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ORIGINAL ARTICLE

Optimization of sample sizes, sampling intervals, and control limits of the $\overline{X}\&S$ chart system monitoring independent quality characteristics

M. Shamsuzzaman · Z. Wu

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Abstract This article designs an $\overline{X}\&S$ control chart system for monitoring process shifts in mean and standard deviation in a multistage manufacturing system. Each of the $\overline{X}\&S$ chart combination in a chart system monitors one of the critical quality characteristics (dimensions) of a product. The design algorithm optimally allocates the detection power of the chart system among different stages as well as between the $\overline{X}$ chart and S chart within each stage based on the values of certain parameters (e.g., process capability, magnitude of the process shift) that would affect the performance of the chart system. Meanwhile, the sample sizes, sampling intervals, and control limits of the $\overline{X}\&S$ charts are also optimized. The optimization design is carried out using false alarm rate and inspection capacity as constraints. Consequently, the performance of the system as a whole is improved without requiring additional cost and effort for inspection. The results of the comparative studies show that from an overall viewpoint, the optimal $\overline{X}\&S$ chart system is more effective (in terms of reduction in detection time) than the traditional 3-sigma $\overline{X}\&S$ chart system as well as a suboptimal $\overline{X}\&S$ chart system by about 53 and 26 %, respectively. Some useful guidelines have been brought forth to aid the users to adjust the sample sizes, sampling intervals, and control limits of the charts in a system.

Keywords Quality control · Statistical process control · Control chart system · Mean and standard deviation shift

M. Shamsuzzaman (✉)
Industrial Engineering and Engineering Management Department, College of Engineering, University of Sharjah, PO Box 27272, Sharjah, United Arab Emirates
e-mail: mshamsuzzaman@sharjah.ac.ae

Z. Wu
School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore 639798, Singapore
e-mail: mzwu@ntu.edu.sg

1 Introduction

The Shewhart $\overline{X}\&S$ (or $\overline{X}\&R$) control charts are widely used in industries for monitoring process shifts. While the $\overline{X}$ chart monitors the mean shifts δ_μ, the S (or R) chart controls the standard deviation shifts δ_σ. Usually, the sample size n is chosen as 4, 5, or 6. However, when monitoring critical processes or detecting small shifts, the value of n needs to be moderately large. For such cases, the R chart loses statistical efficiency and the S chart is preferable. The sampling interval h is usually decided according to the concept of rational subgroups. Recently, since on-line measurement and distributed computing systems are becoming the norm in statistical process control (SPC) applications [1], the sampling interval may be much smaller than the working shift and the notation of rational subgrouping is not enforced. The 3-sigma control limits are commonly adopted, but they can be adjusted to satisfy different requirements on false alarm rate. The traditional Shewhart charts are easy to design and operate. Unfortunately, their performance is unsatisfactory from either the statistical or economic viewpoint, especially for small or moderate process shifts. Many modifications to the Shewhart $\overline{X}\&S$ charts have been reported [2, 3]. However, most of the previously proposed models for designing $\overline{X}\&S$ charts only consider different single processes and develop algorithms for the design of individual $\overline{X}\&S$ control charts.

The fabrication of a product usually includes many process stages. The integration of all these stages results in a *multi-stage manufacturing system*. For example, in the manufacturing of a mechanical part, each stage corresponds to the machining of a dimension (see Fig. 1). Some of the dimensions are critical to the overall quality of the product and the corresponding processes have to be monitored by the control charts. A $\overline{X}\&S$ *control chart system* is the combination of all the $\overline{X}\&S$ charts that are used to monitor the means and standard deviations of the critical quality characteristics in a

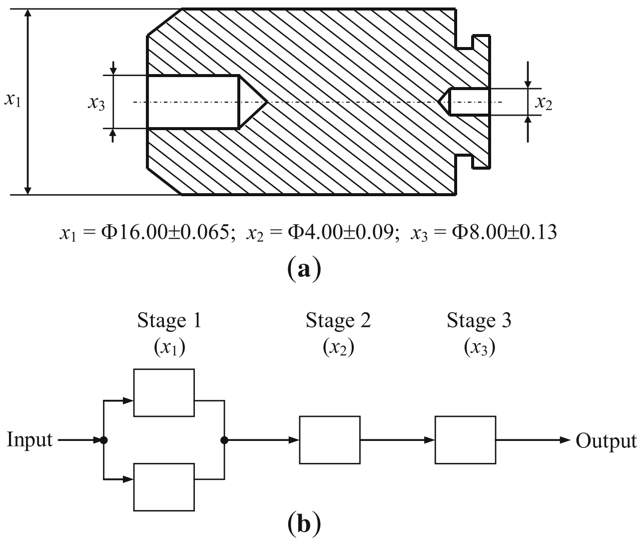

Fig. 1 **a** Dimensions of the part. **b** Process stages

manufacturing system. Due to the differences in production rates and other factors, some stages may have more than one parallel streams or machines. In some applications, a single chart or a group of charts is used to monitor the outputs from all the streams of a stage. However, in this article, a separate set of $\overline{X}\&S$ charts will be applied to the output of each individual stream. This scenario helps to detect and diagnose the out-of-control stream [4]. Usually, the parallel streams in a stage have the same mean, standard deviation, and target [5–7], and, therefore, identical control charts are applied to each of them.

