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ORIGINAL ARTICLE

Effect of phase I estimation error on the monitoring of simple linear profiles in phase II

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Abstract The cases where a functional relationship between a response variable and one or more explanatory variables is used to monitor the quality of a product or process appear to be common in practice. This relationship usually is referred to as profile with parameters that are rarely known and have to be estimated prior to starting online monitoring. Phase II initiates after a statistically in-control condition is established, and parameters of the underlying distribution are estimated. It is obvious that phase II analysis of control charting is affected by errors in the estimated parameters. In this paper, the effect of estimation error on phase II analysis of simple linear profiles is studied. The in-control and out-of-control performance of the exponentially weighted moving average (EWMA)-3 method are evaluated using average run length (ARL) criterion. Overall, the in-control ARL in average decreases about 96.7 % when we have $m=100$ samples in phase I to estimate the parameters. This value reduces to 2.4 % for $m=900$ for a fixed sample size of $n=4$. The behavior of the EWMA-3 chart in the out-of-control situation totally depends on the direction of the occurring shifts and whether the ill-estimated parameter is overestimated or underestimated.

Keywords Estimation error · Exponentially weighted moving average · Statistical process control · Simple linear profiles · Average run length

1 Introduction

For many industrial applications, the quality of a process may be better represented by a functional relationship among one or more response variables and one or more predictors. This functional relationship is called profile. Statistical profile monitoring, a relatively new subarea of statistical quality control, deals with monitoring schemes when the quality of a process or product is better characterized by a linear or nonlinear profile. Most of work in the profile monitoring literature has focused on phase II which is the ongoing monitoring of a quality characteristic over time. Two monitoring schemes are proposed by Kang and Albin [7] for phase II analysis of simple linear profiles. Their first method consists of a Hotelling T^2 control chart for monitoring linear profile parameters, and their second method is based on a combined exponentially weighted moving average (EWMA) and range (R) chart to monitor average residuals between a sample and a reference line. A EWMA-3 approach consisting of three individual charts for monitoring intercept, slope, and standard deviation has been suggested by Kim et al. [8]. Zou et al. [23] developed a control chart based on a change-point model for monitoring simple linear profiles. A multivariate EWMA control scheme has been developed by Zou et al. [22] in order to monitor linear profiles in phase II. A self-starting control chart based on recursive residuals has been proposed by Zou et al. [24] for monitoring simple linear profiles. Most of the literature in phase II profile monitoring considers fixed explanatory variables. However, in some cases we might deal with random predictors. A detailed study has been performed by

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Noorossana et al. [13] on the performance of different monitoring schemes when having random explanatory variables. When the explanatory variables are random, a part of this randomness might be carried out by measurement error. Noorossana and Zerehsaz [14] developed some methods to account for the measurement error in the explanatory variable.

Phase I analysis of simple linear profiles is the retrospective analysis of the data and it is usually performed with the purpose of estimating the parameters of the baseline model in a statistically in-control situation. There are several methods in the profile monitoring literature that can be applied for phase I control of profiles. Mahmoud and Woodall [10] have applied the well-known F-test existing in the multiple linear regression literature to monitor simple linear profiles. A change-point method is suggested by [9] in order to perform retrospective analysis of simple linear profiles. All these studies are based on the fact that error terms in the underlying functional model are independently and identically distributed normal random variables. It is obvious that these assumptions may fail under certain conditions. When any of these assumptions fails to hold then one cannot apply the usual methods. A linear mixed-effect model is suggested by Jensen et al. [3] to monitor simple linear profiles in phase I when independence assumption of the random errors does not hold. The proposed method accounts for within autocorrelation structure. The problem of within autocorrelation in simple linear profiles in phase II has been addressed by Soleimani et al. [17]. A change-point approach is developed by Yeh and Zerehsaz [21] for phase I control of linear profiles with individual observations. For a more thorough literature review in this context, please refer to Woodall [19].

As it was stated earlier, quality of the estimates obtained in the phase I analysis could affect the performance of control schemes in phase II. Many authors have contributed to the investigation of the effect of parameter estimation in control charting. A comprehensive study has been done by Quesenberry [15] on the effect of sample size on the estimated limits of $\overline{X}$ and X control charts. An estimator for process standard deviation is proposed in order to improve performance of $\overline{X}$ control chart (Del Castillo [1]). A robust control chart is developed by Wu et al. [20] to deal with the problem of estimation error. Jones et al. [6] investigated the effect of estimation error on the performance of EWMA control chart. Jones [5] provides a new design procedure in order to modify the performance of EWMA control chart with estimated parameters. Shu et al. [16] studied the performance of cause-selecting control charts with estimated parameters. Jensen et al. [4] provide a good literature review on estimation error issues.

Having a detailed study in the literature of estimation error, we can notice that small amounts of error in the estimated parameters in phase I can significantly affect the charting performance in phase II. This issue might reveal itself as either numerous false alarms causing additional cost for finding the nonexistent root causes for out-of-control observations or deteriorated detection power of an out-of-control sample. Depending on the nature of the process, each of these problems might overweigh the other in the sense of loss of time and cost. To the best of our knowledge, the effect of estimation error on the performance of common control charts used for monitoring simple linear profiles has not been studied. In this paper, we tried to perform a thorough study on the effect of estimation error on the performance of EWMA-3 control scheme. The reason for considering this control scheme is that EWMA-3 appears to be one of the most effective methods for phase II monitoring of simple linear profiles.

The next section provides a brief description on EWMA-3 control chart. The repercussion of each parameter estimate on the performance of EWMA-3 method is investigated in Section 3. In Section 4, the role of reference sample size (m) and subgroup size (n) on the performance of EWMA-3 control scheme in the presence of estimation error is investigated. An illustrative example is provided in Section 5 to show how estimation error can affect performance of EWMA-3 chart. Our concluding remarks are provided in the final section.

