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Determining the Optimum Process Mean Under Quality Loss Function

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Kapur et al. and Wen and Mergen have addressed the problems of products having specification limits but the manufacturing process not being capable of meeting the specifications in the short term. Kapur et al. report that inspection in an on-line quality control system is carried out as a short-term approach to reduce variance of the items shipped to the customers. Wen and Mergen balance the costs of products being out-of-specification by setting the optimal process mean in the short term. In this paper, we propose a modified Wen and Mergen's cost model with a linear and quadratic asymmetrical for the measurement of the quality loss of products which are within specification, for determining the optimum process mean.

Keywords: Process mean; Process standard deviation; Specification limits; Taguchi's quality loss function; Target value

1. Introduction

The traditional concept of conformance to specifications is that items should meet the specification limits. Taguchi [1] has presented the quadratic quality loss function for reducing the deviation from the target value. The objective of this quality improvement method is to minimise the total losses to society, where society includes both producers and consumers.

Kapur et al. [2–5] and Wen and Mergen [6] have addressed the methods for quality improvement in the short term. Kapur and Wang [2, p. 28] said "suppose we can't improve the present process, then a short term approach to decrease variance of the units shipped to the customer is to put specification limits on the process and truncate the distribution by inspection". Wen and Mergen [6] described a method for setting the optimal process mean when the process is not capable of meeting specification limits in the short term. In some cases, changing (or improving) the process may not be an economically feasible option, at least in the short term, because the financial resources may not be available to make the necessary improvement. Wen and Mergen's [6] proposed technique assumes that the process mean can be changed easily, but not the process standard deviation. They select the optimum process mean based on balancing the cost of not meeting the upper specification limit (T_U) and the lower specification limit (T_L).

Li [7–9], Li and Chirng [10], Maghsoodloo and Li [11], and Li and Chou [12] adopt a quadratic and linear quality loss function for unbalanced tolerance design. They proposed a constant process standard deviation and constant process coefficient of variation models. However, they have not considered the different costs for the products at the specification limits and out-of-specification.

The advantage of the quadratic loss function is that we can evaluate losses in terms of process mean and process standard deviation. Thus, in order to reduce the expected losses, we have to reduce bias (deviation of process mean from the target value) and process standard deviation. If the quality characteristic concentrates on the target value with the minimum standard deviation, then the product has minimal quality loss. However, the regular quadratic loss function is patently inappropriate in some situations. Trietsch [13, p. 69] remarks that "One such case occurs when the expected cost of exceeding the tolerance limits is not equal to the right and to the left of the target. Missing by cutting too much, for instance, may imply scrap, while cutting too little only causes rework. When this is the case one possible response is fitting a loss function that is not symmetric, and not necessarily quadratic."

Taguchi's [1] definition of quality is that quality is the loss imparted to society from the time the product is shipped. Quality loss functions assign measurable penalities that are proportional to the distance a quality characteristic is away from its desired target value. When compared to the trational definition in regard to the quality of conformance, the quality loss function approach implies that merely meeting specifications is not sufficient. Wen and Mergen [6] have neglected to consider the quality loss for products within specification in the model. In this paper, we propose a modified Wen and

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Mergen's [6] cost model with a linear and quadratic asymmetrical quality loss of products within specification for determining the optimum process mean. Finally, the solution procedure and a numerical example are included.

2. Modified Wen and Mergen's Cost Model

The mathematical programming model makes the following three assumptions:

1. Quality characteristic, X, is normally distributed with unknown mean μ and known variance σ^2.
2. The quality characteristic is nominal-is-best.
3. The target value, T, is the middle value of the specifications, i.e. $T = (T_U + T_L)/2$.

According to Wen and Mergen [6, p. 508], the total loss per item is

$$C_T = D_U \int_{T_U}^{\infty} f(x)\mathrm{d}x + D_L \int_{-\infty}^{T_L} f(x)\mathrm{d}x \quad (1)$$

where,

T_U = upper specification limit

T_L = lower specification limit

C_T = total loss per item due to exceeding T_U and T_L

D_U = monetary loss per item of exceeding T_U

D_L = monetary loss per item of staying below T_L

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ -\infty < x < \infty.$$

Case 1. Quadratic Asymmetrical Quality Loss Function

Define a quadratic asymmetric quality loss function X for the nominal-is-best quality characteristic. We can use the coefficients k_1 and k_2 for the two directions of deviation from the target value. If tolerances for both sides are Δ_1 and Δ_2, the quality loss at the specification limit is defined as A_1 and A_2, respectively. Then the quadratic asymmetric quality loss function is

$$L(x) = \begin{cases} k_1(T - x)^2 & (x \leq T) \\ k_2(x - T)^2 & (x \geq T) \end{cases} \quad (3)$$

where $k_1 = A_1/\Delta_1^2$ and $k_2 = A_2/\Delta_2^2$.

