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ORIGINAL ARTICLE

Geometric error modeling and compensation for five-axis CNC gear profile grinding machine tools

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Abstract The relative position and orientation deviation between grinding wheel and workpiece caused by geometric errors affect the machining accuracy of five-axis CNC gear profile grinding machine tools directly. Therefore, geometric error modeling and compensation are presented according to homogeneous transformation and differential motion matrix based on the multi-body system theory for the accuracy enhancement of the machine tools. Firstly, the open kinematic chain and the ideal homogeneous transformation matrix from workpiece to grinding wheel are established according to the topological structure of the CFXZAY-type, five-axis gear profile grinding machine tools and the basic homogeneous transformation matrix between coordinate frames. Secondly, the homogenous transformation matrices of linear pairs and rotary pairs with geometric errors are calculated, and the relationships of the error propagation from workpiece to grinding wheel are acquired as well on the basis of 37 geometric error components analysis. The geometric error models including position and orientation errors of grinding wheel in workpiece coordinate system are obtained with matrix multiplication by using small-angle approximation and ignoring the second-order and high-order error terms. Then, the Jacobian is obtained by using transforming differential motion matrix of each motion axis to compensate the integrated error components of the grinding wheel, which can make the compensation effective and convenient with using the corresponding homogeneous transformation matrix. Finally, error measurement, error compensation, and machining experiments are carried out on a five-axis CNC gear profile grinding machine tool SKMC-1200W/10 to verify the applicability and effectiveness of the proposed error modeling, error compensation, and research approach.

Keywords Geometric errors · Profile grinding · Error components · Error compensation · Topological errors

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1 Introduction

With the rapid development of technology and science, continuously improved product, the demands of machining precision requirements are increasingly high. Precision machining and ultra-precision machining technology have become the research hot spot of manufacturing technology and one of the most important development directions. As the typical high-precision machining equipment, five-axis CNC gear profile grinding machine tools are widely used in gear machining that needs high precision and high efficiency. Nevertheless, the machining accuracy of the machine tools is influenced by many factors, such as geometric errors, thermal errors, servo tracking error, and vibration error. Research showed that geometric errors and thermal errors take part in 60% of the machining errors [1]. Because of the stability, repeatability, and measurability, modeling and compensation of geometric errors are an effective way to improve the machining accuracy of machine tools [2].

Recently, many scholars have conducted a number of studies in regards to the geometric error modeling of machine tools and developed many theories and methods, such as screw theory, product of exponential method (POE), multi-body system theory, differential transform theory, matrix summation approach, and orthogonal polynomials. Zhao et al. presented a linear geometric error model expressed as error twists of every axis and proposed the position errors and orientation errors of five-axis machine tools based on screw theory, though the squareness errors were not included in the model [3]. He et al. presented a generic error model for serial-robot calibration in explicit form based on POE formula, which was proved to be complete, minimal, and continuous by simulation [4]. Fu et al. established an integrated POE model of multi-axis machine tools, which combines the method based on the definition of geometric errors to obtain the three twists of each axis and the adjoint matrix through coordinate system transformation, and verified the effectiveness and accuracy of the model by experiments [5]. Chen et al. proposed a geometric error model with differential transform theory for multi-axis CNC machine tools, and they applied a Jacobian matrix to describe the relationship between the tool pose and the compensation error vector for compensation on a five-axis machine tool to test and verity the model [6]. Lin et al. implemented the kinematic equation into six components with clear physical meaning using the matrix summation approach for geometric error model of five-axis machine tools to reduce the calculations substantially and make model manageable and understandable [7]. Fan et al. presented a model that included thermal error and geometric error of five-axis machine tools based on orthogonal polynomials and transformed the polynomial regression into multiple linear regressions with the application of the orthogonal polynomials to calculate it easily [8]. However, the monohydric orthogonal polynomial table and the variance table were needed in this method. Moreover, the 30 basic error components of the machine tools are confronted with large computational quality in the modeling process.

Among them, homogeneous transformation matrices of multi-body system theory are popular used for the modeling of machine tools [9]. Multi-body system theory is highly effective for the major approaches of geometric error modeling, which is established with the multiplication of homogeneous transformation matrices. The modeling process is convenient, intuitive, and universal to the matrices which contain ideal and actual error matrices of every axis. What is more, the model could reflect the geometric error influences of every axis on the integrated errors with homogeneous coordinate transformation according to the kinematic transmission chain of machine tools. Thus, the error models are realistically significant to the design, manufacture, and machining accuracy of machine tools. Kong et al. developed an integrated kinematic error model for ultra-precision raster milling by using the surface generation mechanism based on HTM of multi-body system theory, which made the machining error budgets of multi-axis machining systems more convenient and applicable [10]. Okafor et al. established a geometric error model by using homogeneous coordinate transformation according to rigid body kinematics and small-angle approximation of the errors for three-axis machine tools with linear error, angular error, and squareness error [11]. Jun et al. provided a parameterized geometric error model using the homogeneous transformation matrices based on the rigid body kinematics to analysis the errors and the methodology of online measurement to improve the machining accuracy acceptably [12]. Chen et al. established a volumetric error model including 37 error components based on homogeneous transformation matrix and rigid body kinematics with high accuracy and effectiveness for five-axis machine tools design [13]. Zhu et al. regarded five-axis machine tools as rigid multi-body system and presented an effective and applicable integrated geometric error model, which did not require special measurement instruments, including 21 translational error parameters associated with linear-motion axes and 6 angular error parameters for each rotation axis [14]. The corresponding prototype software system was also developed to validate the model. Fan et al. developed a universal kinematic error modeling method and a corresponding analysis method for NC machine tools based on multi-body system and derived the essential condition for precision machining through mathematical equations. [15].

Geometric errors of machine tools are the major contributor of total errors, and error compensation technique is one of the most effective methods to improve the machining accuracy. Lei and Hsu presented a spherical test algorithm to estimate the measured link errors, analyzed the singularity problems, and investigated a real-time error compensation method to improve the overall position accuracy of five-axis CNC machine tools dramatically [16]. Huang et al. merged iterative compensation algorithm into the post-processor through NC code modification, which was calculated by utilizing the Newton method, and generated an effective geometric error-compensated NC program of a specific five-axis machine tool [17]. Ahn et al. developed a volumetric error compensation method to minimize the overall volumetric error in simultaneous cutting using weighted least squares for a multi-axis machining center [18]. Khan et al. put forward an efficient algorithm with the systematic error model using recursive method to compensate the overall effect of all position-independent and position-dependent geometric errors of five-axis machine tools by modifying the NC codes [19]. Habibi et al. selected tool deflection estimation model and geometrical error analyzing methods to develop complementary algorithms for the errors compensation and then generated compensated tool path NC program by tracing the initial tool path to compensate geometrical errors and improve the accuracy of machined features using the software [20]. Cui et al. investigated the framework of error compensation software system and the algorithms related to error prediction as positioning, linear, and circular interpolation movement error compensation to improve the movement accuracy of CNC machine tools [21]. Peng et al. proposed a geometric error compensation of universal post-processing algorithm with iteration methods to reduce the difficulty of inverse kinematic models and improve the calculation speed and the accuracy of compensation [22]. However, besides singularity problems, the differential calculation including is also huge and complex.

Homogeneous transformation between different coordinate frames, which is used to compute the error contribution of every axis to the position and orientation accuracy of end effector, is popular for geometric error modeling of multi-axis milling machine tools. Moreover, it is also used to build Jacobian matrix for error compensation of the machine tools with the models. While in the field of multi-axis gear profile grinding machine tools, the researches and studies on the corresponding theory and method for geometric error modeling and compensation are way out of sufficiency. It not only promotes the machining quality of workpiece but also improves the processing technology progress significantly [23].

Therefore, a systematic error model is established to integrate the geometric errors with homogeneous transformation between different frames based on multi-body system theory to improve the machining accuracy of CNC gear profile grinding machine tools. First, the transforming matrices of every axis in ideal status between coordinate frames will be established according to the theory of homogeneous transformation. Next, the geometric error components of CNC gear profile grinding machine tools will be analyzed, and the transformation matrix of linear pairs and rotary pairs sequentially can be proposed based on the topological structure and forward kinematics of the tools. So, the integrated geometric error model of the machine tools in workpiece coordinate system, including position errors and orientation errors of grinding wheel, will be established according to forward kinematics. Then, Jacobian matrix of the machine tools will be calculated to compensate the error components in grinding wheel coordinate frame based on the integrated geometric error model. Finally, experiments will be demonstrated on a five-axis CNC gear profile grinding machine tool to verify the accuracy of the error modeling and compensation.