Even though many manufacturing systems consist of a series of process stages, the literature on the design of control chart systems that monitor multistage manufacturing systems is still limited. Many authors developed the group control chart for monitoring the output from multiple streams of a single stage [5–7]. Williams and Peters [8] presented an np-control scheme for a multistage production process. Hawkins [9] discussed multivariate quality control based on regression-adjusted variables. Several papers have been published studying the multistage processes and the diagnosis problems [10–14]. However, none of these approaches considered the multistage manufacturing processes as a whole and designed the charting parameters in an integrative and optimal manner.

Since the processes in different stages in a manufacturing system have different precision and other characteristics (e.g., process capability, magnitudes of shifts), therefore, if all the charts in a chart system are designed in an integrative and optimal manner, the performance of the chart system as a whole will be improved significantly. The designs of $\overline{X}$ control chart systems for monitoring mean shift δ_μ in multistage manufacturing systems have been reported in the literature [15–18]. Some recent articles proposed control chart systems for time-between-events monitoring in a multistage manufacturing system [19, 20]. Wu and Shamsuzzaman [21] proposed algorithm for the design of $\overline{X}\&S$ control chart system for monitoring independent quality characteristics in a multistage manufacturing system, where the control limits of each chart in the chart system were optimized, but the sample sizes and the sampling intervals of the charts were determined as for the traditional system. The traditional design method for $\overline{X}\&S$ control charts selects the sample size just for the administrative convenience or based on the general production rate [22]. Consequently, it is unlikely that they will achieve excellent performance.

This article develops an $\overline{X}\&S$ control chart system for monitoring the mean and standard deviation of the processes in different stages in a manufacturing system. It designs control limits, as well as sample sizes and sampling intervals, of all the $\overline{X}$ charts and S charts in a system in an integrative and optimal manner. The in-control and out-of-control system conditions are related to the process capability index C_{pk}. The objective is to speed up the signalling of out-of-control cases, on condition that neither the false alarm rate nor the required inspection capacity is greater than the specified value.

2 Design of the $\overline{X}\&S$ control chart system

2.1 Assumptions

1. The quality characteristic x_i (e.g., the dimension of a mechanical part) in each process is normally distributed as $\sim N(\mu_0, \sigma_0^2)$. Moreover, μ_0 is equal to the central value between the lower and upper specification limits LSL and USL.
2. The g_i parallel streams (or machines) in the ith stage of a manufacturing system have the same mean, standard deviation, and target and use the identical control charts (with same sample size, sampling interval, and control limits). This assumption is adopted by many researchers studying the multiple stream systems [5–7].
3. The time required to inspect a unit in a stage is substantially smaller than the sampling interval of that stage.

2.2 Input parameters

The design of an $\overline{X}\&S$ control chart system requires the following input parameters:

s	The number of process stages in the control chart system
g_i	The number of streams or machines in the ith stage
LSL_i	The lower specification limit in the ith stage
USL_i	The upper specification limit in the ith stage
$\mu_{0,i}$	The in-control process mean in the ith stage

$\sigma_{0,i}$ The in-control process standard deviation in the ith stage
t_i The time required to inspect a unit in the ith stage
τ The minimum allowable in-control average time to signal (ATS_0) of the chart system
R The available inspection capacity of the manufacturing system
$c_{\min,i}$ The out-of-control (or critical) value of the process capability index in the ith stage
q_i The probability of occurrence of the out-of-control cases in the ith stage

Most of the above parameters can be obtained from manufacturing records or can be estimated. The numbers s and g_i can be decided from the corresponding process planning. The values of LSL_i and USL_i can be found from the engineering drawings of the mechanical part. The distribution parameters $\mu_{0,i}$ and $\sigma_{0,i}$ are usually estimated from the data observed in the pilot runs or the process capability studies. The value of t_i can be easily estimated from historical data or the results of a field test. The specification τ is decided based on the trade-off between the false alarm rate and the detection power. It is desirable to make the actual or resultant ATS_0 equal to τ so that the requirement on false alarm rate is met and the potential power of the control chart system is fully utilized in the meantime. The available inspection capacity R indicates the time that the manufacturing system allows to spend on SPC activities. It depends on the total time that the operators are engaged with the quality inspection. The value of R may be a fraction if an operator spends only part of his time in SPC and larger than one if more than one operator is deployed to the quality inspection in a system. In order to make full use of the available capacity, it is desired to have the actual or required inspection capacity r equal to the specified value R.