2 The EWMA-3 method

This method presented by Kim et al. [8] seems to be one of the most appropriate approaches for phase II monitoring of simple linear profiles (Woodall et al. [18]). The EWMA-3 control chart has two advantages over the other methods. First, this chart outperforms the other control charts in detecting the shifts in the parameters. Second, one may easily identify which of the parameters has contributed to the out-of-control condition. The basic idea of applying this method originates from the fact that if we code the independent variables in a simple linear regression model, the intercept and slope estimators will be independent from each other. Hence, we will be able to apply three control charts to monitor the intercept, slope, and standard deviation separately.

The underlying regression model is expressed as

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \quad \mathrm{i} = 1, 2 \dots n \tag{1}$$

where β_0 and β_1 are the regression coefficients, x_is are the explanatory variables, and ε_is are independent normal random variables with mean zero and constant

Table 1 Properties of the EWMA-3 control chart

$EWMA_I$				$EWMA_S$				$EWMA_N$		
L_I	LCL_I	UCL_I	ARL_0	L_S	LCL_S	UCL_S	ARL_0	L_N	UCL_N	ARL_0
3.0156	12.4974	13.5026	584.4	3.0109	1.7756	2.2244	584.3	4.0734	1.3578	584.6

variance σ^2. The use of least square error method leads to the following estimators for the simple linear profile parameters

$$\widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1 \overline{x}, \quad \widehat{\beta}_1 = \frac{Sxy}{Sxx} \tag{2}$$

where $Sxx = \sum_{i=1}^{n} (x_i - \overline{x})^2$ and $Sxy = \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})$. By coding x^i to $x_i' = x_i - \overline{x}$, we can use three separate control charts for monitoring intercept, slope, and standard deviation. In this case, the model introduced in Eq. (1) can be rewritten as

$$y_i = A_0 + A_1 x_i' + \varepsilon_i \qquad \text{i} = 1, 2 \ldots n \tag{3}$$

where $A_0 = \beta_0 + \beta_1 \overline{x}$, $\beta^1 = A^1$, and $x_i' = x_i - \overline{x}$. This is true because we can write

$$\begin{aligned} y_i &= \beta_0 + \beta_1 x_i + \varepsilon_i + \beta_1 \overline{x} - \beta_1 \overline{x} \\ &= \beta_0 + \beta_1 \overline{x} + \beta_1 (x_i - \overline{x}) + \varepsilon_i = A_0 + A_1 x', \end{aligned}$$

The three EWMA statistics are calculated as

$$\begin{aligned} EWMA_I(j) &= \theta \widehat{A}_{0j} + (1-\theta)\, EWMA_I(j-1) \quad j = 1, 2 \\ EWMA_S(j) &= \theta \widehat{A}_{1j} + (1-\theta)\, EWMA_S(j-1) \quad j = 1, 2 \\ EWMA_N(j) &= \max\{\theta(\text{MSE}_j - \sigma^2) + (1-\theta) EWMA_N(j-1), 0\} \quad j = 1, 2 \end{aligned} \tag{4}$$

where $\text{EWMA}_I(0) = A_0$, $\text{EWMA}_S(0) = A_1$, $\text{EWMA}_N(0) = 0$, and θ is a smoothing constant. Small values of θ increase the power of EWMA chart for detecting small shifts in the parameters (Montgomery [11]). Also, $\widehat{A}_{0j} = \overline{y}$ and $\widehat{A}_{1j} = \widehat{\beta}_1 = \frac{Sxy}{Sxx}$. The upper and lower control limits for the three EWMA control charts are defined as

$$\begin{aligned} LCL_I &= A_0 + L_I \sqrt{\frac{\theta}{2-\theta} \frac{\sigma^2}{n}} \\ LCL_I &= A_0 - L_I \sqrt{\frac{\theta}{2-\theta} \frac{\sigma^2}{n}} \\ UCL_S &= A_1 + L_S \sqrt{\frac{\theta}{2-\theta} \frac{\sigma^2}{Sxx}} \\ LCL_S &= A_1 - L_S \sqrt{\frac{\theta}{2-\theta} \frac{\sigma^2}{Sxx}} \\ UCL_N &= +L_N \sqrt{\frac{\theta}{2-\theta} \text{var}(\text{MSE}_j)} \end{aligned} \tag{5}$$

where L_I, L_S, and L_N are positive constants that can be adjusted to give a desired in-control ARL, and $\text{var}(\text{MSE}_j) = \frac{2\sigma^4}{n-2}$ (Kim et al. [8]).

3 Effect of parameter estimation on the performance of EWMA-3 control

In this section, we investigate the effect of estimation error on the performance of EWMA-3 scheme when only one parameter is subject to estimation error in phase I analysis. This investigation helps us to understand the magnitude of the effect of the estimation error on the control chart scheme and take precaution measures in advance.

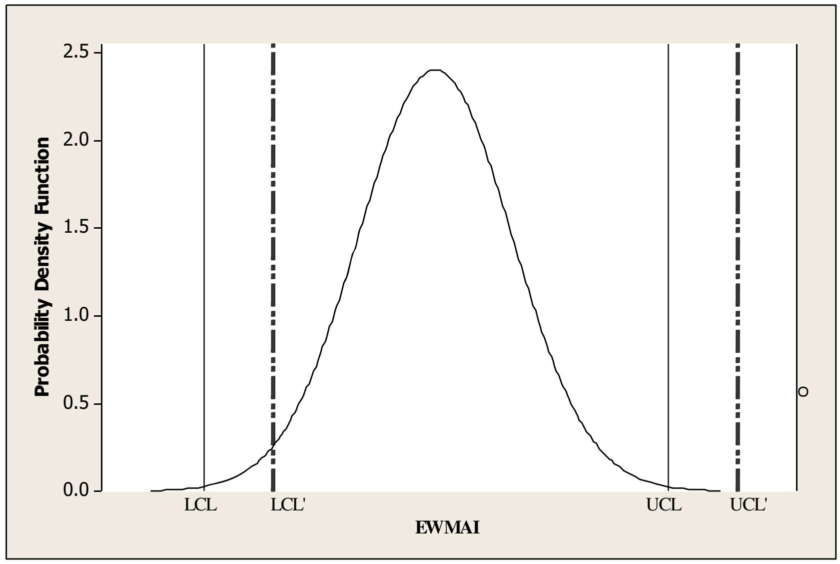
3.1 Estimation error in the intercept

In the EWMA-3 control scheme, neither the EWMA_S nor the EWMA_N chart undergoes changes when the intercept parameter