2 Geometric error modeling with homogeneous transformation

2.1 Homogeneous transformation matrix between coordinate frames

There are two spatial coordinate systems S_n and S_m, as shown in Fig. 1. The coordinates of an arbitrary point P in coordinate systems S_n and S_m are (x_n, y_n, z_n) and (x_m, y_m, z_m), respectively. When the coordinate system S_n translates along its X_n axis by

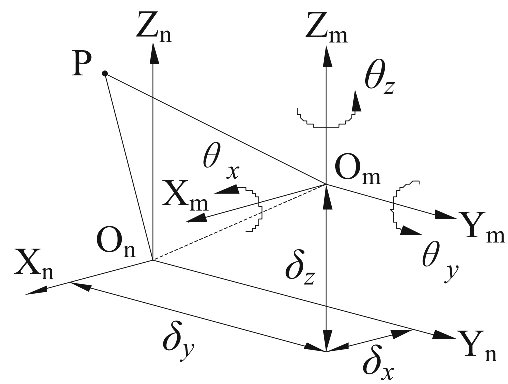

Fig. 1 Homogeneous transformation between coordinate frames

δ_x to coordinate system S_m, the relationship between (x_n, y_n, z_n) and (x_m, y_m, z_m) of point P can be represented as

$$\begin{bmatrix} x_n \\ y_n \\ z_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \delta_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_m \\ y_m \\ z_m \\ 1 \end{bmatrix} = {}^{\mathrm{m}}_{n}M^{n}(X) \cdot \begin{bmatrix} x_m \\ y_m \\ z_m \\ 1 \end{bmatrix} \quad (1)$$

where ${}^{\mathrm{m}}_{n}M^{n}$ (X) is the homogeneous transformation matrix of S_n translates along its X_n axis by δ_x to S_m.

That is,

$${}^{\mathrm{m}}_{n}M^{n}(X) = \begin{bmatrix} 1 & 0 & 0 & \delta_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

Similarly, the homogeneous transformation matrices ${}^{\mathrm{m}}_{n}M^{n}$ (Y) and ${}^{\mathrm{m}}_{n}M^{n}$ (Z) of S_n translate along its Y_n axis and Z_n axis by δ_y and δ_z to S_m, respectively, can be represented as

$${}^{\mathrm{m}}_{n}M^{n}(Y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

$${}^{\mathrm{m}}_{n}M^{n}(Z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \delta_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

When coordinate system S_n rotates around its X_n axis by θ_x to coordinate system S_m, the relationship between (x_m, y_m, z_m) and (x_n, y_n, z_n) of the point P can be represented as

$$\begin{bmatrix} x_n \\ y_n \\ z_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_m \\ y_m \\ z_m \\ 1 \end{bmatrix} = {}^{\mathrm{m}}_{n}M^{n}(A) \cdot \begin{bmatrix} x_m \\ y_m \\ z_m \\ 1 \end{bmatrix} \quad (5)$$

where ${}^{\mathrm{m}}_{n}M^{n}$ (A) is the homogeneous transformation matrix of S_n rotates around its X_n axis by θ_x to S_m.

That is,

$$ {}_{n}^{\mathrm{m}}M^{n}(A)=\begin{bmatrix}1&0&0&0\\0&\cos\theta_x&-\sin\theta_x&0\\0&\sin\theta_x&\cos\theta_x&0\\0&0&0&1\end{bmatrix} \tag{6}$$

Similarly, the homogeneous transformation matrices ${}_{n}^{\mathrm{m}}M^{n}$ (B) and ${}_{n}^{\mathrm{m}}M^{n}$ (C) of S_n rotates around its Y_n axis and Z_n axis by θ_y and θ_z to S_m, respectively, can be represented as

$$ {}_{n}^{\mathrm{m}}M^{n}(B)=\begin{bmatrix}\cos\theta_y&0&\sin\theta_y&0\\0&1&0&0\\-\sin\theta_y&0&\cos\theta_y&0\\0&0&0&1\end{bmatrix} \tag{7}$$

$$ {}_{n}^{\mathrm{m}}M^{n}(C)=\begin{bmatrix}\cos\theta_z&-\sin\theta_z&0&0\\\sin\theta_z&\cos\theta_z&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \tag{8}$$

In a word, the coordinate system S_n translates along X_n axis, Y_n axis, and Z_n axis by δ_x, δ_y, and δ_z, respectively. Then, it rotates around X_n axis, Y_n axis, and Z_n axis by θ_x, θ_y, and θ_z to coordinate system S_m, respectively. So, the homogeneous transformation matrix can be obtained as follows:

$$ {}_{n}^{\mathrm{m}}M^{n}={}_{n}^{\mathrm{m}}M^{n}(X)\cdot{}_{n}^{\mathrm{m}}M^{n}(Y)\cdot{}_{n}^{\mathrm{m}}M^{n}(Z)\cdot{}_{n}^{\mathrm{m}}M^{n}(A)\cdot{}_{n}^{\mathrm{m}}M^{n}(B)\cdot{}_{n}^{\mathrm{m}}M^{n}(C)$$
$$=\begin{bmatrix}\cos\theta_y\cdot\cos\theta_z&-\cos\theta_y\cdot\sin\theta_z&\sin\theta_y&\delta_x\\\sin\theta_x\cdot\sin\theta_y\cdot\cos\theta_z+\cos\theta_x\cdot\sin\theta_z&\cos\theta_x\cdot\cos\theta_z-\sin\theta_x\cdot\sin\theta_y\cdot\sin\theta_z&-\sin\theta_x\cdot\cos\theta_y&\delta_y\\\sin\theta_x\cdot\sin\theta_z-\cos\theta_x\cdot\sin\theta_y\cdot\cos\theta_z&\cos\theta_x\cdot\sin\theta_y\cdot\sin\theta_z+\sin\theta_x\cdot\cos\theta_z&\cos\theta_x\cdot\cos\theta_y&\delta_z\\0&0&0&1\end{bmatrix} \tag{9}$$

When θ_x, θ_y, θ_z are small enough, the second-order and high-order minuteness can be ignored by using small-angle approximation. After that, Eq. (9) can be represented as

$$ {}_{n}^{\mathrm{m}}M^{n}=\begin{bmatrix}1&-\theta_z&\theta_y&\delta_x\\\theta_z&1&-\theta_x&\delta_y\\-\theta_y&\theta_x&1&\delta_z\\0&0&0&1\end{bmatrix} \tag{10}$$

Generally, ${}_{n}^{\mathrm{m}}M^{n}$ represents the homogenous transformation matrix of coordinate system S_m relative to S_n, and it is an important matrix used for geometric error modeling of machine tools.

2.2 Forward kinematics of five-axis CNC gear profile grinding machine tools

Forward kinematics is widely used to devote the kinematic relationship between tool and workpiece of machine tools. As a typical five-axis machine tool, CNC gear profile grinding machine tool can be treated as an open kinematic chain from workpiece to grinding wheel. The topological structure of CFXZAY-type gear profile grinding machine tool, which consists of three linear axes and two rotary axes, is shown as Fig. 2. The order of its open kinematic chain is workpiece → C axis → framework → X axis → Z axis → A axis → Y axis → grinding wheel.

According to Fig. 2 and Eq. (8), the homogenous transformation matrix ${}_{F}^{C}M^{F}$ of C axis relative to framework in ideal status can be represented as

$$ {}_{F}^{C}\mathrm{M}^{F}=\begin{bmatrix}\cos\gamma&-\sin\gamma&0&0\\\sin\gamma&\cos\gamma&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

where symbol γ represents the ideal rotation angle of C axis relative to its zero position.

Thus, the transformation matrix ${}_{F}^{C}M^{F}$ of framework relative to C axis can be represented as

$$ {}_{C}^{F}M^{C}=\left({}_{F}^{C}M^{F}\right)^{-1}$$

Similarly, the homogenous transformation matrices of X axis relative to framework, Z axis relative to X axis, A axis relative to Z axis, and Y axis relative to A axis in ideal status can be obtained as follows:

$$ {}_{F}^{X}M^{F}=\begin{bmatrix}1&0&0&x\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ {}_{X}^{Z}M^{X}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&z\\0&0&0&1\end{bmatrix},\ {}_{Z}^{A}M^{Z}=\begin{bmatrix}1&0&0&0\\0&\cos\alpha&-\sin\alpha&0\\0&\sin\alpha&\cos\alpha&0\\0&0&0&1\end{bmatrix},\ {}_{A}^{Y}M^{A}=\begin{bmatrix}1&0&0&0\\0&1&0&y\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

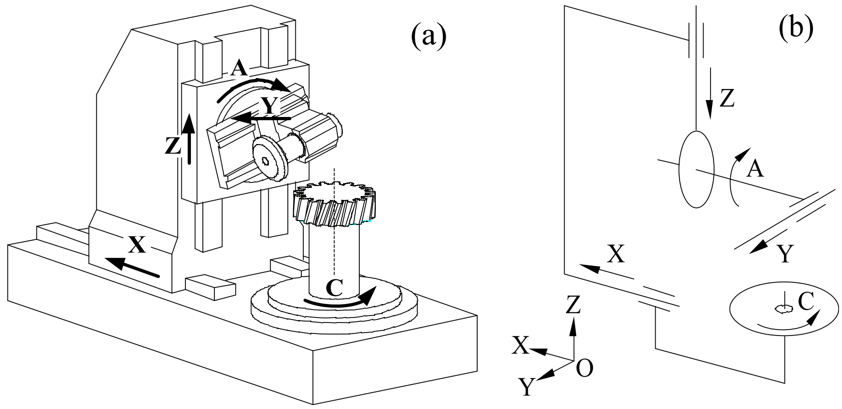

Fig. 2 The geometrical structure (**a**) and the topological structure (**b**) of the CFXZAY-type, five-axis gear profile grinding machine tools

where symbols x, y, and z represent the ideal movement distance of X axis, Y axis, and Z axis relative to their zero positions, respectively, and symbol α represents the ideal rotation angle of A axis relative to its zero position.