In this article, the process status is related to a single capability index C_{pk} instead of two parameters mean shift δ_μ and standard deviation shift δ_σ [23].

$$C_{pk} = \min\left(\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right) \quad (1)$$
$$\mu = \mu_0 + \delta_\mu\sigma_0, \quad \sigma = \delta_\sigma\sigma_0$$

When the process is in-control, $\delta_\mu=0$ (i.e., $\mu=\mu_0$) and $\delta_\sigma=1$ (i.e., $\sigma=\sigma_0$). The most attractive feature of the index C_{pk} is that this single index can oversee both the centering and spread of a quality characteristic with reference to specification limits. In addition, the index C_{pk} contains more managerial meaning than the process shifts δ_μ and δ_σ. Especially, the quality assurance (QA) engineers, as well as the operators and managers at different levels, are more familiar with the process capability index C_{pk} and feel more comfort to specify and manipulate it [23]. When process shifts (δ_μ and/or δ_σ) take place in a process due to some assignable causes, C_{pk} will decrease. When δ_μ and/or δ_σ increase to a critical level, C_{pk} will decrease to the critical value $c_{\min}$. Consequently, the product quality will degrade significantly and the number of nonconforming parts per million (PPM) value of the product will increase to an intolerable level. Thus, the process is considered out of control and a signal should be produced quickly when $C_{pk}=c_{\min}$. Usually, $c_{\min}$ is decided in accordance with some considerations, such as quality requirement, manufacturing cost, and whether the process is an existing one or whether the product is critical to safety [4]. A default value of one may be used for $c_{\min}$. It results in 2,700 defects in one million products.

Finally, the probability q_i that an out-of-control case takes place in the ith stage may have to be estimated from the historical data of the out-of-control cases. For example, if in a manufacturing system, 10 out of 25 out-of-control cases occurred in the first stage, then $q_1\approx 10/25=0.4$. If such historical data are not available, it may be reasonable to estimate q_i by the following formula [15]:

$$q_i = g_i \Big/ \sum_{j=1}^{s} g_j \quad (2)$$

This means the probability that the out-of-control case happens in the ith stage is proportional to the number of parallel streams in this stage provided that the probability of out-of-control for each stream is the same. In Eq. (2), the denominator denotes the total number of streams in the system, and the numerator denotes the number of streams in stage i. Therefore, for any given number of streams in the system, the probability q_i in stage i will be increased as the number of stream g_i is increased.

2.3 Optimization model

Based on the above specifications, the design of the $\overline{X}\&S$ control chart system can be conducted by using the following optimization model:

$$\text{Minimize}: \quad T \quad (3)$$

$$\text{Subject to}: \quad ATS_0 \geq t \quad (4)$$

$$r \leq R \quad (5)$$

with respect to n_i, h_i, $LCL_{\bar{x},i}$, $UCL_{\bar{x},i}$, and $UCL_{s,i}$ (i=1, 2,…, s)where the objective function T is called the *average informative time* (the average time required to detect an out-of-control condition). The calculation and interpretation of T will be discussed shortly. The parameters $LCL_{\bar{x},i}$ and $UCL_{\bar{x},i}$ are the lower and upper control limits of the $\overline{X}$ chart monitoring mean shift $\delta_{\mu,i}$ in the ith stage, whereas $UCL_{s,i}$ is the upper control limit of the S chart monitoring increasing standard deviation shift $\delta_{\sigma,i}$ in the ith stage. Since control charts are most often used to detect a deterioration in process quality [24], therefore, only the increasing standard deviation shift $\delta_{\sigma,i}$ is handled in this article and the lower control limit $\mathrm{LCL}_{s,i}$ of the S chart is fixed as zero. It will enhance the power of the S chart for detecting the increasing hazardous $\delta_{\sigma,i}$ shifts. The in-control ATS_0 of the chart system in constraint (5) is calculated by Eq. (13) in Appendix 1, and the actual or required inspection capacity r of the system in constraint (6) is calculated by Eq. (15) in Appendix 2. The average informative time T can be calculated by [21]:

$$T = \sum_{i=1}^{s} \left(\frac{1}{p_i}\cdot h_i\right)\cdot q_i \tag{6}$$

where p_i is called the *informative power* of stage i, that is, the power generated by the $\overline{X}\&S$ charts in a stream (or machine) in stage i when out-of-control case happens in this stream (or more specifically, when C_{pk} in this stream decreases to $c_{\min,i}$). The ratio $1/p_i$ is actually the average run length (ARL) and h_i is the sampling interval in stage i. The informative power p_i is calculated by Eq. (31 in Appendix 3, and the probability q_i that an out-of-control case takes place in the ith stage is calculated by Eq. (15). The average informative time T is equal to the out-of-control average time to signal (ATS) of the system if the informative power p_i is replaced by the overall power P_i. Like the out-of-control ATS, the smaller is T, the more effective is the chart system in detecting out-of-control cases. It is noted that when an out-of-control case occurs in a particular stream (called the *out-of-control stream*) in stage i, the overall power P_i of the chart system is the union of p_i and $p_{i,\mathrm{depend}}$.

