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ORIGINAL ARTICLE

Optimal condition-based maintenance policy for a partially observable system with two sampling intervals

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Abstract In this paper, we propose an optimal Bayesian control policy with two sampling intervals minimizing the long-run expected average maintenance cost per unit time for a partially observable deteriorating system. Unlike the previous optimal Bayesian approaches which used periodic sampling models with equidistant intervals, a novel sampling methodology is proposed which is characterized by two sampling intervals and two control thresholds. The deterioration process is modeled as a 3-state continuous time hidden-Markov process with two unobservable operating-states and an observable failure state. At each sampling epoch, the multivariate observation data provides only partial information about the actual state of the system. We start observing the system with a longer sampling interval. If the posterior probability that the system is in the warning state exceeds a warning limit, observations are taken more frequently, i.e., the sampling interval changes to a shorter one, and if the posterior probability exceeds a maintenance limit, the full inspection is performed, followed possibly by preventive maintenance. We formulate the maintenance control problem in a partially observable Markov decision process (POMDP) framework to find the two optimal control limits and two sampling intervals. Also, the mean residual life (MRL) of the system is calculated as a function of the posterior probability. A numerical example is provided and comparison of the proposed scheme with several alternative sampling and maintenance control strategies is carried out.

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Nomenclature

X_t	State of the system at time t
q_{ij}	Instantaneous transition rate
$P_{ij}(t)$	Transition probability function from state i to state j
ν_i	Exponential distribution parameter of the sojourn time in state i
ξ	Failure time of the system
Δ_1	Longer sampling interval
Δ_2	Shorter sampling interval
$Y_{n_1\Delta_1}$	Residual of the observation at time $n_1\Delta_1$
μ_i	Mean vector of normally distributed observation in state i
Σ_i	Covariance matrix of normally dis tributed observation in state i
$f(y \mid i)$	Conditional density of the observation vector given the state is i
W	Warning limit
M	Maintenance control limit
(h, I)	Inspection state
PM	Preventive maintenance state
T_I	Inspection time
T_M	Preventive maintenance time
T_F	Failure replacement time
τ	Expected sojourn time
C	Expected cost
K	Number of subintervals of [0, 1]
$\Pi_{n_1\Delta_1}$	Posterior probability at time $n_1\Delta_1$

$f_\xi(t)$	Probability density of the time to failure
$E(\xi)$	Expected time to failure
MRL	Mean residual life of the system
$R(t \mid \Pi_{n_1\Delta_1+n_2\Delta_2})$	Conditional reliability function
$h(t)$	Hazard rate function

1 Introduction

Technical systems in modern production and manufacturing industries deteriorate and fail as the result of usage and age which often leads to high production cost and low system availability. To avoid costly failures, preventive maintenance (PM) is commonly performed while the system is still operational. Traditional PM models can be classified into three different categories: (i) age replacement models [1, 30], (ii) the block replacement models [3, 27], and (iii) the periodic maintenance models [2, 12]. Such traditional models are simple and easy to implement; however, they do not take into account the information obtained from condition monitoring (CM), resulting in inaccurate failure predictions and low maintenance cost reduction [10].

The state-of-the-art maintenance program that overcomes the above problem of the traditional PM models which do not utilize CM information is known as the condition-based maintenance (CBM). The CBM program collects information through condition monitoring and recommends maintenance actions based on the observed data. In the CBM, the maintenance actions are performed only when the observed data indicate severe system deterioration. When applying CBM, the average cost is significantly reduced by eliminating the unnecessary maintenance operations [7, 10, 19, 33]. In the CBM, the data are usually collected in three different ways: (i) continuously, (ii) periodically with equal and constant sampling intervals, and (iii) non-periodically, i.e., using different and time-varying sampling intervals. Most of the current CBM data collection procedures belong to category (ii), i.e., the CBM data are collected at equidistant sampling epochs. Considering regular sampling, the main focus of the CBM approaches in [13, 14, 19, 31] was to determine the optimal maintenance policy that minimizes or maximizes a given objective function. However, when the sampling is costly, it is important to determine jointly the optimal times when the sampling/inspection should be performed, and how to use this information for making maintenance decisions. Recently, [18] has formulated and analyzed this joint optimization problem in a discrete setting and established the form of the optimal policy. An early work which demonstrates the benefits of non-periodic sampling of a deteriorating system with N fully observable states is [22]. Under reasonable monotonicity assumptions, the authors partially characterized the form of the optimal policy and proved that the equidistance sampling is not optimal and the time between two consecutive samples should monotonically decrease as the system deteriorates.

The same observation has been made in the quality control literature. For example, [34] is a recent contribution in the quality control area which proposed an adaptive single control chart for jointly monitoring both the process mean and variability. Based on an exponentially weighted moving average procedure, two different sampling intervals are used, i.e., a longer sampling interval when the process is in control, and a shorter sampling interval when the chart indicates the possibility of the out of control condition. Similarly, [23] proposed a sequential probability ratio test chart (SPRT) with a variable sampling interval scheme, which uses a longer sampling interval when the process is likely to be in control. On the other hand, a shorter sampling interval is adopted when there is an indication that the process shifts to an out-of-control condition. The results showed a significant improvement in the overall efficiency of the chart with two sampling intervals when compared with the SPRT chart with a single sampling interval.

Reference [29] developed a statistical adaptive process control mechanism for a system with two dependent process steps. An adaptive sampling interval Z_x control chart is designed to monitor the quality variable corresponding to the first process step, while another adaptive sampling interval Z_e control chart is used to monitor the second process step. The longer sampling interval is used when both sample statistics (Z_x and Z_e) fall inside the warning limits. The shorter sampling time is used instead, when both sample statistics fall outside the warning limits but inside the control limits. More related references can be found in [25, 26] and [4].

However, the above-mentioned non-periodic sampling interval approaches have been applied only to traditional (non-Bayesian) control charts and have not yet been considered for Bayesian control charts which were proved to be optimal in quality control by [5] for a univariate attribute process control and by [20] for a multivariate variable process control.