Therefore, based on the kinematic chain order of the machine tool, the ideal homogenous transformation matrix of grinding wheel relative to workpiece can be obtained as

$$
\begin{aligned}
&{}_{w}^{g}M^{w} = {}_{w}^{C}M^{w}\cdot{}_{C}^{F}M^{C}\cdot{}_{F}^{X}M^{F}\cdot{}_{X}^{Z}M^{X}\cdot{}_{Z}^{A}M^{Z}\cdot{}_{A}^{Y}M^{A}\cdot{}_{Y}^{g}M^{Y}\\
&=\begin{bmatrix}\cos\gamma & \cos\alpha\cdot\sin\gamma & -\sin\alpha\cdot\sin\gamma & y\cdot\cos\alpha\cdot\sin\gamma+x\cdot\cos\gamma\\ -\sin\gamma & \cos\alpha\cdot\cos\gamma & -\sin\alpha\cdot\cos\gamma & y\cdot\cos\alpha\cdot\cos\gamma - x\cdot\sin\gamma\\ 0 & \sin\alpha & \cos\alpha & z+y\cdot\sin\alpha\\ 0 & 0 & 0 & 1\end{bmatrix}
\end{aligned}
\tag{11}
$$

where ${}_{w}^{g}M^{w}$ represents the forward kinematics of the machine tool, which can obtain the ideal grinding wheel position and orientation in coordinate system of workpiece, and ${}_{w}^{C}M^{w}$ and ${}_{Y}^{g}M^{Y}$ are identity matrices because of the workpiece and the grinding wheel are attached with C axis and Y axis, respectively.

2.3 Geometric error modeling for the machine tools

According to the structure and components of the machine tools, there are 33 geometric error components for three linear axes and two rotary axes, which are defined based on ref. [17] and shown as Table 1. Moreover, Fig. 3ac– shows the six basic error components of X axis and C axis, three squareness errors of linear axis, respectively. Δ_{jk} and ε_{jk} ($j = x, y, z$; $k = x, y, z, a, c$) are the linear errors and angular errors of axis k along the j direction, respectively.

Based on the homogenous transformation matrix, geometric error modeling of every axis could be transformed to grinding wheel coordinate system. In this way, the error effect of every axis to geometric errors of the machine tool can be obtained, and then, the integrated error of it in grinding wheel coordinate system can be calculated by accumulating the errors together. After that, to obtain the grinding wheel position and orientation errors, forward kinematics is integral to transform the integrated errors to workpiece coordinate system.

2.3.1 Homogenous transformation matrix of linear pairs

When geometric errors of X axis exist, the homogenous transformation matrix ${}_{F}^{X}M^{F}{}_{e}$ of X axis relative to framework with errors can be represented as

$$
{}_{F}^{X}M^{F}{}_{e} = \begin{bmatrix} {}_{F}^{X}R_{3\times3} & {}_{F}^{X}P_{3\times1}\\ 0_{1\times3} & 1_{1\times1}\end{bmatrix} \tag{12}
$$

where ${}_{F}^{X}R_{3\times3}$ and ${}_{F}^{X}P_{3\times1}$ are the rotary and translation transformation matrices of X axis relative to framework, respectively. They are represented as

$$
{}_{F}^{X}R_{3\times3} = \begin{bmatrix} 1 & -\varepsilon_{zx} & \varepsilon_{yx}\\ \varepsilon_{zx} & 1 & -\varepsilon_{xx}\\ -\varepsilon_{yx} & \varepsilon_{xx} & 1\end{bmatrix} \tag{13}
$$

$$
{}_{F}^{X}P_{3\times1} = \boldsymbol{x} + \boldsymbol{\delta}_{x}\cdot\boldsymbol{s}_{x} \tag{14}
$$

where symbol $\boldsymbol{x}$ presents the given motion of guide rail of the machine tool along X axis, $\boldsymbol{x} = [x, 0, 0]^{T}$; symbol $\boldsymbol{\delta}_{x}$ is the linear error of X axis along the X, Y, and Z directions respectively, $\boldsymbol{\delta}_{x} = [\delta_{xx}, \delta_{yx}, \delta_{zx}]^{T}$; and symbol $\boldsymbol{s}_{x}$ is the squareness error of linear axis to the reference axis planes Y-X and Z-X, $\boldsymbol{s}_{x} = [0, s_{yx}, s_{zx}]^{T}$.

Table 1 Geometric error components of the machine tools

Motion axis	Linear error	Angular error	Squareness error
X	δ_{xx}, δ_{yx}, δ_{zx}	ε_{xx}, ε_{yx}, ε_{zx}	
Y	δ_{xy}, δ_{yy}, δ_{zy}	ε_{xy}, ε_{yy}, ε_{zy}	s_{xy}, s_{zy}
Z	δ_{xz}, δ_{yz}, δ_{zz}	ε_{xz}, ε_{yz}, ε_{zz}	s_{xz}
A	$\delta_{x\alpha}$, $\delta_{y\alpha}$, $\delta_{z\alpha}$	$\varepsilon_{x\alpha}$, $\varepsilon_{y\alpha}$, $\varepsilon_{z\alpha}$	
C	δ_{xc}, δ_{yc}, δ_{zc}	ε_{xc}, ε_{yc}, ε_{zc}	

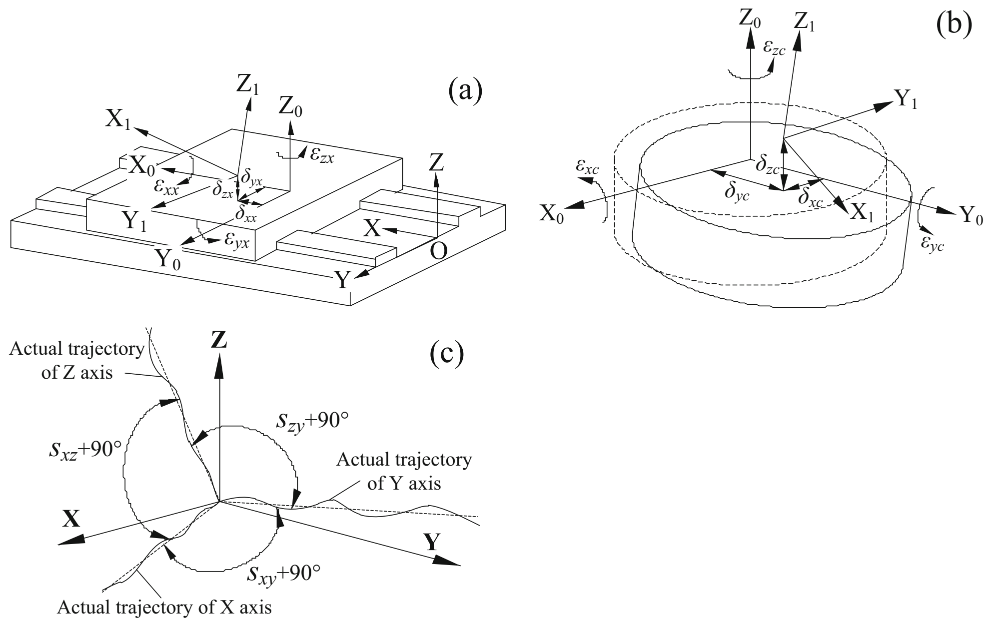

Fig. 3 **a** The six basic error components of *X* axis. **b** The six basic error components of *C* axis. **c** The three squareness errors of linear axis

Equations (13) and (14) into Eq. (12) can be obtained as

$$ {}^{X}_{F}M^{F}_{e} = \begin{bmatrix} 1 & -\varepsilon_{zx} & \varepsilon_{yx} & x+\delta_{xx} \\ \varepsilon_{zx} & 1 & -\varepsilon_{xx} & \delta_{yx} \\ -\varepsilon_{yx} & \varepsilon_{xx} & 1 & \delta_{zx} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15} $$