$$P_i = p_i \cup p_{i,\mathrm{depend}} \tag{7}$$

where p_i is the power generated by the out-of-control stream where mean shift $\delta_{\mu,i}$ and/or standard deviation shift $\delta_{\sigma,i}$ takes place. The power $p_{i,\mathrm{depend}}$ is the union of the power generated by the $\overline{X}\&S$ charts in other streams and/or other stages rather than the out-of-control stream and mainly results from the dependency between the critical quality characteristics x_i. However, in many processes, especially the mechanical parts machining processes, the critical characteristics (dimensions of the mechanical part) are often independent of each other (e.g., Fig. 1) and the power $p_{i,\mathrm{depend}}$ is not a major concern; the reason is that the tolerance chains of a machine part are independent and each of them usually contains only one functional dimension [17]. In case more than one functional dimension resides in one tolerance chain and the outputs of preceding stage have an effect on the following stages, the functional dimension may be interdependent. The consideration of both the induced mean and standard deviation shifts ($\delta_{\mu,i}$ and $\delta_{\sigma,i}$) pertaining to the interdependency between the quality characteristics in different process stages makes the situation more complicated and needs to be discussed separately. The optimization model proposed in this article minimizes the informative time T (considering only p_i) rather than the out-of-control ATS (considering both p_i and $p_{i,\mathrm{depend}}$).

2.4 Optimization search

The optimization search is quite difficult due to the large number of variables n_i, h_i, $\mathrm{LCL}_{\overline{x},i}$, $\mathrm{UCL}_{\overline{x},i}$, and $\mathrm{UCL}_{s,i}$ (i= 1, 2, …, s). Some of the variables are integers and others are fractions. The general strategy is to search from point to point until the improvement is negligible [25]. Like most of the optimization strategies employed in SPC, the algorithm proposed in this article makes no attempt to secure a global optimal solution. Instead, it focuses on deriving a convenient and systematic procedure for identifying a satisfactory and workable solution that could be adopted in practice [26]. The optimal values of n_i, h_i, $\mathrm{LCL}_{\overline{x},i}$, $\mathrm{UCL}_{\overline{x},i}$, and $\mathrm{UCL}_{s,i}$ are sought in a three-level optimization search. The high-level search optimizes n_i and h_i. The mid-level search distributes the detection power (or type I error α) of the chart system among the s stages, and the low-level search allocates the power between the $\overline{X}$ chart and S chart within each stage (i.e., optimize the control limits $\mathrm{LCL}_{\overline{x},i}$, $\mathrm{UCL}_{\overline{x},i}$, and $\mathrm{UCL}_{s,i}$).

The high-level search is carried out by using a method proposed by Lam et al. [16] (see Appendix 2). The purpose is to determine the optimal values of n_i and h_i that will make the required inspection capacity (r) equal to the specified value (R) of the manufacturing system (i.e., satisfy constraint (6)). For a set of (n_i, h_i, i=1, 2, …, s) determined in the high level, the mid-level search [15] identifies the optimal set of (α_1, α_2,…, α_s) so that the in-control ATS_0 of the system is exactly equal to the specification τ (i.e., satisfy constraint (5)). Here, α_i is the type I error probability of a stream in the ith stage. Finally, for a given set of (n_i, h_i, α_i) determined in the high- and mid-levels, the control limits of the $\overline{X}\&S$ charts of each stream are optimized in low-level search [21]. The low-level search loops all s stages. For each stage, an allocating factor w_i ($0\le w_i \le 1$) is used to allocate the type I error α_i to $\alpha_{\overline{x},i}$ (for the $\overline{X}$ chart) and $\alpha_{s,i}$ (for the S chart). Then, the type I error α_i of the joint $\overline{X}\&S$ charts is [3, 21]

$$\alpha_i = \alpha_{\overline{X},i} + \alpha_{s,i} - \alpha_{\overline{X},i}\cdot\alpha_{s,i}$$
$$\text{or, } \alpha_{s,i} = \frac{(1-w_i)\alpha_i}{1-w_i\alpha_i}, \text{ where, } \alpha_{\overline{X},i} = w_i\alpha_i \tag{8}$$

Then, the control limits can be determined.