In this paper, a multivariate Bayesian control chart with two different sampling intervals is considered for a CBM application. We start monitoring the system with a longer sampling interval. A new sample is taken less frequently during the time when the system is in its healthier state, i.e., when the posterior probability is below the warning limit, and the system is monitored more frequently when the system degrades and the posterior probability exceeds a warning limit. Finally, if the posterior probability exceeds a maintenance limit, the full inspection is performed followed by preventive maintenance if the true alarm occurs. We compare the performance of the proposed maintenance policy with two other maintenance policies: (i) the Bayesian control policy which takes a sample at regular sampling epochs, and (ii) the well-known age-based replacement policy. The deterioration process is modeled as a 3-state continuous-time hidden-Markov process. States 0 and 1 are not observable, representing good (healthy) and warning (unhealthy) system condition, respectively. Only the failure state 2 is assumed to be observable. Upon system failure, corrective maintenance is performed. In addition, based on the proposed model and the posterior probability statistic, analytical expressions are derived for the conditional reliability and the mean residual life functions of the system. The proposed methodology is advantageous over its traditional counterparts as it dynamically updates the estimates of the reliability when a new observation becomes available.

Finally, we note that the proposed model has several potential applications. For example, there is a close relation between maintenance application and medical decision making in the health care industry, such as monitoring and treating breast cancer. The question that arises naturally at this point is when and how often to screen a patient who could develop breast cancer. Also, the cost associated with annual screening must be compared with the medical cost of cancer treatments. The main issue with the existing methodologies is that the screening is not performed at the right time or, on the other hand, too infrequently or too often. The methodology proposed in this paper can be applied to such healthcare situations where the proposed method provides decisions that minimize the total expected medical costs, while maximizing the effectiveness of screening and treatment over the lifetime of the patient.

The remainder of the paper is organized as follows. Section 2 summarizes the assumptions and provides the details of the problem and model formulations. In Section 3, we present the Bayesian control scheme for the cost minimization problem. In Section 4, a computational procedure is developed in the SMDP framework based on the policy iteration algorithm to compute the optimal sampling intervals, the optimal Bayesian control limits, and the minimum average maintenance cost. Section 5 deals with the computation of the mean residual life. In Section 6, a numerical example is provided to demonstrate the effectiveness of the proposed sampling and maintenance scheme by comparing the newly developed policy with two different policies. Finally, Section 7 concludes the paper.

2 Model formulation

Consider the deteriorating system which is characterized by a continuous-time homogeneous hidden-Markov chain $\{X_t : t \geq 0\}$ with the state space $S = \{0, 1, 2\}$ with two unobservable operational states 0 and 1 representing the good (healthy) and warning (unhealthy) state, respectively, and an observable failure state (state 2). Note that using only two operational states is sufficient in most practical applications (see, e.g., [16]). In many cases, the system deterioration is gradual, but to detect a severe system condition, it is important to define only two distinct phases. The first phase is the normal phase where the observations behave approximately as a stationary process, i.e., the process stationarity is not grossly violated. Once the system degradation has passed a certain level, the behavior of the observations changes substantially. We refer to this phase as the warning state. The instantaneous transition rates for the state process are given by

$$q_{ij} = \lim_{h \to o} \frac{P(X_h = j \mid X_0 = i)}{h} < +\infty, \qquad i \neq j$$
$$q_{ii} = -\sum_{i \neq j} q_{ij}, \tag{1}$$

where $i, j \in \{0, 1, 2\}$, and the state transition rate matrix Q can be written as follows:

$$Q = \begin{bmatrix} -(q_{01} + q_{02}) & q_{01} & q_{02} \\ 0 & -q_{12} & q_{12} \\ 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

where q_{01}, q_{02}, and q_{12} are the instantaneous transition rates of the Markov process. We assume that the state process is non-decreasing with probability 1, i.e., $q_{ij} = 0$ for all $j < i$ and the failure state (state 2) is absorbing.

The system can make transitions from state 0 to state 1 with probability p_{01}, or from state 0 to state 2 with probability p_{02}. The system is assumed to start in the healthy state (state 0), i.e., $P(X_0 = 0) = 1$. It is assumed that the sojourn times in state 0 and 1 are exponentially distributed. The transition probability matrix is obtained by explicitly solving the Kolmogorov backward differential equations ([28]) and it is given by

$$P = [P_{ij}(t)]$$
$$= \begin{bmatrix} e^{-\nu_0 t} & \frac{q_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})}{\nu_0 - \nu_1} & 1 - e^{-\nu_0 t} - \frac{q_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})}{\nu_0 - \nu_1} \\ 0 & e^{-\nu_1 t} & 1 - e^{-\nu_1 t} \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

where $\nu_0 = q_{01} + q_{02}$, $\nu_1 = q_{12}$.

Let $\xi = \inf\{t \in R^+ : X_t = 2\}$ be the observable failure time of the system. We start monitoring the system using longer sampling interval Δ_1 and switch to shorter sampling interval Δ_2 when the posterior probability exceeds a warning limit which indicates increased system degradation. The information obtained at time $n_1\Delta_1$ or $n_1\Delta_1 + n_2\Delta_2$ is denoted by $Y_{n_1\Delta_1}$, and $Y_{n_1\Delta_1+n_2\Delta_2}$, $n_1, n_2 \in \mathbb{N}$, respectively, where $\Delta_1 > \Delta_2$. While the system is in state 0 (healthy state), the observations follow $N_d(\mu_0, \Sigma_0)$, which is a d-dimensional normal distribution with mean vector μ_0 and covariance matrix Σ_0, and when the system is in state 1 (warning state), the observations follow $N_d(\mu_1, \Sigma_1)$, where $\mu_0, \mu_1, \Sigma_0, \Sigma_1$ are assumed to be known model parameters which can be estimated from the data together with the hidden Markov model parameters [15]. Given that the state is equal to i, where $i = \{0, 1\}$, the conditional density of the observation vector is given by

$$f(y|i) = \frac{1}{\sqrt{(2\pi)^d |\Sigma_i|}} \exp\Big(-1/2(y-\mu_i)^T \Sigma_i^{-1}(y-\mu_i)\Big). \quad (4)$$

After collecting an observation sample and processing the new information, one of the following three actions $A(i) \in \{1, 2, 3\}$ must be taken at each decision epoch:

1. Continue and take a new sample after Δ_1 time units.
2. Continue and take a new sample after Δ_2 time units.
3. Stop operation and perform full system inspection, followed possibly by preventive maintenance.

