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ORIGINAL ARTICLE

Prediction of chatter stability for milling process using Runge-Kutta-based complete discretization method

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Abstract On the basis of the classical Runge-Kutta method and the complete discretization method, a Runge-Kutta-based complete discretization method (RKCDM) is proposed in the paper to predict the chatter stability of milling process, in which the regenerative effect is taken into consideration. Firstly, the dynamics model of milling process is simplified as a 2-DOF vibration system in the two orthogonal directions, which can be expressed as coefficient-varying periodic differential equations with a single time delay. Then, all parts of the delay differential equation (DDE), including delay term, time-domain term, parameter matrices, and most of all the differential terms are discretized using the classical fourth-order Runge-Kutta iteration method to replace the direct integration scheme used in the classical semi-discretization method (C-SDM) and the classical complete discretization scheme with the Euler method (C-CDSEM), which can simplify the complexity of the discretization iteration formula greatly. Lastly, the Floquet theory is adopted to predict the stability of milling process by judging the eigenvalues of the state transition matrix corresponding to certain cutting conditions. Comparing RKCDM with C-SDM and C-CDSEM, the numerical simulation results show that RKCDM has the highest convergence rate, computation accuracy, and computation efficiency. As dichotomy search rather than sequential search is used in the algorithm, the calculation time for obtaining the stability lobe diagrams (SLDs) is greatly reduced. As a result, it is practical to determine the optimal chatter-free cutting conditions for milling operation in shop floor applications.

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Keywords Milling process · Stability lobe diagram · Semi-discretization method · Complete discretization method · Runge-Kutta method

1 Introduction

In milling process, machining chatter has a negative effect on the machined surface quality, the cutting tool, and even the machine tool. Prediction of chatter stability is an important and effective way for optimal selection of spindle speed and cutting depths to avoid chatter and improve production efficiency. Based on the classical dynamic model considering the regenerative effect described by a delay differential equation (DDE) with time varying coefficients [1], the stability lobe diagrams (SLDs) can be obtained to find the critical cutting depths and the corresponding spindle speeds [2, 3]. It contains stability boundaries, below which vibration dies down and above which vibration grows to finite amplitudes eventually.

Besides experimental methods [4] and experimental-analytical methods [5], numerical algorithms to predict SLDs have been developed. The basic way to predict SLDs is to transform DDEs from time domain to frequency domain using the Laplace Transform. And then the critical axial cutting depths and the corresponding spindle speeds are calculated utilizing real and image part of the characteristic equation of the system in frequency domain under the premise of the giving radial depth of cut. Utilizing this method, Altintas and Budak [6] presented an analytical method (ZOA method) for predicting milling stability lobe diagrams based on the mean value of Fourier series of the dynamic milling coefficients. Li et al. [7] proposed an analytical method to predict the stability lobe diagrams for orbital drilling using a numerical approach to calculate the average directional cutting force coefficients, which is represented by the ratio of the axial to the tangential feed per tooth with regard to the spindle speed.

The ZOA is efficient and fast, but it is unable to predict the existence of the additional stability regions and period doubling bifurcations in case of low immersion milling. To overcome this problem, several other methods such as multi-frequency solution (MFS), temporal finite element analysis (TFEA), semi-discretization method (SDM), full-discretization method (FDM), and direct numerical solution method (DNSM) have been developed. The MFS was first explored by Budak and Altintas [8], and then extended by Merdol and Altintas [9] which considers harmonics of the tooth spacing angle and spread of the transfer function with the harmonics of the tooth passing frequencies. The TFEA was presented by Bayly et al. [10, 11], which can be utilized to predict both the milling stability and the surface location error, but its prediction accuracy is challenged when the radial immersion ratio approximates to one. SDM and the so-called 1st DM (first-order semi-discretization method) were developed by Insperger and Stepan [12, 13], and it is an efficient numerical method for stability analysis of linear-delayed systems which can be widely adopted as benchmarks methods to evaluate other methods of predicting milling stability in time domain. On the basis of direct integration scheme, Ding et al. [14] presented a FDM to predict milling stability which has higher prediction efficiency without sacrificing any prediction accuracy compared with SDM. They [15] also developed a semi-analytical method, the numerical integration methods (NIM), to predict milling stability on the basis of integration methods of integral equations. The authors [16] proposed a time-domain DNSM to obtain SLDs for milling operation, in which a fourth-order Runge–Kutta method was utilized to solve the dynamic differential equations, and several chatter detection criterions were also applied to the predicted time-domain data to determine the stability of milling process.

Recently, a lot of researches have focused on presenting new methods with an aim to gain both high computational efficiency and high convergence rate. Li et al. [17] presented a complete discretization scheme with the Euler's method (CDSEM) for solving the one-DOF and two-DOF motion equations of milling process. When compared with SDM and FDM, the benchmark results of one-DOF and two-DOF milling stability prediction show that CDSEM can obtain acceptable precision in most ranges, and CDSEM is faster than FDM. Ding et al. [18] proposed two Runge-Kutta-based semi-analytical methods, the classical fourth-order Runge-Kutta method (CRKM) and the generalized Runge-Kutta method (GRKM) to predict the stability of milling process considering the regenerative effect. Simulation results show that the convergence rate of the CRKM is lower than expectation, and the GRKM is of high convergence rate and high computation efficiency.

In all the discretization methods such as SDM, FDM, CDSEM, CRKM, and GRKM, the algorithm used to obtain the SLDs of milling process needs to iterate the process of determining the stability of certain cutting conditions for both discrete available axial depths of cut and available discrete spindle speeds. Therefore, the process to obtain the SLDs is very time-consuming and not suitable for practical applications. On the basis of the classical Runge-Kutta method and the complete discretization scheme, this paper presents a Runge-Kutta based complete discretization method (RKCDM) for predicting the stability of milling process, in which an iterative dichotomy search other than a sequential search was used to determine the stable borders which results in a decreased calculation amount by an order of magnitude.

