Int J Adv Manuf Technol (2016) 86:1627–1637
DOI 10.1007/s00170-015-8307-4

ORIGINAL ARTICLE

# An optimization design of the combined Shewhart-EWMA control chart

M. Shamsuzzaman[1] · M. B. C. Khoo[2] · S. Haridy[3] · I. Alsyouf[1]

Received: 2 August 2015 / Accepted: 22 December 2015 / Published online: 9 January 2016


**Abstract** This article presents an optimization design of the combined Shewhart $\overline{X}$ chart and exponentially weighted moving average (EWMA) chart ($\overline{X}$ &EWMA chart in short) used in statistical process control (SPC). The design algorithm not only optimizes the charting parameters of the $\overline{X}$ chart and EWMA chart, but also optimizes the allocation of detection power between the two charts' elements, based on the loss function, so that the best overall performance can be achieved. The optimization design is carried out under the constraints on the false alarm rate and available resources. While the optimization design effectively improves the overall performance of the $\overline{X}$ &EWMA chart over the entire process shift range, it does not increase the difficulty of understanding and implementing this combined chart. A $2^k$ factorial experiment shows that the optimal $\overline{X}$ &EWMA chart outperforms the main competitor, the basic $\overline{X}$ &EWMA chart, by about 50 %, on average. Moreover, this article provides the SPC practitioners with a design table to facilitate the designs of the $\overline{X}$ &EWMA charts. From this design table, the users can directly find the optimal values of the charting parameters according to the design specifications.

**Keywords** Manufacturing industries · Quality control · Statistical process control · Shewhart-EWMA chart · Random process shift · Loss function

✉ M. Shamsuzzaman
mshamsuzzaman@sharjah.ac.ae

M. B. C. Khoo
mkbc@usm.my

S. Haridy
salah@pmail.ntu.edu.sg

I. Alsyouf
ialsyouf@sharjah.ac.ae

[1] Department of Industrial Engineering and Engineering Management, College of Engineering, University of Sharjah, PO Box 27272, Sharjah, United Arab Emirates

[2] School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Minden, Penang, Malaysia

[3] Benha Faculty of Engineering, Benha University, Benha 13512, Egypt

## 1 Introduction

Control charts in SPC are widely used for monitoring process variation over time in manufacturing industries and service sectors. Following the pioneering work by Shewhart [1], many new charts and their applications have been reported [2–4]. Among all the charts, the Shewhart $\overline{X}$ (or $X$) chart has been studied extensively by researchers and practitioners. However, the Shewhart-type charts are relatively insensitive to small process shifts [5]. Two types of control charts such as cumulative sum (CUSUM) charts [6] and exponentially weighted moving average (EWMA) charts [7] are recommended for detecting shifts of small and moderate sizes.

The design and application of CUSUM and EWMA charts in industries and other sectors is increasing rapidly [8–10]. The performance of the EWMA chart is approximately equivalent to that of the CUSUM chart, but the former is often superior to the latter for detecting larger shifts [5]. However, the use of a single EWMA control chart is efficient in detecting small changes in process shifts but relatively less efficient when the changes are large due to inertia problem [11, 12]. Woodall and Mahmoud [12] compared the signal resistance values (i.e., the inertia) for several types of univariate and multivariate charts and concluded that Shewhart limits should be used with EWMA charts, especially when the smoothing parameter is small. Many other studies suggested using the combined Shewhart-EWMA chart to guard against the inertia problem [13–15].

Many new algorithms for the design of the Shewhart-EWMA charts have been reported in the literature. Lucas and Succucci [14] examined the run length properties of the EWMA charts and explained several extensions including a fast initial response (FIR) feature and a combined Shewhart-EWMA charting scheme. Klein [16] evaluated a group of $\overline{X}$ -EWMA charts and compared them with standard Shewhart runs rules schemes, in terms of their *ARL* profiles. Albin et al. [17] recommended using combined *X*-EWMA chart, as both charts can be plotted on a single graph. They also showed that the combined scheme is able to detect large and small shifts in the process mean and large shifts in the process standard deviation. Tolley and English [18] provided an economic design of the combined $\overline{X}$ &EWMA chart based on Duncan's [19] model. They compared the cost performance of the single EWMA chart with that of the combined $\overline{X}$ &EWMA chart and concluded that the combined chart is not a well-behaved scheme under a constrained in-control *ARL* ($ARL_0$). Some researchers [20, 21] considered variable charting parameters in order to improve the performance of the Shewhart-EWMA chart. However, none of the abovementioned articles considered a specific range of mean shifts, instead, a single or a pair of mean shifts was considered to design the charting schemes. Moreover, the EWMA weighting factor $\lambda$ was determined subjectively. In practice, the value of $\lambda$ not only depends on the prespecified size of process shift but also on a given $ARL_0$ [14].

The process shift (e.g., mean shift $\delta$) is a random variable and has different probability distributions for different processes. In most practical applications, it will not be possible to specify the exact size of the process shift to be expected [22]. It means that a control chart designed based on one or few $\delta$ values may have optimal performance for the particular value(s) of $\delta$, but may work unsatisfactorily for other values of $\delta$. It highlights the importance of making the control scheme effective over a distribution range of $\delta$.

This article proposes an algorithm for the optimization design of the combined $\overline{X}$ &EWMA chart for monitoring the entire mean shift range. The charting parameters, such as the sample size, sampling interval, control limits, and weighting factor are also optimized. Throughout this article, it is assumed that the quality characteristic *x* is normally and independently distributed with known in-control mean $\mu_0$ and standard deviation $\sigma_0$. When a mean shift occurs, the process mean $\mu$ will change accordingly, that is,

$$\mu = \mu_0 + \delta\sigma_0 \quad (1)$$

where $\delta$ is the mean shift, in terms of $\sigma_0$. When the process is in control, $\delta = 0$. The shift in process standard deviation has not been considered (i.e., $\sigma \equiv \sigma_0$) in the discussion of the $\overline{X}$ &EWMA chart. The effectiveness of a control chart is usually measured by the average time to signal (*ATS*). The out-of-control *ATS* is the average time required to signal an out-of-control case, whereas the in-control $ATS_0$ is the average time required to produce a false alarm. In the optimization design of a control chart, the frequently used objective function is to minimize the out-of-control *ATS* at one or a few specified mean shift values [23, 24]. However, the objective of this study is to improve the performance of the charts across the shift range rather than the effectiveness at one or a few particular points. Many authors [25, 26] used the loss function to measure the effectiveness of a chart for monitoring random process shift.