When the system is new or renewed, the posterior probability that it is in the warning state is equal to zero, and we take the first sample after Δ_1 time units. If the posterior probability that the system is in state 1 exceeds a warning limit W, observations are taken more frequently, i.e., the sampling interval changes to a shorter one having the length of Δ_2 time units. If the posterior probability exceeds a maintenance limit M, the system is stopped and the full inspection is performed, followed possibly by preventive maintenance. We consider the following cost structure:

1. C_S: Sampling cost incurred every time we observe the system.
2. C_I: Inspection cost incurred when full inspection is performed, which takes T_I time units.
3. C_P: Preventive maintenance cost when preventive maintenance is performed, which takes T_P time units.
4. C_F: Failure (replacement) cost incurred when corrective maintenance is performed, which takes T_F time units.

We make a reasonable assumption that $C_F \geq C_P + C_I$. Note that, if the failure cost is less than the cost of preventive maintenance, the optimal action is always to do corrective maintenance only, i.e., system replacement upon failures.

From renewal theory, for any stationary policy δ, determined by the sampling intervals (Δ_1, Δ_2), warning limit W and the maintenance limit M, the long-run expected average cost per unit time is calculated as the expected total cost (TC) incurred in one cycle divided by the expected cycle length (CL), where a cycle is completed when either preventive or corrective maintenance is carried out or when the false alarm occurs, which brings the system to as-good-as-new condition. The objective is to minimize the long-run expected average cost per unit time defined as follows:

$$\frac{E_\delta(TC)}{E_\delta(CL)}. \quad (5)$$

Let

$$T_w = \inf\{n_1\Delta_1 : \Pi_{n_1\Delta_1} \geq W\}, \quad (6)$$

represent the first time when the posterior probability exceeds the warning limit W. Also, let

$$T_M = \inf\{n_1\Delta_1 + n_2\Delta_2 : \Pi_{n_1\Delta_1+n_2\Delta_2} \geq M\}, \quad (7)$$

represent the first time when the posterior probability exceeds the maintenance limit M. Note that n_2 can be equal to zero.

The total number of samples in a replacement cycle is $\mathrm{N} = \max\{n_1 + n_2 : n_1\Delta_1 + n_2\Delta_2 \leq T_M \wedge \xi\}$. Based on the cost definition given above, the total cost per cycle is calculated as follows:

$$\begin{aligned} TC = {} & C_S.N + C_I I_{(\xi > T_M)} + C_P I_{(\xi > T_M, X_{T_M}=1)} \\ & + C_F I_{(T_M > \xi)}. \end{aligned} \quad (8)$$

The cycle length CL is given by the following equation:

$$\begin{aligned} CL = {} & \min(T_M, \xi) + T_I I_{(\xi > T_M)} \\ & + T_P I_{(\xi > T_M, X_{T_M}=1)} + T_F I_{(\xi < T_M)}. \end{aligned} \quad (9)$$

The terms on the right-hands side (RHS) of Eq. 8 represent the total sampling cost, inspection cost, preventive maintenance cost, and failure cost. In the next section, we introduce Bayesian control chart and present the formulas for the calculation of the posterior probabilities of the system being in the warning state and derive a formula for the calculation of the conditional reliability function.

3 The CBM policy based on Bayesian control chart and two sampling intervals

The Bayesian control chart monitors the posterior probability that the system is in the warning state. For the proposed CBM policy, the observations are taken using longer sampling interval Δ_1, and if the posterior probability that the system is in state 1 exceeds the warning limit W, observations are taken more frequently, i.e., the sampling interval changes to a shorter one, Δ_2. Once the posterior probability exceeds the maintenance threshold M, full system inspection is performed followed possibly by preventive maintenance. We assume that after an inspection, repair, or replacement, a new system cycle begins.

Next, the posterior probability that the system is in an unhealthy state is presented separately for longer sampling interval Δ_1, and shorter sampling interval Δ_2. The posterior probability for longer sampling interval Δ_1 is calculated as follows:

$$\Pi_{n_1\Delta_1} = P(X_{n_1\Delta_1} = 1 \mid \xi > n_1\Delta_1, Y_1, ..., Y_{n_1\Delta_1}) = \frac{\theta_1}{\theta_2 + \theta_1}, \tag{10}$$

where

$$\theta_1 = f(Y_{n_1\Delta_1} \mid 1)(P_{01}(\Delta_1)(1-\Pi_{(n_1-1)\Delta_1}) + P_{11}(\Delta_1)\Pi_{(n_1-1)\Delta_1}),$$

and

$$\theta_2 = f(Y_{n_1\Delta_1} \mid 0)P_{00}(\Delta_1)(1-\Pi_{(n_1-1)\Delta_1}),$$

where $\Pi_{n_1\Delta_1}$ is the probability that the system is in the warning state given all available information until time $n_1\Delta_1$, which represents sufficient information for decision-making. The term $P_{ij}(\Delta_1)$ represents the transition probability of the state process for the longer sampling interval Δ_1. The terms $f(Y_{n_1\Delta_1} \mid 0)$ and $f(Y_{n_1\Delta_1} \mid 1)$ are the conditional densities of the observation vectors at sampling epoch $n_1\Delta_1$ given the hidden state (see Eq. 4). Similarly, for the shorter sampling interval, the posterior probability for $n_2 > 0$ is given by

$$\Pi_{n_1\Delta_1+n_2\Delta_2} = P(X_{n_1\Delta_1+n_2\Delta_2} = 1 \mid \xi > n_1\Delta_1+n_2\Delta_2, Y_1, ..., Y_{n_1\Delta_1+n_2\Delta_2}) = \frac{\theta_1'}{\theta_2' + \theta_1'}, \tag{11}$$

where

$$\theta_1' = f(Y_{n_1\Delta_1+n_2\Delta_2} \mid 1)(P_{01}(\Delta_2)(1 - \Pi_{n_1\Delta_1+(n_2-1)\Delta_2}) + P_{11}(\Delta_2)\Pi_{n_1\Delta_1+(n_2-1)\Delta_2}),$$

and

$$\theta_2' = f(Y_{n_1\Delta_1+n_2\Delta_2} \mid 0)P_{00}(\Delta_2)(1 - \Pi_{n_1\Delta_1+(n_2-1)\Delta_2}).$$