Due to the existence of squareness error s_{zx}, there is a displacement error $x{\cdot}s_{zx}$ in the minus X direction when the movement distance of Z axis from its zero position is z, including six basic error components of Z axis. Therefore, similar to Eq. (15), the transformation matrix ${}^{Z}_{X}M^{X}_{e}$ of Z axis relative to X axis with geometric errors is represented as

$$ {}^{Z}_{X}M^{X}_{e} = \begin{bmatrix} 1 & -\varepsilon_{zz} & \varepsilon_{yz} & -x\cdot s_{xz}+\delta_{xz} \\ \varepsilon_{zz} & 1 & -\varepsilon_{xz} & \delta_{yz} \\ -\varepsilon_{yz} & \varepsilon_{xz} & 1 & z+\delta_{zz} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16} $$

When the movement distance of Y axis from its zero position is y, the displacement errors in the minus X and Z directions are $y{\cdot}s_{xy}$ and $y{\cdot}s_{zy}$, respectively. So, the transformation matrix ${}^{Y}_{A}M^{A}_{e}$ of Y axis relative to A axis with geometric errors is represented as

$$ {}^{Y}_{A}M^{A}_{e} = \begin{bmatrix} 1 & -\varepsilon_{zy} & \varepsilon_{yy} & -y\cdot s_{xy}+\delta_{xy} \\ \varepsilon_{zy} & 1 & -\varepsilon_{xy} & y+\delta_{yy} \\ -\varepsilon_{yy} & \varepsilon_{xy} & 1 & -y\cdot s_{zy}+\delta_{zy} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17} $$

2.3.2 Homogenous transformation matrix of rotary pairs

The transformation matrices with geometric errors of C axis relative to framework with geometric errors can be represented as

$$ {}^{C}_{F}M^{F}_{e} = {}^{C}_{F}R_{1} \times {}^{C}_{F}R_{2} \tag{18} $$

where ${}^{C}_{F}R_{1}$ and ${}^{C}_{F}R_{2}$ are the rotary and prismatic transformation matrices of C axis relative to framework, respectively. ${}^{C}_{F}R_{1} = {}^{C}_{F}M^{F}$.

According to Eq. (10) and the components of the machine tool, the matrix ${}^{C}_{F}R_{2}$ can be represented as

$$ {}^{C}_{Z}R_{2} = \begin{bmatrix} 1 & -\varepsilon_{zc} & \varepsilon_{yc} & \delta_{xc} \\ \varepsilon_{zc} & 1 & -\varepsilon_{xc} & \delta_{yc} \\ -\varepsilon_{yc} & \varepsilon_{xc} & 1 & \delta_{zc} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19} $$

Then, the error homogenous transformation matrix of C axis relative to framework with geometric errors is obtained as

$$ {}^{C}_{F}M^{F}_{e} = \begin{bmatrix} \cos\gamma-\varepsilon_{zc}\sin\gamma & -\varepsilon_{zc}\cos\gamma-\sin\gamma & \varepsilon_{yc}\cos\gamma+\varepsilon_{xc}\sin\gamma & \delta_{xc}\cos\gamma-\delta_{yc}\sin\gamma \\ \sin\gamma+\varepsilon_{zc}\cos\gamma & -\varepsilon_{zc}\sin\gamma+\cos\gamma & \varepsilon_{yc}\sin\gamma-\varepsilon_{xc}\cos\gamma & \delta_{xc}\sin\gamma+\delta_{yc}\cos\gamma \\ -\varepsilon_{yc} & \varepsilon_{xc} & 1 & \delta_{zc} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{20} $$

Thus, the transformation matrix ${}^{F}_{C}M^{C}_{e}$ of framework relative to C axis with geometric errors can be presented as

$$ {}^{F}_{C}M^{C}{}_{e} = \left({}^{C}_{F}M^{F}{}_{e}\right)^{-1} $$

Similarly, the transformation matrix ${}^{A}_{Z}M^{Z}{}_{e}$ of A axis relative to Z axis with geometric errors is calculated as

$$ \begin{aligned} {}^{A}_{Z}M^{Z}{}_{e} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\varepsilon_{z\alpha} & \varepsilon_{y\alpha} & \delta_{x\alpha} \\ \varepsilon_{z\alpha} & 1 & -\varepsilon_{x\alpha} & \delta_{y\alpha} \\ -\varepsilon_{y\alpha} & \varepsilon_{x\alpha} & 1 & \delta_{z\alpha} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\varepsilon_{z\alpha} & \varepsilon_{y\alpha} & \delta_{x\alpha} \\ \varepsilon_{z\alpha}\cos\alpha+\varepsilon_{y\alpha}\sin\alpha & \cos\alpha-\varepsilon_{x\alpha}\sin\alpha & -\varepsilon_{x\alpha}\cos\alpha-\sin\alpha & \delta_{y\alpha}\cos\alpha-\delta_{z\alpha}\sin\alpha \\ \varepsilon_{z\alpha}\sin\alpha-\varepsilon_{y\alpha}\cos\alpha & \sin\alpha+\varepsilon_{x\alpha}\cos\alpha & -\varepsilon_{x\alpha}\sin\alpha+\cos\alpha & \delta_{y\alpha}\sin\alpha+\delta_{z\alpha}\cos\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{21} $$

2.3.3 Grinding wheel position and orientation error

As an end effector of open kinematic chain for the machine tool, grinding wheel influences the machining accuracy of workpiece directly. The integrated errors—position and orientation errors of grinding wheel relative to workpiece—can be obtained by geometric error modeling. In the ideal situation, there are no geometric errors between the grinding wheel coordinate system S_g and the workpiece coordinate system S_w. According to the order of kinematic chain, the homogenous transformation matrix of grinding wheel relative to workpiece can be obtained as Eq. (11). However, because of the existence of geometric errors, the homogenous transformation matrix ${}_{w}^{g}{}^{w}M_e$ of grinding wheel relative to workpiece with geometric errors can be represented as

$$\begin{aligned}{}_{w}^{g}M^{w}{}_{e} &= {}_{w}^{F}M^{w}{}_{e} \cdot {}_{F}^{g}M^{F}{}_{e} \\ &= \left({}_{w}^{C}M^{w}{}_{e} \cdot {}_{C}^{F}M^{C}{}_{e}\right) \\ &\quad \cdot \left({}_{F}^{X}M^{F}{}_{e} \cdot {}_{X}^{Z}M^{X}{}_{e} \cdot {}_{Z}^{A}M^{Z}{}_{e} \cdot {}_{A}^{Y}M^{A}{}_{e} \cdot {}_{Y}^{g}M^{Y}{}_{e}\right)\end{aligned} \tag{22}$$

In other words, ${}_{w}^{g}M^{w}{}_{e}$ can be treated as an error motion matrix superimposed on the ideal motion in practice. That is,

$${}_{w}^{g}M^{w}{}_{e} = {}_{w}^{g}E^{w} \cdot {}_{w}^{g}M^{w} \tag{23}$$

where ${}_{w}^{g}E^{w}$ is the integrated error matrix of grinding wheel relative to workpiece. According to the theory of small error assumption and Eq. (10), ${}_{w}^{g}E^{w}$ can be represented as

$${}_{w}^{g}E^{w} = \begin{bmatrix} 1 & -\eta_z & \eta_y & p_x \\ \eta_z & 1 & -\eta_x & p_y \\ -\eta_y & \eta_x & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{24}$$

where p_x, p_y, and p_z are the position errors and η_x, η_y, and η_z are the orientation errors of grinding wheel in workpiece coordinate system.

As a result, by using small-angle approximation and ignoring the second-order and high-order error terms, the geometric errors of the machine tool can be obtained as follows:

$$\begin{aligned}p_x = {}& \left(\delta_{xx}+\delta_{xy}+\delta_{xz}+\delta_{x\alpha}+\delta_{xc}\right)\cdot\cos\gamma-\left(\delta_{yx}+\delta_{yz}+\delta_{yc}\right)\cdot\sin\gamma+\left[\left(\varepsilon_{zy}+\delta_{z\alpha}\right)\cdot\sin\alpha-\left(\delta_{yy}+\delta_{y\alpha}\right)\cdot\cos\alpha\right]\cdot\sin\gamma+x \\ &\cdot\left[\varepsilon_{zx}+\varepsilon_{zz}+\left(\varepsilon_{zy}+\varepsilon_{z\alpha}\right)\cdot\cos\alpha+\left(\varepsilon_{yy}+\varepsilon_{y\alpha}\right)\cdot\sin\alpha\right]\cdot\sin\gamma+y\cdot\left[\left(\varepsilon_{zy}-s_{xy}\right)\cdot\cos\gamma-\left(\varepsilon_{xy}+s_{yz}\right)\cdot\sin\alpha\cdot\sin\gamma\right]-z \\ &\cdot\left[\varepsilon_{yz}\cdot\cos\gamma-\left(\varepsilon_{zy}+\varepsilon_{z\alpha}\right)\cdot\sin\alpha\cdot\cos\gamma+\left(\varepsilon_{yy}+\varepsilon_{y\alpha}\right)\cdot\cos\alpha\cdot\cos\gamma+\left(\varepsilon_{xy}+\varepsilon_{xz}+\varepsilon_{x\alpha}\right)\cdot\sin\gamma\right]\end{aligned}$$