The rest of this paper is organized as follows. Section 2 sets up the mathematical model of dynamic milling system and the derivation of RKCDM. Section 3 gives the detailed algorithm of RKCDM. Section 4 shows the numerical results of the proposed RKCDM, compares it with other numerical methods and lists some discussion. Some conclusions are drawn in Section 5.

2 Complete discretization method based on the classical Runge-Kutta method

2.1 Dynamics modeling of 2-DOF milling process

The governing equations of a 2-DOF milling model with couple and delay differential equations read

$$\begin{pmatrix}\ddot{x}(t)\\ \ddot{y}(t)\end{pmatrix}+\begin{pmatrix}2\varsigma_x\omega_{\mathrm{n}x} & 0\\ 0 & 2\varsigma_y\omega_{\mathrm{n}y}\end{pmatrix}\begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\end{pmatrix}$$
$$+\begin{pmatrix}\omega_{\mathrm{n}x}^2+\dfrac{a_{\mathrm{p}}\omega_{\mathrm{n}x}^2h_{xx}(t)}{k_x} & \dfrac{a_{\mathrm{p}}\omega_{\mathrm{n}x}^2h_{xy}(t)}{k_x}\\ \dfrac{a_{\mathrm{p}}\omega_{\mathrm{n}y}^2h_{yx}(t)}{k_y} & \omega_{ny}^2+\dfrac{a_{\mathrm{p}}\omega_{\mathrm{n}y}^2h_{yy}(t)}{k_y}\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$$
$$=\begin{pmatrix}\dfrac{a_{\mathrm{p}}\omega_{\mathrm{n}x}^2h_{xx}(t)}{k_x} & \dfrac{a_{\mathrm{p}}\omega_{\mathrm{n}x}^2h_{xy}(t)}{k_x}\\ \dfrac{a_{\mathrm{p}}\omega_{\mathrm{n}y}^2h_{yx}(t)}{k_y} & \dfrac{a_{\mathrm{p}}\omega_{\mathrm{n}y}^2h_{yy}(t)}{k_y}\end{pmatrix}\begin{pmatrix}x(t-T)\\ y(t-T)\end{pmatrix} \quad (1)$$

where ω_{nx}, ζ_x, k_x, ω_{ny}, ζ_y, and k_y are the natural frequency, relative damping, and stiffness in x and y directions; a_p is the axial depth of cut; K_t is the tangential specific cutting force; T is tooth passing period; $x(t\text{-}T)$ and $y(t\text{-}T)$ are the delayed terms; the time varying directional coefficients read

$$\begin{aligned} h_{xx}(t) &= \sum_{j=1}^{N} -g_j(t)\left[\sin 2\phi_j(t) + K_r\left(1-\cos 2\phi_j(t)\right)\right] \\ h_{xy}(t) &= \sum_{j=1}^{N} -g_j(t)\left[\left(1+\cos 2\phi_j(t)\right) + K_r\sin 2\phi_j(t)\right] \\ h_{yx}(t) &= \sum_{j=1}^{N} g_j(t)\left[\left(1-\cos 2\phi_j(t)\right) + K_r\sin 2\phi_j(t)\right] \\ h_{yy}(t) &= \sum_{j=1}^{N} g_j(t)\left[\sin 2\phi_j(t) - K_r\left(1+\cos 2\phi_j(t)\right)\right] \end{aligned} \quad (2)$$

where N is the number of the teeth, K_r is the ratio of the radial specific cutting force to the tangential specific cutting force, ϕ_j(t) is angular position of tooth j defined as

$$\phi_j(t) = (2\pi\Omega/60)t + 2\pi(j-1)/N \quad (3)$$

where Ω is the spindle speed in r/min. The function $g(\phi_j(t))$ is a screen function used to determine whether tooth j is in cutting or not, which is defined as

$$g\left(\phi_j(t)\right) = \begin{cases} 1 & \text{if } \phi_{st} < \phi_j(t) < \phi_{ex} \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

where ϕ_{st} and ϕ_{ex} denote the start and exit angles of the jth tooth, respectively. For up milling, $\phi_{st}=0$ and $\phi_{ex}=\arccos(1\text{-}2a_e/D)$; for down milling, $\phi_{st}=\arccos(2a_e/D\text{-}1)$ and $\phi_{ex}=\pi$, where a_e/D denotes the radial immersion ratio (a_e is the radial depth of cut, and D is the diameter of the cutting tool).

2.2 Determine stability using classical Runge-Kutta method

By transforming Eq. (1) to the space state form, the following expression can be got

$$\dot{\mathbf{u}}(t) = \mathbf{A}(t)\mathbf{u}(t) + \mathbf{B}(t)\mathbf{u}(t-\tau) \quad (5)$$

Where $\mathbf{A}(t)$, $\mathbf{B}(t)$ are time periodic coefficient matrices, $A(t+T)=A(t)$, $B(t+T)=B(t)$, T is the time period, and τ is the time delay, where $T=\tau$ for the single delay milling process. $A(t)$ and $B(t)$ can be expressed as

$$\mathbf{A}(t) = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_{nx}^2 - \dfrac{a_p\omega_{nx}^2 h_{xx}(t)}{k_x} & -\dfrac{a_p\omega_{nx}^2 h_{xy}(t)}{k_x} & -2\varsigma_x\omega_{nx} & 0 \\ -\dfrac{a_p\omega_{ny}^2 h_{yx}(t)}{k_y} & -\omega_{ny}^2 - \dfrac{a_p\omega_{ny}^2 h_{yy}(t)}{k_y} & 0 & -2\varsigma_y\omega_{ny} \end{Bmatrix}$$

$$\mathbf{B}(t) = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{a_p\omega_{nx}^2 h_{xx}(t)}{k_x} & \dfrac{a_p\omega_{nx}^2 h_{xy}(t)}{k_x} & 0 & 0 \\ \dfrac{a_p\omega_{ny}^2 h_{yx}(t)}{k_y} & \dfrac{a_p\omega_{ny}^2 h_{yy}(t)}{k_y} & 0 & 0 \end{Bmatrix} \quad \mathbf{u}(t) = \begin{pmatrix} x(t) \\ y(t) \\ \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \quad (6)$$