## 2 Optimization design

### 2.1 Specifications

To design an $\overline{X}$ &EWMA chart, only the following three parameters need to be specified:

- $\tau$ Minimum allowable in-control $ATS_0$ for the chart
- $R$ Maximum allowable inspection rate
- $\delta_{max}$ Upper bound of the mean shift $\delta$

The value of $\tau$ is decided according to the requirements on the false alarm rate and the detection power. If the cost of handling the false alarms is high, a larger $\tau$ should be used to reduce the false alarm frequency. Otherwise, $\tau$ may be set to a lower value in order to increase the detection effectiveness. The inspection rate *R* is defined as the number of inspected units per unit time when the process is in control. Its value is decided according to the available resources (operators and measuring instruments) and can be estimated from the field test during the pilot runs. The upper bound of the mean shift $\delta_{max}$ is the maximum possible mean shift in a process and can be decided based on the information on out-of-control cases. However, if such kind of process information is not available, $\delta_{max}$ may be set as six directly. The numerical studies in Sect. 3 show that, when $\delta \geq 6$, almost all charts have a similar *ATS* value very close to 0.5 (which is the minimum possible value of the steady-state *ATS* [27], and therefore, a further comparison of the chart's performance beyond ($\delta = 6$) is actually unimportant.

### 2.2 Optimization model

The statistic $S_i$ to be plotted and updated for the EWMA chart is

$$S_i = \lambda \cdot \bar{x}_i + (1-\lambda) \cdot S_{i-1} \quad i = 1,\ 2, \ldots, \quad (2)$$

where $\lambda$ ($0<\lambda\leq 1$) is the weighting factor. The sample mean $\overline{x}_i$ is the average of the measurements in the $i$th sample. The value of $S_0$ (i.e., at $i=0$) is the process target (i.e., $S_0=\mu_0$). The $\overline{X}$ &EWMA combination will produce an out-of-control signal if $S_i$ falls beyond the control limits of the EWMA chart and/or the current value of the sample average $\overline{x}_i$ exceeds the control limits of the $\overline{X}$ chart.

The design algorithm of the $\overline{X}$ &EWMA chart is formulated by the following optimization model:

$$Minimize: \quad ML, \tag{3}$$

$$\text{Subject to}: \ ATS_0 \geq \tau, \tag{4}$$

$$r \leq R \tag{5}$$

Design variables: $\lambda$, $n$, $h$, $H$, $UCL$.

where $ML$ is the mean loss per out-of-control case and $r$ is the actual (or resultant) inspection rate. The constraint on inspection rate $R$ ensures that the use of the optimization model will not need extra inspection resources [2, 28]. The calculation of $ML$ will be explained shortly. For simplicity, the focus of this article is on the studies of the combination of the $\overline{X}$ &EWMA chart for detecting increasing process shifts in the mean. As a result, an upper-sided EWMA chart with an upper control limit ($H$) and an $\overline{X}$ chart with an upper control limit ($UCL$) are combined. A symmetrical $\overline{X}$ &EWMA chart for detecting decreasing mean shifts can be designed straightforwardly. The optimization model optimizes $\lambda$, $n$, $h$, $H$, and $UCL$ in order to minimize $ML$, provided that the constraints on $ATS_0$ and $r$ are all satisfied.

When $\sigma$ is constant ($\sigma=\sigma_0$), the loss $L$ incurred by a given mean shift $\delta$ can be determined by [26],

$$L(\delta) = \sigma^2 + (\mu-\mu_0)^2 = \sigma_0^2\left(1+\delta^2\right) \tag{6}$$

Moreover, since the quality cost is proportional to $ATS$, the overall loss $ML$ can be calculated as follows [29],

$$ML = \int_0^{\delta_{\max}} L(\delta)\cdot ATS(\delta)\cdot f(\delta)\cdot d\delta = \frac{\sigma_0^2}{\delta_{\max}}\int_0^{\delta_{\max}} \left(1+\delta^2\right)\cdot ATS(\delta)\cdot d\delta \tag{7}$$

where $ATS(\delta)$ is produced by the control chart at $\delta$, and $\delta_{max}$ is the upper bound of the mean shift $\delta$ that is important to the users. Furthermore, since it is generally assumed that all mean shifts within the range ($0 < \delta\leq\delta_{\max}$) occur with equal probabilities [30], a uniform distribution of $\delta$ is implied. The index $ML$ is the average value of the loss function per out-of-control case over the probability distribution of the random mean shift $\delta$. It is a comprehensive measure of the overall charting performance as it considers all the contributors to the quality cost including the time to signal and the magnitude of $\delta$. The value of $ML$ is acquired by the integration across the whole shift range. This integration can be computed accurately by a numerical method, such as the Legendre-Gauss Quadrature. The $ATS(\delta)$ of the $\overline{X}$ &EWMA chart is calculated by the formulae presented in the Appendix.

### 2.3 Optimization search algorithm

Among the five design variables, $\lambda$, $n$, $h$, $H$, and $UCL$, the parameters $\lambda$, $n$, and $UCL$ are treated as independent design variables. The sampling interval $h$ depends on $n$, that is,

$$h = n\big/R. \tag{8}$$

Equation (8) ensures that the constraint on the inspection rate $r$ (constraint (5)) is satisfied, and meanwhile, the available resources are fully utilized. When the inspection rate $R$ is given, an optimal combination of $n$ and $h$ will result in the minimum value of $ML$. The control limits $H$ and $UCL$ are determined so that the resultant in-control $ATS_0$ of the $\overline{X}$ &EWMA combination is equal or very close to $\tau$ (constraint (4)).

The optimization search is conducted through a three-level search as outlined in the following:

1. Specify $\tau$, $R$ and $\delta_{max}$.
2. Initialize $ML_{min}$ as a large number, say $10^7$ ($ML_{min}$ is used to store the minimum value of $ML$).
3. At the first level, the optimal value of $n$ is searched from one with a step size of one. For a given value of $n$, the sampling interval $h$ is calculated by Eq. (8). It ensures the satisfaction of constraint (5).
4. At the second level, the optimal value of $\lambda$ is searched in the range of ($0<\lambda\leq 1$).
5. At the third level, for a given set of values of ($n$, $h$, $\lambda$), search for the optimal value of $UCL$ with a starting value of $UCL_{\overline{X}}$, which is the upper control limit of an individual $\overline{X}$ chart that meets ($ATS_0=\tau$). It is noted that the $UCL$ of the $\overline{X}$ &EWMA chart cannot be smaller than $UCL_{\overline{X}}$; otherwise, the constraint of ($ATS_0\geq\tau$) will be violated. Next, for a given set of values of ($n$, $h$, $\lambda$, $UCL$),
   - (a) Determine the control limit $H$ that ensures the satisfaction of the constraint of ($ATS_0\geq\tau$) by the $\overline{X}$ &EWMA chart.
   - (b) When the values of all five charting parameters ($n$, $h$, $\lambda$, $UCL$, and $H$) are preliminarily determined, calculate the objective function $ML$ by Eq. (7).
   - (c) If the calculated $ML$ is smaller than the current $ML_{min}$, replace the latter by the former and the current values of ($n$, $h$, $\lambda$, $UCL$, and $H$) are stored as a temporary optimal solution.
6. At the end of the entire three-level search, the optimal $\overline{X}$ &EWMA chart that produces the minimum $ML$ and satisfies the constraints ($ATS_0\geq\tau$) and ($r\leq R$), is identified.

The corresponding optimal values of ($n$, $h$, $\lambda$, $UCL$, and $H$) are also finalized.

A computer program in C language has been developed to carry out the optimization design of the $\overline{X}$ &EWMA control chart. Usually, an optimal solution is obtained within few seconds of CPU time by using a personal computer. The program can be obtained on request from the authors.

