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ORIGINAL ARTICLE

An improved full-discretization method for chatter stability prediction

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Abstract
An improved full-discretization method (IFDM) based on the golden search is presented in this brief paper to predict stability lobe diagram (SLD). To begin with, the mathematical model of milling dynamics considering the regenerative chatter is expressed as a state space form. With the time delay being separated equally into a limited amount of elements, the time series expression is obtained by interpolating the integral nonhomogeneous term using linear approximation. Then, 2^N order algorithm is adopted to resolve the exponential term into a real matrix, which avoids the exponential matrix that has to be calculated each time in scanning the plane comprised of axial cutting depth and spindle speed. Lastly, the golden search instead of traditional sequential search is applied to seek the crucial axial cutting depths corresponding to different spindle speeds, which can improve computational efficiency remarkably. The verifications with two classic benchmark examples demonstrate that the proposed method has higher computational efficiency.

Keywords Milling process · Chatter stability prediction · Improved full-discretization method

1 Introduction

The milling chatter caused by improper selection of spindle speed and cutting depth has a seriously negative effect on machined efficiency, so it is of significance to make an accurate prediction of chatter stability prior to processing. Since Sridhar [1] proposed a simulation way to research on chatter stability diagram (SLD) of machine tool system in 1980s, many works have been done to promote the development of prediction technique.

At present, the existing approach for chatter stability forecast can be classified into two categories: one is analytic method and the other is numerical method. Altintas et al. [2] developed a zero-order analytical (ZOA) method using mean value of Fourier series of dynamic milling forces, and opened the first step of analytic algorithm. ZOA is fast and effective, but it can be not applicable for low radial immersion milling [13]. Later, Merdol et al. [3] presented a multi-frequency solution via bringing more harmonic frequencies of cutting forces into dynamic model. In order to enhance prediction accuracy, an improved multi-frequency is proposed in [4]. Ozturk [5] and Sun [6] achieved chatter free for the five axis ball end milling using the analytical method. Li Z et al. [7] extended ZOA into cutting stable region forecast of helical milling operation.

With the rapid growth of computational mathematics, quite a few numerical algorithms of constructing the high accuracy SLD have been developed, especially in recent years. Balachandran et al. [8] presented a unified mechanics-based dynamic model and studied the impact of nonlinear properties on the stability of the milling process. Li H et al. [9] put forward a new time domain criterion for the chatter stability analysis of the dynamic milling process. Insperger et al. [10] proposed the semi-discretization method (SDM) by approximating delay term of delayed differential equation (DDE), the discretization error of which comes up to $O(h^2)$. Zhou et al. developed a high order full-discretization method for prediction of milling stability in [11]. Long and his co-authors [12] analyzed the influence of the loss-of-contact effect and the tool and time delay effect on the stability region. Ding et al. [13] interpolated the time delay term and time periodic part of DDE and proposed a full-discretization method (FDM), which demonstrates higher arithmetic speed in comparison with SDM. Subsequently, Insperger et al. evaluated relations of SDM and FDM in [14], and pointed out that FDM is an improved SDM method since not every element in dynamic equation was discretized. Balachandran et al. [15] investigated

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the effect of variable spindle speed on stability in up- and down-milling processes. Zhang et al. [16] utilized a numerical differentiation method to predict chatter stability for high speed milling. Tang et al. [17] studied on SLD using an updated full-discretization method. Dai et al. [18] applied precise integration method for the chatter stability prediction of five axis ball-end milling. Li M et al. [19] brought forward a complete discretization scheme with discretizing all parts of the response of the system, including delay term, time domain term, parameter matrices using Euler's method (CDSEM), which is more efficient compared with FDM and SDM, while the discretization error run up to O(h^2). Li Z et al. [20] adopted the classical fourth-order Runge-Kutta method to approximate the terms of the DDE and promoted a Runge-Kutta-based complete discretization method (RKCDM), whose discretization error attains to O(h^5).

According to combing stability prediction methods, the following agreements can be reached:

- Firstly, the prediction accuracy of numerical solution is popular than ZOA method because the nonlinear factors in the milling process can be considered into the mathematics model [13].
- Secondly, FDM can achieve the prediction of chatter stability without any discretization error loss [13], while as pointed in [14, 19], the exponential term must be calculated every time in sweeping the range of spindle spend, which limits the calculation efficiency of FDM.
- Lastly, by interpolating the exponential term in the solution using the differential method, CDSEM exhibits higher calculation efficiency than FDM, but there is discretization error presented in CDSEM, which affects the prediction accuracy of stability [20].