Now, we would like to include the quadratic asymmetric quality loss within specification in Wen and Mergen's model [6]. The modified model is as follows:

Minimise

$$C_{T_1} = C_T + \int_{T_L}^{T} k_1(T - x)^2 f(x)\mathrm{d}x + \int_{T}^{T_U} k_2(x - T)^2 f(x)\mathrm{d}x \quad (4)$$

where k_1 *and* k_2 are constants called quality loss coefficient.

From the Appendix, Eq. (4) can be rewritten as

Minimise

$$\begin{aligned} C_{T_1} = {} & D_U\left[1 - \Phi\left(\frac{T_U - \mu}{\sigma}\right)\right] + D_L\Phi\left(\frac{T_L - \mu}{\sigma}\right) + k_1\left\{\left[(\sigma^2 + \mu^2)\Phi\left(\frac{T - \mu}{\sigma}\right) - \sigma(T + \mu)\phi\left(\frac{T - \mu}{\sigma}\right)\right.\right. \\ & \left. - (\sigma^2 + \mu^2)\Phi\left(\frac{T_L - \mu}{\sigma}\right) + \sigma(T_L + \mu)\phi\left(\frac{T_L - \mu}{\sigma}\right)\right] - 2T\left[-\sigma\phi\left(\frac{T - \mu}{\sigma}\right) + \mu\Phi\left(\frac{T - \mu}{\sigma}\right)\right. \\ & \left.\left. + \sigma\phi\left(\frac{T_L - \mu}{\sigma}\right) - \mu\Phi\left(\frac{T_L - \mu}{\sigma}\right)\right] + T^2\left[\Phi\left(\frac{T - \mu}{\sigma}\right) - \Phi\left(\frac{T_L - \mu}{\sigma}\right)\right]\right\} \\ & + k_2\left\{\left[(\sigma^2 + \mu^2)\Phi\left(\frac{T_U - \mu}{\sigma}\right) - \sigma(T_U + \mu)\phi\left(\frac{T_U - \mu}{\sigma}\right)\right.\right. \\ & \left. - (\sigma^2 + \mu^2)\Phi\left(\frac{T - \mu}{\sigma}\right) + \sigma(T + \mu)\phi\left(\frac{T - \mu}{\sigma}\right)\right] - 2T\left[-\sigma\phi\left(\frac{T_U - \mu}{\sigma}\right) + \mu\Phi\left(\frac{T_U - \mu}{\sigma}\right)\right. \\ & \left.\left. + \sigma\phi\left(\frac{T - \mu}{\sigma}\right) - \mu\Phi\left(\frac{T - \mu}{\sigma}\right)\right] + T^2\left[\Phi\left(\frac{T_U - \mu}{\sigma}\right) - \Phi\left(\frac{T - \mu}{\sigma}\right)\right]\right\} \end{aligned} \quad (5)$$

where $\Phi(z)$ is the cumulative distribution function for the standard normal random variable with density function $\phi(z)$,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) \ (-\infty < z < \infty) \quad (6)$$

Case 2. Linear Asymmetrical Quality Loss

Define a linear asymmetric quality loss function X for the nominal-is-best quality characteristic. We can use the coefficients k_1 and k_2 for the two directions of deviation from the target value. If tolerances for both sides are Δ_1 and Δ_2, the quality loss at the specification limit is defined as A_1 and A_2, respectively. Then the linear asymmetric quality loss function is

$$L(x) = \begin{cases} k_1(T - x) & (x \leq T) \\ k_2(x - T) & (x \geq T) \end{cases} \quad (7)$$

where $k_1 = A_1/\Delta_1$ and $k_2 = A_2/\Delta_2$.