We further need to simplify the posteriors given by Eqs. 10–11. From Eq. 4, the ratio of the conditional density of the observation vector in healthy state over the conditional density of the observation vector in warning state is a ratio of two normal densities which has the following representation

$$\frac{f(y|0)}{f(y|1)} = \frac{(2\pi)^d|\Sigma_0|^{-1/2}\exp\left[-1/2(y-\mu_0)'\Sigma_0^{-1}(y-\mu_0)\right]}{(2\pi)^d|\Sigma_1|^{-1/2}\exp\left[-1/2(y-\mu_1)'\Sigma_1^{-1}(y-\mu_1)\right]} = h\exp[1/2((Y_{n_1\Delta_1}-B)'A(Y_{n_1\Delta_1}-B) + C)], \tag{12}$$

where

$$A = \Sigma_1^{-1} - \Sigma_0^{-1},$$

$$B = \left(\Sigma_1^{-1} - \Sigma_0^{-1}\right)^{-1}\left(\Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0\right),$$

$$C = \left(\mu_1{}'\Sigma_1^{-1}\mu_1 - \mu_0'\Sigma_0^{-1}\mu_0\right) - B'\left(\Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0\right),$$

and $$h = (|\Sigma_1| \cdot |\Sigma_0|^{-1})^{1/2}. \tag{13}$$

Using Eqs. 12 and 13, the posterior probability in Eq. 10 simplifies to

$$\Pi_{n_1\Delta_1} = \frac{D^1_{\Pi_{(n_1-1)\Delta_1}}}{h\exp[1/2(V_{n_1\Delta_1}+C)]D^0_{\Pi_{(n_1-1)\Delta_1}} + D^1_{\Pi_{(n_1-1)\Delta_1}}}, \tag{14}$$

where

$$D^0_{\Pi_{(n_1-1)\Delta_1}} = P_{00}(\Delta_1)(1 - \Pi_{(n_1-1)\Delta_1}), \tag{15}$$

$$D^1_{\Pi_{(n_1-1)\Delta_1}} = P_{01}(\Delta_1)(1-\Pi_{(n_1-1)\Delta_1}) + P_{11}(\Delta_1)\Pi_{(n_1-1)\Delta_1}, \tag{16}$$

$$V_{n_1\Delta_1} = (Y_{n_1\Delta_1} - B)^T A(Y_{n_1\Delta_1} - B). \tag{17}$$

The terms h, A, B, and C are defined in Eq. 13.

The same approach is applied for the shorter sampling interval Δ_2 which is omitted here to save the space. For the development of the computational algorithm in the SMDP framework presented in Section 4, the calculation of the conditional reliability function is required. Lemma 1 below provides the formula for the reliability function.

Lemma 1 *For any $t \in R+$, the conditional reliability function is given by*

$$R(t \mid \Pi_{n_1\Delta_1+n_2\Delta_2}) = (1 - \Pi_{n_1\Delta_1+n_2\Delta_2})(1 - P_{02}(t)) + \Pi_{n_1\Delta_1+n_2\Delta_2}(1 - P_{12}(t)). \tag{18}$$

The proof of Lemma 1 is in Appendix.

In the next section, we develop the computational algorithm in the SMDP framework, which will be used to find the optimal CBM policy.

4 Computational algorithm in the SMDP framework

In this section, we develop a computational algorithm in the SMDP framework. We start monitoring the system with longer sampling interval, Δ_1. Suppose that at the sampling epoch $n_1\Delta_1$, the system has not failed, i.e., $\xi > n_1\Delta_1$, and we compute the posterior probability using Eq. 10. We partition the posterior probability interval [0, 1] into K subintervals. For a fixed large K, we define the coded value of π as k, $1 \leq k \leq K$, if the current value of the posterior probability is in the interval $\left[\frac{k-1}{K}, \frac{k}{K}\right)$. We define the set $L_1 = \{(i,1) : i < W\}$, where the first component indicates that the coded posterior probability is below the warning limit, i.e., $i < W$, and the second component indicates that the sampling interval is Δ_1. If the posterior probability exceeds the warning limit, the shorter sampling interval will be used and the SMDP is defined to be in state $(j,2)$, where j is the current coded value of the posterior probability above the warning limit, and the second component indicates that the next sampling interval is Δ_2. We define the set $L_2 = \{(j,2) : W \leq j < M\}$. We note that $W, M \in \{1, 2, ..., K\}$, $W \leq M$. We define the set $L_3 = \{(h, I) : M \leq h\}$, where I indicates that the full inspection is initiated when the system enters a state in this set.

Similarly, if the posterior probability is above the maintenance limit, and after full system inspection the system is found to be in unhealthy state, the SMDP is defined to be in state PM, where PM represents the preventive maintenance state. We define the set $L_4 = \{PM\}$.

Finally, the SMDP is defined to be in state F upon observable system failure. We define the set $L_5 = \{F\}$.

Thus the state space for the SMDP is given by $L = \{0\} \cup L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$. For the cost minimization problem, the SMDP is determined by the following quantities [28]:

1. $P_{r,k}$ = the probability that the system will be in state $k \in L$ at the next decision epoch given the current state is $r \in L$.
2. τ_r = the expected sojourn time until the next decision epoch given the current state is $r \in L$.
3. C_r = the expected cost incurred until the next decision epoch given the current state is $r \in L$.

Using quantities defined above, for a fixed warning limit W and maintenance limit M, the long-run expected average cost $g(W, M)$ can be obtained by solving the following system of linear equations:

$$u_r = C_r - g(W, M)\tau_r + \sum_{k \in L} P_{r,k} u_k, \quad \text{for} \quad r \in L$$
$$u_0 = 0. \tag{19}$$

Next, computation of the transition probabilities will be considered.