Confronted with the conditions described above, it is urgent to seek a new method to strike a balance between FDM and CDSEM. In this paper, an improved full-discretization method (IFDM) is proposed to predict the chatter stability for the milling process, which can give consideration to calculation accuracy and efficiency at the same time. Hence, the rest of this paper is organized as follows. Section 2 shows the algorithm of IFDM. Section 3 makes the verification of IFDM in computational efficiency, convergence rate and prediction accuracy using one and two DOF milling benchmark examples. Some conclusions are drawn in Section 4.

2 Algorithm of the improved full-discretization method

Similar as FDM, the delay dynamic equation established from the milling process is expressed as state space form:

$$\dot{v}(t) = A_0 v(t) + A(t)v(t) - A(t)v(t-T) \tag{1}$$

where A_0 is a constant matrix consisting of the model parameters of the cutter system; $A(t)$ and $B(t)$ are periodic coefficient matrices.

Using $f(t)$ to represent $A(t)v(t) - A(t)v(t-T)$, the general solution for Eq. (1) can be obtained as

$$v(t) = e^{A_0\cdot(t-t_p)} v(t_{p+1}) + \int_{t_p}^{t} e^{A_0\cdot(t-\delta)} \cdot f(\delta) d\delta \tag{2}$$

Afterwards, dividing the period T into m parts equally, namely $T = m\cdot\tau$, the integral nonhomogeneous term $f(t)$ in the interval of $[t_p, t_{p+1}]$ can be approximated as

$$f(t) = r_0 + r_1(t - t_{p+1}) \tag{3}$$

where $r_0 = A(t_{p+1})v(t_{p+1}) - A(t_{p+1})v(t_{p+1} - m\cdot\tau)$

$$r_1 = \frac{1}{\tau}\left[A(t_{p+1})v(t_{p+1}) - A(t_{p+1})v(t_{p+1} - m\cdot\tau) - A(t_p)v(t_p) + A(t_p)v(t_p - m\cdot\tau)\right]$$

In order to make the expression concise, r_0 and r_1 are further exhibited as

$$r_0 = A_{p+1}v_{p+1} - A_{p+1}v_{p+1-m} \qquad r_1 = \frac{1}{\tau}\left(A_{p+1}v_{p+1} - A_{p+1}v_{p+1-m} - A_p v_p + A_p v_{p-m}\right)$$

Substituting Eq. (3) into Eq. (2), $v(t_{p+1})$ can be written as

$$v(t_{p+1}) = T_1\left[v(t_p) + A_0^{-1}r_0 - A^{-1}\tau r_1 + A_0^{-2}r_1\right] - A_0^{-1}\left(r_0 + A_0^{-1}r_1\right) \tag{4}$$

where T_1 equals to $e^{A_0\cdot\tau}$. Differing from exponential matrix existing directly in iteration formula of FDM, the next will utilize 2^N algorithm to decompose the exponential term into a real matrix.

Firstly, dividing τ into $\Lambda = 2^{20}$ parts equally, T_1 can be converted into:

$$T_1 = e^{A_0\cdot\tau} = \left(e^{A_0\cdot\frac{\tau}{\Lambda}}\right)^{\Lambda} = \left(e^{A_0\cdot\Delta t}\right)^{\Lambda} \tag{5}$$

Since $\Delta t = \frac{\tau}{\Lambda}$ is an extremely small time interval, $e^{A_0\cdot\Delta t}$ can be expanded with the truncated Taylor expansion:

$$\begin{aligned} e^{A_0\cdot\Delta t} &\approx I + A_0\Delta t + (A_0\Delta t)^2/2! + (A_0\Delta t)^3/3! \\ &+ (A_0\Delta t)^4/4! \\ &= I + T_a \end{aligned} \tag{6}$$

Combining Eq. (5) and Eq. (6), the following can be get

$$T_1 = (I + T_a)^{2^N} = \left[(I + T_a)^2\right]^{2^{N-1}} \tag{7}$$

Then, by multiplying N times, T_1 can be represented as the sum of identity matrix and the non-identity matrix.