Table 2 ARL values with regard to EWMA-3 control chart under intercept shifts from β_0 to $\beta_0+\lambda\sigma$ with different amounts of intercept estimation

		λ										
The estimated β_0	Percentile	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
3	50	200	59.1	16.2	7.9	5.1	3.8	3.1	2.6	2.3	2.1	1.9
2.96	35	183.5	42.8	12.8	6.1	3.8	2.6	1.9	1.5	1.2	1	1
2.99	45	191	52.4	13.9	6.6	4	2.7	2	1.5	1.3	1	1
3.01	55	193.2	63.4	17.8	8	6.2	5.8	4.1	3.6	2.8	2.7	2
3.04	65	184	79.2	19.9	9.7	6.4	5	4.2	3.7	3.3	3.1	2.9

Fig. 1 Probability density function of $EWMA_I$ statistic (control limits based on estimated intercept parameter, control limits based on actual intercept parameter)



Probability density function of $EWMA_I$ statistic (–··–· control limits based on estimated intercept parameter, ——— control limits based on actual intercept parameter)

is estimated with error. The estimation error in the intercept parameter has no effect on the expected value and standard deviation of $EWMA_I$ statistic. This issue, however, may affect both the $EWMA_I(0)$ and its control limits. Assume that instead of considering the true parameter β_0, we incorrectly apply $\beta_0^{'}$ in the computation of control limits. As pointed out earlier, in the EWMA-3 chart, the intercept parameter is computed as $A_0 = \beta_0 + \beta_1\bar{x}$. As a result, the amount of lower and upper control limits could be obtained by

$$LCL_I = A_0^{'} - L_I\sigma\sqrt{\frac{\theta}{(2-\theta)n}}, \quad UCL_I =, A_0^{'} + L_I\sigma\sqrt{\frac{\theta}{(2-\theta)n}},$$

where $A_0^{'} = \beta_0^{'} + \beta_1\bar{x}$ is the ill-estimated intercept. Apparently, the performance of EWMA-3 control chart would be influenced largely because the control limits of $EWMA_I$ chart depart from their real values in the presence of estimation error. To evaluate the behavior of EWMA-3 approach in this case, a simulation study has been carried out in MATLAB software. We assume that the underlying model is given by the following relationship

$$y_{ij} = 3 + 2x_i + \varepsilon_{ij} \qquad i = 1, 2, \ldots 4 \quad j = 1, 2,$$

where the independent variable x_i takes values 2, 4, 6, and 8 and ε_{ij}s follow a normal distribution with mean zero and constant variance $\sigma^2=1$. Centering the x_i variables leads to the new intercept parameter $A_0 = \beta_0 + \bar{x}(\beta_1) = 13$. Consequently, the control limits will be constructed using A_0 in the known parameter situation. The in-control ARL with known parameters is set to be roughly 200, and the smoothing constant is considered 0.2. These values give us a reasonable detection power for small to moderate shifts while having an appropriate false-alarm rate for a EWMA chart. The out-of-control situation is simulated based on sustained changes in the parameters of the profile. That is, the shifted parameter remains out of control until the end of simulation. Both ARL_0 and ARL_I values are computed using 10,000 simulations. To be more specific, in order to compute the ARL values, we start simulating the samples while a counter counts the number of in-control observations prior to receiving an out-of-control point. We repeat this operation 10,000 times and take the average of these 10,000 run length values. The resulting average gives us the ARL value. Additional specifications of the simulation study can be found in Table 1 (known parameters).

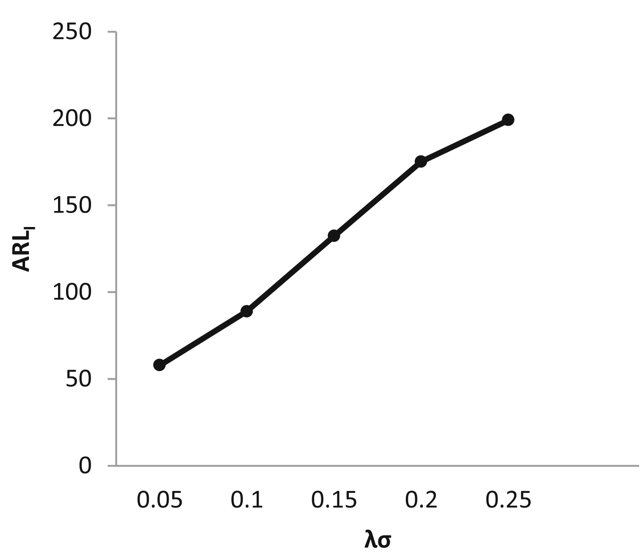


Fig. 2 Out-of-control performance of EWMA-3 chart when the intercept parameter is estimated

Table 3 ARL values with regard to EWMA-3 control chart under slope shifts from β_1 to $\beta_1+\beta\sigma$ with different amounts of intercept estimation

β											
The estimated β_0	Percentile	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
3	50	101.6	36.5	17.0	10.3	7.2	5.5	4.5	3.8	3.3	2.9
2.96	35	76.7	28.4	13.8	8.2	5.6	4.1	3.2	2.6	2.1	3.8
2.99	45	94.1	33.4	15.3	8.8	6	4.4	3.4	2.7	2.2	3.9
3.01	55	110.9	39.9	19	11.7	8.3	6.6	5.5	4.8	4.3	3.9
3.04	65	128.9	47.5	21	12.5	8.8	6.8	5.7	4.9	4.4	4

Tables 2 and 4 give ARL values for shifts in the intercept, slope, and standard deviation parameters when β_0 parameter is estimated inaccurately. It is assumed that there exists a set of m=30 historical data each containing n=4 pairs of (x_i, y_i) observations to use for estimating the parameters in phase I. Furthermore, it is assumed that the values for the slope and error variance are known or are estimated without any error.