4.1 Computing the transition probabilities

The SMDP transition probabilities for the states defined above are calculated as follows:

1. For i and k below the warning limit:

$$\begin{aligned} &P_{(i,1),(k,1)} = \\ &P\left(\frac{k-1}{K} \leq \Pi_{n_1\Delta_1} < \frac{k}{K}, \xi > n_1\Delta_1 | \xi > (n_1-1)\Delta_1, i\right) \\ &= P\left(\frac{k-1}{K} \leq \Pi_{n_1\Delta_1} < \frac{k}{K} \mid \xi > n_1\Delta_{n_1}, i\right) \\ &\quad \times P(\xi > n_1\Delta_1 \mid \xi > (n_1-1)\Delta_1, i) \\ &= P\left(\frac{k-1}{K} \leq \Pi_{n_1\Delta_1} < \frac{k}{K} \mid \xi > n_1\Delta_1, i\right) R(\Delta_1 \mid i). \end{aligned} \tag{20}$$

2. Similarly, for k below and j above the warning limit:

$$\begin{aligned} &P_{(k,1),(j,2)} \\ &= P\left(\frac{j-1}{K} \leq \Pi_{n_1\Delta_1} < \frac{j}{K} | \xi > n_1\Delta_1, k\right) R(\Delta_1 | k). \end{aligned} \tag{21}$$

3. The transition probability from state $(j,2)$ to state $(g,2)$ where $j, g < M$, and the shorter sampling interval Δ_2 is used, is given by

$$\begin{aligned} &P_{(j,2),(g,2)} \\ &= P\left(\frac{g-1}{K} \leq \Pi_{n_1\Delta_1+n_2\Delta_2} < \frac{g}{K} | \xi > n_1\Delta_1 + n_2\Delta_2, j\right) \\ &\quad \times R(\Delta_2 \mid j). \end{aligned} \tag{22}$$

   Note that the sampling interval will be Δ_2 after the switching point even if the posterior probability falls below the warning limit.
4. When the posterior probability exceeds the maintenance limit M, full system inspection is performed and the transition probability is given by

$$\begin{aligned} &P_{(i,1),(h,I)} \\ &= P\left(\frac{h-1}{K} \leq \Pi_{n_1\Delta_1} < \frac{h}{K} |\xi > n_1\Delta_1, i\right) R(\Delta_1 \mid i), \end{aligned} \tag{23}$$

   and

$$\begin{aligned} &P_{(j,2),(h,I)} \\ &= P\left(\frac{h-1}{K} \leq \Pi_{n_1\Delta_1+n_2\Delta_2} < \frac{h}{K} |\xi > n_1\Delta_1 + n_2\Delta_2, j\right) \\ &\quad \times R\big(\Delta_2 \mid j\big). \end{aligned} \tag{24}$$

5. The system can make transition from state (h, I) either to state 0 which indicates false alarm and the system being in healthy state, or to state PM, which indicates

true alarm and the PM is initiated. The corresponding transition probabilities are given by

$$P_{(h,I),0} = 1 - \frac{h-0.5}{K},$$
$$P_{(h,I),PM} = \frac{h-0.5}{K}. \quad (25)$$

6. When the system is in the observable failure state F, mandatory corrective maintenance is performed to bring the system to healthy state. Therefore, the remaining transition probabilities are

$$\begin{aligned} P_{(i,1),F} &= 1 - R(\Delta_1 \mid \Pi_{n_1\Delta_1}) \\ &= (1 - \Pi_{n_1\Delta_1})P_{02}(t) + \Pi_{n_1\Delta_1} P_{12}(t), \\ P_{(j,2),F} &= 1 - R(\Delta_2 \mid \Pi_{n_1\Delta_1+n_2\Delta_2}) \\ &= (1 - \Pi_{n_1\Delta_1+n_2\Delta_2})P_{02}(t) \\ &\quad + \Pi_{n_1\Delta_1+n_2\Delta_2} P_{12}(t), \\ P_{PM,0}, &= 1, \\ P_{F,0} &= 1. \end{aligned} \quad (26)$$

The second term on the RHS of Eq. 20 is calculated using the conditional reliability given by Eq. 18, while the first term on the RHS of Eq. 20 is given by Eq. 27 as follows:

$$P\left(\frac{k-1}{K} \le \Pi_{n_1\Delta_1} < \frac{k}{K} \mid \xi > n_1\Delta_1, i\right) =$$
$$P(a < V_{n_1\Delta_1} \le b | X_{n_1\Delta_1=0}) \left[\frac{D^0_{\Pi_{(n_1-1)\Delta_1}}}{D^1_{\Pi_{(n_1-1)\Delta_1}} + D^0_{\Pi_{(n_1-1)\Delta_1}}}\right]$$
$$+ P(a < V_{n_1\Delta_1} \le b | X_{n_1\Delta_1=1}) \left[\frac{D^1_{\Pi_{(n_1-1)\Delta_1}}}{D^1_{\Pi_{(n_1-1)\Delta_1}} + D^0_{\Pi_{(n_1-1)\Delta_1}}}\right], \quad (27)$$

where

$$a = 2\ln\left[\frac{(1-\frac{k}{K})D^1_{\Pi_{(n_1-1)\Delta_1}}}{\frac{k}{K}D^0_{\Pi_{(n_1-1)\Delta_1}}} h\right] - C \text{ and}$$
$$b = 2\ln\left[\frac{(1-\frac{k-1}{K})D^1_{\Pi_{(n_1-1)\Delta_1}}}{\frac{k-1}{K}D^0_{\Pi_{(n_1-1)\Delta_1}}} h\right] - C.$$

