the parametric copula family has been correctly specified

3.2.2 Results concerning the sample size and computational complexity

The results given in Figs. 1, 2, 3, 4, 5, 6, 7, 8 and 9 show that, as predicted and required by the asymptotic convergence of the different copula processes, parameter estimates for all parametric copulas improve with increasing sample size. In addition to this, the decrease in bias is approximately linear. In all configurations, however, the bias of the ML-estimates are always lower than the biases of the MD-estimators even for the largest sample size $n = 500$. This underlines the previously formulated empirical optimality of maximum-likelihood in this study.

Similarly, the time in seconds needed for computing the parameter estimates, as expected, increases with increasing sample size. Again, the ML-estimator yields the best results with a negligible average computation time of under one second per estimation. For the MD-estimators, results are again mixed. The estimators $\hat{\theta}_n^{K,\cdots}$ based on Kendall's dependence function require considerably more computation time than the other two estimators. The estimators $\hat{\theta}_n^{Ros,\cdots}$ based on Rosenblatt's transform on the other hand just like the ML-estimator require only a negligible computation time.

3.2.3 Effect of the parameter location

Again, as can be seen form Figs. 1, 2, 3, 4, 5, 6, 7, 8 and 9, the results for the ML- and MD-estimators differ considerably. For all configurations of samples sizes and parametric copulas, the bias of the ML-estimations is relatively robust over the respective parameter intervals. Thus the previously found low bias and negligible computation time of the ML-estimator are also independent of a change of the parameter location.

For the minimum-distance estimators, results with respect to the influence of the parameter location on the estimation bias are strongly dependent on the parametric copula family: for the elliptical copulas, the estimation bias of all MD-estimators is strongly negative for the extreme left part of the parameter interval ($\theta = -0.9$) and increases linearly to a strongly positive bias at $\theta = 0.9$. Thus, estimation bias seems to increase with increasing dependence expressed by the copula. In the middle part of the parameter range between $\theta = 0.3$ and $\theta = 0.5$, the bias of the MD-estimators are lowest and almost negligible. For the Archimedean copulas and the MD-estimators, absolute bias is relatively low for the initial parameter values with the mean bias increasing with increasing values of the true parameters. In addition to these findings, the MD-estimates for the archimedean copulas seem to be rather unstable with respect to estimation bias.

3.2.4 Effect of the type of statistic

Concerning the effect of the type of statistic on the estimation bias, one can see from Figs. 1, 2, 3, 4, 5, 6, 7, 8 and 9 that on average there are no significant differences between the CvM-, the KS and L_1-statistic. Especially for the archimedean copulas, however, the L_1-variant seems to yield slightly better results than the other two statistics. In contrast to the CvM- and KS-statistic, the L_1-variant also requires slightly more computation time. Besides these minor differences, the selection of the empirical process underlying the MD-estimator rather than the choice of statistic seems to be of importance for an unbiased parameter estimation.

4 Conclusion

In this paper, a comprehensive Monte Carlo simulation study on the performance of minimum-distance and maximum-likelihood estimators for bivariate parametric copulas was presented. By comparing ten different estimators for five parametric copulas in a total of around 800, 000 repetitions, this paper constitutes the largest Monte Carlo experiment on minimum-distance estimators for copulas carried out to date.

The central finding is that when excluding the influence of the marginals on the estimation of copula parameters, maximum-likelihood yields the best results concerning bias, efficiency and computational complexity. Minimum-distance estimators on the other hand yield considerably higher biases at greater computational cost. As a result, this study does not find any evident reason why one would be inclined to use an MD-estimator instead of classical maximum-likelihood. Furthermore, maximum-likelihood seems to yield stable estimates with regard to the parameter location whereas minimum-distance estimates are highly dependent on the location of the true copula parameter. The L_1-variant used in the minimum-distance estimators in this study yielded slightly better results than the corresponding Cramér-von-Mises or Kolmogorov-Smirnov statistics. The choice of test statistic, however, seems to be of no importance compared to the choice of underlying copula process. Finally, results for the different kinds of parametric copula families are mixed. Minimum-distance estimators seem to produce better results for elliptical copulas in contrast to the appalling performance when used for estimating the parameters of archimedean copulas.

In future work, it would be interesting to see whether the empirical optimality of the ML-estimator holds true when the simulated data is further influenced by parametric marginals or when the data is contaminated.

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Appendix A: empirical observators for the MD-estimators based on Kendall's dependence function

Goal: To compute an approximation to the distances $\rho_k^{L_1}$, ρ_K^{CvM} and ρ_K^{KS} dependent on a given sample $\mathbf{U}$, a hypothesised parametric copula family C and a parameter θ.
Algorithm:

(1) Choose $m \geq n$.
(2) Generate a random sample $\mathbf{U}_\mathbf{1}^*, \ldots, \mathbf{U}_\mathbf{m}^*$ of size m from the copula C_θ.
(3) Compute the transformed data

$$V_i^* \equiv \frac{1}{m} \sum_{j=1}^{m} \mathbf{1}(U_j^* \leq U_i^*), \quad i \in \{1, \ldots, m\}.$$

(4) Approximate $K_{\hat{\theta}}$ by

$$B_m(t) \equiv \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}(V_i^* \leq t), \quad t \in [0; 1].$$

(5) Approximate $\rho_k^{L_1}$, ρ_K^{CvM} and ρ_K^{KS} by

$$\hat{\rho}_k^{L_1} \equiv \frac{n}{m} \sum_{i=1}^{m} \left| K_n(V_i^*) - B_m(V_i^*) \right|$$

$$\hat{\rho}_k^{CvM} \equiv \frac{n}{m} \sum_{i=1}^{m} \left\{ K_n(V_i^*) - B_m(V_i^*) \right\}^2$$

$$\hat{\rho}_k^{KS} \equiv \max_{i \in \mathbb{N}_m} \left| K_n(V_i^*) - B_m(V_i^*) \right|$$

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