$$For(i=1;i\leq N;i++)\quad T_a=2T_a+T_a\times T_a;T_1=I+T_a; \tag{8}$$

Setting $f_0=T_1A_0^{-1}-A_0^{-1}$, $f_1=(T_1A_0^{-2}-T_1A^{-1}\tau-A_0^{-2})/\tau$, $f_2=f_0+f_1$, Eq. (4) can be displayed as

$$\begin{aligned} v_{p+1} &= \left(T_1-f_1A_p\right)v_p+f_2A_{p+1}v_{p+1}-f_2A_{p+1}v_{p+1-m} \\ &\quad +f_1A_pv_{p-m} \end{aligned} \tag{9}$$

If $(I-f_2A_{p+1})$ is reversible, Eq. (9) can be presented as

$$\begin{aligned} v_{p+1} &= f_3\left(T_1-f_1A_p\right)v_p-f_3f_2A_{p+1}v_{p+1-m} \\ &\quad +f_3f_1A_pv_{p-m} \end{aligned} \tag{10}$$

where $f_3=(I-f_2A_{p+1})^{-1}$

Since no index matrix factor is included in iteration formula, Eq. (10) is much streamlined and simplified than FDM. To acquire state transition matrix, a n(m + 1) dimensional vector v_p is defined as

$$y_p=col\left(v_p,v_{p-1}\cdots v_{p+1-m},v_{p-m}\right)$$

The discrete map is illustrated as

$$y_{p+1}=C_py_p \tag{11}$$

where the coefficient matrix C_p can be constructed as follows:

$$C_p=\begin{bmatrix} PK & 0 & 0 & \cdots & 0 & RK1 & RK2 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{12}$$

where PK, $RK1$, and $RK2$ equals to $f_3(T_1-f_1A_p)$, $-f_3f_2A_{p+1}$, and $f_3f_1A_p$, respectively. C_p is determined by coupling Eq. (12) from $i=0$ to $k-1$ one by one.

$$v_p=\Phi v_0 \tag{13}$$

Lastly, the same as FDM and CDSEM, Floquet theory is also applied to judge the eigenvalues of the transition matrix Φ. However, instead of sequential search used in [13, 19], the golden search is carried out to seek the crucial cutting depths at different spindles via using the data including the modal parameters of cutter system, specific cutting forces coefficients, and cutter geometry. The detailed flow chart is revealed in Fig. 1.

Remark The most important differences between the proposed method and famous FDM, CDSEM are enumerated in two points. For one thing, FDM shows higher calculation accuracy without any discretization error loss in contrast to some algorithms such as SDM, CDSEM [13], but the exponential matrixes existing in parameters terms Φ_0, Φ_1, Φ_2, and Φ_3 must be calculated every times while sweeping axial cut depths and spindle speeds, which are very complicated and time-costing. To solve this problem, CDSEM applied Euler's method to discrete the exponential term, which improves the calculation efficiency, but makes the discretization error reach O(h^2) [20]. In the proposed method, the time series expression is attained by interpolating the integral rather than partial nonhomogeneous term. A real matrix is acquired by 2^N algorithm to replace the exponential term in iteration formula, which really discretizes all parts in contrast to FDM. For another, the golden search is utilized to seek the crucial axial depth of cut at different spindle speeds, which overcomes the shortcoming of sequential search used in classic methods that every point in the plane composed of spindle spend and axial cutting depth must be scanned. These are the primary reasons that the proposed method has far higher calculation efficiency than FDM and CDSEM, which will be demonstrated in next section using two classic examples.

3 Verification and comparison

One and two degrees of freedom (DOF) milling dynamic models will be used as benchmarks to verify the characteristics of IFDM, where all the parameters are coming from FDM in [13]. Besides, the programs in this paper run in a personal computer [Intel Core (TM) i5-4460, 3.2GHz, 3.2GB].