$$\begin{aligned}p_y = {}& \left(\delta_{xx}+\delta_{xy}+\delta_{xz}+\delta_{x\alpha}+\delta_{xc}\right)\cdot\sin\gamma+\left(\delta_{yx}+\delta_{yz}+\delta_{yc}\right)\cdot\cos\gamma+\left[\delta_{y\alpha}\cdot\cos\alpha-\left(\varepsilon_{zy}+\delta_{z\alpha}\right)\cdot\sin\alpha\right]\cdot\cos\gamma-x \\ &\cdot\left[\varepsilon_{zx}+\varepsilon_{zz}+\left(\varepsilon_{yy}+\varepsilon_{y\alpha}\right)\cdot\sin\alpha+\left(\varepsilon_{zy}+\varepsilon_{z\alpha}\right)\cdot\cos\alpha\right]\cdot\cos\gamma+y\cdot\left[\left(\varepsilon_{zy}-s_{xy}\right)\cdot\sin\gamma+\left(\varepsilon_{xy}+s_{yz}\right)\cdot\sin\alpha\cdot\cos\gamma\right] \\ &+z\cdot\left\{\left(\varepsilon_{xy}+\varepsilon_{xz}+\varepsilon_{x\alpha}\right)\cdot\cos\gamma-\varepsilon_{yz}\cdot\sin\gamma+\left[\left(\varepsilon_{zy}+\varepsilon_{z\alpha}\right)\cdot\sin\alpha-\left(\varepsilon_{yy}+\varepsilon_{y\alpha}\right)\cdot\cos\alpha\right]\cdot\sin\gamma+\left(\varepsilon_{zx}+\varepsilon_{zz}+\varepsilon_{z\gamma}\right)\cdot\sin\alpha\cdot\cos\alpha\cdot\sin\gamma\right\}\end{aligned}$$

$$\begin{aligned}p_z = {}& \delta_{zx}+\delta_{zz}+\delta_{zc}+\left(\delta_{yy}+\delta_{y\alpha}\right)\cdot\sin\alpha+\left(\varepsilon_{zy}+\delta_{z\alpha}\right)\cdot\cos\alpha-x\cdot\left[s_{zx}-\varepsilon_{yx}-\varepsilon_{yz}-\left(\varepsilon_{yy}+\varepsilon_{y\alpha}\right)\cdot\cos\alpha+\left(\varepsilon_{zy}+\varepsilon_{z\alpha}\right)\cdot\sin\alpha\right]-y \\ &\cdot\left(\varepsilon_{x\alpha}+s_{yz}\cdot\cos\alpha\right)\end{aligned}$$

$$\eta_x = \left[-\left(\varepsilon_{yx}+\varepsilon_{yc}+\varepsilon_{yz}\right)-\left(\varepsilon_{yy}+\varepsilon_{y\alpha}\right)\cdot\cos\alpha+\left(\varepsilon_{zy}+\varepsilon_{z\alpha}\right)\cdot\sin\alpha\right]\cdot\sin\gamma+\left(\varepsilon_{xx}+\varepsilon_{xy}+\varepsilon_{xz}+\varepsilon_{x\alpha}+\varepsilon_{xc}\right)\cdot\cos\gamma$$

$$\eta_y = \left[\left(\varepsilon_{yx}+\varepsilon_{yz}+\varepsilon_{yc}\right)+\left(\varepsilon_{y\alpha}+\varepsilon_{yy}\right)\cdot\cos\alpha-\left(\varepsilon_{z\alpha}+\varepsilon_{zy}\right)\cdot\sin\alpha\right]\cdot\cos\gamma+\left(\varepsilon_{xx}+\varepsilon_{xy}+\varepsilon_{xz}+\varepsilon_{x\alpha}+\varepsilon_{xc}\right)\cdot\sin\gamma$$

$$\eta_z = \left(\varepsilon_{zx}+\varepsilon_{zz}+\varepsilon_{zc}\right)+\left(\varepsilon_{yy}+\varepsilon_{y\alpha}\right)\cdot\sin\alpha+\left(\varepsilon_{zy}+\varepsilon_{z\alpha}\right)\cdot\cos\alpha-\left[\left(\varepsilon_{yx}+\varepsilon_{yz}+\varepsilon_{yc}\right)\cdot\sin\gamma-\left(\varepsilon_{xx}+\varepsilon_{xy}+\varepsilon_{xz}+\varepsilon_{x\alpha}+\varepsilon_{xc}\right)\cdot\cos\gamma\right]\cdot\sin\alpha\cdot\sin(\alpha-\gamma)$$

3 Error compensation for geometric errors

On the basis of the geometric error model, error compensation is the compulsory method to improve the machining accuracy of five-axis CNC gear profile grinding machine tools. As hardware compensation method has many defects, software compensation method, which has gradually replaced the traditional hardware compensation, costs lower, and it is easier to realize. Compared to the traditional software compensation methods, Jacobian has the advantages of easy to use, fast calculation speed, high precision, good application, etc. Jacobian of the machine tools can be obtained by using transforming differential motion matrix of every motion axis relative to grinding wheel based on the analysis of error modeling process. After that, the integrated error components of grinding wheel can be compensated by the Jacobian instead of geometric errors relative to workpiece with much superiority and practical value.

Differential transformation between different coordinate systems, which can be represented as a 6 × 6 motion matrix, is an important basis of robotics [2]. Based on homogenous transformation matrix between different coordinate systems, the transforming differential motion matrix of coordinate system *m* relative to coordinate system *n* can be represented as [24]

$$D\left[{}^{m}_{n}M^{n}\right]=\begin{bmatrix} n_x & n_y & n_z & (p\times n)_x & (p\times n)_y & (p\times n)_z \\ o_x & o_y & o_z & (p\times o)_x & (p\times o)_y & (p\times o)_z \\ a_x & a_y & a_z & (p\times a)_x & (p\times a)_y & (p\times a)_z \\ 0&0&0& n_x & n_y & n_z \\ 0&0&0& o_x & o_y & o_z \\ 0&0&0& a_x & a_y & a_z \end{bmatrix} \tag{25}$$

Differential motion matrix can be used for calculating the influence of differential errors of grinding wheel, including differential translational errors and rotational errors, in workpiece coordinate system.

In accordance with the open kinematic chain order of the machine tools and Eq. (11), the homogenous transformation matrices of grinding wheel relative to the other motion parts also can be represented as

$$\begin{aligned} {}^{g}_{C}M^{C} &= {}^{F}_{C}M^{C}\cdot{}^{X}_{F}M^{F}\cdot{}^{Z}_{X}M^{X}\cdot{}^{A}_{Z}M^{Z}\cdot{}^{Y}_{A}M^{A}\cdot{}^{g}_{Y}M^{Y} \\ {}^{g}_{F}M^{F} &= {}^{X}_{F}M^{F}\cdot{}^{Z}_{X}M^{X}\cdot{}^{A}_{Z}M^{Z}\cdot{}^{Y}_{A}M^{A}\cdot{}^{g}_{Y}M^{Y} \\ {}^{g}_{X}M^{X} &= {}^{Z}_{X}M^{X}\cdot{}^{A}_{Z}M^{Z}\cdot{}^{Y}_{A}M^{A}\cdot{}^{g}_{Y}M^{Y} \\ {}^{g}_{Z}M^{Z} &= {}^{A}_{Z}M^{Z}\cdot{}^{Y}_{A}M^{A}\cdot{}^{g}_{Y}M^{Y} \\ {}^{g}_{A}M^{A} &= {}^{Y}_{A}M^{A}\cdot{}^{g}_{Y}M^{Y} \\ {}^{g}_{Y}M^{Y} &= {}^{g}_{Y}M^{Y} \end{aligned} \tag{26}$$

Based on Eqs. (25) and (26), differential motion matrix of every axis relative to the grinding wheel can be calculated readily and conveniently. Therefore, the differential motion matrix of *C* axis relative to grinding wheel can be obtained as follows:

$$D\left[{}^{g}_{C}M^{C}\right]=\begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & (-y\sin\alpha+z\cos\alpha)\sin\gamma & (-y\sin\alpha+z\cos\alpha)\cos\gamma & -y\cos\alpha-z\sin\alpha \\ \cos\alpha\cos\gamma & \cos\alpha\cos\gamma & -\sin\alpha & x\sin\alpha\sin\gamma-z\cos\gamma & x\sin\alpha\cos\gamma+z\sin\gamma & x\cos\alpha \\ \sin\alpha\sin\gamma & \sin\alpha\cos\gamma & \cos\alpha & -x\cos\alpha\sin\gamma+y\cos\gamma & -x\cos\alpha\cos\gamma-y\sin\gamma & x\sin\alpha \\ 0&0&0& \cos\gamma & -\sin\gamma & 0 \\ 0&0&0& \cos\alpha\sin\gamma & \cos\alpha\cos\gamma & -\sin\alpha \\ 0&0&0& \cos\alpha\sin\gamma & \sin\alpha\cos\gamma & \cos\alpha \end{bmatrix}$$