As it is known, the sample mean of the intercept estimator $\overline{\hat{\beta}}_0$ follows a normal distribution with mean (median) β_0 and variance $\frac{\sigma^2\left(\frac{1}{n}+\frac{\bar{x}^2}{S_{xx}}\right)}{m}$. Various percentiles of the $\overline{\hat{\beta}}_0$ distribution have been considered in our study. Hence, the first column of Tables 2 and 4 includes different percentiles of $\overline{\hat{\beta}}_0$ distribution. These amounts, indeed, represent the ill-estimated intercept. The 50th percentile is the true intercept parameter.

First, consider the in-control situation. Whereas the parameter is either overestimated or underestimated, the control limits of EWMA$_I$ chart deviate from their actual amounts; a large percent of EWMA$_I$'s distribution would be located out of the control limits. Figure 1 demonstrates this fact. Hence, in this situation, more false alarms will be received from the EWMA-3 chart. This can be observed in Table 2.

The behavior of EWMA-3 approach in the out-of-control state depends generally on the fact that the intercept parameter is overestimated or underestimated. Suppose that the estimated intercept is given by $\beta_0'=\beta_0+\delta$ where δ is an either positive or negative constant. Providing that an underestimation occurs ($\delta<$ 0), the control limits will drift toward the left, and the right side of EWMA$_I$ distribution will fall outside the upper control limit. Clearly, when the intercept parameter shifts from β_0 to $\beta_0+\lambda\sigma$, the expected value of EWMA$_I$ statistic will be equal to $A_0+\lambda\sigma$. This shift is detected more rapidly than the known parameter case as the EWMA$_I$ distribution moves to the right direction. In other words, a bigger percentage of the EWMA$_I$ distribution will be placed beyond UCL_I by enlargement of the intercept parameter.

Performance of the EWMA$_I$ control chart in the overestimation case is entirely associated with the kind of relationship between $\lambda\sigma$ and δ. According to Fig. 1, the existence of a positive shift in the intercept imposes the distribution of EWMA$_I$ statistic to transfer toward the right direction. When $\lambda\sigma$ is smaller than δ a remarkable portion of the distribution will fall inside the control limits when the shift size increases. This implies that the detection power of EWMA-3 control chart will be even more diminished for the larger shift sizes. A simulation analysis is performed in order to illustrate this point. We assume that δ parameter equals +0.25; hence, we have $A_0' = \beta_0' + \beta_1\bar{x} = 13.25$. The ARL values have been calculated for different intercept shifts. Again, we assume that the underlying model is given by the following relationship:

$$y_{ij} = 3 + 2x_i + \varepsilon_{ij} \qquad i = 1, 2, \ldots 4 \quad j = 1, 2,$$

x_i takes values 2, 4, 6, and 8 and ε_{ij}s follow a normal distribution with mean zero and constant variance σ^2=1.

Table 4 ARL values with regard to EWMA-3 control chart under standard deviation shifts from σ to $\gamma\sigma$ with different amounts of intercept estimation

γ											
The estimated β_0	Percentile	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
3	50	31.6	11.7	6.5	4.5	3.4	2.7	2.3	2.1	1.8	1.7
2.96	35	30.6	10.6	5.4	3.4	2.4	1.8	1.4	1.1	1	1
2.99	45	30.2	10.7	5.3	3.3	2.3	1.8	1.4	1.1	1	1
3.01	55	30.9	10.6	5.4	3.4	2.4	1.8	1.4	1.1	1	1
3.04	65	30.3	10.3	5.3	3.4	2.4	1.8	1.4	1.1	1	1

Table 5 ARL values with regard to EWMA-3 control chart under intercept shifts from β_0 to $\beta_0+\lambda\sigma$ with different amounts of slope estimation

		λ										
The estimated β_1	Percentile	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
2	50	200	59.1	16.2	7.9	5.1	3.8	3.1	2.6	2.3	2.1	1.9
1.93	35	146.4	31.6	11.4	6.4	4.5	3.5	2.8	2.4	2.2	2	1.8
1.97	45	190.2	46.5	14.1	7.3	4.9	3.7	3	2.5	2.2	2	1.9
2.03	55	188.2	70.9	18.1	8.3	5.3	3.9	3.1	2.6	2.3	2.1	1.9
2.07	65	146.9	105.8	24.2	9.8	5.9	4.2	3.3	2.7	2.4	2.1	1.9

Figure 2 shows that the performance of EWMA-3 chart deteriorates regularly. The ARL_I criterion reaches its maximum amount when $\lambda\sigma=\delta$. This situation is exactly similar to the in-control state.

There is a possibility that the shift size $\lambda\sigma$ could be larger than the estimation error δ. In this case, the EWMA-3 chart may discover the intercept shifts more slowly compared to the case that the parameter is known. The ability of this chart improves as the shift size gets larger (see Table 2). Table 3 shows the ARL values for shifts in slope from β_1 to $\beta_1+\beta\sigma$. It is obvious that the expected value of $EWMA_I$ chart moves from A_0 to $A_0+\bar{x}\beta\sigma$. In this case, the results can be interpreted in a similar manner as the previous situation when one encounters intercept shifts. Results in Table 4 indicate that EWMA-3 approach can detect shifts in the standard deviation quicker than the known parameter case. This is because the $EWMA_I$ chart provides more alarms when estimating β_0 parameter with error.

