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ORIGINAL ARTICLE

The reconstruction of a semi-discretization method for milling stability prediction based on Shannon standard orthogonal basis

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Abstract In order to increase the calculation speed of the semi-discretization method (SDM) without accuracy loss, this paper reconstructs the SDM for predicting the stability lobes of the dynamic milling process, mainly considering the regenerative effect. The model of the dynamic milling process is expressed as the linear delay-differential equations (DDE). The fast calculation method is established by reconstructing the SDM based on the Shannon standard orthogonal basis (SSOB). First, the delay term of DDE is constructed without information loss based on Shannon interpolation functions, and SSOB is derived. Secondly, the closed form expression for the transition matrix of the system is constructed based on the SSOB, and the stability limit is predicted based on the Floquet theory. The transition matrix-based SDM and SSOB are theoretically compared, and it shows that the SDM is a special case of the method based on SSOB when the SSOB is regarded as the average in the sampling interval. The fast calculation method is established by using the variable sampling numbers during the period of the delay time in which the variable sampling numbers are determined by the condition which is used to construct the SSOB. Finally, this proposed fast method is used to the one and two degrees of freedom milling model, and the results show that the calculation accuracy is not reduced, and the calculation speed based on the proposed method can be improved nearly five times on the one degree of freedom model and 2.6 times on the two degrees of freedom model, compared to the semi-discretization method.

Keywords Milling stability · Shannon standard orthogonal basis · Sampling interval · Time delay

1 Introduction

Chatter in the cutting process is an undesirable phenomenon which leads to many negative effects such as the reduction of surface finish quality and life of cutters, poor dimensional inaccuracy, the limitation to machine productivity, etc. [1–6]. Ever since the 1950s, the different types of chatter were found, such as regenerative effects, mode coupling effects, frictional effects, thermo-mechanical effects, etc. [7]. Regenerative effects are considered as the most common cause of chatter in milling operation. For this reason, many researchers studied the milling stability based on the regenerative effects. In general, these methods of the milling stability analysis are divided into five types:

(1) Time response simulation (TRS) method. Its essence is to obtain the dynamic time response of delay-differential equations (DDEs) and to determine the milling stability according to the convergence of vibration amplitude, such as the studies of Tlusty [8, 9], Smith and Tlusty [10], Campomanes and Altintas [11], and Davies [12, 13]. The TRS method can consider the nonlinear influence of the machining process, such as out of the cutting for the cutter, etc.
(2) Frequency domain analysis (FDA) method. Its essence is to transform the delay-differential equations to the

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Laplace domain and let the real part of the root of the characteristic equation be zero to obtained the stability limit based on cutting parameters, such as the studies of Altintas and Budak [14] (zero-order solution method) and Merdol and Altintas [15, 16] (multi-frequency solution method). In the FDA method, the zero-order solution has a high calculation speed; the multi-frequency solution has a high calculation precision.

(3) Semi-discretization method (SDM). Its essence is to discrete the delay term of delay-differential equations and use the Floquet theory to determine the stability under different cutting parameters, such as the studies of Insperger and Stepan [17, 18], Dombovari and Altintas [19], K. Ahmadi, and F. Ismail [20]. The semi-discretization method has a high calculation precision which is the same as the multi-frequency solution method.

(4) Full-discretization method (FDM). Its essence is that the response of the system is calculated via the direct integration scheme with the help of discretizing the time period [15]. The method has a high calculation speed, such the studies of Ye Ding [21, 22], T. Insperger [23], and Yilong Liu [24].

(5) Temporal finite element analysis (TFEA). Its essence is that the discrete-time equations are obtained by the use of temporal finite elements while the tool is in the cut. The approximate solution during the cut is required to match the exact solution for free vibration of the tool at the beginning and end of each cut, such as the studies of P.V. Bayly [25] and B.P. Mann [26]. The TFEA method has a high calculation speed and precision for a small radial depth of cut.

The above methods play a very vital role in prompting the development of milling stability analysis. In this paper, a fast calculation for predicting milling stability is proposed by reconstructing the semi-discretization method based on the Shannon standard orthogonal basis, and the benchmark examples for the one and two degrees of freedom milling model are used to verify the validity of the proposed method. The proposed method is an impetus for the further development of the semi-discretization method.

2 Theoretical basic and reconstruction of the semi-discretization method

2.1 Shannon sampling theorem

The sampling theorem was first proposed in 1928 by Harry Nyquist, and Claude Elwood Shannon, the founder of the information theory, described in detail the theorem in 1948. The derivation process of the theorem is not described here; the main contents are as follows:

Assuming continuous function $f(t)\in L^2(R)$ ((where $L^2(R)$ is the square integrable functions space in which functions satisfy the requirement: $\int_{-\infty}^{+\infty}|f(t)|^2 dx<+\infty$), when the Fourier transform of continuous function $f(t)$ has the following property: $F(\omega)=0$, $|\omega|>L$ (where L is a real that L is called cut-off frequency) and the sampling interval satisfies $\Delta\le\pi/L$, $f(t)$ can be reconstructed without any loss of information through the following reconstruction formula:

$$f(t)=\sum_{n\in Z} f(n\Delta)\frac{\sin\Delta^{-1}\pi(t-n\Delta)}{\Delta^{-1}\pi(t-n\Delta)} \tag{1}$$

where $\frac{\sin\Delta^{-1}\pi(t-n\Delta)}{\Delta^{-1}\pi(t-n\Delta)}$ is called the Shannon interpolation function; the reconstruction process of $f(t)$ is shown in Fig. 1.

2.2 Existing problem of the semi-discretization method

The delayed term of the delay-differential equations in the SDM is approximated by the average value of the adjacent sampling values as shown in Fig. 2. The approximation is as follows:

$$x(t-T)\approx\frac{x(t_i-T+\Delta)+y(t_i-T)}{2}=\frac{(x_{i-m+1}+x_{i-m})}{2}\,(t\in[t_i,t_{i+1}])$$

As can be seen from Fig. 2, when $t=t_{i-m+\varsigma}$ ($\varsigma\in[0,1]$), there exists a large error E_S compared to the actual value $x_{i-m+\varsigma}$ of the delayed term $x(v)$.