In order to solve Eq. (5) using the Runge-Kutta method, the period T should be firstly divided into m intervals with a time step Δt, $T=m\Delta t$, and m is the discrete number during one tooth passing period. The classical fourth-order Runge-Kutta method is an explicit form, written as [18]

$$\begin{aligned} \mathbf{u}_{i+1} &= \mathbf{u}_i + \frac{\Delta t}{6}(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4) \\ \mathbf{K}_1 &= f(t_i, \mathbf{u}_i) \\ \mathbf{K}_2 &= f\left(t_i + \frac{\Delta t}{2}, \mathbf{u}_i + \frac{\Delta t}{2}\mathbf{K}_1\right) \\ \mathbf{K}_3 &= f\left(t_i + \frac{\Delta t}{2}, \mathbf{u}_i + \frac{\Delta t}{2}\mathbf{K}_2\right) \\ \mathbf{K}_4 &= f(t_i + \Delta t, \mathbf{u}_i + \Delta t\mathbf{K}_3) \end{aligned} \quad (7)$$

where u_i represents $\mathbf{u}(i\Delta t)$, u_{i+1} denotes $u((i+1)\Delta t)$, t_i denotes idt, and the variable i is an integer satisfying $0 \le i \le m$. To solve the Eq. (5) using the Runge-Kutta method, the following equations can be deduced in sequence:

$$\begin{aligned} \mathbf{K}_1 &= \mathbf{A}_i\mathbf{u}_i + \mathrm{B}_i\mathbf{u}_{i-m} \\ \mathbf{K}_2 &= \mathbf{F}_{i,2}\mathbf{u}_i + \mathbf{F}_{i-m,2}\mathbf{u}_{i-m} + \mathbf{F}_{i-m+1,2}\mathbf{u}_{i-m+1} \\ \mathbf{K}_3 &= \mathbf{F}_{i,3}\mathbf{u}_i + \mathbf{F}_{i-m,3}\mathbf{u}_{i-m} + \mathbf{F}_{i-m+1,3}\mathbf{u}_{i-m+1} \\ \mathbf{K}_4 &= \mathbf{F}_{i,4}\mathbf{u}_i + \mathbf{F}_{i-m,4}\mathbf{u}_{i-m} + \mathbf{F}_{i-m+1,4}\mathbf{u}_{i-m+1} \end{aligned} \quad (8)$$

where

$$\begin{aligned} \mathbf{A}_i &= \frac{1}{\Delta t}\int_{t_i}^{t_{i+1}} \mathbf{A}(t)\mathrm{d}t \\ \mathbf{B}_i &= \frac{1}{\Delta t}\int_{t_i}^{t_{i+1}} \mathrm{B}(t)\mathrm{d}t \end{aligned} \quad (9)$$

$$\begin{aligned} \mathbf{F}_{i,2} &= \mathbf{A}_{i+0.5} + 0.5\Delta t\mathbf{A}_{i+0.5}\mathbf{A}_i \\ \mathbf{F}_{i-m,2} &= 0.5\Delta t\mathbf{A}_{i+0.5}\mathbf{B}_i + 0.5\mathbf{B}_{i+0.5} \\ \mathbf{F}_{i-m+1,2} &= 0.5\mathbf{B}_{i+0.5} \\ \mathbf{F}_{i,3} &= \mathbf{A}_{i+0.5} + 0.5\Delta t\mathbf{A}_{i+0.5}\mathbf{F}_{i,2} \\ \mathbf{F}_{i-m,3} &= 0.5\Delta t\mathbf{A}_{i+0.5}\mathbf{F}_{i-m,2} + 0.5\mathbf{B}_{i+0.5} \\ \mathbf{F}_{i-m+1,3} &= 0.5\Delta t\mathbf{A}_{i+0.5}\mathbf{F}_{i-m+1,2} + 0.5\mathbf{B}_{i+0.5} \\ \mathbf{F}_{i,4} &= \mathbf{A}_{i+1} + \Delta t\mathbf{A}_{i+1}\mathbf{F}_{i,3} \\ \mathbf{F}_{i-m,4} &= \Delta t\mathbf{A}_{i+1}\mathbf{F}_{i-m,3} \\ \mathbf{F}_{i-m+1,4} &= \Delta t\mathbf{A}_{i+1}\mathbf{F}_{i-m+1,3} + \mathbf{B}_{i+1} \end{aligned} \quad (10)$$

By substituting Eqs. (8) into (7), the following iterative formula of $\mathbf{u}(t)$ can be obtained

$$\mathbf{u}_{i+1} = \mathbf{F}_i\mathbf{u}_i + \mathbf{F}_{i-m}\mathbf{u}_{i-m} + \mathbf{F}_{i-m+1}\mathbf{u}_{i-m+1} \tag{11}$$

where the coefficients F_i, F_{i-m}, and F_{i-m+1} can be expressed as

$$\begin{aligned} \mathbf{F}_i &= \mathbf{I} + \frac{\Delta t}{6}\mathbf{A}_i + \frac{\Delta t}{3}\mathbf{F}_{i,2} + \frac{\Delta t}{3}\mathbf{F}_{i,3} + \frac{\Delta t}{6}\mathbf{F}_{i,4} \\ \mathbf{F}_{i-m} &= \frac{\Delta t}{6}\mathbf{B}_i + \frac{\Delta t}{3}\mathbf{F}_{i-m,2} + \frac{\Delta t}{3}\mathbf{F}_{i-m,3} + \frac{\Delta t}{6}\mathbf{F}_{i-m,4} \\ \mathbf{F}_{i-m+1} &= \frac{\Delta t}{3}\mathbf{F}_{i-m+1,2} + \frac{\Delta t}{3}\mathbf{F}_{i-m+1,3} + \frac{\Delta t}{6}\mathbf{F}_{i-m+1,4} \end{aligned} \tag{12}$$

where I is an $n \times n$ identity matrix, and n is the dimension of vector $\mathbf{u}(t)$.