3.2 Estimation error in the slope

Estimation error in the slope parameter may influence both the $EWMA_I$ and $EWMA_S$ control limits. Since $EWMA_I$ control limits are computed based on A_0 parameter where $A_0=\beta_0+\beta_1\bar{x}$, then slope estimation could affect the lower and upper control limits of $EWMA_I$ statistic. The $EWMA_S$ control limits are given by

$$LCL_S=\beta_1^{'}-L_S\sigma\sqrt{\frac{\theta}{(2-\theta)Sxx}}\ ,UCL_S$$
$$=\beta_1^{'}+L_S\sigma\sqrt{\frac{\theta}{(2-\theta)Sxx}},$$

where $\beta'_1=\beta_1+\delta$ is the slope parameter subject to estimation error. The in-control ARL decreases in this situation. Table 5 shows the ARL values for a possible range of intercept shifts. Apparently, the $EWMA_S$ chart produces more false alarms except when no error exists in estimating the slope parameter. Underestimating the slope parameter moves the control limits of $EWMA_I$ and $EWMA_S$ to the left direction. Hence, the out-of-control ARL values will be smaller than the ARL values when no estimation error is present. Similar to what we discussed in Subsection 3.1, in the overestimation condition, the ARL_I values are affected by the relationship between δ and $\lambda\sigma$ parameters.

The charting performance of EWMA-3 control chart under the slope parameter shifts from β_1 to $\beta_1+\beta\sigma$ is tabulated in Table 6. It is obvious that the expected values of the $EWMA_I$ and $EWMA_S$ statistics in this case can be expressed as $\beta_0+\beta\sigma\bar{x}$ and $\beta_1+\beta\sigma$, respec-

Table 6 ARL values with regard to EWMA-3 control chart under slope shifts from β_1 to $\beta_1+\beta\sigma$ with different amounts of slope estimation

		β									
The estimated β_1	Percentile	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
2	50	101.6	36.5	17.0	10.3	7.2	5.5	4.5	3.8	3.3	2.9
1.93285	35	50.3	20.4	11	7.1	5.2	4.1	3.4	2.9	2.5	2.3
1.97246	45	78.9	29.1	14	8.4	5.9	4.5	3.6	3	2.7	2.4
2.02754	55	125.7	43.5	18.2	10.1	6.7	5	3.9	3.3	2.8	2.5
2.06715	65	172	66.3	25	12.7	7.9	5.6	4.3	3.5	3	2.6

Table 7 ARL values with regard to EWMA-3 control chart under standard deviation shifts from σ to $\gamma\sigma$ with different amounts of slope estimation

		γ									
The estimated β_1	Percentile	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
2	50	33.5	12.7	7.2	5.1	3.9	3.2	2.8	2.5	2.3	2.1
1.93285	35	27.6	10.3	5.8	3.9	3	2.5	2.1	1.9	1.7	1.6
1.97246	45	29.8	10.7	5.9	4	3	2.5	2.1	1.9	1.7	1.6
2.02754	55	30	10.6	5.8	4	3	2.5	2.1	1.9	1.7	1.6
2.06715	65	27.5	10.4	5.8	3.9	3	2.5	2.1	1.9	1.7	1.6

tively. As it is known, the associated control limits for both statistics change since the slope parameter is estimated with error. That is, we will apply $\beta'_1=\beta_1+\delta$ parameter to establish the control limits of EWMA-3 method instead of β_1. In the EWMA$_S$ chart, as long as $\delta>0$, parameter is greater than $\beta\sigma$ where $\beta>0$, the detection power will be reduced as the shift size increases. The worst state happens when we have $\delta=\beta\sigma$. Now, suppose that we have $\delta<\beta\sigma$. The EWMA$_S$ chart may be less capable to detect β_1 shifts in comparison to the known parameter case. On the other hand, the ability of this chart will improve when the magnitude of shifts becomes larger. It is noticeable that the same situation may occur in the EWMA$_I$ chart with this difference that in this chart, the relationship between δ and $\beta \sigma\bar{x}$ parameters has to be considered to describe the results.

If the slope parameter is underestimated, regardless of the fact that the process is in or out of control, both EWMA$_I$ and EWMA$_S$ charts yield more alarms. Based on what was stated, estimation error in β_1 leads to more false alarms in the in-control state. As it is known, standard deviation shifts are detected barely by the EWMA$_N$ chart implying that the other two charts are not affected by changes in the error variance. The EWMA-3 chart is able to discover the standard deviation changes quicker compared to the case that the slope parameter is estimated faultlessly inasmuch as the EWMA$_I$ and EWMA$_S$ charts alarm more frequently. Results in Table 7 can easily reflect this fact.

3.3 Estimation error in variance

Estimation of standard deviation is more serious compared to the other two parameters since this parameter affects the control limits of all three charts. We assume that $\sigma'^2 = \delta\sigma^2$ is used incorrectly instead of σ^2. Thus, the control limits may be given by

$$LCL_I = A_0 - L_I\sigma'\sqrt{\frac{\theta}{(2-\theta)n}},\ \ UCL_I = A_0 + L_I\sigma'\sqrt{\frac{\theta}{(2-\theta)n}},$$

$$LCL_S = A_1 - L_S\sigma'\sqrt{\frac{\theta}{(2-\theta)Sxx}},\ \ UCL_S = A_1 + L_S\sigma'\sqrt{\frac{\theta}{(2-\theta)Sxx}},$$

$$UCL_N = L_N\sqrt{\frac{\theta}{2-\theta}\frac{2\sigma'^4}{n-2}}$$

It is worth mentioning that the expected value of EWMA$_N$ statistic can be computed as

$$E[EWMA_N] = \sigma^2 - \sigma'^2 = (1-\delta)\sigma^2,$$

Table 8 ARL values with regard to EWMA-3 control chart under intercept shifts from β_0 to $\beta_0+\lambda\sigma$ with different amounts of variance estimation

		λ										
σ^2	Percentile	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
1	50	200	59.1	16.2	7.9	5.1	3.8	3.1	2.6	2.3	2.1	1.9
0.7198	35	123.7	44.5	13.9	7.2	4.8	3.6	2.9	2.5	2.2	2	1.8
0.87156	45	163	52.6	15.1	7.5	4.9	3.7	3	2.5	2.2	2	1.9
1.11636	55	211.5	61.1	16.3	7.9	5.1	3.8	3.1	2.6	2.3	2.1	1.9
1.31803	65	284.7	71	17.6	8.3	5.3	3.9	3.2	2.7	2.3	2.1	1.9

Table 9 ARL values with regard to EWMA-3 control chart under slope shifts from β_1 to $\beta_1+\beta\sigma$ with different amounts of variance estimation