3.1 One DOF milling dynamic system

The dynamic equation for a single DOF milling system can be represented as

$$\ddot{x}(t)+2\xi\omega_n\dot{x}(t)+\omega_n^2x(t)=-\frac{wh(t)}{m_t}(x(t)-x(t-T)) \tag{14}$$

where ξ is the relative damping, ω_n is the angular natural frequency, w is axial depth of cut, and m_t is the modal mass. T is the time delay, which equals to the tooth passing period, namely $T=60/(N_fn)$; N_f is the number of teeth and n is the spindle speed (r/min); $h(t)$ is the cutting force coefficient:

$$h(t)=\sum_{j=1}^{N}g\left(\phi_j(t)\right)\left[K_t\cos\left(\phi_j(t)\right)+K_n\sin\left(\phi_j(t)\right)\right] \tag{15}$$

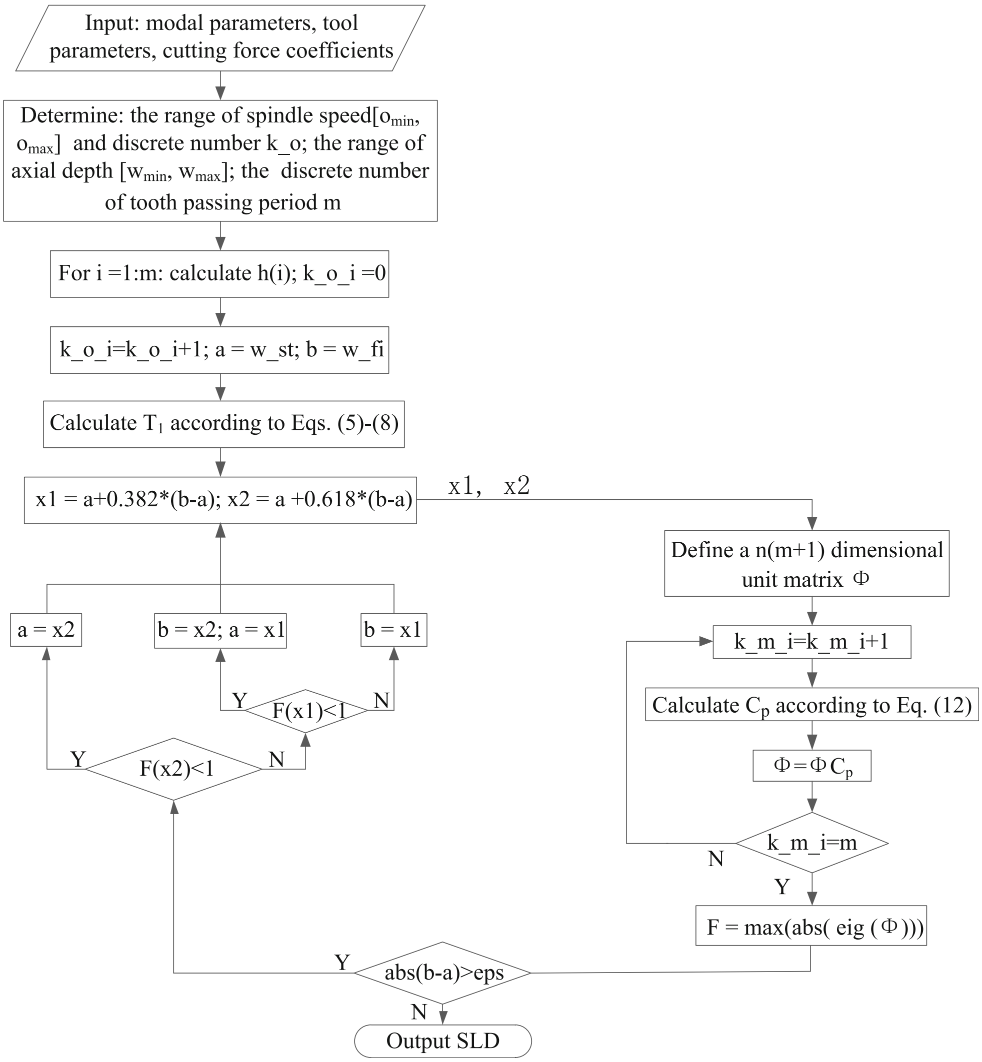


Fig. 1 Flow chart for obtaining SLD of the milling process using IFDM

where K_t and K_n are the tangential and the normal cutting force coefficients, respectively, and $\phi_j(t)$ is the angular position of the jth tooth:

$$\phi_j(t) = (2\pi \cdot n/60)\cdot t + (j-1)\cdot 2\pi/N_f \tag{16}$$

The function $g(\phi_j(t))$ is used to determine whether tooth j is in cutting or not, which is defined as

$$g(\phi_j) = \begin{cases} 1 & \phi_{\text{in}} < \phi_j < \phi_{\text{out}} \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where ϕ_{in} and ϕ_{out} is the start and exit angles of the jth tooth, separately. For up-milling, ϕ_{in} and ϕ_{out} equal to 0 and arcos(1-2a_e/D), separately; for down-milling, ϕ_{in} and ϕ_{out} equal to arcos(1-2a_e/D) and π, separately, where a_e is the radial depth of cut, and D is the diameter of the cutter.