To obtain the transition matrix, we first define a new $n \times (m+1)$ dimensional vector z_i as

$$\mathbf{z}_i = \mathrm{col}(\mathbf{u}_i, \mathbf{u}_{i-1}, \ldots, \mathbf{u}_{i-m+1}, \mathbf{u}_{i-m}) \tag{13}$$

Accordingly, the resulting discrete map is expressed as

$$\mathbf{z}_{i+1} = \mathbf{D}_i\mathbf{z}_i \tag{14}$$

where the coefficient matrix D_i can be constructed as a (2 m + 4)-dimensional matrix

$$\mathbf{D}_i = \left\{ \begin{array}{ccccccccccc} \mathbf{F}_{i,11} & \mathbf{F}_{i,12} & \mathbf{F}_{i,13} & \mathbf{F}_{i,14} & 0 & \cdots & 0 & \mathbf{F}_{i-m+1,11} & \mathbf{F}_{i-m+1,12} & \mathbf{F}_{i-m,11} & \mathbf{F}_{i-m,12} \\ \mathbf{F}_{i,21} & \mathbf{F}_{i,22} & \mathbf{F}_{i,23} & \mathbf{F}_{i,24} & 0 & \cdots & 0 & \mathbf{F}_{i-m+1,21} & \mathbf{F}_{i-m+1,22} & \mathbf{F}_{i-m,21} & \mathbf{F}_{i-m,22} \\ \mathbf{F}_{i,31} & \mathbf{F}_{i,32} & \mathbf{F}_{i,33} & \mathbf{F}_{i,34} & 0 & \cdots & 0 & \mathbf{F}_{i-m+1,31} & \mathbf{F}_{i-m+1,32} & \mathbf{F}_{i-m,31} & \mathbf{F}_{i-m,32} \\ \mathbf{F}_{i,41} & \mathbf{F}_{i,42} & \mathbf{F}_{i,43} & \mathbf{F}_{i,44} & 0 & \cdots & 0 & \mathbf{F}_{i-m+1,41} & \mathbf{F}_{i-m+1,42} & \mathbf{F}_{i-m,41} & \mathbf{F}_{i-m,42} \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \end{array} \right\} \tag{15}$$

where $F_{i,hj}$, $F_{i-m+1,hj}$ and $F_{i-m,hj}$ are the elements of matrices F_i, F_{i-m+1} and F_{i-m} in the hth row and jth column. The $(2m+4)$-dimensional transition matrix Φ is determined by coupling Eq. (15) for $i = 0, 1, \ldots, k-1$:

$$\Phi = \mathbf{D}_{k-1}\mathbf{D}_{k-2}\cdots\mathbf{D}_2\mathbf{D}_1 \tag{16}$$

According to the Floquet theory, if any of the modules of the eigenvalues of the transition matrix Φ is larger than one,

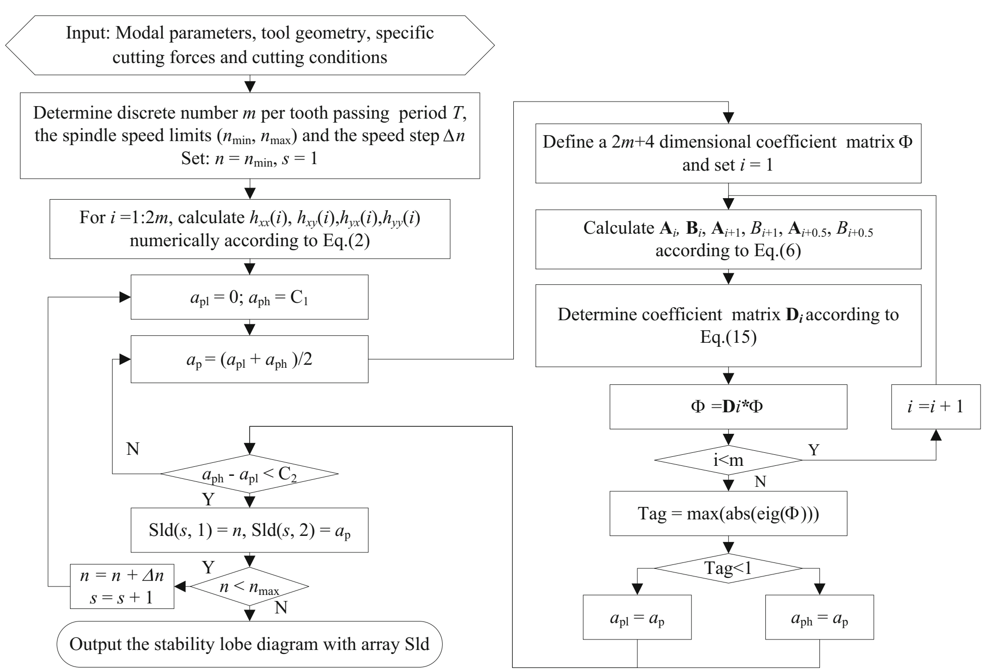
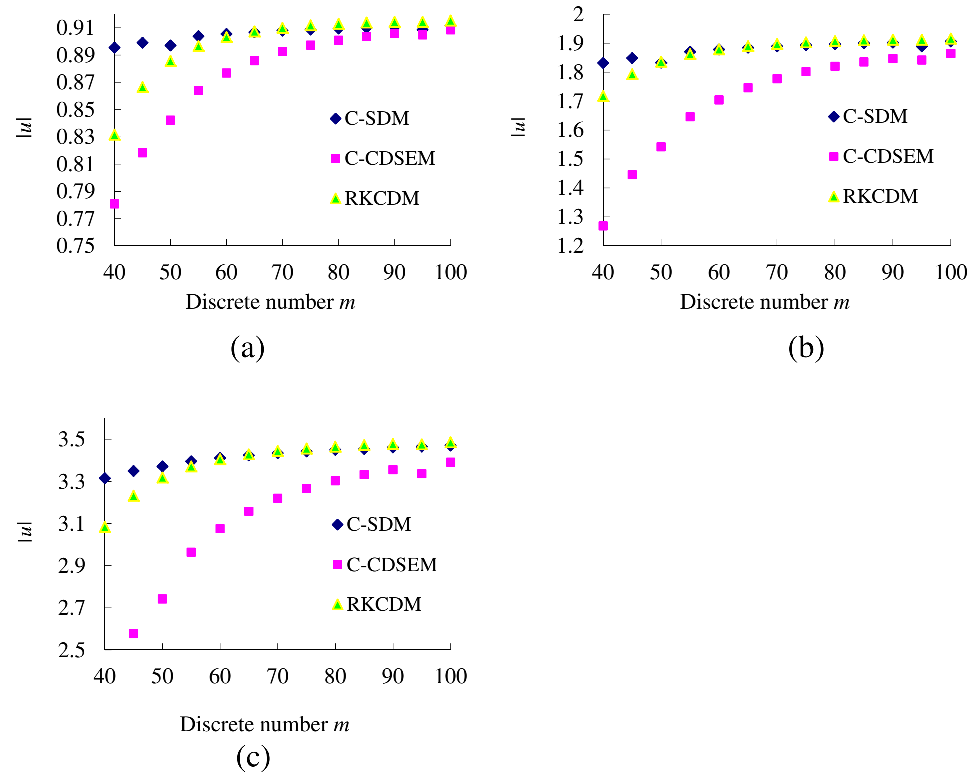