## 3 Comparative studies

This section compares the performance of the following four charts.

1. The basic EWMA chart
   It is an EWMA chart that uses a weighting factor $\lambda$ fixed at 0.1. This $\lambda$ value is widely used for a conventional EWMA chart [5]. Moreover, the sample size $n$ is fixed at one, as ($n=1$) is usually believed to be the most effective from an overall viewpoint [22] (the sampling interval $h$ is determined by Eq. (8)).
2. The optimal EWMA chart [28]
   For this EWMA chart, the charting parameters ($\lambda$, $n$, $h$, $H$) are optimized that satisfies the constraints of ($ARL_0 \geq \tau$ and $r \leq R$) and meanwhile minimizes $ML$.
3. The basic $\overline{X}$ &EWMA chart
   Similar to the conventional EWMA chart, this $\overline{X}$ &EWMA combination uses ($\lambda=0.1$) and ($n=1$). The upper control limit $UCL$ of the $\overline{X}$ chart is selected between 4 and 4.5 according to Lucas and Saccucci [14] (in this study, $UCL$ is set as 4.25), and the control limit $H$ of the EWMA chart is adjusted to make the $ATS_0$ of the $\overline{X}$ &EWMA combination equal to $\tau$.
4. The optimal $\overline{X}$ &EWMA chart
   For this version of the $\overline{X}$ &EWMA chart, the optimal values of the charting parameters ($n$, $h$, $\lambda$, $UCL$, and $H$) are determined by the optimization design proposed in this article.

Without losing generality, the in-control $\mu_0$ and $\sigma_0$ of the quality characteristic $x$ are assumed to be 0 and 1, respectively. To facilitate the comparison, a normalized $ML_{normal}$, for each chart is calculated using the $ML$ value of the optimal $\overline{X}$ &EWMA chart as the norm, that is,

$$ML_{normal} = \frac{ML}{ML_{Opt\ \overline{X}\&\text{EWMA}}}. \tag{9}$$

Obviously, if the value of $ML_{normal}$ of a chart is more than one, the performance of this chart is inferior to that of the optimal $\overline{X}$ &EWMA chart, and vice versa.

The comparison is first conducted under the following general conditions,

$$\tau = 500, R = 1, \delta_{max} = 6. \tag{10}$$

The four charts are designed for this case and the results are shown as follows:

$$\begin{aligned}
&\text{Basic EWMA chart :} && n=1,\ h=1.0,\ \lambda=0.1,\ H=0.62868, \\
& && ML=23.87, ML_{normal}=1.232 \\
&\text{Optimal EWMA chart :} && n=1, h=1.0, \lambda=0.23,\ H=1.04144, \\
& && ML=21.64, ML_{normal}=1.117 \\
&\text{Basic}\overline{X}\&\text{ EWMA chart :} && n=1, h=1.0,\ \lambda=0.1,\ H=0.62895,\ UCL=4.25 \\
& && ML=20.61, ML_{normal}=1.064 \\
&\text{Optimal}\overline{X}\&\text{EWMA :} && n=1,\ h=1.0,\ \lambda=0.09,\ H=0.59246,\ UCL=3.72 \\
& && ML=19.37,\ ML_{normal}=1.000
\end{aligned}$$

The mean ($ATS$) and the standard deviation ($SDTS$) of the time to signal ($TS$) of the four charts are also calculated within the mean shift range of ($0<\delta\leq\delta_{max}$), and the results are displayed in Table 1. The curves of the normalized $ATS$ (i.e., $ATS/ATS_{Opt\ \overline{X}\&\text{EWMA}}$) of the four charts are illustrated in Fig. 1. It is interesting to observe the following from Table 1 and Fig. 1:

1. Firstly, each of the four charts generates an $ATS_0$ value equal to $\tau$ when the process is in control ($\delta=0$). It ensures that the requirement on the false alarm rate is satisfied.
2. For all charts, the values of $SDTS$ is very small (the largest $SDTS$ is obtained by basic $\overline{X}$ &EWMA chart at $\delta=0$, which is equal to 5.5638 (only 1.1 % of the mean (= $ATS=500$) value)). Except few cases (especially when $\delta$ is zero or very small), the difference between the $SDTS$ values of the optimal $\overline{X}$ &EWMA chart and its competitors is negligible, which shows the consistency of the results obtained by the proposed design algorithm.
3. The performance of the optimal EWMA chart has been improved considerably compared with the basic EWMA

**Table 1** *ATS* and *SDTS* comparison among the control charts ($\tau = 500$, $R = 1$, $\delta_{max} = 6$)

| $\delta$ | Basic EWMA | Optimal EWMA | Basic $\overline{X}$&EWMA | Optimal $\overline{X}$&EWMA |
|---|---|---|---|---|
| 0.00 | (500.00, 0.6635) | (500.00, 3.7406) | (500.00, 5.5638) | (500.00, 4.5457) |
| 0.25 | (79.00, 0.4437) | (112.00, 0.9077) | (79.10, 0.6836) | (77.10, 0.1957) |
| 0.50 | (24.90, 0.2021) | (35.30, 0.0877) | (24.90, 0.0727) | (24.40, 0.1691) |
| 0.75 | (12.40, 0.0692) | (15.40, 0.0778) | (12.40, 0.1105) | (12.40, 0.0393) |
| 1.00 | (7.86, 0.0141) | (8.60, 0.0884) | (7.86, 0.0467) | (7.93, 0.0555) |
| 1.25 | (5.66, 0.0076) | (5.64, 0.0286) | (5.66, 0.0885) | (5.74, 0.0427) |
| 1.50 | (4.39, 0.0009) | (4.10, 0.0061) | (4.39, 0.0090) | (4.46, 0.0204) |
| 1.75 | (3.58, 0.0012) | (3.18, 0.0271) | (3.57, 0.0108) | (3.61, 0.0084) |
| 2.00 | (3.01, 0.0145) | (2.58, 0.0006) | (2.99, 0.0052) | (3.01, 0.0147) |
| 2.25 | (2.59, 0.0011) | (2.16, 0.0067) | (2.57, 0.0194) | (2.56, 0.0107) |
| 2.50 | (2.27, 0.0222) | (1.85, 0.0063) | (2.24, 0.0025) | (2.20, 0.0071) |
| 2.75 | (2.02, 0.0053) | (1.62, 0.0012) | (1.97, 0.0052) | (1.90, 0.014) |
| 3.00 | (1.82, 0.0034) | (1.44, 0.0206) | (1.74, 0.0026) | (1.64, 0.0113) |
| 3.25 | (1.66, 0.0083) | (1.30, 0.0020) | (1.55, 0.0106) | (1.42, 0.0006) |
| 3.50 | (1.52, 0.0103) | (1.17, 0.0033) | (1.37, 0.0013) | (1.23, 0.0026) |
| 3.75 | (1.41, 0.0009) | (1.07, 0.0057) | (1.22, 0.0027) | (1.06, 0.0056) |
| 4.00 | (1.31, 0.0082) | (0.97, 0.0048) | (1.08, 0.0059) | (0.92, 0.0058) |
| 4.25 | (1.23, 0.0075) | (0.89, 0.0039) | (0.96, 0.0046) | (0.80, 0.0070) |
| 4.50 | (1.16, 0.0032) | (0.81, 0.0031) | (0.85, 0.0036) | (0.71, 0.0021) |
| 4.75 | (1.10, 0.0040) | (0.73, 0.0024) | (0.76, 0.0027) | (0.64, 0.0014) |
| 5.00 | (1.03, 0.0046) | (0.67, 0.0017) | (0.69, 0.0081) | (0.59, 0.0010) |
| 5.25 | (0.97, 0.0053) | (0.62, 0.0088) | (0.63, 0.0013) | (0.56, 0.0005) |
| 5.50 | (0.90, 0.0041) | (0.58, 0.0008) | (0.59, 0.0009) | (0.53, 0.0003) |
| 5.75 | (0.84, 0.0035) | (0.55, 0.0005) | (0.55, 0.0005) | (0.52, 0.0002) |
| 6.00 | (0.78, 0.0027) | (0.53, 0.0003) | (0.53, 0.0003) | (0.51, 0.0001) |