β											
σ^2	Percentile	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
1	50	101.6	36.5	17.0	10.3	7.2	5.5	4.5	3.8	3.3	2.9
0.7198	35	69.2	27.8	13.5	8.2	5.7	4.4	3.5	3	2.6	2.3
0.87156	45	83.5	32.1	14.9	8.8	6	4.6	3.7	3.1	2.7	2.4
1.11636	55	104.1	37.1	17.2	11.3	7.4	6.7	4.8	4.2	3.7	3.1
1.31803	65	128.2	41.2	19.8	10.9	8.7	7.5	6.4	5	4.3	3.9

The overestimation of variance clearly widens the control limits leading to large in-control and out-of-control ARL values. The performance of EWMA-3 approach improves when σ^2 parameter is underestimated. Tables 8, 9, and 10 provide the ARL values for a variety of shifts with regard to the intercept, slope, and standard deviation when estimation error exists.

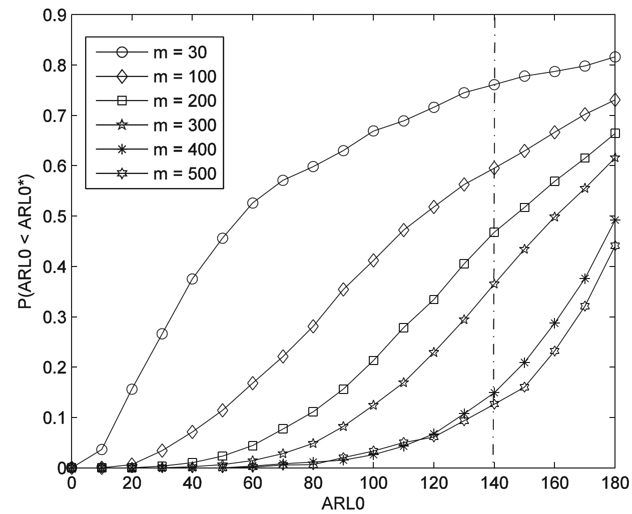
4 Sample size requirements for EWMA-3 control chart under estimation error

The estimation error problem usually involves all the parameters. In essence, when the number of subgroups is not big enough in phase I, the intercept, slope, and standard deviation parameters are determined with error. The interpretation of results is not straightforward in contrast to the case where only one parameter is ill estimated. In this section, we study the in-control performance of EWMA-3 chart characterized by ARL_0 metric. As pointed out earlier, either the overestimation or underestimation of β_0 and β_1 parameters increases the false alarms in the EWMA-3 chart. Underestimating the variance parameter leads to more signals when process is in control. Consequently, the EWMA-3 chart always generates more signals in the presence of estimation error. Note that the abovementioned statement may not be true if the error variance is severely overestimated.

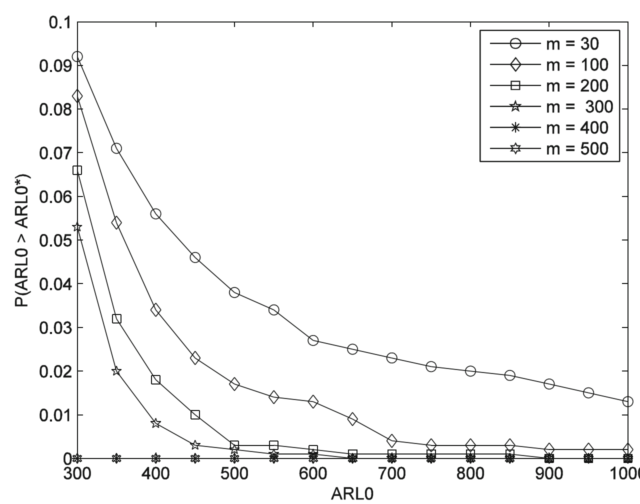
In spite of the fact that the conditional ARL_0 on a given set of β_0, β_1, and σ^2 parameters is a specified constant, the unconditional in-control average run length can be considered as a random variable. As it was stated earlier, the number of subgroups (m) and the sample size (n) can significantly affect the accuracy and precision of the estimates. Thus, it is not surprising that these parameters possibly affect the ARL_0 distribution. Figure 3 demonstrates the empirical distribution of

Table 10 ARL values with regard to EWMA-3 control chart under standard deviation shifts from σ to $\gamma\sigma$ with different amounts of variance estimation

γ											
σ^2	Percentile	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	50	33.5	12.7	7.2	5.1	3.9	3.2	2.8	2.5	2.3	2.1
0.7198	35	21.7	8.6	5	3.5	2.8	2.3	2	1.8	1.6	1.5
0.87156	45	26.3	9.7	5.5	3.8	2.9	2.4	2.1	1.8	1.7	1.5
1.11636	55	35.8	13	6.5	4	3.1	2.5	2.1	1.9	2.6	2.5
1.31803	65	38.8	13.6	7.6	5.3	4.2	3.5	3.1	2.9	2.8	2.5



a $Pr(ARL_0 < ARL_0^*)$



b $Pr(ARL_0 > ARL_0^*)$

Fig. 3 Empirical distribution of ARL_0 variable for various *m*s **a** $P(ARL_0 < ARL_0^*)$ and **b** $P(ARL_0 > ARL_0^*)$

Table 11 Magnitudes of $p(200-\tau \leq ARL_0 \leq 200+\tau)$ for various amounts of τ and m (the in-control ARL in the known parameter case equals 200)

m / τ	10	30	50	70	90	110
m=30	0.021	0.064	0.098	0.149	0.212	0.282
m=100	0.053	0.137	0.232	0.323	0.431	0.57
m=200	0.07	0.219	0.354	0.492	0.642	0.788
m=300	0.099	0.276	0.445	0.615	0.769	0.873
m=400	0.304	0.624	0.791	0.893	0.957	0.985
m=500	0.361	0.679	0.84	0.907	0.96	0.99

unconditional ARL_0 for various amounts of m with four pairs of observations, i.e., $n=4$. The in-control ARL with known parameters is set to 200.