Eq. (14) can be converted into a state space form as

$$\dot{v}(t) = A_0 v(t) + A(t)v(t) - A(t)v(t-T) \tag{18}$$

where A_0, $A(t)$ and $v(t)$ are illustrated as

$$A_0 = \begin{bmatrix} -\xi\omega_n & \dfrac{1}{m_t} \\ m_t(\xi\omega_n)^2 - m_t{\omega_n}^2 & -\xi\omega_n \end{bmatrix} \quad A(t)$$

$$= \begin{bmatrix} 0 & 0 \\ -wh(t) & 0 \end{bmatrix} \quad v(t) = \left[x(t), m_t\dot{x}(t) + m_t\xi\omega_n x(t)\right]^T$$

The parameters for the prediction of stability are given in Table 1, which are the same as [13]. Under the condition that the plane consisting of spindle speed and cutting depth is split into 400 × 200 sized grids, the simulation results and computational times with radial immersion ratios $a_e/D = 1$, 0.1 and 0.05 are displayed in Table 2. Because there is no need to calculate the exponential matrix in the cycle of sweeping every point in the range of cutting depth, the time-consuming sinks nearly 72 and 62% in comparison with FDM and CDSEM, respectively; moreover, the prediction results of IFDM are almost identical to those of FDM, and CDSEM, expect subtle differences.

Table 1 Parameters for stability lobe diagram

f_n(Hz)	ξ	m_t(Kg)	K_t(N/m^2)	K_n(N/m^2)	N_f	Direction	m
922	0.011	0.03993	6×10^8	2×10^8	2	down-milling	40

3.2 Two DOF milling dynamic system

The dynamic equation for two DOF milling system can be shown in Eq. (19).

$$\begin{bmatrix} m_t & 0 \\ 0 & m_t \end{bmatrix}\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix}+\begin{bmatrix} 2m_t\xi\omega_n & 0 \\ 0 & 2m_t\xi\omega_n \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} m_t{\omega_n}^2 & 0 \\ 0 & m_t{\omega_n}^2 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} wh_{xx}(t) & wh_{xy}(t) \\ wh_{yx}(t) & wh_{yy}(t) \end{bmatrix}\begin{bmatrix} -x(t)+x(t-T) \\ -y(t)+y(t-T) \end{bmatrix} \tag{19}$$

where m_t is the modal mass, ξ is the relative damping, ω_n is the angular natural frequency, w is the depth of cut, and T is the time delay, which equals to the tooth passing period, namely $T=60/(N_f n)$, where N_f is the number of teeth and n is the spindle speed (r/min). $h_{xx}(t)$, $h_{xy}(t)$, $h_{yx}(t)$, and $h_{yy}(t)$ are defined as

$$\begin{cases} h_{xx} = \sum\limits_{j=1}^{N_f} g(\phi_j(t))\sin(\phi_j(t))\left[K_t\cos(\phi_j(t))+K_n\sin(\phi_j(t))\right] \\ h_{xy} = \sum\limits_{j=1}^{N_f} g(\phi_j(t))\cos(\phi_j(t))\left[K_t\cos(\phi_j(t))+K_n\sin(\phi_j(t))\right] \\ h_{yx} = \sum\limits_{j=1}^{N_f} -g(\phi_j(t))\sin(\phi_j(t))\left[K_t\sin(\phi_j(t))-K_n\cos(\phi_j(t))\right] \\ h_{yy} = \sum\limits_{j=1}^{N_f} -g(\phi_j(t))\cos(\phi_j(t))\left[K_t\sin(\phi_j(t))-K_n\cos(\phi_j(t))\right] \end{cases} \tag{20}$$