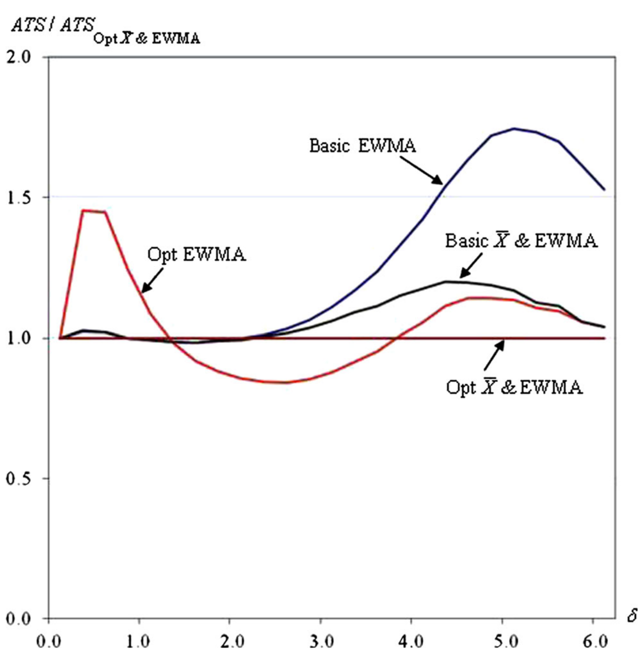


**Fig. 1** Normalized *ATS* curves of the four charts

chart. This is due to the optimization of the charting parameters ($\lambda$, $n$, $h$, $H$). The *ATS* performance of the optimal EWMA chart is even better than other charts over the shift range of ($1.25 \leq \delta \leq 3.50$). However, the overall performance of the optimal EWMA chart is still inferior to the $\overline{X}$ &EWMA charts.

(4) The basic $\overline{X}$ &EWMA chart is more effective than the basic EWMA chart due to the combined effect of the $\overline{X}$ chart and the EWMA chart. It is reflected by the smaller *ATS* values of the basic $\overline{X}$ &EWMA chart, especially when ($\delta \geq 1.75$). The *ATS* performance of the optimal EWMA chart is better than the basic $\overline{X}$ &EWMA chart when ($\delta \geq 1.50$), it highlights the importance of the optimization design.

5. The *ATS* values of the optimal $\overline{X}$ &EWMA chart is often equal to the minimum across the shift range (except for a range of ($1.00 \leq \delta \leq 3.50$)). For the moderate shift ($1.00 \leq \delta \leq 2.00$), the *ATS* performance of the basic EWMA and $\overline{X}$ &EWMA charts is slightly better than the optimal $\overline{X}$ &EWMA chart. The basic EWMA chart is especially sensitive toward small and moderate shifts. It

is also true for the basic $\overline{X}$ &EWMA chart, because it uses same weighting factor $\lambda$ fixed at 0.1 and obtains similar upper control limit ($H$) for the EWMA chart. On the other hand, the optimal EWMA chart shows better performance for the shift range of ($1.25 \leq \delta \leq 3.50$) compared with the optimal $\overline{X}$ &EWMA chart. However, the overall performance of the optimal $\overline{X}$ &EWMA chart is better than all of the competing charts.

It is well known that, no chart will give a better performance than other charts for all the shifts [31]. For this, it is suggested to design a control chart that will have an excellent overall performance in a broad shift domain [30, 32]. Consequently, it is more appropriate to compare the $ML$ (Eq. (7)) values of the charts in order to make an accurate and objective conclusion about the relative effectiveness among the charts. The values of $ML_{normal}$ (as listed above) of the four charts clearly indicate that the basic $\overline{X}$ &EWMA chart outperforms the basic EWMA chart and the optimal EWMA chart. However, the optimal $\overline{X}$ &EWMA chart is more effective than the basic EWMA chart, the optimal EWMA chart and the basic $\overline{X}$ &EWMA chart by about 23.2, 11.7, and 6.4 %, respectively, for this particular case. The combination of the $\overline{X}$ and EWMA charts plus the optimization design makes this scheme very effective from an overall viewpoint.

Next, the performance of the four charts is further studied by a $2^3$ factorial experiment [5]. The three parameters $R$, $\delta_{max}$, and $\tau$ are used as the input factors, and $ML_{normal}$ (Eq. (9)) is taken as the response. Each of the three factors $R$, $\delta_{max}$, and $\tau$ varies at two levels, resulting in eight runs (i.e., eight combinations of the values of the three factors). Since the overall loss $ML$ is deterministic, there is only a single design replicate for each run. The low and high levels for each factor are decided below.

$$\begin{array}{ccc} R & 2 & 10 \\ \delta_{max} & 3.0 & 6.0 \\ \tau & 300 & 1200 \end{array} \tag{11}$$

For each run, the four control charts are designed and each of them produces an $ATS_0$ equal to $\tau$. In each of the eight runs, the relative detection effectiveness of the charts is similar to that revealed in Table 1. The $ML_{normal}$ values under each of the eight runs are calculated and enumerated in Table 2. It can be seen that the optimal $\overline{X}$ &EWMA chart is always the most effective chart in all of the runs.

The average, $\overline{ML}_{normal}$, of the $ML_{normal}$ values for a chart over the eight runs is calculated. The values of $\overline{ML}_{normal}$ indicate that, from an overall viewpoint (over different combinations of the $\tau$, $R$, $\delta_{max}$), the optimal $\overline{X}$ &EWMA chart is more effective (in terms of $ML$) than the basic EWMA chart, the optimal EWMA chart and the basic $\overline{X}$ &EWMA chart by about 57.29, 22.07, and 50.10 %, respectively. The improvement in effectiveness of the optimal $\overline{X}$ &EWMA chart compared to the other three charts over eight runs are further tested through paired $t$ tests [5]. The results are also enumerated at the bottom of Table 2. It shows that the improvement in effectiveness of the optimal $\overline{X}$ &EWMA chart compared to the basic EWMA chart ($p$ value = 0.011), the optimal EWMA chart ($p$ value = 0.002) and the basic $\overline{X}$ &EWMA chart ($p$ value = 0.039) are all significant at 0.05 level.