In general, the semi-discretization method has the following problems:

- When the sampling interval $[t_i,t_{i+1}]$ is large, the approximation of the delayed term exist large information loss.
- The numbers of sampling in a delay time are arbitrarily defined; there are no rules to determine the sampling numbers.
- When the sampling numbers increase, it will produce larger information redundancy, resulting in the increase of the computing time.

Based on the above reasons, this paper reconstructs the SDM without loss of information based on the Shannon standard orthogonal basis. The reconstruction process determines the minimum sampling intervals which do not exist information loss and redundancy of the delayed term from the view of the theoretical analysis, and the sampling interval can change with the change of the delay term. This leads to a big improvement of the computing efficiency. In the following section, the reconstruction process is introduced in detail.

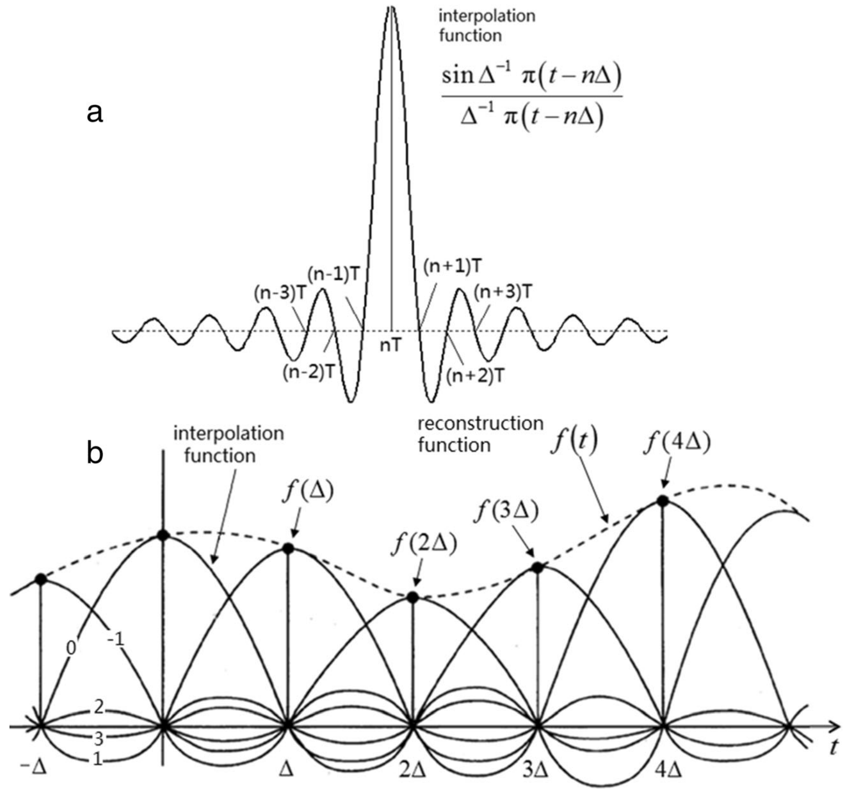


Fig. 1 Function reconstruction process based on the Shannon interpolation function: **a** interpolation function, **b** the reconstruction of function $f(t)$; the values −1, 0, 1, 2, 3 respectively represent the waveform in interval $[-\Delta, 0]$ of the interpolation function respectively in the sampling point $-\Delta$, 0, Δ, 2Δ, 3Δ.

2.3 The reconstruction of the semi-discretization method based on the Shannon orthogonal basis

The dynamic model of the milling process can be described as delay-differential equations; the state-space form is as follows:

$$dx(t)/dt = A(t)x(t) + B(t)x(t-T) \tag{2}$$

where $t \in R$ (R denotes all real), $A(t)$ and $B(t)$ are periodic functions, i.e., $A(t)=A(t+T)$, $B(t)=B(t+T)$, and T is the delay time.

2.3.1 Approximation of the delay term based on the Shannon interpolation function

The delay time T is divided into m equal parts, and the length of the time interval $[t_i, t_{i+1}]$ is Δ; the delay term $x(t-T)$ is reconstructed in the time interval $[t_i, t_{i+1}]$ based on Eq. (1), as follows:

$$\begin{aligned} x(t-T) &= \sum_{i-m+1}^{i-m} x(t_i) \times \frac{\sin\Delta^{-1}\pi(t-i\times\Delta)}{\Delta^{-1}\pi(t-i\times\Delta)} \\ &= x_{i-m+1} \times \frac{\sin\Delta^{-1}\pi(t-(i-m+1)\times\Delta)}{\Delta^{-1}\pi(t-(i-m+1)\times\Delta)} + x_{i-m} \times \frac{\sin\Delta^{-1}\pi(t-(i-m)\times\Delta)}{\Delta^{-1}\pi(t-(i-m)\times\Delta)}, \end{aligned} \tag{3}$$

$t \in [t_i, t_{i+1}],\ i \in R$

Note: Equation (3) only considers the waveforms of sampling point t_i and t_{i+1}, without taking into consideration the response of interpolation functions at other sampling points. The influence at other sampling points on the interval $[t_i, t_{i+1}]$ will be explained in Section 4.

2.3.2 The solution of delay-differential equations

In the time interval $[t_i, t_{i+1}]$, the solutions of Eq. (2) consist of homogenous x_{ih} and particular x_{ip} solutions.

The derivation of homogenous solution x_{ih}:

$$\frac{dx}{x} = A \times dt \tag{4}$$

$$\int_{t_i}^{t} \frac{dx}{x} = \int_{t_i}^{t} A \times dt \tag{5}$$

$$In|x|\big|_{t_i}^{t} = A \times (t-t_i) \tag{6}$$

$$In|x(t)| - In|x(t_i)| = A \times (t-t_i) \tag{7}$$

$$In|x(t)| = A \times (t-t_i) + In|x(t_i)| \tag{8}$$

$$|x(t)| = e^{A\times(t-t_i)+In|x(t_i)|} \tag{9}$$

$$x(t) = \pm\left(e^{A\times(t-t_i)} \times e^{\pm Inx(t_i)}\right) \tag{10}$$

$$x(t) = x(t_i) \times e^{A\times(t-t_i)}, \text{ i.e., } x_{ih} = x(t_i) \times e^{A\times(t-t_i)} \tag{11}$$

The derivation of particular solution x_{ip}:

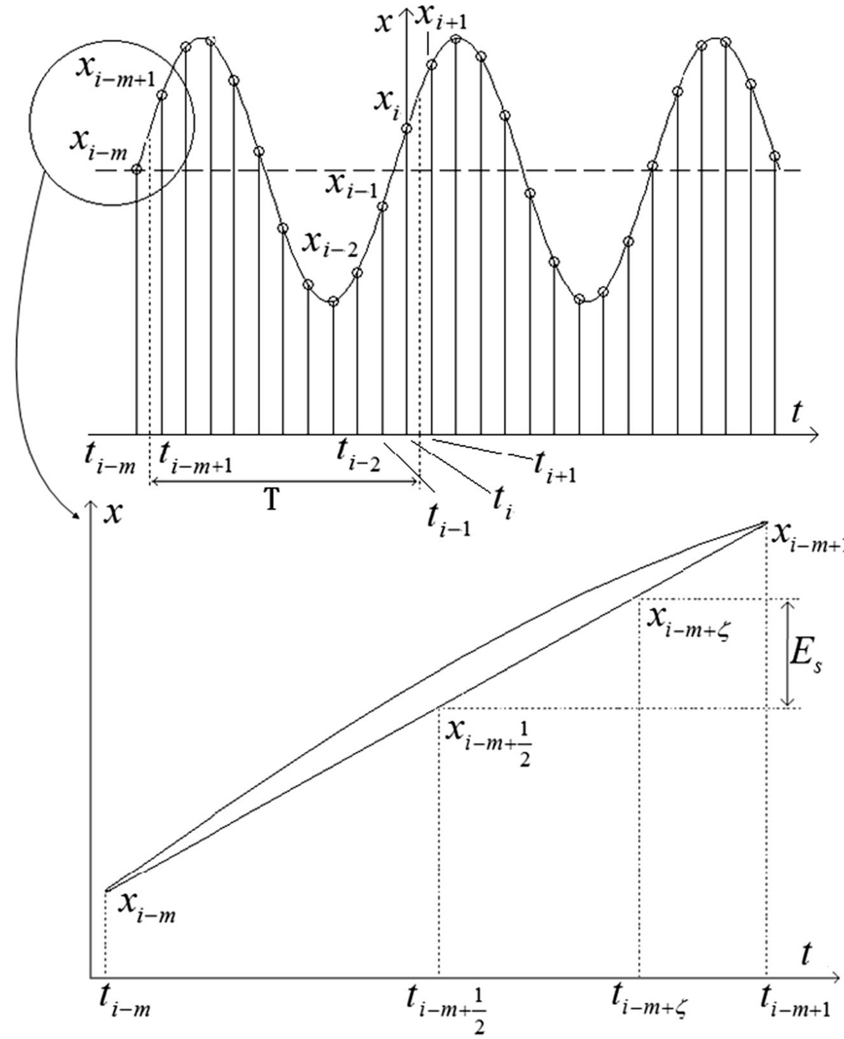


Fig. 2 The approximation of the delayed term $x(t-T)$ ($t\in[t_i,t_{i+1}]$)

Assuming that $e^{A\times}(t-t_i)\times u_1(t)$ is the solution of Eq. (2), its derivation can be obtained, as follows:

$$A\times e^{A\times(t-t_i)}\times u_1(t)+e^{A\times(t-t_i)}\times\frac{du_1(t)}{dt},t\in[t_i,t_{i+1}],i\in R \tag{12}$$

By substituting x_{ip} into Eq. (2), it yields:

$$\begin{aligned}&e^{A\times(t-t_i)}\times\frac{du_1(t)}{dt}+A\times e^{A\times(t-t_i)}\times u_1(t)\\&=A\times e^{A\times(t-t_i)}\times u_1(t)+B\times x(t-T),t\in[t_i,t_{i+1}],i\in R\end{aligned} \tag{13}$$

$$e^{A\times(t-t_i)}\times\frac{du_1(t)}{dt}=B\times x(t-T),t\in[t_i,t_{i+1}],i\in R \tag{14}$$

$$\frac{du_1(t)}{dt}=e^{-(A\times(t-t_i))}\times B\times x(t-T),t\in[t_i,t_{i+1}],i\in R \tag{15}$$

Substituting Eq. (3) into Eq. (15) yields:

$$\begin{aligned}\frac{du_1(t)}{dt}=e^{-(A\times(t-t_i))}\times B\times\Big(&x_{i-m+1}\times\frac{\sin\Delta^{-1}\pi\left(\left(t-\left(i-m+1\right)\times\Delta\right)\right.}{\Delta^{-1}\pi(t-(i-m+1)\times\Delta)}\\&+x_{i-m}\times\frac{\sin\Delta^{-1}\pi(t-(i-m)\times\Delta)}{\Delta^{-1}\pi(t-(i-m)\times\Delta)}\Big)\end{aligned} \tag{16}$$

$$\int_{t_i}^{t}\frac{du_1(t)}{dt}dt=\int_{t_i}^{t}\left[\begin{aligned}&e^{-(A\times(t-t_i))}\times B\times\Big(x_{i-m+1}\times\frac{\sin\Delta^{-1}\pi(t-(i-m+1)\times\Delta)}{\Delta^{-1}\pi(t-(i-m+1)\times\Delta)}\\&+x_{i-m}\times\frac{\sin\Delta^{-1}\pi(t-(i-m)\times\Delta)}{\Delta^{-1}\pi(t-(i-m)\times\Delta)}\Big)\end{aligned}\right]dt \tag{17}$$

$$u_1(t)-u_1(t_i)=\int_{t_i}^{t}\left[\begin{aligned}&e^{-(A\times(t-t_i))}\times B\times\Big(x_{i-m+1}\times\frac{\sin\Delta^{-1}\pi(t-(i-m+1)\times\Delta)}{\Delta^{-1}\pi(t-(i-m+1)\times\Delta)}\\&+x_{i-m}\times\frac{\sin\Delta^{-1}\pi(t-(i-m+1)\times\Delta)}{\Delta^{-1}\pi(t-(i-m+1)\times\Delta)}\Big)\end{aligned}\right]dt \tag{18}$$

$$\int_{t_i}^{t}\left[\begin{aligned}&e^{-(A\times(\tau-t_i))}\times B\times\Big(x_{i-m+1}\times\frac{\sin\Delta^{-1}\pi(\tau-(i-m+1)\times\Delta)}{\Delta^{-1}\pi(\tau-(i-m+1)\times\Delta)}\\&+\quad x_{i-m}\times\quad\frac{\sin\Delta^{-1}\pi(\tau-(i-m)\times\Delta)}{\Delta^{-1}\pi(\tau-(i-m)\times\Delta)}\Big)\end{aligned}\right]d\tau$$

$$\tau\in[t_i,t] \tag{19}$$

Denoting $\frac{\sin\Delta^{-1}\pi(\tau-(i-m+1)\times\Delta)}{\Delta^{-1}\pi(\tau-(i-m+1)\times\Delta)}$ as $\phi(\tau-(i-m+1))$, $\frac{\sin\Delta^{-1}\pi(\tau-(i-m)\times\Delta)}{\Delta^{-1}\pi(\tau-(i-m)\times\Delta)}$ as $\phi(\tau-(i-m))$