Now, we would like to include the linear asymmetric quality loss within specifications in Wen and Mergen's model [6]. The modified model is as follows:

Minimise

$$C_{T_2} = C_T + \int_{T_L}^{T} k_1(T - x)f(x)\mathrm{d}x + \int_{T}^{T_U} k_2(x - T)f(x)\mathrm{d}x \quad (8)$$

Equation (8) can be rewritten as

Minimise

$$\begin{aligned} C_{T_2} = {} & D_U\left[1 - \Phi\left(\frac{T_U - \mu}{\sigma}\right)\right] + D_L\Phi\left(\frac{T_L - \mu}{\sigma}\right) + k_1T\left[\Phi\left(\frac{T - \mu}{\sigma}\right) - \Phi\left(\frac{T_L - \mu}{\sigma}\right)\right] \\ & - k_1\left[-\sigma\phi\left(\frac{T - \mu}{\sigma}\right) + \mu\Phi\left(\frac{T - \mu}{\sigma}\right)\right] + \sigma\phi\left(\frac{T_L - \mu}{\sigma}\right) - \mu\Phi\left(\frac{T_L - \mu}{\sigma}\right)\Bigg] \\ & - k_2T\left[\Phi\left(\frac{T_U - \mu}{\sigma}\right) - \Phi\left(\frac{T - \mu}{\sigma}\right)\right] + k_2\left[-\sigma\phi\left(\frac{T_U - \mu}{\sigma}\right) + \mu\Phi\left(\frac{T_U - \mu}{\sigma}\right)\right. \\ & \left. + \sigma\phi\left(\frac{T - \mu}{\sigma}\right) - \mu\Phi\left(\frac{T - \mu}{\sigma}\right)\right] \end{aligned} \quad (9)$$

3. Solution Procedure

According to Johnson and Kotz [14, p. 55], the $\Phi(z)$ can be approximated by the following formula:

$$\Phi(z) \cong 1 - (a_1 t + a_2 t^2 + a_3 t^3)\phi(z) \quad (10)$$

where

$$t = 1/(1+0.33267z) \quad (11)$$

$$a_1 = 0.4361836 \quad (12)$$

$$a_2 = -0.1201676 \quad (13)$$

$$a_3 = 0.9372980 \quad (14)$$

For the given k_1, k_2, T_L, T, and T_U, we can adopt a direct search method for finding the optimal process mean of the above model (5) and (9).

4. Numerical Example and Sensitivity Analysis

Numerical Example

Consider the example given in Wen and Mergen [6, p. 506]. Assume that the quality characteristic has a normal distribution with a standard deviation $\sigma = 0.00173$, and unknown mean μ. Let the target value, T, for this quality characteristic be 29.997. The monetary loss (per unit) of exceeding upper specification limit, T_U, is $D_U = 1.5$. The monetary loss (per unit) of below lower specification limit, T_L, is $D_L = 0.1$. We have $T_L = 29.995$, $T_U = 29.999$, and the tolerance zone $\Delta_1 = \Delta_2 = 0.002$.

The optimum process mean for Wen and Mergen's model [6] is $\mu = 29.995$ with $C_T = 0.0655775$.

Case 1. Quadratic Asymmetrical Quality Loss Function

Assume that different costs occur for a product at the specification limits and out-of-specification. The cost of products out-of-specification is greater than that of products at the specification limits. Suppose that the quadratic asymmetric quality loss at the specification limits is defined as $A_1 = 3.25 \times 10^{-5}$ and $A_2 = 9.52 \times 10^{-4}$. Hence, we have $k_1 = A_1/\Delta_1^2 = 8$ and $k_2 = A_2/\Delta_2^2 = 238$. Hence, the modified Wen and Mergen's model [6] is Eq. (5). By solving the above model (5), we find the value of μ that minimises C_{T_1} is 29.996 with $C_{T_1} = 0.06783595$.

Case 2. Linear Asymmetrical Quality Loss Function

Suppose that the linear asymmetric quality loss at the specification limits is defined as $A_1 = 0.016$ and $A_2 = 0.476$. Hence, we have $k_1 = A_1/\Delta_1 = 8$ and $k_2 = A_2/\Delta_2 = 238$. Hence, the modified Wen and Mergen's model [6] is Eq. (9). By solving the above model (9), we find the value of μ that minimises C_{T_2} is 29.994 with $C_{T_2} = 0.0826416$.