Finally, the effects of the input parameters ($\tau$, $R$, and $\delta_{max}$) on the overall loss ($ML$) and chart parameters ($\lambda$, $n$, $h$, $H$, $UCL$) of the optimal $\overline{X}$ &EWMA chart are studied based on the $2^3$ factorial experiment. Due to the absence of replications, no error term is computed, and the third order interaction is

**Table 2** Comparison of the four charts in $2^3$ experiment

| Factor combination | Input parameter values | | | $ML_{normal}$ | | |
|---|---|---|---|---|---|---|
| | $R$ | $\delta_{max}$ | $\tau$ | Basic EWMA | Optimal EWMA | Basic $\overline{X}$ &EWMA |
| (1) | 2.0 | 3.0 | 300.0 | 1.173 | 1.059 | 1.169 |
| $R$ | 10.0 | 3.0 | 300.0 | 1.686 | 1.259 | 1.697 |
| $\delta_{max}$ | 2.0 | 6.0 | 300.0 | 1.265 | 1.162 | 1.084 |
| $R*\delta_{max}$ | 10.0 | 6.0 | 300.0 | 1.353 | 1.241 | 1.144 |
| $\tau$ | 2.0 | 3.0 | 1200.0 | 1.489 | 1.174 | 1.493 |
| $R*\tau$ | 10.0 | 3.0 | 1200.0 | 2.704 | 1.253 | 2.862 |
| $\delta_{max}*\tau$ | 2.0 | 6.0 | 1200.0 | 1.320 | 1.217 | 1.113 |
| $R*\tau*\delta_{max}$ | 10.0 | 6.0 | 1200.0 | 1.593 | 1.401 | 1.446 |
| | $\overline{ML}_{normal}$ | | | 1.5729 | 1.2207 | 1.5010 |
| | Difference[a] | | | +4.54 | +1.739 | +3.94 |
| | $p$-value | | | 0.011 | 0.002 | 0.039 |

[a] Difference = $\overline{ML}$ of a chart over eight runs—$\overline{ML}$ of the optimal $\overline{X}$ &EWMA chart over eight runs; positive values indicate preference to optimal $\overline{X}$ &EWMA chart

pooled to compute the noise term. An analysis of variance is performed to identify the significant main and interaction effects, based on a 0.05 level of significance. Table 3 shows the $p$ values of the main and interaction effects. The “+” sign in parentheses represents a positive main or interaction effect, while the “–” sign represents a negative main or interaction effect. Interactions marked with “*” are interactions which result in an opposite sign for the first factor at different levels of the second one.

As shown in Table 3, the maximum allowable inspection rate $R$ has significant negative effect on the $ML$ value of the optimal $\overline{X}$ &EWMA chart ($p$ value = 0.043), i.e., the optimal $\overline{X}$ &EWMA chart reduces $ML$ to a significant degree when the $R$ value is at its high level. This is because the larger $R$ results in a larger sample size ($n$), which makes the control chart more powerful in detecting out-of-control cases, and thus reduces the loss to a significant degree. There are quite a few significant main and interaction factors for the chart parameters. It can be seen that $\delta_{max}$ has significant positive effect on the control limits $H$ ($p$ value = 0.001) and $UCL$ ($p$ value = 0.012) of the optimal $\overline{X}$ &EWMA chart, i.e., a larger $\delta_{max}$ results in wider control limits and vice versa. On the other hand, $\tau$ ($p$ value = 0.006) and $R$ ($p$ value = 0.004) have significant negative effects on $H$, i.e., smaller $\tau$ or $R$ results in larger $H$. The two factor interactions $R*\delta_{max}$ ($p$ value = 0.011) and $\delta_{max}*\tau$ ($p$ value = 0.038) have significant negative effects on $H$, this shows that at low levels of $R$ and $\tau$, there is a larger increase in $H$ when $\delta_{max}$ increases. However, the interaction $R*\tau$ has significant positive effect toward $H$ ($p$ value = 0.017), which means that at a high level of $R$, the increase in $H$ is larger when $\tau$ increases.

## 4 Design table

A design table (Table 4) is provided to facilitate the designs of the $\overline{X}$ &EWMA charts. The charting parameters of the optimal $\overline{X}$ &EWMA charts are displayed in the design table according to different specified values of $\tau$ (=300, 400, …, 1200), $R$ (=4, 10), and $\delta_{max}$ (=3, 6). The users can select $\tau$, $R$, and $\delta_{max}$ values that are closest to the desired $ATS_0$, $R$, and $\delta_{max}$ values in their application, and then directly pick up the charting parameter values from the design table.

In addition, the value of the $ML_{normal}$ of the corresponding basic $\overline{X}$ &EWMA chart in each design is also displayed. The $ML_{normal}$ value will reveal the potential benefit that can be acquired by using the optimal $\overline{X}$ &EWMA chart.

From the design table, it is found that, for any given value of $\tau$, $R$, and $\delta_{max}$, the $ML_{normal}$ value of the basic $\overline{X}$ &EWMA chart is always larger than one. This indicates the consistent superiority of the optimal $\overline{X}$ &EWMA chart over the basic $\overline{X}$ &EWMA chart.

Since the designs of the $\overline{X}$ &EWMA charts are carried out under the standard condition ($\mu_0 = 0$ and $\sigma_0 = 1$), the users of the design table have to standardize the quality characteristic $x$ to $z$.

$$z = \frac{x - \mu_0}{\sigma_0}. \tag{12}$$

The design table only contains the charting parameter values of one-sided control charts. But, the application of the design table can be easily extended to the designs of two-sided charts. The users only have to double the specified $\tau$ value when the two-sided charts are to be designed, and then make use of the design table in the same way as the one-sided chart. The control limits of both the $\overline{X}$ and EWMA charts are symmetrical, and the parameter $\lambda$ is the same for both one-sided and two-sided charts.

## 5 Example

A production line is producing a special type of shaft. The diameter $x$ of the shaft is an important quality characteristic. The process mean can be easily adjusted to the nominal value (10 mm) at the center between the lower and upper specification limits of the quality characteristic $x$. In phase I operation, it is found that the

**Table 3** $p$ Values of the main and interaction effects in the analysis of variance

| Factor combination | Chart parameter | | | | | Loss |
|---|---|---|---|---|---|---|
| | $p$ value ($\lambda$) | $p$ value ($n$) | $p$ value ($h$) | $p$ value ($H$) | $p$ value ($UCL$) | $p$ value ($ML$) |
| $R$ | 0.070 (−) | 0.205 (+) | 0.205 (−) | 0.004 (−) | 0.135 (−) | 0.043 (−) |
| $\delta_{max}$ | 0.126 (−) | 0.058 (−) | 0.161 (−) | 0.001 (+) | 0.012 (+) | 0.126 (−) |
| $R*\delta_{max}$ | 0.205 (−) | 0.205 (−) | 0.314 (+) | 0.011 (−) | 0.135 (+) | 0.259 (+) |
| $\tau$ | 0.205 (−) | 0.126 (+) | 0.314 (+) | 0.006 (−) | 0.312 (−) | 0.114 (+) |
| $R*\tau$ | 0.205 (+) | 0.500 (+) | 0.500 (−) | 0.017 (+) | 0.500 (+) | 0.191 (−) |
| $\delta_{max}*\tau$ | 0.126 (−) | 0.126 (−) | 0.314 (−) | 0.038 (−) | 0.062 (+) | 0.322 (−) |