Similarly, the differential motion matrices of other axes relative to grinding wheel can be calculated as follows:

$$D\left[{}^{g}_{X}M^{X}\right]=\begin{bmatrix} 1&0&0&0&-y\sin\alpha+z\cos\alpha&-y\cos\alpha-z\sin\alpha \\ 0&\cos\alpha&-\sin\alpha&-z&0&0 \\ 0&\sin\alpha&\cos\alpha&y&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&\cos\alpha&-\sin\alpha \\ 0&0&0&0&\sin\alpha&\cos\alpha \end{bmatrix}$$

$$D\left[{}^{g}_{Z}M^{Z}\right]=\begin{bmatrix} 1&0&0&0&-y\sin\alpha&-y\cos\alpha \\ 0&\cos\alpha&-\sin\alpha&0&0&0 \\ 0&\sin\alpha&\cos\alpha&y&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&\cos\alpha&-\sin\alpha \\ 0&0&0&0&\sin\alpha&\cos\alpha \end{bmatrix}$$

$$D\left[{}^{g}_{A}M^{A}\right]=\begin{bmatrix} 1&0&0&0&0&-y \\ 0&1&0&0&0&0 \\ 0&0&1&y&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}$$

$$D\left[{}^{g}_{Y}M^{Y}\right]=I_{6\times 6} \tag{27}$$

It is generally known that only the motion axes of machine tools can be controlled for error compensation. After that, the influence of motion error of *Y* axis to grinding wheel can be calculated as

$$\begin{aligned} {}^{g}_{Y}E^{Y} &= D\left[{}^{g}_{Y}M^{Y}\right]\cdot[0,\ \delta_Y,\ 0,\ 0,\ 0,\ 0]^T \\ &= [0,\ \delta_Y,\ 0,\ 0,\ 0,\ 0]^T \end{aligned}$$

where δ_Y represents the motion error of *Y* axis and ${}^{g}_{Y}E^{Y}$ is the corresponding influence of the error on grinding wheel.

Similarly, the influences of the other motion axes to grinding wheel can be calculated as follows:

$$\begin{aligned} {}^{g}_{A}E^{A} &= D\left[{}^{g}_{A}M^{A}\right]\cdot[0,\ 0,\ 0,\ \varepsilon_A,\ 0,\ 0]^T \\ &= [0,\ 0,\ 0,\ y\varepsilon_A,\ \varepsilon_A,\ 0]^T \\ {}^{g}_{Z}E^{Z} &= D\left[{}^{g}_{Z}M^{Z}\right]\cdot[0,\ 0,\ \delta_Z,\ 0,\ 0,\ 0]^T \\ &= [0,\ -\delta_Z\sin\alpha,\ \delta_Z\cos\alpha,\ 0,\ 0,\ 0]^T \\ {}^{g}_{X}E^{X} &= D\left[{}^{g}_{X}M^{X}\right]\cdot[\delta_X,\ 0,\ 0,\ 0,\ 0,\ 0]^T \\ &= [\delta_X,\ 0,\ 0,\ 0,\ 0,\ 0]^T \end{aligned}$$

$$ {}^{g}_{C}E^{C} = D\left[{}^{g}_{C}M^{C}\right]\cdot[0,\quad 0,\quad 0,\quad 0,\quad 0,\quad \varepsilon_C]^T $$
$$ = [-\varepsilon_C(y\cos\alpha + z\sin\alpha),\ x\varepsilon_C\cos\alpha,\ x\varepsilon_C\sin\alpha,\ 0,\ -\varepsilon_C\sin\alpha,\ \varepsilon_C\cos\alpha]^T $$

Therefore, the integrated error component matrix E of grinding wheel in its coordinate system caused by the motion errors of axes can be represented as the sum errors of the axes; that is,

$$ \begin{aligned} E &= {}^{g}_{Y}E^{Y} + {}^{g}_{A}E^{A} + {}^{g}_{Z}E^{Z} + {}^{g}_{X}E^{X} + {}^{g}_{C}E^{C} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & -y\cos\alpha - z\sin\alpha \\ 0 & 1 & -\sin\alpha & 0 & x\cos\alpha \\ 0 & 0 & \cos\alpha & 0 & x\sin\alpha \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1 & -\sin\alpha \\ 0 & 0 & 0 & 0 & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \delta_X \\ \delta_Y \\ \delta_Z \\ \varepsilon_A \\ \varepsilon_C \end{bmatrix} \\ &= J_M \cdot [\delta_X,\quad \delta_Y,\quad \delta_Z,\quad \varepsilon_A,\quad \varepsilon_C]^T \end{aligned} \tag{28} $$

where J_M represents the Jacobian matrix of the machine tools, which is obtained by using the differential motion matrices of the axes relative to grinding wheel to compensate the integrated error components of grinding wheel conveniently.

Though J_M is a non-square matrix, the inverse of it can be calculated directly by using the pseudo-inverse method. In this way, Eq. (28) can be rewritten as

$$ [\delta_X,\quad \delta_Y,\quad \delta_Z,\quad \varepsilon_A,\quad \varepsilon_C]^T = \left({J_M}^T \cdot J_M\right)^{-1} \cdot {J_M}^T \cdot E \tag{29} $$

Moreover, based on Eq. (27) and the influences of the motion axes to grinding wheel, E is represented as

$$ E = [\delta_{xg},\quad \delta_{yg},\quad \delta_{zg},\quad \varepsilon_{xg},\quad \varepsilon_{yg},\quad \varepsilon_{zg}]^T \tag{30} $$

where symbols δ_{xg}, δ_{yg}, δ_{zg}, ε_{xg}, ε_{yg}, and ε_{zg} represent the integrated error components of grinding wheel along the X, Y, and Z directions, respectively.

So, the compensation values for integrated error components of grinding wheel with Eq. (29) can be obtained as

$$ \begin{aligned} \delta_X &= \delta_{xg} + \left[y\cdot\left(\varepsilon_{xg} - y\cdot\varepsilon_{yg}\right)\cdot\sin\alpha + \varepsilon_{zg}\cdot\left(y^2+1\right)\cdot\cos\alpha\right]\cdot\left(y\cdot\cos\alpha + z\cdot\sin\alpha\right)\Big/\left(y^2+\cos^2\alpha\right) \\ \delta_Y &= \delta_{yg} + \delta_{zg}\cdot\tan\alpha - x\cdot\left[y\cdot\left(\varepsilon_{xg} + y\cdot\varepsilon_{yg}\right)\cdot\tan\alpha - \varepsilon_{zg}\cdot\left(y^2+1\right)\right]\cdot\Big/\left(y^2+\cos^2\alpha\right) \\ \delta_Z &= \delta_{zg}\cdot\csc\alpha - x\cdot\left[y\cdot\left(\varepsilon_{xg} + y\cdot\varepsilon_{yg}\right)\cdot\tan\alpha - \varepsilon_{zg}\cdot\left(y^2+1\right)\right]\cdot\sin\alpha\Big/\left(y^2+\cos^2\alpha\right) \\ \varepsilon_A &= \left[y\cdot\varepsilon_{xg} + \left(\varepsilon_{yg} + \varepsilon_{zg}\cdot\tan\alpha\right)\cdot\cos^2\alpha\right]\Big/\left(y^2+\cos^2\alpha\right) \\ \varepsilon_C &= \left[y\cdot\left(\varepsilon_{xg} - y\cdot\varepsilon_{yg}\right)\cdot\sin\alpha + \varepsilon_{zg}\cdot\left(y^2+1\right)\cdot\cos\alpha\right]\Big/\left(y^2+\cos^2\alpha\right) \end{aligned} \tag{31} $$

It can be seen that compensation values can be obtained conveniently and effectively with the Jacobian matrix based on differential motion matrix. Then, the geometric error model with homogeneous transformation matrix and the error compensation with differential motion matrix for five-axis CNC gear profile grinding machine tools are shown as Fig. 4.

4 Experiments and results

To verify the accuracy and effectiveness of the geometric error model based on homogenous transformation matrix and the error compensation based on differential motion matrix, grinding and compensation experiments are carried out on a five-axis CNC gear profile grinding machine tool SKMC-1200W/10, whose strokes are X900Y400Z800 (unit mm) and A[−45, 45] C[0, 360] (unit deg) for linear axis and rotation axis, respectively. And the repetitive positioning precisions of the motion axes are 3 μm and 1 mdeg, respectively.