Eq. (20) can be transformed into a state space form:

$$\dot{v}(t) = A_0 v(t) + A(t)v(t) - A(t)v(t-T) \tag{21}$$

Table 2 Comparison of SLDs from different methods for single DOF milling model

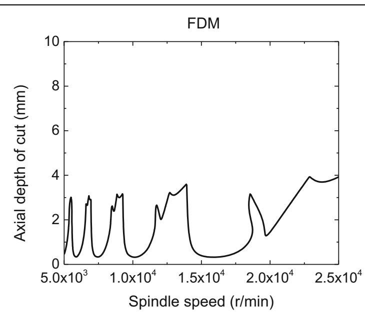
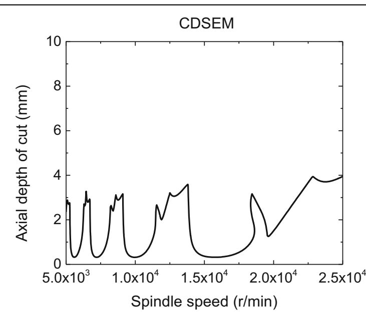
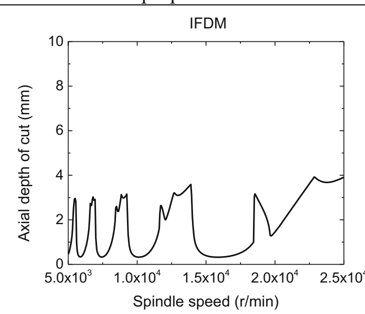
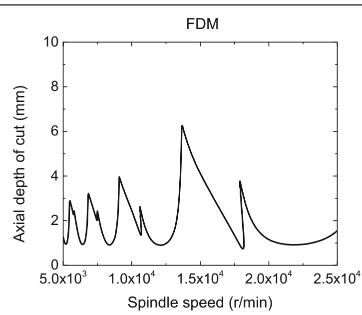
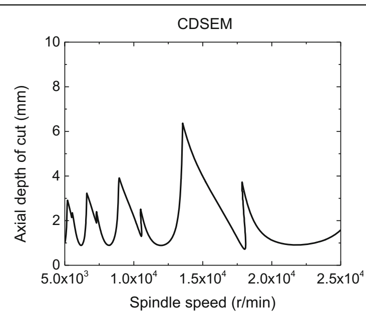
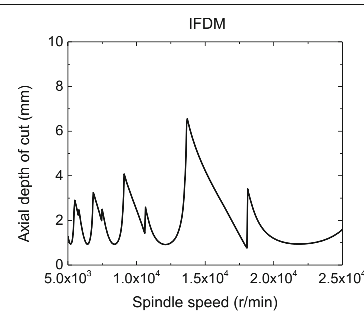
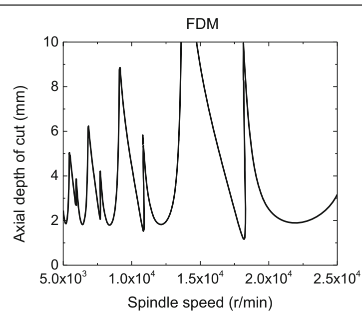
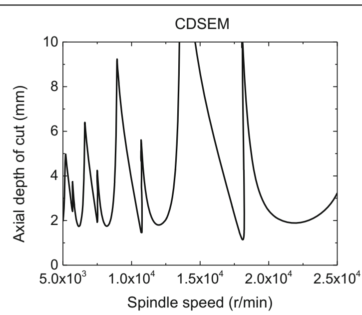