**Table 4** Design table

| $\tau$ | $R$ | $\delta_{max}$ | Basic $\overline{X}$ &EWMA | | | | | | Optimal $\overline{X}$ &EWMA | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $n$ | $h$ | $\lambda$ | $H$ | $UCL$ | $ML_{normal}$ | $n$ | $h$ | $\lambda$ | $H$ | $UCL$ |
| 300 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.70146 | 4.250 | 1.3253 | 4 | 1.00 | 0.05 | 0.18784 | 1.794 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.70146 | 4.250 | 1.1042 | 1 | 0.25 | 0.04 | 0.41596 | 3.588 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.77040 | 4.250 | 1.6966 | 5 | 0.50 | 0.04 | 0.16589 | 1.605 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.77040 | 4.250 | 1.1444 | 1 | 0.10 | 0.03 | 0.40898 | 3.588 |
| 400 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.72376 | 4.250 | 1.3999 | 5 | 1.25 | 0.05 | 0.16987 | 1.638 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.72376 | 4.250 | 1.1131 | 1 | 0.25 | 0.04 | 0.43107 | 3.662 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.79098 | 4.250 | 1.8575 | 6 | 0.60 | 0.04 | 0.15365 | 1.495 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.79098 | 4.250 | 1.1783 | 1 | 0.10 | 0.02 | 0.33273 | 3.662 |
| 500 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.74062 | 4.250 | 1.4816 | 5 | 1.25 | 0.05 | 0.17699 | 1.663 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.74062 | 4.250 | 1.1229 | 1 | 0.25 | 0.04 | 0.44249 | 3.719 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.80671 | 4.250 | 1.9987 | 7 | 0.70 | 0.04 | 0.14355 | 1.406 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.80671 | 4.250 | 1.2133 | 1 | 0.10 | 0.02 | 0.34011 | 3.719 |
| 600 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.75413 | 4.250 | 1.5405 | 6 | 1.50 | 0.05 | 0.16142 | 1.537 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.75413 | 4.250 | 1.1337 | 1 | 0.25 | 0.04 | 0.45149 | 3.765 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.81938 | 4.250 | 2.1237 | 8 | 0.80 | 0.05 | 0.15512 | 1.331 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.81938 | 4.250 | 1.2472 | 1 | 0.10 | 0.02 | 0.34510 | 3.765 |
| 700 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.76534 | 4.250 | 1.6165 | 6 | 1.50 | 0.05 | 0.16577 | 1.5526 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.76534 | 4.250 | 1.1469 | 1 | 0.25 | 0.03 | 0.38780 | 3.803 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.83016 | 4.250 | 2.279 | 8 | 0.80 | 0.04 | 0.13824 | 1.345 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.83016 | 4.250 | 1.2816 | 1 | 0.10 | 0.02 | 0.35088 | 3.803 |
| 800 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.77505 | 4.250 | 1.6625 | 7 | 1.75 | 0.05 | 0.15285 | 1.450 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.77505 | 4.250 | 1.1606 | 1 | 0.25 | 0.03 | 0.39345 | 3.836 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.83946 | 4.250 | 2.3902 | 9 | 0.90 | 0.05 | 0.14964 | 1.279 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.83946 | 4.250 | 1.3154 | 1 | 0.10 | 0.02 | 0.35504 | 3.836 |
| 900 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.78349 | 4.250 | 1.700 | 8 | 2.00 | 0.06 | 0.15996 | 1.366 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.78349 | 4.250 | 1.1742 | 1 | 0.25 | 0.03 | 0.39836 | 3.865 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.84764 | 4.250 | 2.5352 | 9 | 0.90 | 0.05 | 0.15206 | 1.288 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.84764 | 4.250 | 1.3486 | 1 | 0.10 | 0.02 | 0.35865 | 3.865 |
| 1000 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.79098 | 4.250 | 1.7699 | 8 | 2.00 | 0.05 | 0.14506 | 1.376 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.79098 | 4.250 | 1.1877 | 1 | 0.25 | 0.03 | 0.40269 | 3.891 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.85494 | 4.250 | 2.633 | 10 | 1.00 | 0.05 | 0.14416 | 1.230 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.85494 | 4.250 | 1.3812 | 1 | 0.10 | 0.02 | 0.36185 | 3.891 |
| 1100 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.79772 | 4.250 | 1.8002 | 9 | 2.25 | 0.06 | 0.15275 | 1.305 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.79772 | 4.250 | 1.2011 | 1 | 0.25 | 0.03 | 0.40657 | 3.914 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.86160 | 4.250 | 2.7738 | 10 | 1.00 | 0.05 | 0.14597 | 1.238 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.86160 | 4.250 | 1.4140 | 1 | 0.10 | 0.02 | 0.36472 | 3.914 |
| 1200 | 4.0 | 3.0 | 1 | 0.25 | 0.10 | 0.80385 | 4.250 | 1.8249 | 10 | 2.50 | 0.06 | 0.14446 | 1.244 |
| | | 6.0 | 1 | 0.25 | 0.10 | 0.80385 | 4.250 | 1.2143 | 1 | 0.25 | 0.03 | 0.41007 | 3.935 |
| | 10.0 | 3.0 | 1 | 0.10 | 0.10 | 0.86770 | 4.250 | 2.8621 | 11 | 1.10 | 0.05 | 0.13895 | 1.186 |
| | | 6.0 | 1 | 0.10 | 0.10 | 0.86770 | 4.250 | 1.4464 | 1 | 0.10 | 0.02 | 0.36731 | 3.935 |

distribution of $x$ can be well approximated by a normal distribution and the standard deviation of $x$ is very close to 1.0 mm. The QA engineer desired an $ATS_0$ close to 300 h, specified the inspection rate as 4.0 units per hour, and set the upper bound of the mean shift as 6.