The equivalent transformation of Eq. (19) is as follows:

$$\int_{t_i}^{t}\left[\begin{array}{l} e^{-(A\times(\tau-t_i))}\times B\times\Big(x_{i-m+1}\times\phi\Big(\tau(i-m+1)\Big)\\ +x_{i-m}\times\phi(\tau-(i-m))\Big)\end{array}\right]d\tau$$
$$=x_{i-m+1}\times\int_{t_i}^{t}e^{-(A\times(\tau-t_i))}\times B\times\phi(\tau-(i-m+1))d\tau$$
$$+x_{i-m}\times\int_{t_i}^{t}e^{-(A\times(\tau-t_i))}\times B\times\phi(\tau-(i-m))d\tau \tag{20}$$

The equivalent transformation of Eq. (18) is as follows:

$$u_1(t)=x_{i-m+1}\times\int_{t_i}^{t}e^{-(A\times(\tau-t_i))}\times B\times\phi(\tau-(i-m+1))d\tau$$
$$+x_{i-m}\times\int_{t_i}^{t}e^{-(A\times(\tau-t_i))}\times B\times\phi(\tau-(i-m))d\tau+C \tag{21}$$

where $C=u_1(t_i)$.

Depending on Eq. (21), x_{ip} is expressed as follows:

$$x_{ip}=e^{A\times(t-t_i)}\times\Big[x_{i-m+1}\times\int_{t_i}^{t}e^{-(A\times(\tau-t_i))}\times B\times\phi(\tau-(i-m+1))d\tau$$
$$+x_{i-m}\times\int_{t_i}^{t}e^{-(A\times(\tau-t_i))}\times B\times\phi(\tau-(i-m))d\tau\Big] \tag{22}$$

Finally, the solution of Eq. (2) in the time interval $[t_i,t_{i+1}]$ can be expressed as

$$x(t)=x_{ih}+x_{ip}$$
$$=x(t_i)\times e^{A\times(t-t_i)}+e^{A\times(t-t_i)}\times\left[x_{i-m+1}\times\int_{t_i}^{t}\left[\begin{array}{l}e^{-(A\times(\tau-t_i))}\times B\times\\ \phi(\tau-(i-m+1))d\tau\end{array}\right]\right.$$
$$\left.+x_{i-m}\times\int_{t_i}^{t}e^{-(A\times(\tau-t_i))}\times B\times\phi(\tau-(i-m))d\tau\right],t\in[t_i,t_{i+1}],i\in R \tag{23}$$

When $t=t_{i+1}$, Eq. (23) yields

$$x(t_{i+1})=x_{i+1}=x(t_i)\times e^{A\times(\Delta)}+e^{A\times(\Delta)}\times$$
$$\Big[x_{i-m+1}\times\int_{t_i}^{t_{i+1}}e^{-(A\times(\tau-t_i))}\times B\times\phi(\tau-(i-m+1))d\tau+x_{i-m} \tag{24}$$
$$\times\int_{t_i}^{t_{i+1}}e^{-(A\times(\tau-t_i))}\times B\times\phi(\tau-(i-m))d\tau\Big]$$

where

$$\phi(\tau-(i-m+1))=\frac{\sin\frac{1}{\Delta}\pi(\tau-(i-m+1)\times\Delta)}{\frac{1}{\Delta}\pi(\tau-(i-m+1)\times\Delta)}\quad,\tau\in[t_i,t_{i+1}].$$

Denoting $\tau=\mu+t_i$,,Eq. (24) can be rearranged as follows:

$$x(t_{i+1})=x_{i+1}=x_i\times e^{A\times(\Delta)}+e^{A\times(\Delta)}$$
$$\times\Big[x_{i-m+1}\times\int_0^{\Delta}e^{-(A\times\mu)}\times B\times\phi(\mu-(i-m+1))d\mu,0\leq\mu\leq\Delta$$
$$+x_{i-m}\times\int_0^{\Delta}e^{-(A\times\mu)}\times B\times\phi(\mu-(i-m))d\mu\Big] \tag{25}$$

where $\int_0^{\Delta}e^{-(A\times\mu)}\times B\times\phi(\mu-(i-m+1))d\mu$ is equal to

$$\int_0^{\Delta}\Big(e^{-(A\times\mu)}\times B\times\phi(\mu)\Big)d\mu,$$

$\int_0^{\Delta}e^{-(A\times\mu)}\times B\times\phi((t_i+\mu)-(i-m))d\mu$ is equal to

$\int_0^{\Delta}\Big(e^{-(A\times\mu)}\times B\times\phi(\mu)\Big)d\mu.$

The equivalent transformation of Eq. (18) is as follows:

$$x(t_{i+1})=x_{i+1}=x_i\times e^{A\times(\Delta)}+e^{A\times(\Delta)}$$
$$\times\left[x_{i-m+1}\times\int_0^{\Delta}e^{-(A\times\mu)}\times B\times\phi(\mu)d\mu+x_{i-m}\times\int_0^{\Delta}e^{-(A\times\mu)}\times B\times\phi(\mu)d\mu\right] \tag{26}$$

2.3.3 State expression-based orthogonal basis function

From Section 2.1, we can know that when the sampling interval satisfies $\Delta\leq\pi/L$, the functions $f(t)$ can be reconstructed without any loss of information through Eq. (1). When $L=\pi$, that is, the sampling frequency is equal to two times the highest frequency of the signal,, the sampling interval Δ is equal to one, and Eq (1) can be transformed as follows:

$$f(t)=\sum_{n\in Z}f(n)\frac{\sin\pi(t-n)}{\pi(t-n)},n\in Z \tag{27}$$

Denoting $\phi_1(\mu) = \frac{\sin\pi\mu}{\pi\mu}$, by using the Parseval theorem forfunction $\phi_1(\mu)$, yields:

$$< \phi_1(\mu-n), \phi_1(\mu-m) >= \int_{-\infty}^{+\infty} \phi_1(\mu-n)\overline{\phi}_1(\mu-m)du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\Phi 1(\omega)|^2 \mathrm{e}^{-\mathrm{i}\omega(n-l)}\, dx = \delta(n-l) \tag{28}$$

where $\delta(n)$ are unit impulse sequences,

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{29}$$

According to Eqs. (27) and (28), we can know that$\{\phi_1(\mu-n);\ n\in Z\}$ are the standard orthogonal basis in the functions space $\{ f(x)\ ;\ F(\omega)=0, |\omega| > \pi\}$.