Imhof [9] presented a method for the calculation of the cumulative distribution function of $V_{n_1\Delta_1} \mid X_{n_1\Delta_1}$. The author showed that an indefinite quadratic form in normal vectors can be expressed as a linear combination of independent noncentral chi-square variables with positive and negative coefficients. Equation 28 can be simplified as

$$P\left(\frac{k-1}{K} \le \Pi_{n_1\Delta_1} < \frac{k}{K} \mid \xi > n_1\Delta_1, Y_1, ..., Y_{(n_1-1)\Delta_1}, i\right)$$
$$= T_0\left(\frac{k-1}{K}, \frac{k}{K} | \xi \ge n_1\Delta_1, i\right)\left[\frac{D^0_{\Pi_{(n_1-1)\Delta_1}}}{D^1_{\Pi_{(n_1-1)\Delta_1}} + D^0_{\Pi_{(n_1-1)\Delta_1}}}\right]$$
$$+T_1\left(\frac{k-1}{K}, \frac{k}{K} | \xi \ge n_1\Delta_1, i\right)\left[\frac{D^1_{\Pi_{(n_1-1)\Delta_1}}}{D^1_{\Pi_{(n_1-1)\Delta_1}} + D^0_{\Pi_{(n_1-1)\Delta_1}}}\right], \quad (28)$$

where

$$T_0\left(\frac{k-1}{K}, \frac{k}{K} \mid \xi \ge n_1\Delta_1, i\right) = F_0(b) - F_0(a),$$
$$T_1\left(\frac{k-1}{K}, \frac{k}{K} \mid \xi \ge n_1\Delta_1, i\right) = F_1(b) - F_1(a). \quad (29)$$

$F_i(.)$ is the cumulative distribution function of $V_{n_1\Delta_1}$ given the state is i, for $i = 0, 1$.

4.2 Computing the expected costs and the mean sojourn times

The expected cost incurred until the next sampling epoch for state $(i, 1)$ where $(i, 1) < W$ and the longer sampling interval Δ_1 is used, is given by

$$\begin{aligned} C_{(i,1)} &= \mathrm{E}(Cost \mid (i,1)) \\ &= \mathrm{E}(Cost \mid \xi \le n_1\Delta_1) \times P(\xi \le n_1\Delta_1) \\ &+ \mathrm{E}(Cost \mid \xi > n_1\Delta_1) \times P(\xi > n_1\Delta_1) \\ &= C_S \times R(\Delta_1 \mid \Pi_{n_1\Delta_1}). \end{aligned} \quad (30)$$

The expected cost incurred until the next sampling epoch for state $(j, 2)$ where $W \le (j, 2) < M$ and shorter sampling interval Δ_2 is used, is given by

$$\begin{aligned} C_{(j,2)} &= \mathrm{E}(Cost \mid (j,2)) \\ &= \mathrm{E}(Cost \mid \xi \le n_1\Delta_1 + n_2\Delta_2) \times P(\xi \le n_1\Delta_1 + n_2\Delta_2) \\ &+ \mathrm{E}(Cost \mid \xi > n_1\Delta_1 + n_2\Delta_2) \times P(\xi > n_1\Delta_1 + n_2\Delta_2) \\ &= C_S \times R(\Delta_2 \mid \Pi_{n_1\Delta_1+n_2\Delta_2}). \end{aligned} \quad (31)$$

The expected cost incurred until the next sampling epoch for state h such that $h \ge M$, when the full system inspection is performed, is given by

$$C_{(h,I)} = \mathrm{E}(Cost \mid (h, I)) = C_I. \quad (32)$$

The expected cost incurred until the next sampling epoch for state PM is given by

$$C_{PM} = \mathrm{E}(Cost \mid PM) = C_P, \quad (33)$$

and finally, the expected cost in the failure state F is C_F. Using the conditional reliability function, the mean sojourn time for state $(i, 1)$ is derived as follows:

$$\tau_{(i,1)} = \int_0^{\Delta_1} R(t \mid \Pi_{n_1\Delta_1})dt = (1-\Pi_{n_1\Delta_1}) \times \left[\frac{1-e^{-\nu_0\Delta_1}}{\nu_0} + \frac{q_{01}}{\nu_0-\nu_1} \times \left(\frac{\nu_0(1-e^{-\nu_1\Delta_1})-\nu_1(1-e^{-\nu_0\Delta_1})}{\nu_0\nu_1}\right)\right] + \Pi_{n_1\Delta_1}\frac{1-e^{-\nu_1\Delta_1}}{\nu_1}. \tag{34}$$

The mean sojourn time for state $(j, 2)$ is obtained as follows:

$$\tau_{(j,2)} = \int_0^{\Delta_2} R(t \mid \Pi_{n_1\Delta_1+n_2\Delta_2})dt = (1-\Pi_{n_1\Delta_1+n_2\Delta_2}) \times \left[\frac{1-e^{-\nu_0\Delta_2}}{\nu_0} + \frac{q_{01}}{\nu_0-\nu_1} \times \left(\frac{\nu_0(1-e^{-\nu_1\Delta_2})-\nu_1(1-e^{-\nu_0\Delta_2})}{\nu_1\nu_0}\right)\right] + \Pi_{n_1\Delta_1+n_2\Delta_2}\frac{1-e^{-\nu_1\Delta_2}}{\nu_1}. \tag{35}$$

The mean sojourn time when the posterior probability is above the maintenance limit, i.e., $h \geq M$, and the full system inspection is performed, is given by

$$\tau_{(h,I)} = T_I. \tag{36}$$

The mean sojourn time when the PM action is performed is

$$\tau_{PM} = T_P. \tag{37}$$

Finally, the mean sojourn time for the failure state is given by

$$\tau_F = T_F. \tag{38}$$

This completes our proposed computational algorithm in the SMDP framework. Next, we derive an analytical formula for residual life prediction.

5 Residual life prediction

In this section, we derive the explicit formula for the MRL function in terms of the posterior probability statistic, which is given by the following lemma.