$$k_i = \Phi^{-1}\left(1-0.5\alpha_{\overline{X},i}\right)$$
$$\mathrm{LCL}_{\overline{X},i} = \mu_{0,i}-k_i \times \frac{\sigma_{0,i}}{\sqrt{n_i}}, \quad \mathrm{UCL}_{\overline{X},i} = \mu_{0,i} + k_i \times \frac{\sigma_{0,i}}{\sqrt{n_i}} \tag{9}$$
$$\mathrm{UCL}_{s,i} = \sigma_{0,i}\sqrt{\frac{\left[\chi^2_{n_i-1}\right]^{-1}\left(1-\alpha_{s,i}\right)}{n_i-1}}$$

where $\Phi^{-1}()$ is the inverse function of the cumulative probability function for a standard normal distribution and $\left[\chi^2_{n_i-1}\right]^{-1}$ is that for a chi-square distribution with degrees of freedom of (n_i-1). The control limit coefficient k_i of the $\overline{X}$ chart is calculated based on type I error $\alpha_{\overline{x},i}$. The relationship between k_i and $\alpha_{\overline{x},i}$ can be established easily based on the concept of probability limits as discussed in [4] and used in many articles [15, 16].

Next, based on the tentatively determined values of n_i, h_i, $\mathrm{LCL}_{\overline{x},i}$, $\mathrm{UCL}_{\overline{x},i}$, and $\mathrm{UCL}_{s,i}$, the informative power p_i of stage i is evaluated (see Appendix 3). While the high-level optimization trades off between the sample size n_i and the sampling interval h_i, the mid-level optimization enhances the power (or increase α_i) of the charts in some stages and sacrifice the power of the charts in other stages, and the low-level search seeks the optimal value of w_i and the corresponding optimal set of control limits for each stage, so that the informative power p_i will be maximized. Ultimately, all the three levels aim at minimizing the objective function T (Eq. (6)). The overall optimization search is outlined as follows:

1. Specify s, g_i, LSL_i, USL_i, $\mu_{0,i}$, $\sigma_{0,i}$, t_i, τ, R, $c_{\min,i}$, and q_i.
2. Initialize $T_{\min}$ as a large number, say 10^6. $T_{\min}$ is used to record the minimum T.
3. Carry out the high-level search to determine the optimal values of (n_i, h_i) which ensures that r is equal to R.
4. For a given set of $(n_i, h_i, i=1, 2, \ldots, s)$, carry out the mid-level search to investigate different combinations of $(\alpha_1, \alpha_2, \ldots, \alpha_s)$. If the control charts in a stage acquire a large α_i, they will be more powerful in detecting the out-of-control cases. Each set of $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ will make ATS_0 equal to τ.
5. For a given set of (n_i, h_i, α_i), carry out the low-level search that loops all s stages. For each stage, search the optimal value of w_i $(0 \le w_i \le 1)$ that maximizes the informative power p_i.
6. Then, based on the optimal w_i, determine the control limits $\mathrm{LCL}_{\overline{x},i}$, $\mathrm{UCL}_{\overline{x},i}$, and $\mathrm{UCL}_{s,i}$ by Eqs (8) and (9).
7. When the loop over all s stages is completed, calculate T by Eq. (6). If this T value is smaller than the current $T_{\min}$, replace $T_{\min}$ by T and store the optimal values of n_i, h_i, $\mathrm{LCL}_{\overline{x},i}$, $\mathrm{UCL}_{\overline{x},i}$, and $\mathrm{UCL}_{s,i}$ of all s stages as the temporary optimal solution.
8. When the entire search is completed, the final $T_{\min}$ is the minimum T and the corresponding optimal set of n_i, h_i, $\mathrm{LCL}_{\overline{x},i}$, $\mathrm{UCL}_{\overline{x},i}$, and $\mathrm{UCL}_{s,i}$ $(i=1, 2,\ldots, s)$ are also finalized.

A computer program ***chain_nh_xs.c*** in *C* language has been developed to carry out the optimization design. Usually, an optimal solution is obtained in about 1 min of CPU time by using a personal computer.

3 Performance comparison and sensitivity study

This section compares the performance (measured by the average informative time T) of the optimal $\overline{X}\&S$ chart system with the traditional 3-sigma $\overline{X}\&S$ chart system as well as a suboptimal $\overline{X}\&S$ chart system [21]. Like the optimal $\overline{X}\&S$ chart system, the suboptimal $\overline{X}\&S$ chart system is designed to minimize the average informative time T, but it only optimizes its charting parameters $\mathrm{LCL}_{\overline{x},i}$, $\mathrm{UCL}_{\overline{x},i}$, and $\mathrm{UCL}_{s,i}$ (namely, only the mid- and low-level optimization searches are conducted), and n_i and h_i are determined as for the traditional $\overline{X}\&S$ chart system (i.e., the high-level optimization is omitted). It is also assumed that each chart in the traditional $\overline{X}\&S$ chart system uses a sample size of five, an identical sampling interval and the 3-sigma control limits. This section also studies the effects of six parameters (g_i, q_i, t_i, $\sigma_{0,i}$, USL_i, $c_{min,i}$) on the performance of the optimal $\overline{X}\&S$ chart system. Each of the six parameters has a low value, a nominal value, and a high value, as shown below.