Equation (26) can be transformed into

$$x(t_{i+1}) = x_{i+1} = x_i \times e^{A\times(1)} + e^{A\times(1)} \times \left[x_{i-m+1} \times \int_0^1 e^{-(A\times\mu)} \times B \times \phi_1(\mu)d\mu + x_{i-m} \times \int_0^1 e^{-(A\times\mu)} \times B \times \phi_1(\mu)d\mu\right] \tag{30}$$

The sampling interval $\Delta=1$ means that the delay time T is sampled by $1/(2f_0)$ time interval, and the sampling numbers are $T\times(2f_0)$ during the period of the delay time (where f_0 is the chatter frequency, T is inversely proportional to the spindle speed,,i.e., $T=60/\Omega/N$, Ω is the spindle speed, N is the number of teeth). "$\Delta=1$" is the normalized sampling interval, the actual sampling interval can be expressed as $\Delta_1=1/(2f_0)$.

In the actual sampling process, Eq. (30) is expressed as follows:

$$x(t_{i+1}) = x_{i+1} = x_i \times e^{A\times(\Delta_1)} + e^{A\times(\Delta_1)} \times \left[x_{i-m+1} \times \int_0^{\Delta_1} e^{-(A\times\mu)} \times B \times \phi_1(\mu)d\mu + x_{i-m} \times \int_0^{\Delta_1} e^{-(A\times\mu)} \times B \times \phi_1(\mu)d\mu\right] \tag{31}$$

3 The calculation of milling stability lobes

Let $e^{A\times(\Delta_1)} \times \int_0^{\Delta_1} e^{-(A\times\mu)} \times B \times \phi_1(\mu)d\mu = N$; Eq. (31) can be transformed as follows:

$$x(t_{i+1}) = x_{i+1} = x_i \times e^{A\times(\Delta)} + x_{i-m+1}N + x_{i-m} \times N \tag{32}$$

The discrete-time values of the states can be expressed in matrix form as

$$\begin{bmatrix} x_{i+1} \\ x_{i+2} \\ x_{i+3} \\ \vdots \\ x_{i-m+2} \\ x_{i-m+1} \end{bmatrix} = \begin{bmatrix} e^{A\times(\Delta)} & 0 & 0 & \cdots & 0 & e^{A\times(\Delta)}\times N & e^{A\times(\Delta)}\times N \\ I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I & 0 \end{bmatrix} \times \begin{bmatrix} x_i \\ x_{i+1} \\ x_{i+2} \\ \vdots \\ x_{i-m+1} \\ x_{i-m} \end{bmatrix}$$
$$X_{i+1} = M_i \times X_i \tag{33}$$

To determine the transition matrix P over the delay time T,, yield,

$$X_m = [M_{m-1} \times \cdots \times M_2 \times M_1 \times M_0] \times X_0 = P \times X_0 \tag{34}$$

The stability lobes can be determined by scanning the spindle speed at the different axial depth, depending on Eq. (35) (Floquet theory)

$$\max(|\lambda|) = \begin{cases} < 1, & \text{stable} \\ = 1, & \text{critical stable} \\ > 1, & \text{unstable} \end{cases} \tag{35}$$

$\max(|\lambda|)$ indicates the biggest modulus in all modulus of eigenvalues of the transition matrix P.

4 Theoretical comparison of the SDM and the proposed method

The discrete-time values of the states based on SDM can be expressed in matrix form as

$$\begin{bmatrix} x_{i+1} \\ x_{i+2} \\ x_{i+3} \\ \vdots \\ x_{i-m+2} \\ x_{i-m+1} \end{bmatrix} = \begin{bmatrix} e^{A\times(\Delta)} & 0 & 0 & \cdots & 0 & N_1 & N_1 \\ I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I & 0 \end{bmatrix} \times \begin{bmatrix} x_i \\ x_{i+1} \\ x_{i+2} \\ \vdots \\ x_{i-m+1} \\ x_{i-m} \end{bmatrix}$$
$$N_1 = \frac{1}{2}\left(e^{A(\Delta)} - I\right)A^{-1}B$$
$$N = e^{A\times(\Delta_1)} \times \int_0^{\Delta_1} e^{-(A\times\mu)} \times B \times \phi_1(\mu)d\mu, \quad \int_0^{\Delta_1} \phi_1(\mu)d\mu = 0.59, \tag{36}$$

When considering the influence at other sampling points on the interval $[t_i, t_{i+1}]$, the interpolation function can be expressed as

$$\int_0^{\Delta_1} [\ldots + \phi_1(\mu-\Delta_1) + \phi_1(\mu) + \phi_1(\mu+\Delta_1) + \ldots]d\mu = \int_0^{\Delta_1}\left[\tilde{\phi}(\mu)\right]d\mu = \int_0^{\infty} \phi_1(\mu)d\mu \approx 0.5 \tag{37}$$

Table 1 Comparison of different methods for the single DOF milling model (the *horizontal* and *vertical coordinates* for figures are separately “r/min” and “m”)