Table 1. Effect of k_1 ($\sigma = 0.00173$, $T_L = 29.995$, $T_U = 29.999$, $D_L = 0.1$, $D_U = 1.5$, $T = 29.997$, $k_2 = 238$).

k_1	Quadratic model		Linear model	
	μ	C_{T_1}	μ	C_{T_2}
1	29.995	0.067475	29.994	0.080444
2	29.995	0.067536	29.994	0.080719
3	29.995	0.067597	29.994	0.081024
4	29.995	0.067658	29.994	0.081299
5	29.995	0.067719	29.994	0.081604
6	29.995	0.067780	29.994	0.081879
7	29.995	0.067841	29.994	0.082184
8	29.996	0.067836	29.994	0.082489
9	29.996	0.067775	29.994	0.082764
10	29.996	0.067714	29.994	0.083069

Table 2. Effect of k_2 ($\sigma = 0.00173$, $T_L = 29.995$, $T_U = 29.999$, $D_L = 0.1$, $D_U = 1.5$, $T = 29.997$, $k_1 = 8$).

k_2	Quadratic model		Linear model	
	μ	C_{T_1}	μ	C_{T_2}
235	29.995	0.067879	29.994	0.082428
236	29.995	0.067887	29.994	0.082397
237	29.995	0.067894	29.994	0.082397
238	29.996	0.067836	29.994	0.082489
239	29.996	0.067744	29.994	0.082489
240	29.996	0.067653	29.994	0.082520
241	29.996	0.067561	29.994	0.082520
242	29.996	0.067470	29.994	0.082581
243	29.996	0.067378	29.994	0.082581
244	29.996	0.067287	29.994	0.082581
245	29.996	0.067195	29.994	0.082642

Sensitivity Analysis

For a given set of parameters, the value of μ has little variation for both quadratic and linear models and the quadratic model has a smaller cost than that for the linear model (as shown in Tables 1–7).

Table 3. Effect of σ ($k_1 = 8$, $T_L = 29.995$, $T_U = 29.999$, $D_L = 0.1$, $D_U = 1.5$, $T = 29.997$, $k_2 = 238$).

σ	Quadratic model		Linear Model	
	μ	C_{T_1}	μ	C_{T_2}
0.00103	29.995	0.029168	29.996	0.045166
0.00113	29.996	0.036196	29.996	0.055176
0.00123	29.996	0.032361	29.995	0.060730
0.00133	29.996	0.051543	29.995	0.063934
0.00143	29.995	0.055770	29.995	0.068543
0.00153	29.995	0.042246	29.995	0.074402
0.00163	29.994	0.049561	29.995	0.080078
0.00173	29.993	0.069422	29.994	0.082428
0.00183	29.995	0.056358	29.994	0.085205
0.00193	29.995	0.064101	29.994	0.087769
0.00203	29.995	0.074063	29.993	0.091080

Table 4. Effect of D_U ($\sigma = 0.00173$, $T_L = 29.995$, $T_U = 29.999$, $D_L = 0.1$, $T = 29.997$, $k_1 = 8$, $k_2 = 238$).

D_U	Quadratic model		Linear model	
	μ	C_{T_1}	μ	C_{T_2}
0.5	29.996	0.026592	29.995	0.077209
0.6	29.996	0.030717	29.995	0.078247
0.7	29.996	0.034841	29.995	0.079285
0.8	29.996	0.038965	29.995	0.080322
0.9	29.996	0.043090	29.994	0.081329
1.0	29.996	0.047214	29.994	0.081512
1.1	29.996	0.051339	29.994	0.081696
1.2	29.996	0.055463	29.994	0.081909
1.3	29.996	0.059587	29.994	0.082092
1.4	29.996	0.063712	29.994	0.082275
1.5	29.996	0.067836	29.994	0.082489
1.6	29.995	0.068936	29.994	0.082672

Table 5. Effect of D_L ($\sigma = 0.00173$, $T_L = 29.995$, $T_U = 29.999$, $D_U = 1.5$, $T = 29.997$, $k_1 = 8$, $k_2 = 238$).

D_L	Quadratic model		Linear model	
	μ	C_{T_1}	μ	C_{T_2}
0.1	29.996	0.067836	29.994	0.082489
0.2	29.996	0.096084	29.995	0.137634
0.3	29.996	0.124333	29.995	0.187683
0.4	29.996	0.152581	29.996	0.224121
0.5	29.996	0.180829	29.996	0.252441
0.6	29.996	0.209078	29.996	0.280640
0.7	29.996	0.237326	29.996	0.308838
0.8	29.996	0.265575	29.996	0.337158
0.9	29.997	0.282826	29.996	0.365357
1.0	29.997	0.295266	29.997	0.392090

Table 6. Effect of T_L ($\sigma = 0.00173$, $T_U = 29.999$, $D_L = 0.1$, $D_U = 1.5$, $T = (T_L + T_U)/2$, $k_1 = 8$, $k_2 = 238$).