Fig. 1 Flow chart of obtaining SLDs for milling process using RKCDM

Fig. 2 Effect of the discrete number m on the $|u|$ among C-SDM, C-CDSEM, and RKCDM **a** $a_p = 0.1$ mm, $|u_0| = 0.9165$ (stable) **b** $a_p = 0.5$ mm, $|u_0| = 1.9292$ (unstable) **c** $a_p = 1.0$ mm, $|u_0| = 3.5213$ (unstable)

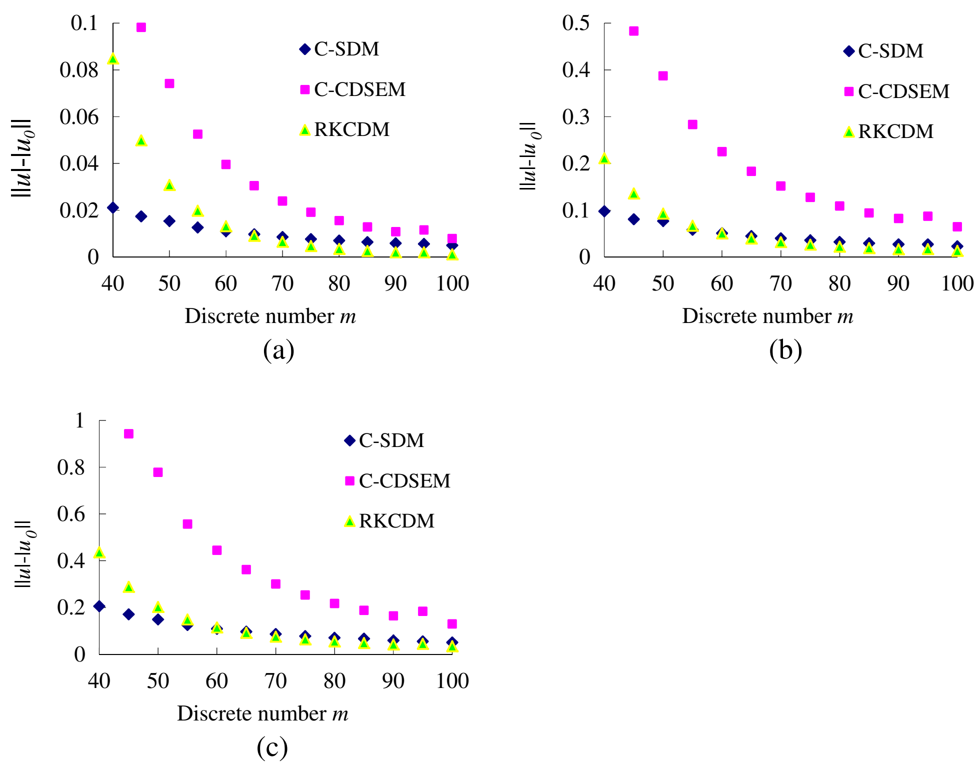


the system is unstable; otherwise, the system is stable. Therefore, the boundary curve dividing stable areas and unstable areas in the stability lobe diagram can serve as a criterion to determine whether chatter occurs or not.

3 Algorithm to obtain SLDs using RKCDM

On the basis of the abovementioned RKCDM for stability analysis of a 2-DOF milling system, a Matlab-based

Fig. 3 Convergence rate comparison among C-SDM, C-CDSEM, and RKCDM **a** $a_p = 0.1$ mm, $|u_0| = 0.9165$ (stable) **b** $a_p = 0.5$ mm, $|u_0| = 1.9292$ (unstable) **c** $a_p = 1.0$ mm, $|u_0| = 3.5213$ (unstable)



simulation module has been developed, which reads the input data such as the modal parameters, tool geometry, specific cutting forces and cutting conditions, and output the predicted SLDs. The flow chart of the RKCDM is shown in Fig. 1. In literature [18], different kinds of interpolation such as the Lagrange interpolation and barycentric Lagrange interpolation were used to determine the middle points of intervals indirectly. In order to improve the calculation accuracy of $A_{i+0.5}$, $B_{i+0.5}$ corresponding to the middle points between discrete point i and $i+1$, $2m$ discrete points rather than m discrete points during one tooth passing time period were used in the paper to calculate the directional cutting force coefficients h_{xx}, h_{xy}, h_{yx}, and h_{yy}, and thus, the even points were be used to calculate the value of A_i, A_{i+1}, B_i, and B_{i+1}, and the odd points were used to calculate the value of $A_{i+0.5}$ and $B_{i+0.5}$.