The position errors and straightness errors of nine-line of the three linear motion axes are measured based on the nine-line method [25] by using Renishaw XL-80 laser interferometer system to calculate the 21 geometric error components. Furthermore, the geometric error components of the two rotary motion axes, A axis and C axis, are identified based on the six-circle method [2] by using Renishaw QC10 ball bar. Then, the identified six basic geometric error components of the five motion axes are shown as Figs. 5, 6, 7, 8, and 9, respectively.

The geometric error component measurement of the machine tool is for the error modeling and error compensation. Generally speaking, five-axis CNC milling machine tools are widely used for machining arbitrary freeform surface. The machining methods of milling profile surface conclude spot-contact fabrication for cured face and surface-contact fabrication for flat surface. However, the five-axis CNC gear profile grinding machine tools are mainly used for the machining of cylindrical helical gear or spur gear based on line-contact machining method. The precision of workpiece is directly dependent on the machining accuracy of the machine tools. So, a cylindrical involute right-hand helical gear workpiece is selected for grinding test to verify the accuracy and effectiveness of the proposed geometric error model and compensation.

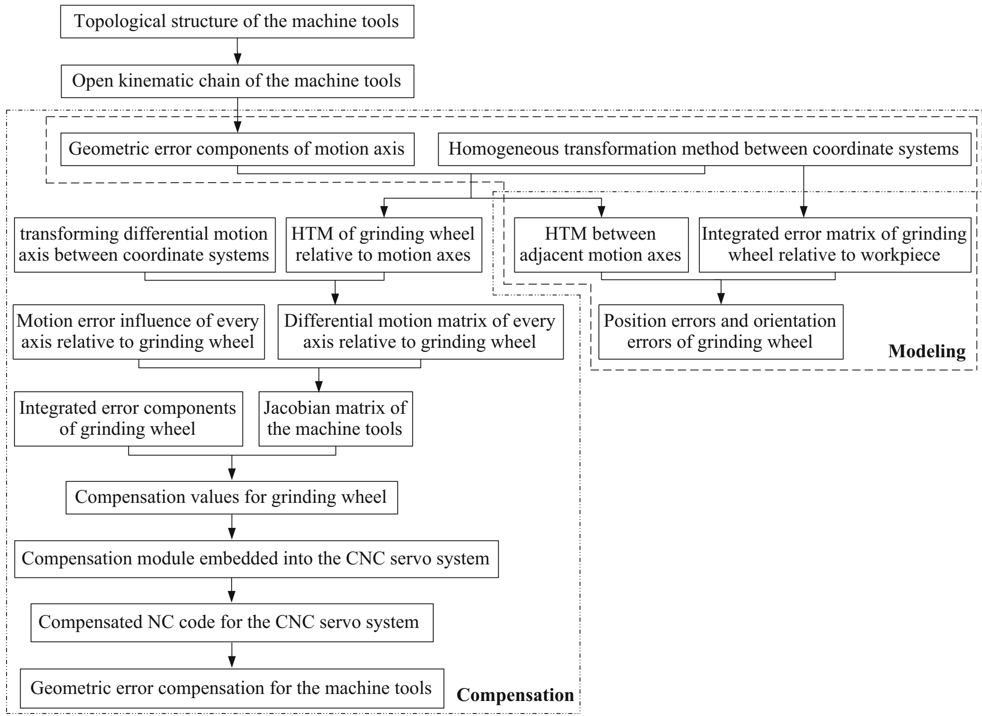

Fig. 4 The flowchart of geometric error model and compensation for five-axis CNC gear profile grinding machine tools

The basic dimension parameters of the gear workpiece are shown as Table 2.

In five-axis milling machining, the arbitrary free-form surface of workpiece with high accuracy is machined by five-axis linkage CNC milling machine center, whose geometric structure and processing method are quite different from five-axis CNC gear profile grinding machine tools. When the surface of gear workpiece is ground, A axis of the machine tools is moved to the corresponding angle, which is equal to the helix angle of the workpiece, and then held in the position. The grinding can be realized through the X axis, Y axis, Z axis, and C axis linkage. Because the geometric error of the grinding wheel path can be obtained by using the error model, the corresponding correction values for the position and orientation errors of the grinding wheel are calculated by using the constructed Jacobian to compensate the integrated error components of the grinding wheel.

The measurement and assessment of the grinded tooth surface errors are also quite different from free-form surface. The machining errors of tooth surface are obtained by comparing the spatial normal distance between theoretical and actual tooth surface in accordance with international standard *ISO 1328–1:2013* [26]. Moreover, a virtual mesh, which forms corresponding grind points, is built on the theoretical tooth surface to facilitate the contrast and analysis. On the same side of the tooth surface, the vertical and horizontal curves formed by grind points are named as tooth alignment and profile curves, respectively. Furthermore, the virtual mesh and the errors between theoretical and actual tooth surface are named as topological mesh and errors, respectively.

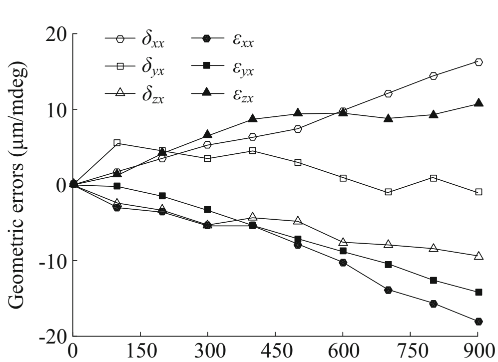

Fig. 5 Geometric error components of X axis

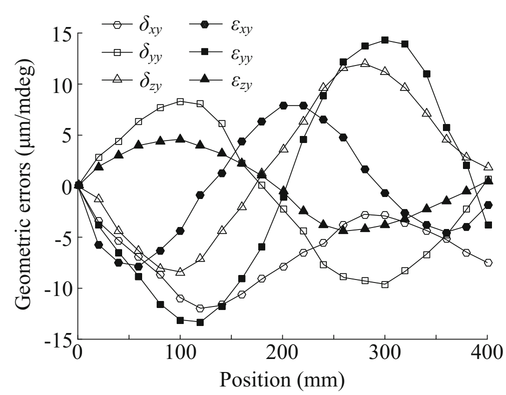

Fig. 6 Geometric error components of Y axis

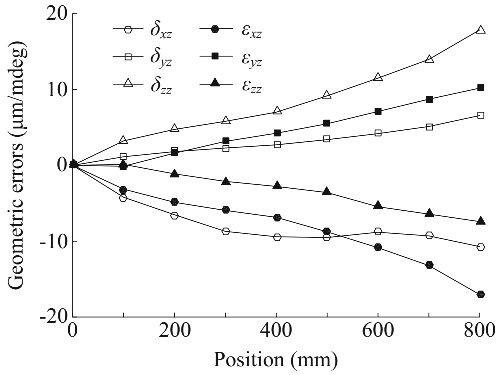

Fig. 7 Geometric error components of *Z* axis

According to the current evaluation standard, the machining accuracy of tooth surface can be evaluated by the topological errors conveniently and efficiently. Therefore, each tooth surface of the workpiece is represented using seven tooth alignment curves and nine tooth profile curves, and the machining topological errors of them are measured on the three coordinate measuring machine (CMM), WENZEL LH1512. The 1st and the 18th tooth are typically selected to represent the gear workpiece. The machining and the error measurement scene are shown as Fig. 10. The machining topological errors are shown as Figs. 11 and 12, respectively. The maximum values of left and right surface topological errors of the 1st and the 18th tooth are −40.6 and −41.4 μm, 56.9 and 57.4 μm without compensation, respectively. After error compensation, the errors of them are significantly reduced to −7.2 and −7.3 μm, −8.4 and −8.2 μm, respectively. The tooth surface accuracy is improved more than 80% on average. In order to show the effects of the error compensation more clearly, the middle curve among the seven tooth alignment curves of each surface of the 1st and the 18th tooth is selected, and the corresponding topological errors of them are shown in Fig. 13. The topological error values of point 1 on left and right tooth surface are reduced from −36.3 and −36.5 μm, −55.1 and −55.3 μm to −7.0 and −7.2 μm, −8.0 and −8.2 μm, respectively. Meanwhile, the values of point 9 on left and right tooth surface are also reduced from −40.5 and −41.0 μm, −56.6 and −57.2 μm to −7.1 and −7.0 μm, −8.2 and −7.9 μm, respectively. Moreover, in the direction of tooth alignment, the topological error values of left and right surface without compensation decrease rapidly from end face to middle face and then increase rapidly from middle face to the other end face. The values after error compensation become much smaller, which represent the improvement of tooth surface precision. Furthermore, the results also show that