T_L	Quadratic model		Linear model	
	μ	C_{T_1}	μ	C_{T_2}
29.991	29.993	0.006852	29.992	0.047729
29.992	29.995	0.027415	29.993	0.051758
29.993	29.995	0.038799	29.994	0.061951
29.994	29.995	0.040296	29.994	0.069580
29.995	29.995	0.076161	29.944	0.082367
29.996	29.993	0.086305	29.994	0.093414
29.997	29.993	0.067022	29.994	0.099632
29.998	29.993	0.083421	29.992	0.099433

5. Conclusions

In modern statistical process control (SPC) methods, we adopt control charts and process capability indices for dealing with the stability of the process and its capability of meeting the specification limits, respectively. If the process is stable but not capable of meeting specifications, then adjustment of the process mean may be carried out as a short-term approach to reduce quality loss. In this paper, we have presented a modified Wen and Mergen's model [6]. The proposed model is a generalisation of Wen and Mergen's model [6]. Further study will assume that the quality characteristic has another distribution and extend the work to the designs of specification limits and process mean.

Table 7. Effect of T_U ($\sigma = 0.00173$, $T_L = 29.995$, $D_L = 0.1$, $D_U = 1.5$, $T = (T_L + T_U)/2$, $k_1 = 8$, $k = 238$).

T_U	Quadratic model		Linear Model	
	μ	C_{T_1}	μ	C_{T_2}
29.996	29.990	0.097830	29.990	0.100058
29.997	29.991	0.085077	29.991	0.099633
29.998	29.993	0.077419	29.993	0.093750
29.999	29.995	0.055237	29.994	0.082611
30.000	29.996	0.019115	29.995	0.070007
30.001	29.998	0.036644	29.996	0.061890
30.002	29.996	0.005877	29.996	0.052429
30.003	29.998	0.028961	29.996	0.048065

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Appendix

The modified Wen and Mergen's cost model [6] including the quadratic asymmetrical quality loss within specifications is as follows:

Minimise

$$C_{T_1} = C_T + \int_{T_L}^{T} k_1\,(T-x)^2 f(x)\mathrm{d}x + \int_{T}^{T_U} k_2(x-T)^2 f(x)\mathrm{d}x$$

$$= D_U\int_{T_U}^{\infty} f(x)\mathrm{d}x + D_L\int_{-\infty}^{T_L} f(x)\mathrm{d}x + \int_{T_L}^{T} k_1(T-x)^2 f(x)\mathrm{d}x + \int_{T}^{T_U} k_2(x-T)^2 f(x)\mathrm{d}x \quad \text{(A1)}$$

where

k_1, k_2 = a constant called quality loss coefficient

T_U = upper specification limit

T_L = lower specification limit

C_{T_1} = total loss per item

D_U = monetary loss per item of exceeding T_U

D_L = monetary loss per item of staying below T_L

$f(x) = \dfrac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right) \; (-\infty < x < \infty)$

T = the middle value of the specifications, $T = (T_U + T_L)/2$

Let $z=\dfrac{x-\mu}{\sigma}$. Equation (A1) can be rewritten as

Minimise

$$C_{T_1} = D_U\int_{(T_U-\mu)/\sigma}^{\infty} \phi(z)\mathrm{d}z + D_L\int_{-\infty}^{(T_L-\mu)/\sigma} \phi(z)\mathrm{d}z$$
$$+ k_1\left[\int_{(T_L-\mu)/\sigma}^{(T-\mu)/\sigma} (\mu+z\sigma)^2\phi(z)\mathrm{d}z - 2T\int_{(T_L-\mu)/\sigma}^{(T-\mu)/\sigma} (\mu+z\sigma)\phi(z)\mathrm{d}z + T^2\int_{(T_L-\mu)/\sigma}^{(T-\mu)/\sigma} \phi(z)\mathrm{d}z\right]$$
$$+ k_2\left[\int_{(T-\mu)/\sigma}^{(T_U-\mu)/\sigma} (\mu+z\sigma)^2\phi(z)\mathrm{d}z - 2T\int_{(T-\mu)/\sigma}^{(T_U-\mu)/\sigma} (\mu+z\sigma)\phi(z)\mathrm{d}z + T^2\int_{(T-\mu)\sigma}^{(T_U-\mu)/\sigma} \phi(z)\mathrm{d}z\right] \quad \text{(A3)}$$

where

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right) \quad -\infty < z < \infty \quad \text{(A4)}$$