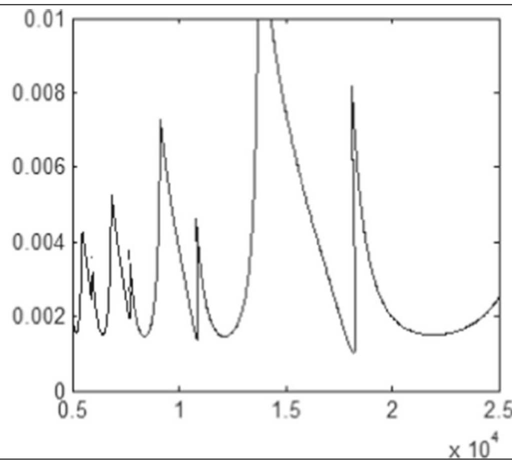
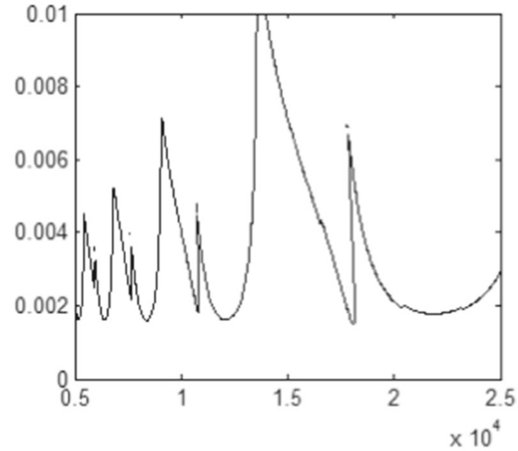
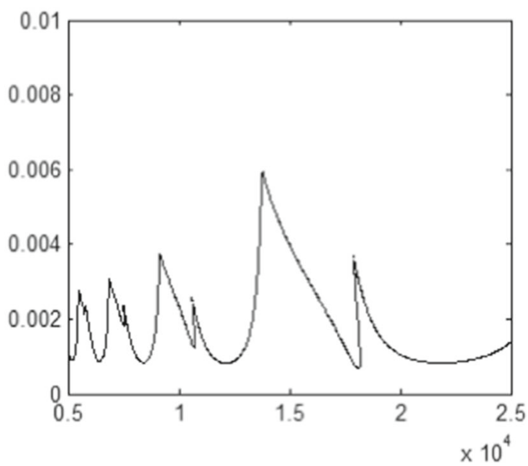
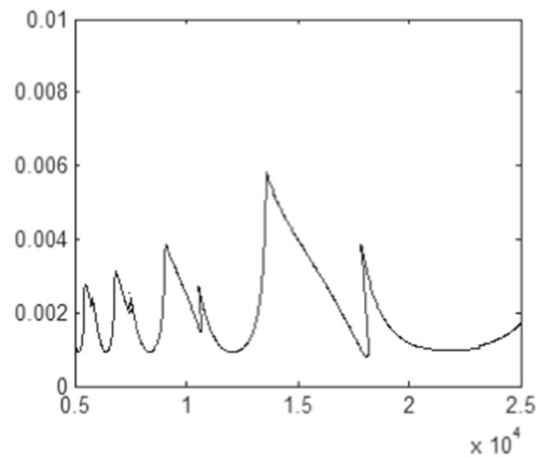
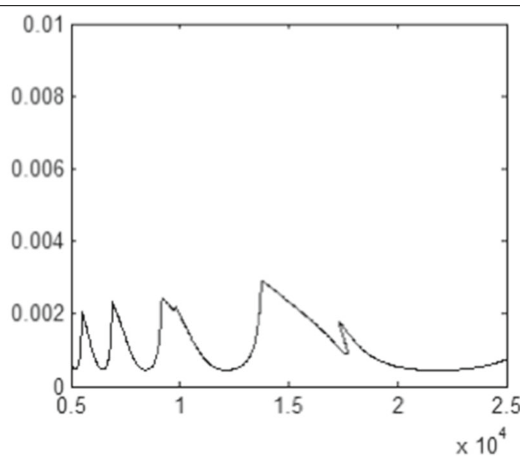
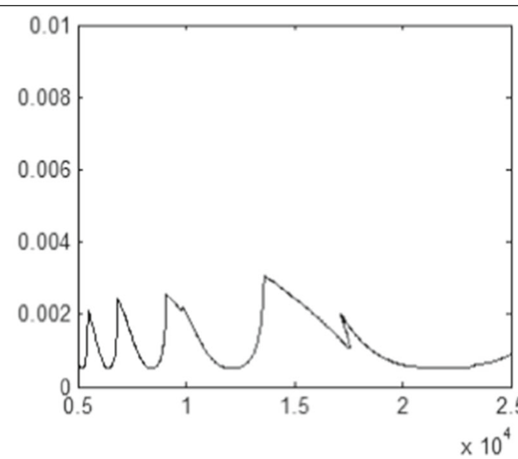
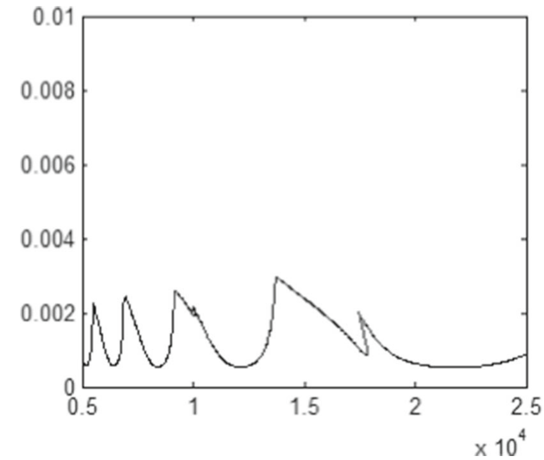
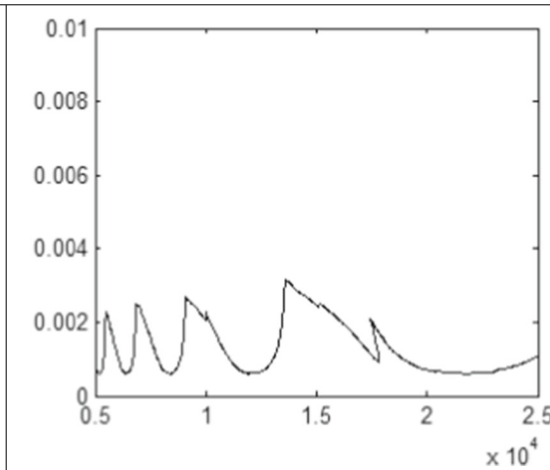
Radial depth of cut ratios	**Semi-discretization method**[12]	**The proposed method**
0.05		
Computational time(s)	547.046	107.39
0.1		
Computational time(s)	545.97	106.112
0.3		
Computational time(s)	553.255	108.389

Table 1 (continued)