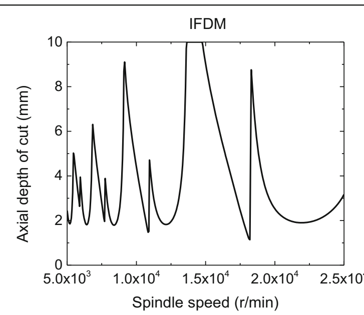
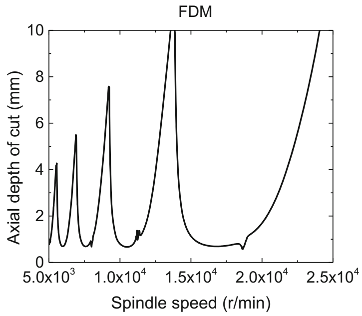
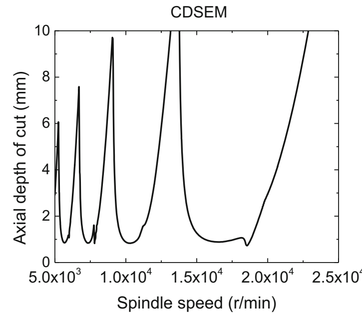
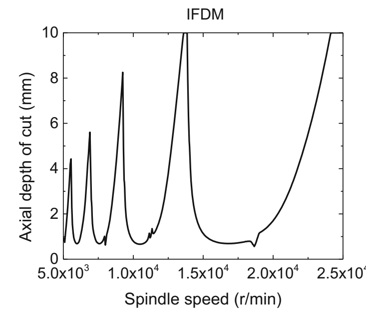
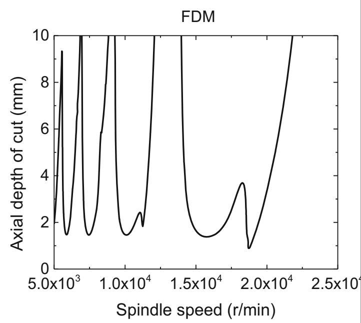
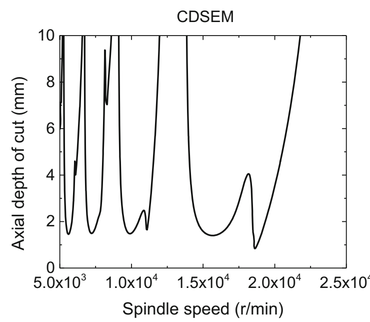
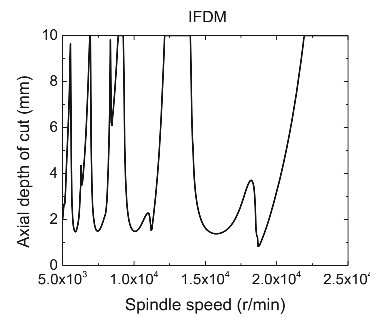
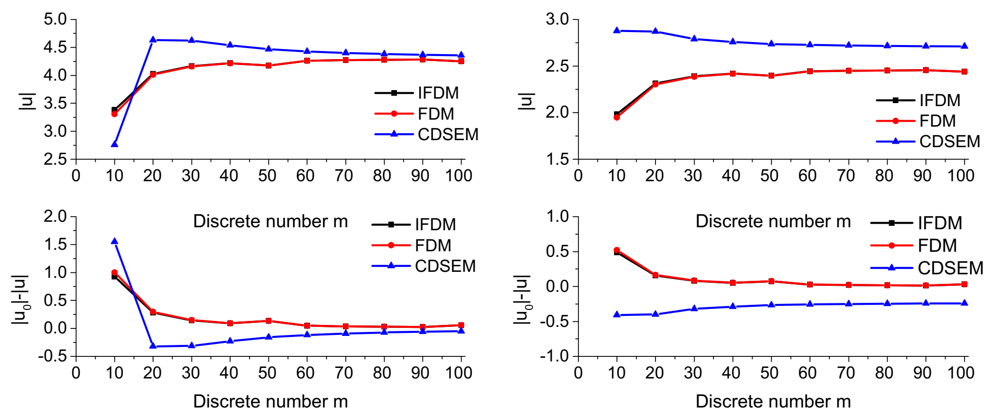
m=40	FDM	CDSEM	The proposed method
a_e/D=1			
Time cost	125.6s	104.1s	37.7s
a_e/D= 0.1			
Time cost	109.8s	79.1s	30.8s
a_e/D = 0.05			
Time cost	1160.0s	78.4s	30.7s