Since a two-sided control chart is to be designed, the specified value of $\tau$ is made equal to 600 (=2 × 300) in order to make use of the design table (see Table 4). From Table 4, corresponding to the row for ($\tau$ = 600), ($R$ = 4.0), and ($\delta_{max}$ = 6), *the charting parameters of the optimal* $\overline{X}$ &EWMA

chart can be found and are listed below, together with its main competitor, the basic $\overline{X}$ &EWMA chart ($LCL$ and $L$ are the lower control limits of the $\overline{X}$ chart and EWMA chart, respectively):

$$\text{Basic}\,\overline{X}\,\&\,\text{EWMA chart}: \quad n=1,\ h=0.25,\ \lambda=0.10,\ L=-0.75413,\ H=0.75413,\ LCL=-4.25,\ UCL=4.25$$

$$\text{Optimal}\ \overline{X}\,\&\ \text{EWMA}: \quad n=1,\ h=0.25,\ \lambda=0.04,\ L=-0.45149,\ H=0.45149,\ LCL=-3.765,\ UCL=3.765$$

In this example, the effectiveness of the two charts (basic $\overline{X}$ &EWMA and optimal $\overline{X}$ &EWMA charts) under a kind of assignable cause is investigated through 30 simulated data. The first 15 data are simulated from a normal distribution with mean $\mu=10$ and standard deviation $\sigma=1$, which means that the process is in-control for the first 15 observations; the last 15 data are simulated from a normal distribution with mean $\mu=11$ and standard deviation $\sigma=1$, which means that the assignable cause occurred at the 16th sample and the last 15 data were drawn from the process when it was out-of-control—that is, after the process has experienced a shift in the mean of $1\sigma$ [5]. In order to standardize the design and operation, the diameter $x$ is converted to $z$ (Eq. (12)), conforming to a standard normal distribution.

$$z_i = \frac{x_i - 10.0}{1.0}$$

The EWMA statistic $S_i$ is calculated (Eq. (2)) for all 30 standardized data and are plotted on the respective control charts (i.e., test statistics $z_i$ are plotted on the $\overline{X}$ chart and $S_i$ are plotted on the EWMA chart) and their effectiveness are compared (see Fig. 2).

Figure 2a (basic $\overline{X}$ &EWMA chart) shows that the EWMA chart alarms the out-of-control condition at the 23rd sample, whereas, Fig. 2b (optimal $\overline{X}$ &EWMA chart) shows that the EWMA chart alarms the out-of-control condition at the 20th sample, which clearly demonstrates the superiority of the optimal $\overline{X}$ &EWMA chart over the basic $\overline{X}$ &EWMA chart. This improvement in detection effectiveness is completely attributable to the optimization design proposed in this article. It is noted that in both cases, the Shewhart $\overline{X}$ charts are not able to identify the out-of-control condition as the $\overline{X}$ chart is relatively insensitive to small shifts. It can be seen from Table 4 that the optimal $\overline{X}$ &EWMA chart is more effective than the basic $\overline{X}$ &EWMA chart by 13.37 % (in terms of $ML$), for this particular example. An on-site computer will aid the implementation of the optimal $\overline{X}$ &EWMA chart. The operators only need to input the sample values of $x$ from a keyboard for each sample (for an automated manufacturing system, data may be collected automatically), then all the computations and plotting will be handled by a computer program.

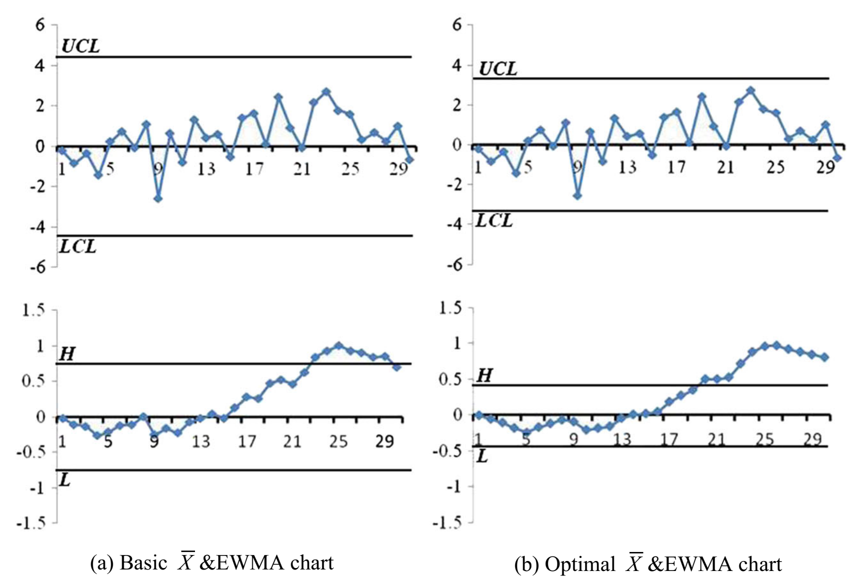


**Fig. 2** Two control charts used in the example

## 6 Conclusions

This article presents an algorithm for the optimization design of the $\overline{X}$ &EWMA chart, which comprises an $\overline{X}$ chart and an EWMA chart. The design algorithm does not only optimize the charting parameters of each of the $\overline{X}$ chart and the EWMA chart but also optimizes the allocation of the detection power between the two individual charts based on the loss function. The optimization design effectively improves the performance of the $\overline{X}$ &EWMA chart over the entire process shift range. The factorial experiment in the comparative studies show that the optimal $\overline{X}$ &EWMA chart is always superior to the basic EWMA chart, the optimal EWMA chart and the basic $\overline{X}$ &EWMA chart under different specifications. The optimal $\overline{X}$ &EWMA chart is more effective than the main competitor, the basic $\overline{X}$ &EWMA chart, by 50.10 %, on average, in terms of *ML*. The optimal $\overline{X}$ &EWMA chart also outperforms the other charts, such as basic EWMA chart and optimal EWMA chart, to a significant degree.

The design of the optimal $\overline{X}$ &EWMA chart is more difficult than that of the traditional EWMA or the combined $\overline{X}$ &EWMA chart; however, its application can be justified by the significant improvement in the overall performance. The studies in this article reveal the importance of optimization design in control charts. The optimization design for the combined $\overline{X}$ &EWMA chart has convincingly achieved higher overall effectiveness. Moreover, the optimal control chart can be implemented as easy as the basic chart.

Finally, some researchers [13–15] suggested using combined $\overline{X}$ &EWMA chart to guard against the inertia problem of the EWMA chart. Future work can study the effect of the optimization design on the inertia property of the $\overline{X}$ &EWMA chart.

**Acknowledgments** This article is a part of an ongoing research project titled "Comparison Study on Effectiveness of EWMA Control Charts for Monitoring Process Mean" conducted by "Sustainable Engineering Asset Management" research group at University of Sharjah, UAE. The authors are grateful to Prof. Wu Zhang (retired), Nanyang Technological University, Singapore, for his kind support throughout the research.

## Appendix: Calculation of the in-control $ATS_0$ and out-of-control $ATS$ of the $\overline{X}$ &EWMA chart

The $\overline{X}$ &EWMA chart can be described by a Markov chain procedure. Suppose that the statistic $S_t$ in Eq. (2) experiences $M$ different transitional states before being absorbed into the out-of-control state. States 0 to $(M-1)$ are in-control states and state $M$ is an out-of-control state. The width $d$ of the interval of each in-control state is given as

$$d = H/(M–0.5). \quad \text{(A1)}$$

The center, $O_i$, of state $i$ is given by

$$O_i = i\,d \quad i = 1, \ldots, M. \quad \text{(A2)}$$

The transition probability $p_{ij}$ from state $i$ to state $j$ of the $\overline{X}$ &EWMA chart is determined as follows:

For $j = 0$,

$$p_{i0} = \begin{cases} \Phi \quad (UCL) \quad , & \text{if } \dfrac{0.5{\cdot}d-(1-\lambda){\cdot}i{\cdot}d}{\lambda} > UCL \\ \Phi\left(\dfrac{0.5{\cdot}d-(1-\lambda){\cdot}i{\cdot}d}{\lambda}\right), & \text{if } \dfrac{0.5{\cdot}d-(1-\lambda){\cdot}i{\cdot}d}{\lambda} < UCL \end{cases} \quad \text{(A3)}$$

For $j > 0$,

$$p_{ij} = \begin{cases} \Phi(u) - \Phi(l)\ , & \text{if } UCL > u \\ \Phi(UCL) - \Phi(l), & \text{if } l < UCL < u \\ 0 \quad , & \text{if } l > UCL \end{cases} \quad \text{(A4)}$$

where $\Phi(.)$ is the standard normal cumulative distribution function, $l = \frac{j{\cdot}d-0.5{\cdot}d-(1-\lambda){\cdot}i{\cdot}d}{\lambda}$, and $u = \frac{j{\cdot}d+0.5{\cdot}d-(1-\lambda){\cdot}i{\cdot}d}{\lambda}$.