According to Fink and Margavio [15, p. 648], Eq. (A3) can be rewritten as

$$C_{T_1} = D_U\left[1-\Phi\left(\frac{T_U-\mu}{\sigma}\right)\right] + D_L\Phi\left(\frac{T_L-\mu}{\sigma}\right) + k_1\left\{\left[M_2\left(\frac{T-\mu}{\sigma}\right) - M_2\left(\frac{T_L-\mu}{\sigma}\right)\right]\right.$$
$$- 2T\left[M_1\left(\frac{T-\mu}{\sigma}\right) - M_1\left(\frac{T_L-\mu}{\sigma}\right)\right] + T^2\left.\left[\Phi\left(\frac{T-\mu}{\sigma}\right) - \Phi\left(\frac{T_L-\mu}{\sigma}\right)\right]\right\}$$
$$+ k_2\left\{\left[M_2\left(\frac{T_U-\mu}{\sigma}\right) - M_2\left(\frac{T-\mu}{\sigma}\right)\right] - 2T\left[M_1\left(\frac{T_U-\mu}{\sigma}\right) - M_1\left(\frac{T-\mu}{\sigma}\right)\right]\right.$$
$$+ T^2\left.\left[\Phi\left(\frac{T_U-\mu}{\sigma}\right)\right] - \Phi\left(\frac{T-\mu}{\sigma}\right)\right]\right\} \quad \text{(A5)}$$

where

$$M_2\left(\frac{T-\mu}{\sigma}\right) = \int_{-\infty}^{(T-\mu)/\sigma} (\mu+z\sigma)^2\phi(z)\mathrm{d}z$$
$$= (\sigma^2+\mu^2)\Phi\left(\frac{T-\mu}{\sigma}\right) - \sigma(T+\mu)\phi\left(\frac{T-\mu}{\sigma}\right) \quad \text{(A6)}$$

$$M_1\left(\frac{T-\mu}{\sigma}\right) = \int_{-\infty}^{(T-\mu)/\sigma} (\mu+z\sigma)\phi(z)\mathrm{d}z$$
$$= -\sigma\phi\left(\frac{T-\mu}{\sigma}\right) + \mu\Phi\left(\frac{T-\mu}{\sigma}\right) \quad \text{(A7)}$$

Substituting Eqs (A6)–(A7) into Eq. (A5), we have

$$C_{T_1} = D_U\left[1-\Phi\left(\frac{T_U-\mu}{\sigma}\right)\right] + D_L\Phi\left(\frac{T_L-\mu}{\sigma}\right) + k_1\left\{\left[(\sigma^2+\mu^2)\Phi\left(\frac{T-\mu}{\sigma}\right) - \sigma(T+\mu)\phi\left(\frac{T-\mu}{\sigma}\right)\right.\right.$$
$$- (\sigma^2+\mu^2)\Phi\left(\frac{T_L-\mu}{\sigma}\right) + \sigma(T_L+\mu)\phi\left.\left(\frac{T_L-\mu}{\sigma}\right)\right] - 2T\left[-\sigma\phi\left(\frac{T-\mu}{\sigma}\right) + \mu\Phi\left(\frac{T-\mu}{\sigma}\right)\right.$$
$$+ \sigma\phi\left(\frac{T_L-\mu}{\sigma}\right) - \mu\Phi\left.\left(\frac{T_L-\mu}{\sigma}\right)\right] + T^2\left.\left[\Phi\left(\frac{T-\mu}{\sigma}\right) - \Phi\left(\frac{T_L-\mu}{\sigma}\right)\right]\right\}$$
$$+ k_2\left\{\left[(\sigma^2+\mu^2)\Phi\left(\frac{T_U-\mu}{\sigma}\right) - \sigma(T_U+\mu)\phi\left(\frac{T_U-\mu}{\sigma}\right)\right.\right.$$
$$- (\sigma^2+\mu^2)\Phi\left(\frac{T-\mu}{\sigma}\right) + \sigma(T+\mu)\phi\left.\left(\frac{T-\mu}{\sigma}\right)\right] - 2T\left[-\sigma\phi\left(\frac{T_U-\mu}{\sigma}\right) + \mu\Phi\left(\frac{T_U-\mu}{\sigma}\right)\right.$$
$$+\sigma\phi\left(\frac{T-\mu}{\sigma}\right) - \mu\Phi\left.\left(\frac{T-\mu}{\sigma}\right)\right] + T^2\left.\left[\Phi\left(\frac{T_U-\mu}{\sigma}\right) - \Phi\left(\frac{T-\mu}{\sigma}\right)\right]\right\} \quad \text{(A8)}$$