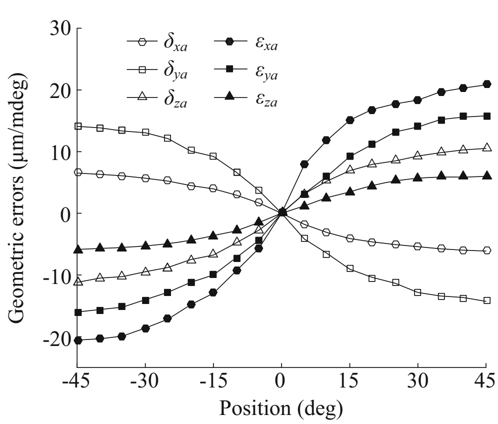

Fig. 8 Geometric error components of *A* axis

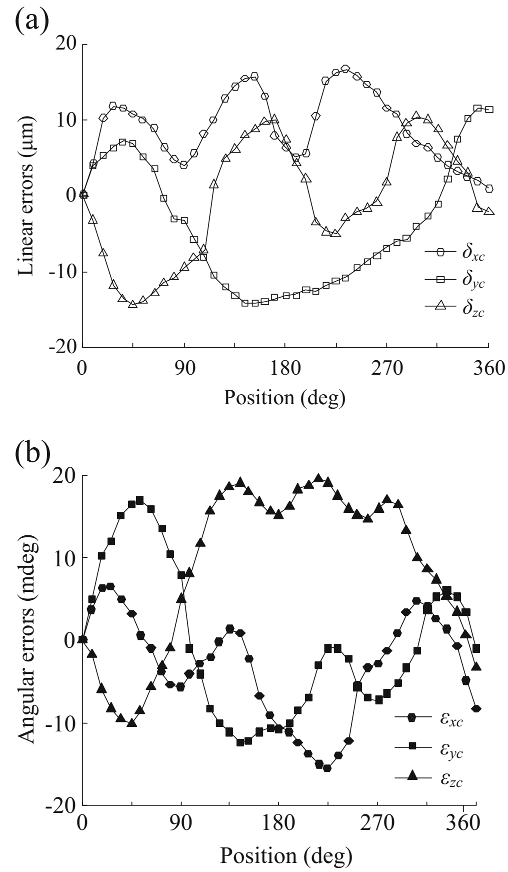

Fig. 9 Geometric error components of *C* axis

Table 2 Basic dimension parameters of the gear workpiece

Parameter	Size
Normal modulus	14 mm
Tooth number	35
Pressure angle	20°
Helix angle	30°
Modification coefficient	0.274

Fig. 10 The machining and the measurement of workpiece

the variation of tooth profile is much less than that of tooth alignment. The deviations of tooth profile slope and tooth thickness are improved a lot. So, it can be seen that the proposed geometric error model and compensation can greatly enhance the machining accuracy of the gear profile grinding machine tool. All in all, the geometric error modeling and compensation-based homogeneous transformation and different motion matrix, the adopted research method, and technical route have the positive consequence of improving the accuracy of five-axis CNC gear profile grinding machine tools.

5 Conclusions

Geometric error is one of the most important factors which affect the machining accuracy of gear profile grinding machine tools, as well as for the precision of workpiece. In view of this situation, homogeneous transformation and differential change between coordinate frames based on the multi-body system theory are employed for geometric error modeling and compensation of five-axis CNC gear profile grinding machine tools. Firstly, the basic homogeneous transformation matrix between coordinate frames is derived for the following modeling process. Secondly, the open kinematic chain from workpiece to grinding wheel is built according to the topological structure of the CFXZAY-type, five-axis gear profile grinding machine tools. And the ideal homogeneous transformation matrix from workpiece to grinding wheel is established to lay the foundation of geometric error modeling. Based on the analysis of 33 geometric error components of the machine tools, the homogenous transformation matrices of linear pairs and rotary pairs with geometric errors are calculated, and the relationships of the error propagation from workpiece to grinding wheel are also acquired at the same time. The position and orientation errors of grinding wheel in workpiece coordinate system are obtained with matrix multiplication by using small-angle approximation and ignoring the second-order and high-order error terms on the basis of the above analysis. Then, the Jacobian of the machine tools is obtained by using transforming differential motion matrix of every motion axis relative to grinding wheel based on the analysis of modeling process

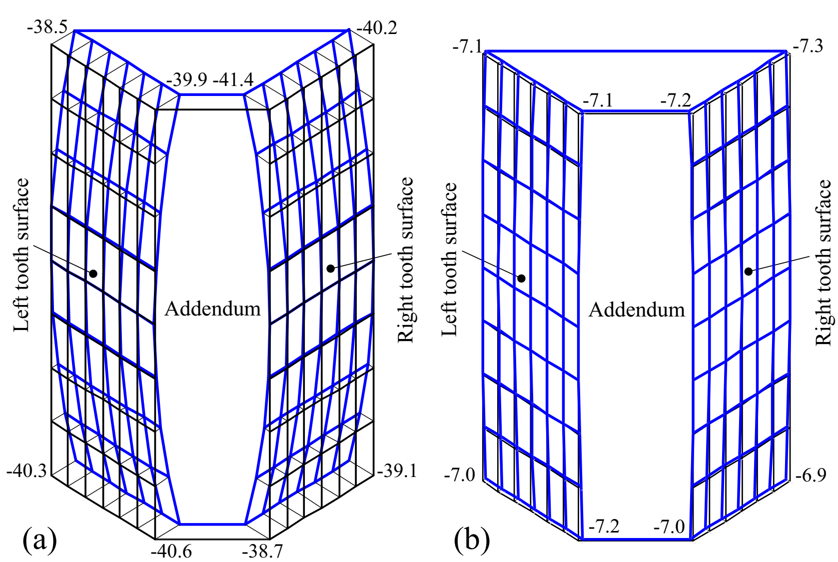

Fig. 11 The machining topological errors of tooth surface of the first tooth without compensation (**a**) and with compensation (**b**) (unit μm). Note that in this figure, the theoretical and the actual topological mesh of tooth surface are represented as *black and blue solid lines*, respectively

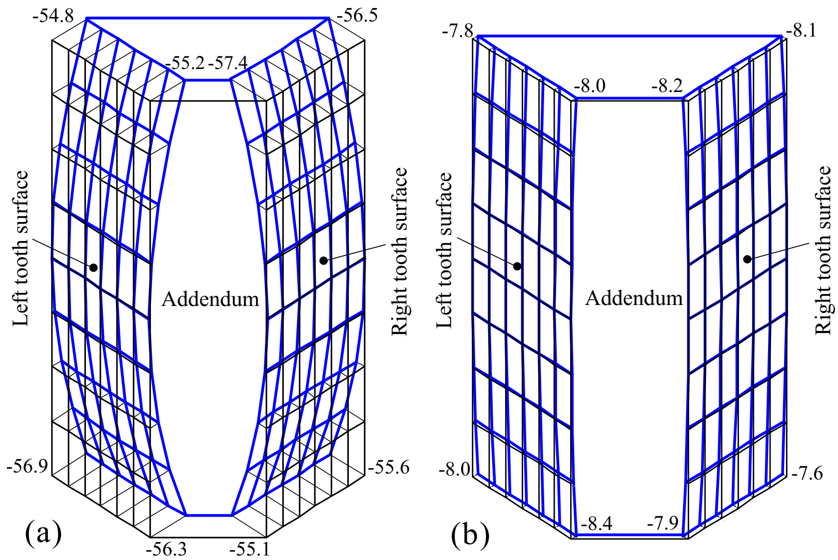

Fig. 12 The machining topological errors of tooth surface of the 18th tooth without compensation (**a**) and with compensation (**b**) (unit μm). Note that in this figure, the theoretical and the actual topological mesh of tooth surface are represented as *black and blue solid lines*, respectively

to compensate the integrated error components of the grinding wheel. Finally, error measurement, error compensation, and machining experiments are carried out on a five-axis CNC gear profile grinding machine tool SKMC-1200W/10. The results not only show that the tooth surface precision of the gear workpiece is improved significantly but also indicate that the proposed error modeling, error compensation, and research approach are applicable and efficient for five-axis CNC gear profile grinding machine tools.

Besides geometric errors, the other error components, such as grinding force errors, thermal errors, grinding wheel wear errors, vibration errors, and controller system errors, also affect the machining accuracy of machine tools to some extent. Therefore, the propagation mechanism, modeling, and compensation of the errors need to be researched and discussed in depth to further improve the accuracy.

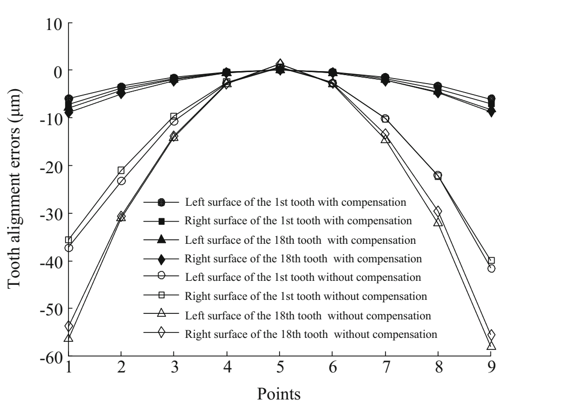

Fig. 13 The tooth alignment topological errors of the 1st and the 18th tooth without/with compensation

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