It is noticeable that for smaller values of m, it is more probable that the EWMA-3 chart triggers alarms quickly with higher probability (see Fig. 3a). Moreover, it is more probable that the ARL_0 variable takes a large value unless the number of subgroups m is as large as possible (see Fig. 3b).

Table 11 gives different probabilities for different ARL_0 values. Let us assume that we estimate the profile parameters using 30 subgroups of size 4. Further, assume that 6.4 % of the distribution of ARL_0 is located between 170 and 230. This amount increases to 67.7 % in case we apply 500 subgroups in order to estimate the parameters. This indicates that by taking larger reference sample size to estimate the parameters, the ARL_0 variable will be located around the real in-control ARL (200 in our case) with higher probability.

Another noteworthy issue when establishing a control chart is the sample size n. So far, we have considered only one value for this parameter, i.e., $n=4$. Figure 4 depicts ARL_0 distribution

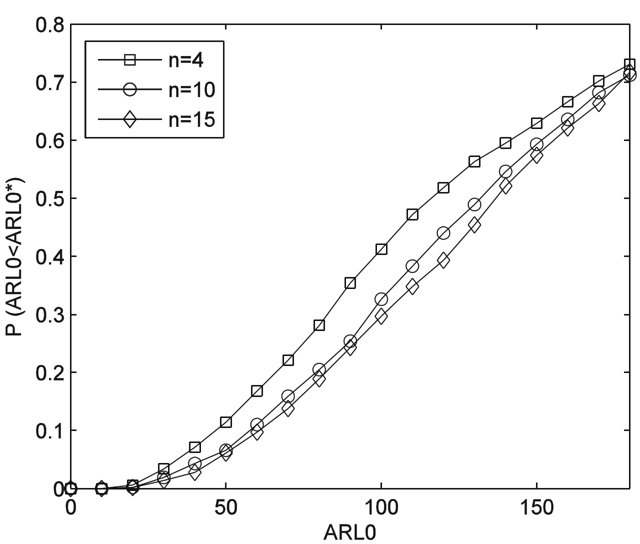


Fig. 4 Empirical distribution of ARL_0 variable for various ns ($m=30$)

Table 12 Average percent increase in early false-alarm rate for different reference sample size values

n	m				
	100	300	500	700	900
4	96.9	28.7	17.6	12.5	2.4
10	90	22.6	14.3	8.9	1.2
15	83	19.7	11.9	6.6	1

for three values of sample size, i.e., n=4, 10, and 15. The number of subgroups is adjusted to 30.

It is clear that the ARL_0 distribution is steeper when the sample size is not large. Apparently, choosing small sample sizes to estimate the parameters enhances the probability of producing frequent false alarms by the EWMA-3 control chart.

According to what has been mentioned by Quesenberry [15], the false-alarm rate in the early observations can be considered as an effective criterion to decide on how large a reference sample size is needed to set up a control chart. He actually computed the amount of increase in the false-alarm rate in the first twenty observations for the $\overline{X}$ charts when parameters are estimated compared to that in the known parameter case. Some other researchers like Jones et al. [6] and Shu et al. [16] have used this metric. It should be pointed out that although false-alarm probability might decrease in some situations, we might see an enlargement in this criterion in average. Table 12 provides the percent average increase of false-alarm rates in the

Table 13 The measured and standard values for different days

Number of sample	Level of x_i	Independent variable (x_i)	Response variable (y_{ij})
1	1	0.76	1.12
1	2	3.29	3.49
1	3	8.89	9.11
2	1	0.76	0.99
2	2	3.29	3.53
2	3	8.89	8.89
3	1	0.76	1.05
3	2	3.29	3.46
3	3	8.89	9.02
4	1	0.76	0.76
4	2	3.29	3.75
4	3	8.89	9.3
5	1	0.76	0.96
5	2	3.29	3.53
5	3	8.89	9.05
6	1	0.76	1.03
6	2	3.29	3.52
6	3	8.89	9.02

first twenty observations for different values of reference sample size m and sample size n. When m is set equal to 900, the probability of false alarms increases about 2.4 %, which is negligible. Obviously, this is not the case for smaller reference sample sizes. It is obvious that the number of observations n affects the percent increase in the early false-alarm rates. However, there is more flexibility to manipulate the reference sample size m compared to the sample size n.

5 An illustrative example

In order to demonstrate how the EWMA-3 chart performs when parameters are subject to estimation error, we consider a real calibration example in an optical imaging system discussed in *NIST/SEMATECH e-Handbook of Statistical Methods* [12]. The dataset includes line widths of photomask reference standards on ten units. Thus, there are 40 measurements available for estimating the calibration relationship in phase I. The estimated profile is $y_{ij} = 0.2817 + 0.9767x_i$, and the standard deviation parameter equals 0.06826 μm. Gupta et al. [2] also consider this example. There exists six other samples each including three different units (lower (level 1), middle (level 2), and upper (level 3)) over the measurement range (see Table 13).

Figure 5 reveals the established EWMA-3 chart. It is obvious that the $EWMA_S$ and $EWMA_N$ charts signal on 4th sample (4th day). The in-control ARL is set to be 200.