As we know, both in the classical SDM algorithm provided by Insperger et al. [12] or the CDSEM algorithm provided by Li et al. [17], the stability must be determined for each discrete axial depth of cut at each given discrete spindle speed, which results in an unbearable computation time. To this end, firstly, an iterative dichotomy search instead of a sequential search was used to determine the stable borders which results in a decreased calculation amount by an order of magnitude. Secondly, a one-dimensional array instead of a two-dimensional array was used to save the SLDs data which results in a decrease of memory space. As a result, the simulation time is greatly reduced and thus makes it possible to apply it in shop floor applications.

4 Simulation result and analysis

Suppose the tooth number of the cutting tool used in the following simulations is 2 ($N=2$). The modal parameters of the dominant mode of the milling system are $\omega_{nx}=922.0$ Hz, $\zeta_x=0.011$, $k_x=5.0\times10^6$ N/m; $\omega_{ny}=922.0$ Hz, $\zeta_y=0.011$, $k_y=5.0\times10^6$ N/m. The tangential and radial specific cutting forces are $K_t=7.96\times10^8$ N/m^2 and $K_r=1.68\times10^8$ N/m^2. All the programs in this paper are executed under the platform of MATLAB 7.6.0 on a laptop (Intel ® Core (TM) 2 Duo CPU,

Table 1 Comparison of SLDs using C-SDM, C-CDSEM, and RKCDM

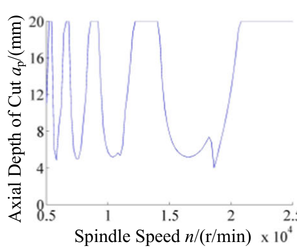
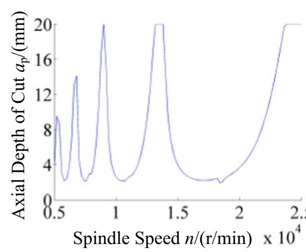
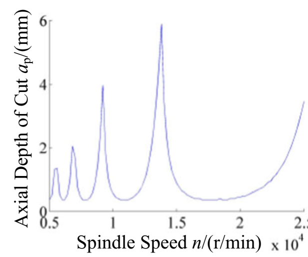
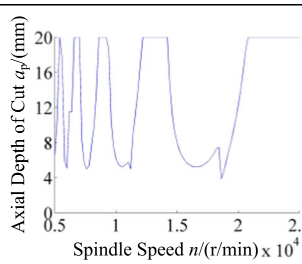
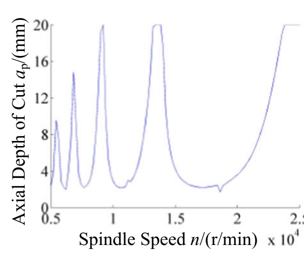
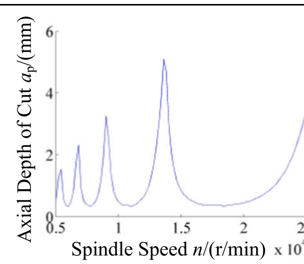
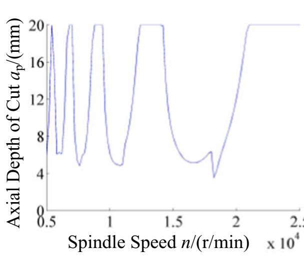
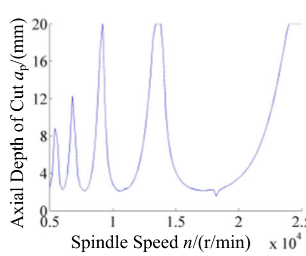
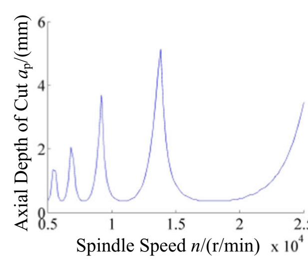
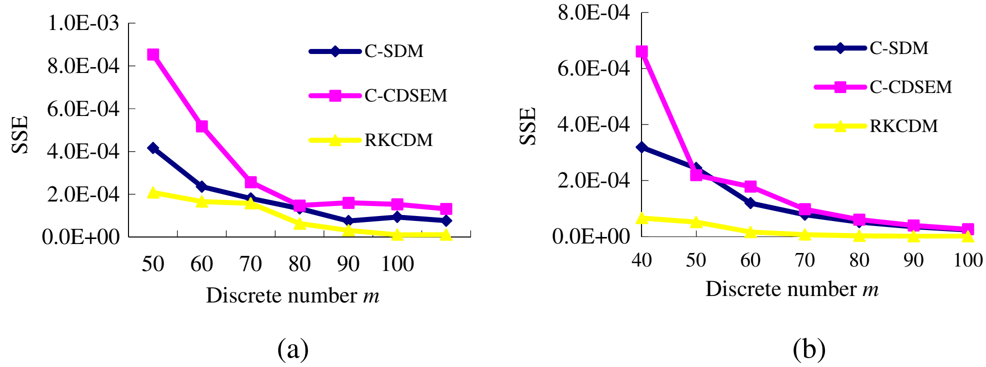
$m=50$	$a_e/D=0.05$	$a_e/D=0.10$	$a_e/D=0.50$
C-SDM			
C-CDSEM			
RKCDM			

Fig. 4 Comparisons of SSE obtained using C-SDM, C-CDSEM, and RKCDM **a** $a_e/D = 0.05$ **b** $a_e/D = 0.10$