Parameter	Low	Nominal	High
g_i	1	1	4
q_i	0.1	0.5	0.9
t_i	0.02	0.04	0.06
$\sigma_{0,i}$	0.01	0.025	0.04
USL_i	$3.2\sigma_{0,i}$	$4.6\sigma_{0,i}$	$6\sigma_{0,i}$
$c_{\min,i}$	1	1.25	1.5

The nominal value denotes the value under normal circumstances. The nominal value of g_i is equal to its low value (=1), because it is assumed that there is usually only one stream in a stage unless the production rate in this stage is much lower than that in other stages. It is also noted that USL_i is expressed as a function of $\sigma_{0,i}$.

Without the loss of generality, the in-control $\mu_{0,i}$ is set at zero. Consequently, $\mathrm{LSL}_i=-\mathrm{USL}_i$. The specifications R (available inspection capacity) and τ (minimum value of ATS_0) of the optimal $\overline{X}\&S$ chart system and the suboptimal $\overline{X}\&S$ chart system are set equal to the R and ATS_0 values of the traditional 3-sigma $\overline{X}\&S$ chart system (i.e., all systems always have the identical R and ATS_0 values).

The performance of a two-stage manufacturing system is studied for six different cases. In each case, all the parameters take their nominal values in both stages, except that one selected parameter called the *active parameter* takes its low value in stage 1 and high value in stage 2. For example, in case 1, the first parameter g_i is designated as the active parameter. Therefore,

$$g_1 = 1,\ \ g_2 = 4, q_1 = q_2 = 0.5, t_1 = t_2 = 0.04, \sigma_{0,1} = \sigma_{0,2} = 0.025, \mathrm{USL}_1 = \mathrm{USL}_2 = 4.6\sigma_{0,i}, c_{\min,1} = c_{\min,2} = 1.25$$

To facilitate the comparison, a ratio B of the average informative time T values resulting from each control chart system is calculated as,

$$B = T/T_{\text{tradition}} \tag{10}$$

where $T_{\text{tradition}}$ is produced by the traditional $\overline{X}\&S$ chart system and T is by each particular chart system. Obviously, if the B value of a chart system in a case is less than 1, this chart system will outperform the traditional $\overline{X}\&S$ chart system when the corresponding active parameter has different values in the two stages.

The B values for all six cases are listed in Table 1 under column (9). It is observed that in all of the six cases, the suboptimal $\overline{X}\&S$ chart system substantially outperforms the traditional $\overline{X}\&S$ chart system; however, the optimal $\overline{X}\&S$ chart system can achieve an even more significant improvement in effectiveness compared to the suboptimal $\overline{X}\&S$ chart system. Among all, the difference in parameters USL_i (the upper specification limits in terms of in-control $\sigma_{0,i}$) and $c_{\min,i}$ (the out-of-control process capability index) promises the greatest potential of using the optimal $\overline{X}\&S$ chart system. For example, a reduction of T by about 65 % compared to the traditional $\overline{X}\&S$ chart system can be achieved when USL_i has different values (USL_1=3.2$\sigma_{0,i}$, USL_2=6$\sigma_{0,i}$ in case 5) and all other parameters are fixed, whereas, in the same case, the suboptimal $\overline{X}\&S$ chart system reduces T by about 44 % compared to the traditional $\overline{X}\&S$ chart system. The average, $\overline{B}$, of the ratio B values for a chart over the six cases is also calculated and enumerated at the bottom of Table 1. The values of $\overline{B}$ indicate that, from an overall viewpoint, the suboptimal $\overline{X}\&S$ chart system is more effective (in terms of reduction in T) than the traditional $\overline{X}\&S$ chart system by about 26.92 %, whereas optimal $\overline{X}\&S$ chart system is more effective than the traditional $\overline{X}\&S$ chart system by about 53.08 %. The further reduction in T (26 % compared to the suboptimal $\overline{X}\&S$ chart system) achieved by the optimal $\overline{X}\&S$ chart system is completely attributable to the optimization of the sample sizes and sampling intervals in addition to the optimization of the control limits.

The actual B value would differ for different systems and circumstances. However, it is believed that the optimal control chart system would be effective in general and beneficial to many real systems. The reason is that, in any manufacturing system, the influential parameters would be in general different among several stages.