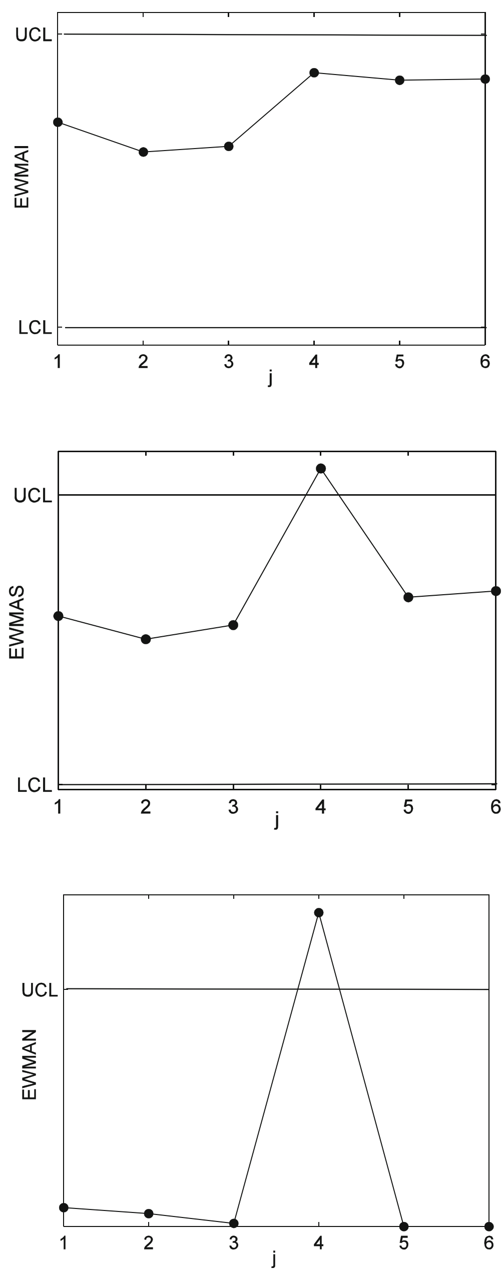


Fig. 5 the $EWMA_3$ chart for monitoring the linear calibration profile

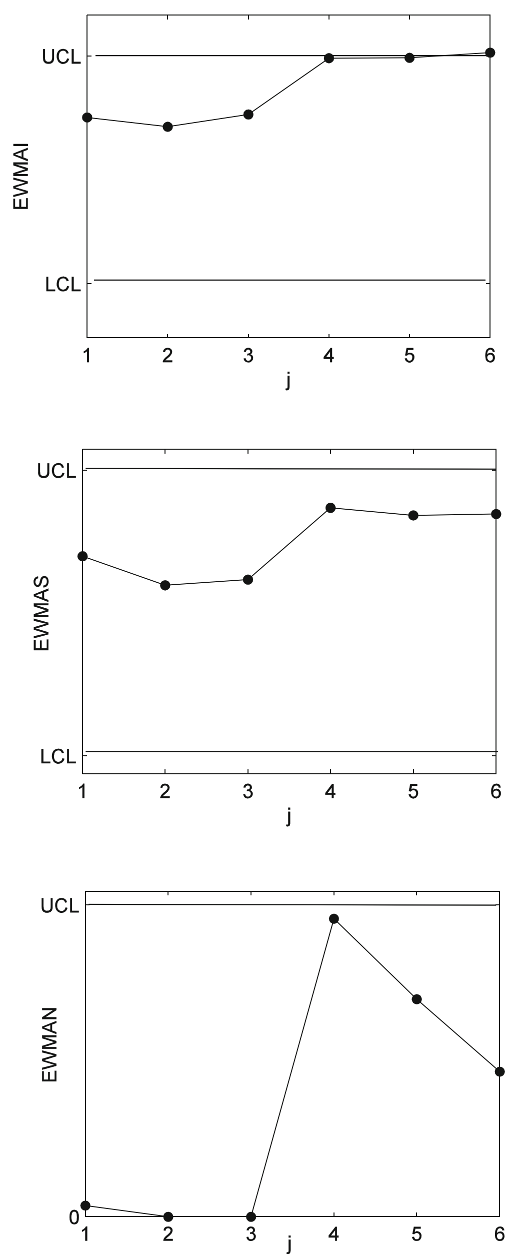


Fig. 6 the $EWMA_3$ control scheme with estimated parameters

In practice, the estimated parameters differ from the actual parameters. Here, we assume that the underlying calibration function $y_{ij} = 0.2817 + 0.9767x_i + \varepsilon_{ij}$ is the real profile, and ε_{ij}s are independent normal random variables with mean zero and constant variance 0.0047. Now, let us assume that the 40 measurements are used to estimate the profile parameters in phase I yielding β_0=0.2428, β_1=0.9799, and σ^2=0.0062. In this situation, the EWMA-3 control scheme will act in a different way. This fact can be seen easily in Fig. 6.

The EWMA$_S$ and EWMA$_N$ charts seem to be in control in all six samples. This, however, is not the case in Fig. 5. The EWMA$_I$ chart, moreover, alarms in the last sampling point. This indicates that the estimation error can change the charting performance of EWMA-3 approach drastically.

6 Conclusion

The main purpose of this study is to evaluate the performance of the EWMA-3 control chart in phase II when the profile parameters are estimated in phase I. Obviously, one of the best solutions for this issue is to find an appropriate value for the reference sample size in phase I. We used different criteria to study both the in-control and out-of-control performance of EWMA-3 chart, and finally provided a framework for choosing both the reference sample size and the number of observations in the profile.

The unfavorable effects of estimation error on the performance of EWMA-3 chart can lead to wrong decisions. The estimation error may lead to two main issues: (1) increase in the false alarms and (2) reduction in the detection power. For the first case, both over- and underestimations of profile parameters increase the false-alarm rate. As an example, when we use 500 samples in phase I to estimate the parameters of a profile with ten points, the false-alarm rate increases about 14 %. This insinuates that both the reference sample size and sample size must be as large as possible in order to reduce the drastic effects of estimation error. For the latter case of out-of-control situation, the performance of the control charts depends on the direction of the shifts and the fact that whether the parameters are underestimated or overestimated. To be more specific, when the intercept parameter is estimated with error, the EWMA$_I$ charting performance in the in-control situation will diminish. Obviously, neither the EWMA$_S$ nor the EWMA$_N$ charts are affected by the ill-estimated intercept parameter. The existence of estimation error in the slope parameter influences both the EWMA$_I$ and EWMA$_S$ charts. In both charts, the false-alarm rate increases under in-control conditions. Finally, the overestimation of the standard deviation parameter leads to a decrease in the ARL values. The underestimation of the standard deviation parameter, conversely, decreases the ARL magnitudes.

Although the linear or nonlinear profiles appear to be increasingly common in practical applications, estimation error can yield misleading results indicating the need for a considerable attention during the estimation phase of control charts.

Conflict of interest The authors declare that they have no conflict of interest.

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