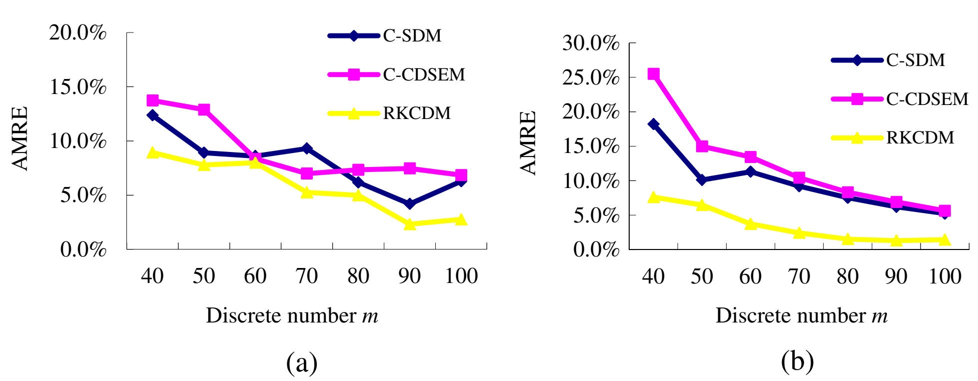
2.4 GHz, 4G). In order to analyze the convergence rate of RKCDM, the radial depth of cut a_e is set as diameter of the cutting tool D to avoid intermittent milling process ($a_e = D$). The spindle speed is selected at $\Omega = 5000$ r/min, and the axial depths of cut a_p are set as 1.0, 0.5, and 0.1 mm, respectively, to reflect the condition of unstable and stable milling process. The simulation results of the classical SDM [12] and the classical CDSEM [17] with the same parameters were also considered for further comparison and analysis, and they were called C-SDM and C-CDSEM, respectively, for the sake of simplicity.

Suppose the maximal module of the eigenvalues of the transition matrix Φ is set as $|u|$, the effect of discrete number m on the obtained $|u|$ using RKCDM, C-SDM, and C-CDSEM are shown in Fig. 2. The figure shows that with the increase m, $|u|$ shows a tendency of monotonically increasing and approaches to the theoretical value infinitely. As $|u|$ is used to determine the stability under certain cutting condition, and thus, the predicted critical axial depth of cut is always larger than the theoretical one, the predicted stable region obtained using these numerical methods is always larger than the theoretical one. It means that the stable region near the border dividing the stable and the unstable region is unreliable.

In order to assess the convergence rate of different numerical methods such as RKCDM, C-SDM, and C-CDSEM, the $|u|$ obtained at the discrete number of 500 ($m = 500$) is set as $|u_0|$ which is regarded as the exact value. The comparisons of convergence rate among RKCDM, C-SDM, and C-CDSEM are shown in Fig. 3. The figure shows that no matter what radial immersion ratio is, the convergence rate of RKCDM is higher than that of C-SDM and C-CDSEM, and the discretization error of the latter two is $O(h^2)$ [19]. It means that the discretization error of the proposed RKCDM is corresponding with the theoretical transaction error of the classical fourth-order Runge–Kutta method ($O(h^5)$). Ding et al. [14] stated that the discretization error of CRKM (classical fourth-order Runge–Kutta method) used for semi-analytical prediction of milling stability was no higher than $O(h^3)$ and attributed it to the fact that two-point barycentric Lagrange interpolation formula was used to approximate the middle points, and the remainder of the two-point interpolation is $O(h^2)$. As a result, the convergence rate and approximation accuracy of CRKM were limited. In the proposed RKCDM, the middle points were calculated directly by discretization method rather than by interpolation method. Therefore, the convergence rate and approximation accuracy of RKCDM is only determined by that of the classical fourth-order Runge–Kutta method. The figure also shows that when m is larger than 65, RKCDM has the minimum approximation error; and when m is less than 65, C-SDM has the minimum approximation error.

The SLDs under different radial immersion ratios $a_e/D = 0.05, 0.1, 0.5$ are predicted and shown in Table 1, in which discrete number $m = 50$ and spindle speed step $\Delta n = 100$ r/min. The simulation results, obtained using C-SDM, C-CDSEM, and RKCDM, demonstrate that the simulation

Fig. 5 Comparisons of AMRE obtained using C-SDM, C-CDSEM, and RKCDM **a** $a_e/D = 0.05$ **b** $a_e/D = 0.10$



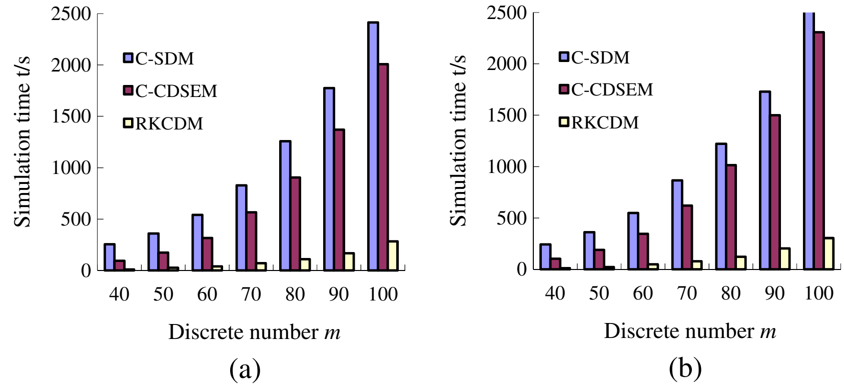


Fig. 6 Comparisons of simulation time using C-SDM, C-CDSEM and RKCDM **a** $a_e/D = 0.05$ **b** $a_e/D = 0.10$

results obtained using RKCDM is almost identical to that obtained using C-SDM and C-CDSEM under different radial immersion ratios, except some no distinct visual difference.