When computing the $ATS_0$, the transition probability $p_{ij}$ is calculated with $\mu = 0$. Based on $p_{ij}$, the in-control transition matrix $\boldsymbol{R_0}$ can be established. It is a $(M \times M)$ matrix excluding the elements associated with the absorbing (or out-of-control) state. The $ATS_0$ value is equal to the first element of the vector $\boldsymbol{V}$ given by the following expression:

$$V = (I–R_0)^{-1} h, \quad \text{(A5)}$$

where $\boldsymbol{I}$ is an identity matrix and $\boldsymbol{h}$ is a vector with all elements equal to the sampling interval $h$.

The transition matrix $\boldsymbol{R}$ for calculating the out-of-control $ATS$ can be established similarly, except that the transition probability $p_{ij}$ in $\boldsymbol{R}$ should be evaluated using the out-of-control $\mu = \mu_0 + \delta\sigma_0$. The out-of-control $ATS$ under the steady-state mode is calculated as the following [27],

$$ATS = B^T\left[(I–R)^{-1} h–0.5h\right], \quad \text{(A6)}$$

where $\boldsymbol{B}$ is the steady-state probability vector with $(\mu = 0)$. It is obtained by first normalizing $\boldsymbol{R_0}$ and then solving the following equation.

$$B = R_0^T B, \quad \text{(A7)}$$

subject to

$$\mathbf{1}^T B = 1, \quad \text{(A8)}$$

where $\mathbf{1}$ is a vector with all elements equal to one.

It is worth emphasizing that all formulae derived in the Appendix have been checked by simulation.

## References


1. Shewhart WA (1926) Quality control charts. Bell Syst Tech J 5(4): 593–603
2. Jiao JR, Helo PT (2006) Optimization design of a CUSUM control chart based on taguchi's loss function. Int J Adv Manuf Tech 35: 1234–1243
3. Zhang HY, Shamsuzzaman M, Xie M, Goh TN (2011) Design and application of exponential chart for monitoring time-between-events data under random process shift. Int J Adv Manuf Tech 57: 849–857
4. Haridy S, Wu Z, Khoo MBC, Yu FJ (2012) A combined synthetic & np scheme for detecting increases in fraction nonconforming. Comput Ind Eng 62:979–988
5. Montgomery DC (2005) Introduction to statistical quality control. Wiley, New York
6. Page ES (1954) Continuous inspection scheme. Biometrika 41: 100–114
7. Roberts SW (1959) Control chart tests based on geometric moving averages. Technometrics 1:239–250
8. Zhao Y, Tsung F, Wang ZJ (2005) Dual CUSUM control scheme for detecting a range of mean shift. IIE Trans 37:1047–1058
9. Shu L, Jiang W, Wu S (2007) A one-sided EWMA control chart for monitoring process means. Commun Stats - Simul Comput 36: 901–920
10. Pastell M, Madsen H (2008) Application of CUSUM charts to detect lameness in a milking robot. Expert Syst Appl 35:2032–2040
11. Yashchin E (1987) Some aspects of the theory of statistical control schemes. IBM J Res Dev 31:199–205
12. Woodall WH, Mahmoud MA (2005) The inertial properties of quality control charts. Technometrics 47:425–436
13. Lucas JM (1982) Combined Shewhart-CUSUM quality control schemes. J Qual Technol 14:51–59
14. Lucas JM, Saccucci MS (1990) Exponentially weighted moving average control schemes: properties and enhancements. Technometrics 32:1–12
15. Capizzi G, Masarotto G (2003) An adaptive exponentially weighted moving average control chart. Technometrics 45:199–207
16. Klein M (1996) Composite Shewhart-EWMA statistical control schemes. IIE Trans 28:475–481
17. Albin SL, Kang L, Shea G (1997) An X and EWMA chart for individual observations. J Qual Technol 29(1):41–48
18. Tolley GO, English JR (2001) Economic designs of constrained EWMA and combined EWMA- $\overline{X}$ control schemes. IIE Trans, 33:429–436
19. Duncan AJ (1956) The economic design of $\overline{X}$ charts used to maintain current control of a process. J Am Stat Assoc 51:228–242
20. Simoes BFT, Epprecht EK, Costa AFB (2010) Performance comparison of EWMA control chart schemes. Qual Technol Quani Manag 7(3):249–261
21. Lin YC, Chou CY (2011) Robustness of the EWMA and the combined $\overline{X}$ -EWMA control charts with variable sampling intervals to non-normality. J Appl Stat 38(3):553–570
22. Reynolds MR Jr, Stoumbos ZG (2004) Should observations be grouped for effective process monitoring. J Qual Technol 36:343–366
23. Shu L, Tsung F, Kapur K (2004) Monitoring and diagnosis with multiple cause-selecting control charts. Qual Eng 16:437–450
24. Yu FJ, Hou JL (2006) Optimization of design parameters for $\overline{X}$ control charts with multiple assignable causes. J Appl Stat 33: 279–290
25. Wu Z, Yang M, Khoo MBC, Castagliola P (2011) What are the best sample sizes for the $\overline{X}$ and CUSUM charts?. Int J Prod Econ 131: 650–662
26. Wu Z, Shamsuzzaman M, Pan ES (2004) Optimization designs of the control charts based on taguchi's loss function and random process shifts. Int J Prod Res 43:379–390
27. Reynolds MR Jr, Amin RW, Arnold JC (1990) CUSUM charts with variable sampling intervals. Technometrics 32:371–384
28. Shamsuzzaman M, Wu Z (2012) Design of EWMA control chart for minimizing the proportion of defective units. Int J Qual Reliab Manag 29(8):953–969
29. Wu Z, Yang M, Jiang W, Khoo MBC (2008) Optimizations designs of the combined Shewhart-CUSUM control charts. Comput Stat Data Anal 53:496–506
30. Sparks RS (2000) CUSUM charts for signalling varying location shifts. J of Qual Tech 32:157–171
31. Reynolds MR Jr, Stoumbos ZG (2006) Comparisons of some exponentially weighted moving average control charts for monitoring the process mean and variance. Technometrics 48:550–567
32. Shu L, Jiang W (2006) A Markov chain model for the adaptive CUSUM control chart. J Qual Technol 38:135–147