The optimal values of n_i, h_i, and α_i of the optimal $\overline{X}\&S$ chart system are also enumerated in Table 1 (columns 4, 5, and 6). The value of α_i indicates the proportion of detection power allocated to stage i. A parameter A_i is calculated for each stage [15] and listed in column 7:

$$A_i = \frac{n_i\alpha_i}{h_i}(i = 1,\ \ 2) \tag{11}$$

These A_i values provide the users with the general guidelines for adjusting the sample size, sampling intervals, and control limits of the charts in different stages. For example, if A_1 is larger than A_2, it means that more detection power should be allocated to the stage 1. That is, n_1 is larger than n_2, h_1 is smaller than h_2, and/or the control limits $\mathrm{LCL}_{\overline{x},1}$, $\mathrm{UCL}_{\overline{x},1}$, and $\mathrm{UCL}_{s,1}$ of the $\overline{X}$ and S charts in stage 1 should be made relatively tighter than that in stage 2. As the general guidelines drawn from Table 1, the control charts in the following stage should be made more powerful where

1. The number of streams is smaller and/or
2. The out-of-control case is more likely to occur and/or
3. The process has larger variance and/or
4. The specification limits are tighter with respect to the in-control standard deviation and/or
5. The out-of-control process capability index is larger

With these guidelines in mind, the users can adjust the sample sizes, sampling intervals, and control limits of the $\overline{X}$ and S charts in different stages rationally and effectively, even if more complicated computation has not been carried out.

4 Examples

4.1 Example 1

A mechanical part as shown in Fig. 1a is to be manufactured through a manufacturing system [17]. Among all process stages, only three of them are functional (or critical to the overall quality of the part) and have to be closely monitored by the $\overline{X}\&S$ charts. Stage 1 turns the outer surface (Φ16.00± 0.065), stage two drills the small hole (Φ4.00±0.09), and stage three bores the large hole (Φ8.00±0.13). The design engineers indicate the dimensions and design tolerances on

Table 1 Performance comparison and sensitivity study

(1) Case	(2) Stage	(3) Active parameter	(4) Optimal n	(5) Optimal h	(6) Optimal α	(7) A	(8) Optimal k		(9) Ratio B	
							$k_{\bar{x}}$	k_s	Suboptimal $\overline{X}$&S chart system	Optimal $\overline{X}$&S chart system
1	1	g_1=1	15	1.773	0.017555	0.1485	2.530	2.655	0.8074	0.5330
	2	g_2=4	11	2.660	0.015318	0.0634	2.527	2.873		
2	1	t_1=0.02	18	2.000	0.009994	0.0900	2.722	2.861	0.8758	0.5322
	2	t_2=0.06	11	3.000	0.024530	0.0899	2.358	2.682		
3	1	p_1=0.1	9	3.180	0.013009	0.0368	2.584	2.964	0.7444	0.4388
	2	p_2=0.9	19	2.650	0.024060	0.1725	2.417	2.508		
4	1	$\sigma_{0,1}$=0.01	16	3.300	0.022255	0.1079	2.445	2.553	0.8758	0.5313
	2	$\sigma_{0,2}$=0.04	17	3.300	0.021228	0.1094	2.462	2.568		
5	1	USL_1=3.2σ_0	25	3.800	0.045189	0.2973	2.070	2.569	0.5590	0.3517
	2	USL_2=6.0σ_0	13	3.800	0.004773	0.0163	2.957	3.169		
6	1	$c_{\min,1}$=1.0	8	2.400	0.001466	0.0049	3.304	3.664	0.5221	0.4281
	2	$c_{\min,2}$=1.5	16	2.400	0.030072	0.2005	2.189	3.173		
$\overline{B}$									0.7308	0.4692

the drawing. Since the process in stage 1 is much more time consuming, two machines are used in parallel. Three sets of $\overline{X}$&S charts as a system are used to carry out the on-line monitoring of the processes in the three machining stages. Furthermore, since two identical machines are used side by side in stage 1, the $\overline{X}$&S control charts in stage 1 have two duplicates, each controls one of the two parallel machines. The block diagram of the control chart system is displayed in Fig. 1b. The specifications of the system are listed below.

- Number of stages in the system: s=3
- Number of streams: g_1=2, g_2=g_3=1
- Lower specification limit (mm): LSL_1=15.935, LSL_2=3.910, LSL_3=7.870
- Upper specification limit (mm): USL_1=16.065, USL_2=4.090, USL_3=8.130
- Process standard deviations (mm): $\sigma_{0,1}$=0.012, $\sigma_{0,2}$=0.023, $\sigma_{0,3}$=0.038
- Inspection time (min): t_1=0.40, t_2=0.35, t_3=0.65
- Out-of-control process capability: $c_{\min,1}$=$c_{\min,2}$=$c_{\min,3}$=1

The probabilities q_i of the out-of-control occurrences are estimated from historical records, which show that the numbers of out-of-control cases occurring in each of the three stages are 6, 8, and 4, respectively. Therefore,

$$\begin{aligned} q_1 &= 6/(6+8+4) = 0.334 \\ q_2 &= 8/(6+8+4) = 0.444 \\ q_3 &= 4/(6+8+4) = 0.222 \end{aligned} \tag{12}$$

Originally, the system uses the traditional 3-sigma $\overline{X}$&S charts for all the processes. Operators take a sample of size five (n_1=n_2=n_3=5) for every 2 h (h_1=h_2=h_3=2) for each chart. As a result, the inspection capacity R of the traditional system is equal to 0.075 (Eq. (15) in Appendix 2), that is, an operator spends 7.5 % of his/her time (per shift per day) on the inspection activities. The in-control ATS_0 of the traditional $\overline{X}$&S chart system is equal to 76.27 h (Eq. (13) in Appendix 1).