Lemma 2 *For any $t \in R+$, the mean residual life is given by*

$$MRL_{n_1\Delta_1+n_2\Delta_2} = \frac{\Pi_{n_1\Delta_1+n_2\Delta_2}(q_{02}-q_{12})+q_{01}+q_{12}}{q_{12}(q_{01}+q_{02})}. \tag{39}$$

Proof

$$\begin{aligned} &\mathrm{MRL}_{n_1\Delta_1+n_2\Delta_2} \\ &= \mathrm{E}\{\xi - n_1\Delta_1 - n_2\Delta_2 \mid \xi > n_1\Delta_1+n_2\Delta_2, \Pi_{n_1\Delta_1+n_2\Delta_2}\} \\ &= \int_0^{\infty} R(t \mid \Pi_{n_1\Delta_1+n_2\Delta_2})dt \\ &= \frac{\Pi_{n_1\Delta_1+n_2\Delta_2}(q_{02}-q_{12})+q_{01}+q_{12}}{q_{12}(q_{01}+q_{02})}, \end{aligned} \tag{40}$$

where $\mathrm{E}\{\cdot\}$ denotes the expectation operator, and the conditional reliability function is given by Eq. 18. □

6 Experimental results

In this section, we illustrate the proposed computational procedure with a numerical example. We assume that the system deterioration follows a continuous-time homogeneous hidden-Markov chain $\{X_t : t \geq 0\}$ with state space $\{0, 1, 2\}$. States 0 and 1 are unobservable, representing the healthy and unhealthy operational states, respectively, and state 2 corresponds to the observable failure state. The sojourn time in state 0 has an exponential distribution with parameter $\nu_0 = q_{01}+q_{02}$, and the sojourn time in state 1 has an exponential distribution with parameter $\nu_1 = q_{12}$. The state parameters are given by $q_{01} = 0.026$, $q_{02} = 0.004$, and $q_{12} = 0.3$. The residual observation process is obtained through CM and it is assumed that each observation vector follows $N_2(\mu_0, \Sigma_0)$ when the system is in the healthy state and $N_2(\mu_1, \Sigma_1)$ when the system is in the unhealthy state with the following parameters:

$$\mu_0 = \begin{pmatrix} 0.21 \\ -0.01 \end{pmatrix} \qquad \Sigma_0 = \begin{pmatrix} 1.5 & 0.61 \\ 0.61 & 1.9 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 0.75 \\ 0.54 \end{pmatrix} \qquad \Sigma_1 = \begin{pmatrix} 1.81 & 1.97 \\ 1.97 & 2.22 \end{pmatrix}.$$

Maintenance time parameters are given by $T_I = 3$, $T_P = 4$, $T_F = 10$, and maintenance cost parameters are $C_S = 5$, $C_I = 10$, $C_P = 500$, $C_F = 3500$. We compute the optimal sampling intervals (Δ_1 and Δ_2) and the control limits W and M that minimize the long-run expected average cost per unit time. Number $K = 30$ defining 30 subintervals of $[0, 1]$ is used to obtain the optimal results which are shown

Table 1 Results for the optimal Bayesian control policy with two sampling intervals

Optimal warning limit (W)	Optimal maintenance limit (M)	Optimal long sampling interval (Δ_1)	Optimal short sampling interval (Δ_2)	Average cost
0.133	0.3666	3	2	55.8432

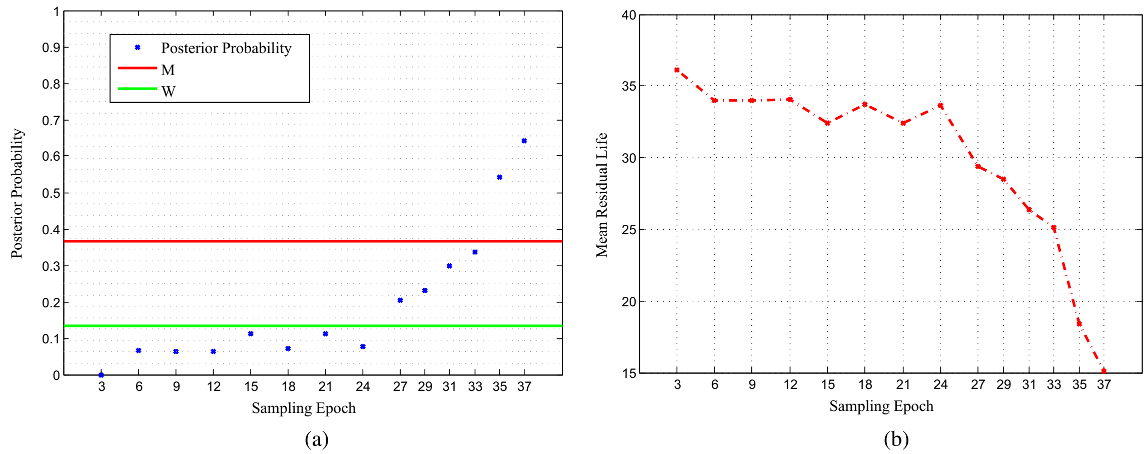

Fig. 1 **a** The illustration of the proposed Bayesian control policy. **b** The mean residual life of the system

in Table 1. The policy iteration algorithm gives the following optimum values: the longer sampling interval $\Delta_1 = 3$, the shorter sampling interval $\Delta_2 = 2$, the warning limit $W = 0.133$, the inspection limit $M = 0.3666$, and the minimum expected average cost is equal to 55.8432. The algorithm took 22.35 s for each run on an Intel Core(TM) i5 CPU with 2.27 GHz. Figure 1a shows the posterior probability plot for the simulated data together with the warning and maintenance limits. Figure 1b shows the mean residual life of the system. It is observed that when the posterior probability exceeds the maintenance limit, the system's residual life starts to decrease significantly. It is assumed that the system starts working from healthy state, i.e., $\Pi_0 = 0$. So the corresponding MRL is equal to:

$$MRL_0 = \frac{q_{01} + q_{12}}{q_{12}(q_{01} + q_{02})} = \frac{1}{\nu_0} + \frac{p_{01}}{\nu_1}, \tag{41}$$

where $p_{01} = \frac{q_{01}}{\nu_0}$. It has also been shown (see [15]) that:

$$f_\xi(t) = p_{01}\frac{\nu_0\nu_1}{\nu_0 - \nu_1}(e^{-\nu_1 t} - e^{-\nu_0 t}) + p_{02}\nu_0 e^{-\nu_0 t}. \tag{42}$$

Using Eq. 42, the expected time to failure of the system is given by

$$\mathrm{E}(\xi) = \int_0^\infty t f_\xi(t) = \frac{1}{\nu_0} + \frac{p_{01}}{\nu_1}, \tag{43}$$

which agrees with Eq. 41. The numerical value of MRL_0 is equal to 36.22.