0.5		
Computational time(s)	561.944	111.49

$$N = e^{A\times(\Delta_1)} \times \int_0^{\Delta_1} e^{-(A\times\mu)} \times B \times \tilde{\phi}(\mu)d\mu \tag{38}$$

Equation (37) is the approximation without the loss of information in the interval $[t_i, t_{i+1}]$. The general approximate can be obtain depending on Eq. (37)

$$\begin{aligned}
N &= e^{A\times(\Delta_1)} \times \int_0^{\Delta_1} e^{-(A\times\mu)} \times B \times \tilde{\phi}(\mu)d\mu \\
&\approx 0.5\ e^{A\times(\Delta)} \times \left(\int_0^{\Delta} e^{-(A\times\mu)}d\mu\right) \times B \\
&= 0.5\ e^{A\times(\Delta)} \times e^{-(A\times\mu)}\Big|_0^{\Delta}\left(-A^{-1}\times B\right) \\
&= 0.5e^{A\times(\Delta)} \times \left(e^{-(A\times\mu)}-I\right) \times \left(-A^{-1}\times B\right) \\
&= 0.5 \times \left(I-e^{-(A\times\mu)}\right) \times \left(-A^{-1}\times B\right) \\
&= \frac{1}{2} \times \left(e^{-(A\times\mu)}-I\right) \times \left(A^{-1}B\right)
\end{aligned} \tag{39}$$

Equation (38) is approximately equal to Eq. (36); this shows that the semi-discretization method is a special case of the reconstructed method proposed in this paper.

If the proposed method is approximated by using Eq. (39), the biggest advantage of the reconstruction process compared with SDM is that the sampling numbers $(60/\Omega/N)\times(2f_0)$ are changed with the change of the spindle speed. When the spindle speed increases, the delay time T decreases and the sampling number decreases, and vice versa.

5 Verification

In this section, two examples for one and two degrees of freedom milling models from reference [27] are utilized. Computer programs of the proposed approach are all written in Matlab 7.0 and implemented on the same computer.

Section 2.3 proposes the reconstruction process of SDM in which the essential is to determine the minimum sampling interval in the delay time, and the sampling numbers will change when the spindle speed changes, and determines the accurate approximation without information less of the delay term $x(t-T)$.

Note: In order to highlight the advantages of the variation of the sampling number during the delay time, the general approximation of the proposed method is used, i.e., Eq. (39), and is compared to the SDM.

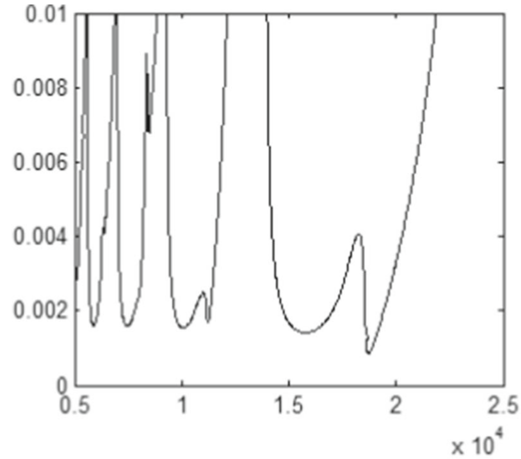
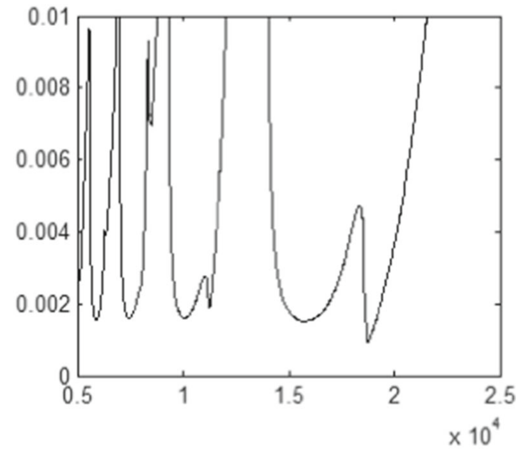
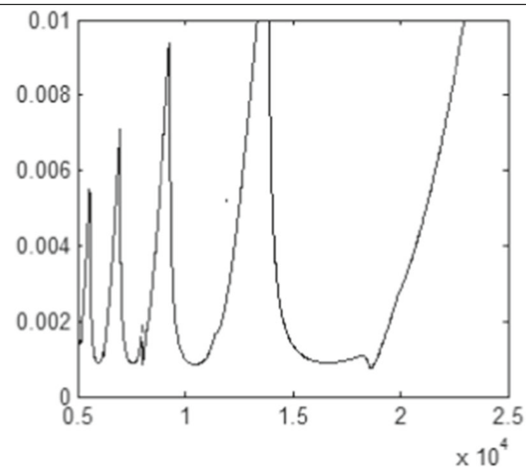
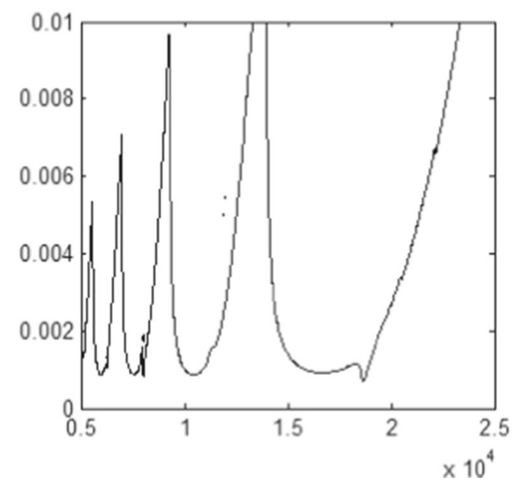
5.1 Single DOF milling model

The state equation of the single DOF milling model is the same as Eq.(2), where $A(t)$ and $B(t)$ can be expressed in the interval $[t_i, t_{i+1}]$ as follows:

$$A_i(t) = \begin{bmatrix} 0 & 1 \\ -\left(\omega_n^2 + \frac{wh_i}{m_t}\right) & -2\zeta w_n \end{bmatrix}, \quad B_i(t) = \begin{bmatrix} 0 & 0 \\ \frac{wh_i}{m_t} & 0 \end{bmatrix}, \quad u(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

where ω_n is the angular natural frequency, h_i is the specific

Table 2 Comparison of different methods for the two DOF milling model (the *horizontal* and *vertical coordinates* for figures are separately "r/min" and "m")