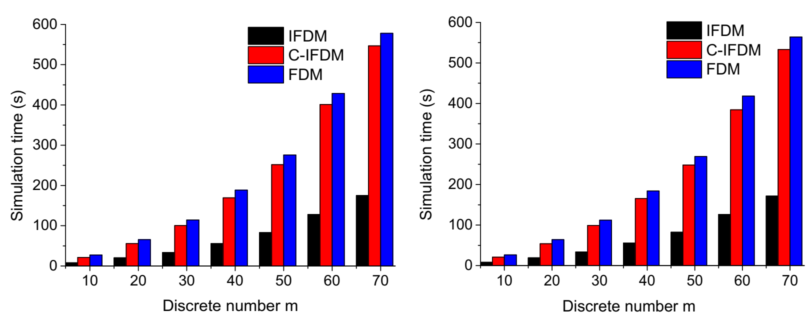
Table 3 Comparison of SLDs from different methods for two DOF milling model

m=40	FDM	CDSEM	The proposed IFDM
$a_e/D=$ 0.1			
Time cost	184.0s	153.0s	55.5s
$a_e/D=$ 0.05			
Time cost	188.3s	156.5s	55.7s



(a) w =1mm, $|u_0|$ = 4.3097 (b) w = 0.5mm, $|u_0|$ = 2.4707

Fig. 2 Comparison of convergence rate among IFDM, FDM, and CDSEM



(a) a_e/D =0.1 (b) a_e/D=0.05

Fig. 3 Comparison of simulation times among IFDM, FDM, and CDSEM where A_0, $A(t)$, and $v(t)$ are exhibited as follows:

$$A_0 = \begin{bmatrix} -\xi\omega_n & 0 & 1/m_t & 0 \\ 0 & -\xi\omega_n & 0 & 1/m_t \\ m_t{\omega_n}^2(\xi^2-1) & 0 & -\xi\omega_n & 0 \\ 0 & m_t{\omega_n}^2(\xi^2-1) & 0 & -\xi\omega_n \end{bmatrix} \quad A(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -wh_{xx}(t) & -wh_{xy}(t) & 0 & 0 \\ -wh_{yx}(t) & -wh_{yy}(t) & 0 & 0 \end{bmatrix}$$

$$v(t) = [x(t), y(t), m_t\dot{x}(t) + m_t\xi\omega_n x(t), m_t\dot{y}(t) + m_t\xi\omega_n y(t)]^T$$

With all parameters for prediction of stability here coming from one DOF milling dynamic system, SLDs attained by IFDM, FDM, and CDSEM at radial immersion ratios $a_e/D = 0.1$ and 0.05 are exposed in Table 3. There is no doubt that the simulation results from IFDM are almost same as those from FDM, and CDSEM except slight differences.

In order to compare the convergence rate of IFDM with FDM, CDSEM, enable the radial cutting depth a_e equal to the diameter of the cutter D to avoid intermittent milling process. Besides, set the spindle speed as $n = 1 \times 10^4$ rpm, and the axial depths of cut w are chosen as 1 and 0.5 mm, respectively. Supposing $|u|$ to represent the crucial eigenvalues of the transition matrix Φ, the $|u_0|$ from FDM at the discrete number equal to 500 ($m = 500$) is regarded as the exact value. Figure 2 reveals the relation of $|u|$, $|u_0| - |u|$and discrete number m. It is no denying that no matter what radial immersion ratio is, the convergence rate of IFDM is almost the same as FDM and higher than CDSEM.

For the sake of estimating the computational efficiency of 2^N algorithm, IFDM with sequential search is named as C-IFDM. The detailed comparisons in simulation time among the proposed IFDM, C-IFDM, FDM, and CDSEM are given in Fig. 3. It is quite clear that under the same conditions, the computational efficiency of IFDM is apparently higher than that of FDM and CDSEM. Taking the discrete number m equal to 40 for an instance, IFDM can be dropped about 64% computation time than CDSEM; 70% than FDM in which 2^N algorithm contributes nearly 10%. Further investigation on the computational efficiency of IFDM and C-IFDM indicates that golden search decreases by about 60% calculation time than sequential search used in FDM.

4 Conclusions

An improved full-discretization method (IFDM) is presented in this paper to make a chatter stability prediction for the milling process. So as to avoid the discretization error caused by numerical difference, the direct integration scheme is applied to acquire the response of the system from the delay dynamic equation at first. In each small time interval, the integral nonhomogeneous term is approximated by means of linear interpolation. After utilizing 2^N algorithm to dissolving exponential term into a real matrix, an iteration formula with the complete discrete form is derived. The crucial axial cutting depths in the range of spindle speeds are sought out by the golden search rather than traditional sequential search. Comparisons with FDM [13] and complete discretization scheme with Euler's method (CDSEM) [19] have been conducted using two classic benchmark examples. The simulation results indicate the proposed method can reduce about 60–70% computational time.

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