Table 2 Results for the optimal Bayesian control policy with one sampling interval

Optimal maintenance limit	Optimal sampling interval	Average cost
0.3	3	62.8623

6.1 Comparison with other policies

In this subsection, we compare the performance of our proposed maintenance policy with other policies: (i) the Bayesian control policy with one sampling interval and (ii) age-based policy.

First, we compare the proposed maintenance policy with a policy using a single sampling interval and a single control limit. All the parameters remain the same as in the previous example. As shown in Table 2, in this case, the minimum long-run expected average cost is equal to 62.8623, which is a significant increase. The optimal sampling interval for this policy is equal to 3.

Next, we compare the proposed optimal Bayesian control policy with the well-known age-based replacement policy which does not take condition monitoring information into account. Consider an age-based policy that initiates preventive maintenance at time $n\Delta$. From renewal theory, the expected average cost per unit time for this policy is given by

$$C(n) = \frac{C_F F(n\Delta) + C_P \bar{F}(n\Delta)}{\int_0^{n\Delta} \bar{F}(s)ds}, \tag{44}$$

where $F(t) = p_{02}(t)$ is the distribution function of ξ and $\bar{F}(t) = 1 - F(t)$. Reference [1] proved that under the age-based replacement policy, the optimal preventive replacement time γ satisfies

$$h(\gamma)\int_0^\gamma R(t \mid 0)dt - (1 - R(\gamma \mid 0)) = \frac{C_P}{C_F - C_P}, \tag{45}$$

where the conditional reliability function is given by

$$R(t \mid 0) = 1 - p_{02}(t) = e^{-\nu_0 t} + \frac{q_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})}{\nu_0 - \nu_1}, \tag{46}$$

and the hazard rate function is given by

$$h(t) = \frac{1}{R(t \mid 0)} \cdot \left(-\frac{dR(t \mid 0)}{dt}\right) = \frac{(\nu_0 - \nu_1)\nu_0 e^{-\nu_0 t} - q_{01}(\nu_0 e^{-\nu_0 t} - \nu_1 e^{-\nu_1 t})}{(\nu_0 - \nu_1)e^{-\nu_0 t} + q_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})}. \quad (47)$$

Based on Eqs. 44–47 and considering the same parameters, the minimum long-run expected average cost using the age-based policy increased to 89.4692, which is considerably higher than the optimal average cost obtained by the proposed maintenance policy with two sampling intervals.

7 Conclusions

In this paper, we have proposed a Bayesian CBM policy with two sampling intervals for a partially observable deteriorating system subject to random failure. The deterioration process is modeled as a 3-state continuous-time hidden-Markov process. States 0 and 1 are not observable, representing good and warning system condition, respectively. Only the failure state 2 is assumed to be observable. Upon system failure, corrective maintenance is performed. The system is subject to CM at discrete time epochs, starting with a longer sampling interval. If at a sampling epoch the posterior probability that the system is in state 1 exceeds a warning limit, observations are taken more frequently, i.e., the sampling interval changes to a shorter one. If the posterior probability exceeds a maintenance limit, the full system inspection is performed followed by preventive maintenance, if the system is found to be in the warning state. The optimal control problem has been formulated and solved in the SMDP framework. The proposed optimal sampling and maintenance policy which is easy to implement has been compared with Bayesian control policy using periodic sampling as well as with the traditional age-based policy, showing considerably lower average maintenance cost. We have considered a hidden Markov model to describe the deterioration process which has been successfully applied to real data (see, e.g, [16]). Further improvement can be expected by considering a hidden semi-Markov model, which is a suitable topic for future research.

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Appendix: Proof of Lemma 1

Proof The reliability function can be obtained by conditioning as follows:

$$\begin{aligned}
&R(t \mid \Pi_{n_1\Delta_1+n_2\Delta_2}) \\
&= P(\xi > n_1\Delta_1+n_2\Delta_2+t \,|\, \xi > n_1\Delta_1+n_2\Delta_2, Y_1, ..., \\
&\qquad Y_{n_1\Delta_1+n_2\Delta_2}, \Pi_{n_1\Delta_1+n_2\Delta_2}) \\
&= P(X_{n_1\Delta_1+n_2\Delta_2+t} \neq 2 \,|\, \xi > n_1\Delta_1+n_2\Delta_2, Y_1, ..., \\
&\qquad Y_{n_1\Delta_1+n_2\Delta_2}, \Pi_{n_1\Delta_1+n_2\Delta_2}) \\
&= P(X_{n_1\Delta_1+n_2\Delta_2+t} \neq 2 \mid X_{n_1\Delta_1+n_2\Delta_2} = 0, \\
&\qquad \xi > n_1\Delta_1+n_2\Delta_2, Y_1, ..., Y_{n_1\Delta_1+n_2\Delta_2}, \Pi_{n_1\Delta_1+n_2\Delta_2}) \\
&\quad \times P(X_{n_1\Delta_1+n_2\Delta_2} = 0 \mid Y_1, ..., Y_{n_1\Delta_1+n_2\Delta_2}, \Pi_{n_1\Delta_1+n_2\Delta_2}) \\
&\quad + P(X_{n\Delta_1+n_2\Delta_2+t} \neq 2 \mid X_{n_1\Delta_1+n_2\Delta_2} = 1, \\
&\qquad \xi > n_1\Delta_1 + n_2\Delta_2, ..., Y_{n_1\Delta_1+n_2\Delta_2}, \Pi_{n_1\Delta_1+n_2\Delta_2}) \\
&\quad \times P(X_{n_1\Delta_1+n_2\Delta_2} = 1 \,|\, Y_1, ..., Y_{n_1\Delta_1+n_2\Delta_2}, \Pi_{n_1\Delta_1+n_2\Delta_2}) \\
&= (1-\Pi_{n_1\Delta_1+n_2\Delta_2})(1-P_{02}(t)) + \Pi_{n_1\Delta_1+n_2\Delta_2}(1-P_{12}(t)).
\end{aligned} \quad (48)$$

□

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