In order to further analyze the prediction accuracy of the proposed RKCDM with that of C-SDM and C-CDSEM, as well as the effect of discrete number m on the prediction accuracy, two technical criterias were used. The first is the sum of squared error (SSE), and the other is the arithmetic mean of relative error (AMRE), which can be expressed as

$$\begin{aligned} SSE &= \sum_{i=1}^{r} \left(a_{\mathrm{p}i} - a_{\mathrm{p}i0}\right)^2 \\ AMRE &= \frac{1}{r}\sum_{i=1}^{r} \frac{\left|a_{\mathrm{p}i} - a_{\mathrm{p}i0}\right|}{a_{\mathrm{p}i0}} \end{aligned} \tag{17}$$

where $a_{\mathrm{p}i}$, $a_{\mathrm{p}i0}$ is the predicted and expected critical axial depth of cut at the ith discrete spindle speed, respectively, and r is the total discrete points within the concerned spindle speed range.

When C-SDM is used as a benchmark method to evaluate other numerical methods to predict milling stability in time domain, under the condition of discrete number per tool passing period $m = 200$, the AMRE between RKCDM and C-SDM is 2.1474 %, and the AMRE between C-CDM and C-SDM is 2.6321 %. Therefore, the critical axial depths of cut under each discrete spindle speed using C-SDM at $m = 200$ can be used as the exact value to evaluate the effect of discrete number m on the prediction accuracy using different numerical methods. The comparisons of SSE among C-SDM, C-CDSEM, and RKCDM under different a_e/D were shown in Fig. 4, and the comparisons of AMRE among C-SDM, C-CDSEM, and RKCDM under different a_e/D were shown in Fig. 5. Figure 4 shows that under the same discrete number per tool passing period m, the SSE obtained using KCDM is less than that obtained using C-SDM and C-CDSEM, and with the increase of the discrete number per tool passing period m, the SSE using RKCDM, C-SDM and C-CDSEM all decreased. Figure 5 shows that the effect of the discrete number per tool passing period m on AMRE is the same as that on SSE. Regardless of the radial immersion ratio a_e/D, when the discrete number per tool passing period m is over than 40, the AMRE obtained using RKCDM is less than 10 %. In other words, the prediction accuracy of SLDs obtained using RKCDM is acceptable as m is over 40. If the AMRE obtained using RKCDM is required less than 5 %, the discrete number per tool passing period should be selected as $m > 60$.

The comparisons of simulation time t between the proposed RKCDM, C-SDM, and C-CDSEM under different discrete number m and radial immersion ratio a_e/D were shown in Fig. 6. The figure shows that with the increase of m, the simulation time t of RKCDM, C-SDM, and C-CDSEM all show an exponential growth tendency, which means that it is not economical to improve the prediction accuracy by only increasing the discrete number m limitlessly. Under the same discrete number m and the same cutting conditions, the

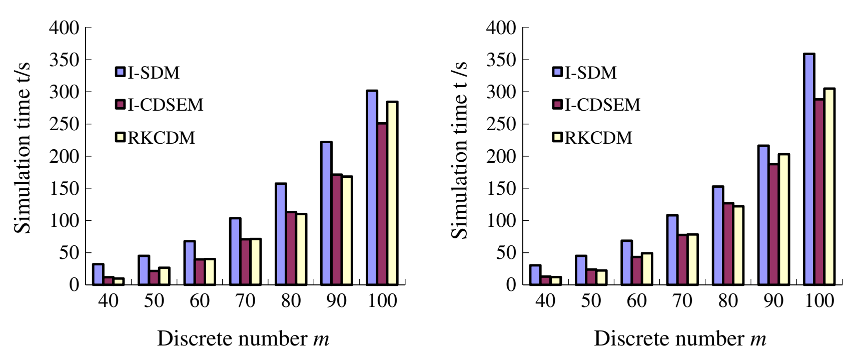


Fig. 7 Comparisons of simulation time using I-SDM, I-CDSEM, and RKCDM

simulation time t using RKCDM is the shortest, while that using C-SDM is the longest, and there is an order of magnitude of difference in simulation speed between them. Most of this difference is due to the fact that the dichotomy search was used in RKCDM while the sequential search is used both in C-SDM and C-CDSEM, and the remaining is due to the different nature of these numerical methods. In order to further reveal the effect of these methods on the simulation time t, both the classical SDM and the classical CDSEM were improved by applying dichotomy search other than sequential search in constructing SLDs and which were called I-SDM and I-FDM respectively. The comparisons of simulation time t between RKCDM, I-SDM and I-CDSEM under a different discrete number m and radial immersion ratio a_e/D were shown in Fig. 7. The figure shows that the simulation time t using I-SDM is the longest and that using the remaining two methods have no significant difference. In short, even compared with I-SDM and I-CDSEM, the proposed RKCDM not only has the highest simulation precision but also has good simulation speed. Moreover, with the application the of the dichotomy search, the prediction of SLDs for milling process becomes practical in engineering applications.

5 Conclusions

This paper focuses on the exact prediction of milling stability using a Runge-Kutta-based complete discretizaton method, and the following conclusions can be drawn from this research.

(1) The dynamic milling system is expressed as coefficient-varying periodic differential equations with a single time delay, all parts of which are discretized using the classical fourth-order Runge–Kutta method, the Floquet theory is adopted to determine the stability of certain milling condition, and when the above process is repeated for all discrete spindle speeds and for all discrete axial depth of cuts, the stability lobe diagrams can be obtained.

(2) Comparisons have been conducted in the aspect of convergence rate, prediction accuracy, and computation efficiency among the proposed method and other numerical stability prediction methods. The simulation results show that the proposed method not only has the highest convergency rate and thus the prediction accuracy but also has the fastest computation speed when compared with the classical semi-discretization method (C-SDM) and the complete discretization scheme with the Euler method (C-CDSEM).

(3) Simulation results indicate that as dichotomy search rather than sequential search is used in constructing the stability lobe diagrams, under the same prediction accuracy, the simulation time of the proposed method can be greatly reduced when compared with other numerical stability prediction methods for milling process. As a result, it can be used in practical engineering applications.

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