Radial depth of cut ratios	Semi-discretization method[12]	The proposed method
0.05		
Computational time(s)	1749.715	671.472
0.1		
Computational time(s)	1765.681	657.854

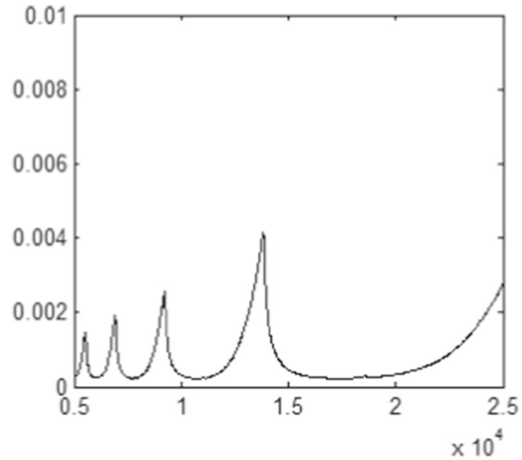
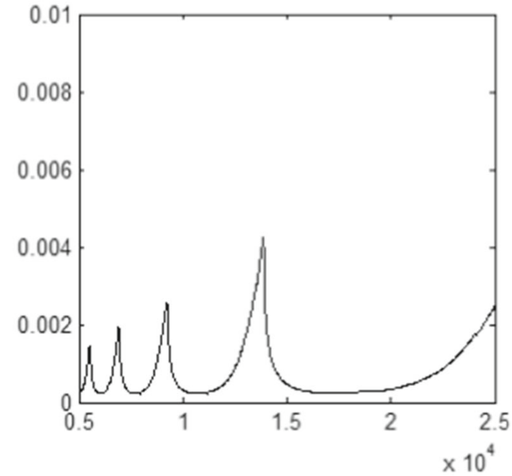
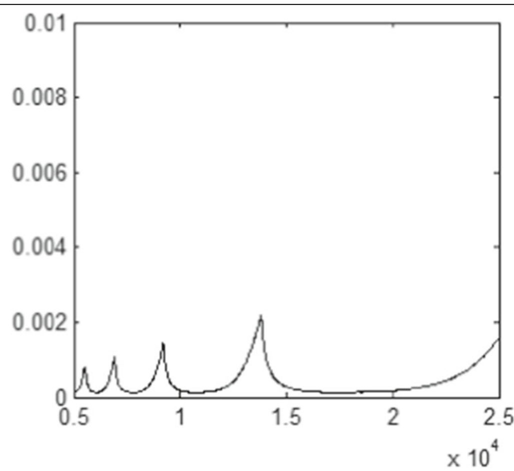
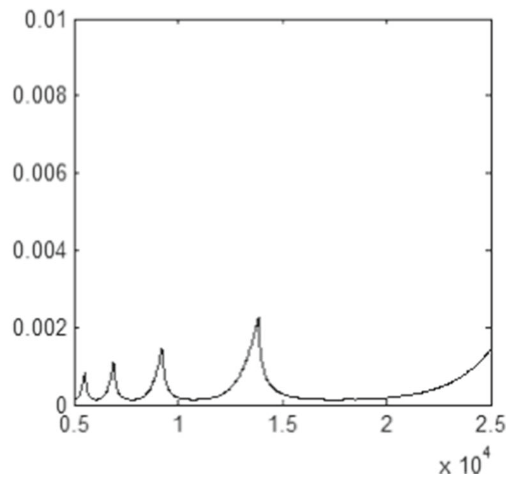
cutting force coefficient that has been determined in reference [27], m_t is the modal mass, w is the depth of cut, and ζ is the relative damping. In this case, the detailed parameters are as follows: ω_n=5793 rad/s, m_t=0.03993 kg, ζ=0.011, $K_t=6\times10^8$N/m^2, $K_n=2\times10^8$N/m^2, the number of teeth N=2, the sampling number is 40, and the radial depth of cut ratios are respectively 0.05, 0.1, 0.3, and 0.5.

The stability lobes for various radial depth of cut ratios at the SDM and the proposed method are shown in Table 1. It can be seen from Table 1 that the calculation time of the proposed method can be reduced nearly five times, and the calculation accuracy is not reduced, and the waveforms-based proposed method and SDM are in good agreement.

5.2 Two DOF milling model

The state equation of the two DOF milling model is the same as Eq. (2). These parameters of the following matrix are equal to Section 5.1; the stability charts for various radial depth of cut ratios at the SDM and the proposed method are shown in Table 2.

Table 2 (continued)

0.3		
Computational time(s)	1814.905	671.44
0.5		
Computational time(s)	1849.748	686.744

$$A_i(t)=\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\left(\omega_n^2+\dfrac{wh_{xxi}}{m_t}\right) & -\dfrac{wh_{xyi}}{m_t} & -2\zeta w & 0\\ -\dfrac{wh_{yxi}}{m_t} & -\left(\omega_n^2+\dfrac{wh_{yyi}}{m_t}\right) & 0 & -2\zeta w_n \end{bmatrix},\quad B_i(t)=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \dfrac{wh_{xxi}}{m_t} & \dfrac{wh_{xyi}}{m_t} & 0 & 0\\ \dfrac{wh_{yxi}}{m_t} & \dfrac{wh_{yyi}}{m_t} & 0 & 0 \end{bmatrix},\quad u(t)=\begin{pmatrix} x(t)\\ y(t)\\ \dot{x}(t)\\ \dot{y}(t) \end{pmatrix}$$

It can be seen from Table 2 that the calculation time of the proposed method can be reduced nearly 2.6 times, and the calculation accuracy also is not reduced, and the waveforms-based proposed method and SDM are in good agreement.

6 Conclusion

In this work, fast calculation without accuracy loss is proposed for predicting the milling stability which is obtained by reconstructing the semi-discretization method based on the

Shannon standard orthogonal basis. The fast calculation process is established by using the variable sampling numbers during the period of the delay time in which the variable sampling numbers are determined by the condition that is used to construct the Shannon standard orthogonal basis. Finally, the validity of the proposed method is verified by two benchmark examples for the one and two degrees of freedom milling model; the results show that the proposed method has high calculation and does not have accuracy loss compared to the SDM. This research is an impetus for the further development of the semi-discretization method.

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