Now, the QA engineer decides to design a suboptimal $\overline{X}$&S chart system [21] and an optimal $\overline{X}$&S chart system in order to improve the effectiveness of the chart system. He decides to maintain the false alarm rate and the required inspection capacity at the same level as in the traditional chart system. Thus, the τ and R values of the suboptimal and the optimal chart systems are specified as 76.27 h and 0.075, respectively. The computer program ***chain_nh_xs.c*** designs the suboptimal $\overline{X}$&S chart system and the optimal $\overline{X}$&S chart system in 36.141 CPU seconds in a personal computer (Pentium IV 2.4 GHz). The results are listed in Table 2.

It is observed that all the chart systems have the same ATS_0 (=76.27 h) and r (=0.075) values. As shown in Table 2, the values of the average informative time T generated by the three chart systems are quite different. They are 24.91, 14.11, and 9.27 h for the traditional $\overline{X}$&S chart system, suboptimal $\overline{X}$&S chart system, and optimal $\overline{X}$&S chart system, respectively. Or, in other words, the suboptimal $\overline{X}$&S chart system reduces T by 43.35 % and the optimal $\overline{X}$&S chart system reduces T by an even high percentage of 62.77 %, compared

Table 2 The three control chart systems in the example 1

System	Stage	Sample size n	Sampling interval (h) h	Control limits (mm)			Type I error probability α	The average informative time (h) T
				$LCL_{\bar{x},i}$	$UCL_{\bar{x}}$	UCL_s		
Traditional $\overline{X}\&S$ chart system	1	5	2.00	15.984	16.016	0.02356	0.006588	24.91
	2	5	2.00	3.969	4.031	0.04516	0.006588	
	3	5	2.00	7.949	8.051	0.07462	0.006588	
Suboptimal $\overline{X}\&S$ chart system	1	5	2.00	15.981	16.019	0.02802	0.000624	14.11
	2	5	2.00	3.973	4.027	0.04838	0.009360	
	3	5	2.00	7.959	8.041	0.08275	0.015704	
Optimal $\overline{X}\&S$ chart system	1	6	4.25	15.984	16.016	0.02465	0.002212	9.27
	2	13	2.86	3.983	4.017	0.03577	0.011061	
	3	16	5.84	7.980	8.020	0.05360	0.048186	

to the traditional $\overline{X}\&S$ chart system. Due to the shortened average informative time T (similar to the out-of-control ATS), the average number of nonconforming products or the quality cost can be considerably reduced in the out-of-control cases, on condition that the false alarm rate is maintained at the specified level.

4.2 Example 2

The production line of a certain antibiotic factory contains three critical processes—filtering, decolorizing, and finishing—that needs to be closely monitored [21, 27]. It was decided to establish the $\overline{X}\&S$ chart system for monitoring the three critical processes. The specifications of the production system are listed below (some data are supplemented by the authors).

Number of stages in the system: $s=3$
Number of streams: $g_1=g_2=g_3=1$
Lower specification limit: $LSL_1=76.522$, $LSL_2=92.235$, $LSL_3=100.610$
Upper specification limit: $USL_1=80.578$, $USL_2=93.965$, $USL_3=104.390$
Process standard deviations: $\sigma_{0,1}=0.507$, $\sigma_{0,2}=0.173$, $\sigma_{0,3}=0.450$
Inspection time (min): $t_1=0.50$, $t_2=0.40$, $t_3=0.75$
Out-of-control process capability: $c_{\min,1}=c_{\min,2}=c_{\min,3}=1$
Probability of occurrence: $q_1=0.545$, $q_2=0.273$, $q_3=0.182$

The traditional 3-sigma $\overline{X}\&S$ and the suboptimal $\overline{X}\&S$ chart systems are designed with sample sizes of ($n_1=n_2=5$, $n_3=10$) and sampling intervals of ($h_1=h_2=240$, $h_3=120$). The computer program ***chain_nh_xs.c*** designs the traditional 3-sigma $\overline{X}\&S$, suboptimal $\overline{X}\&S$ [21], and optimal $\overline{X}\&S$ chart systems in 25.844 CPU seconds in a personal computer (Pentium IV 2.4 GHz). The results are listed in Table 3.

It is noted that all the chart systems have the same ATS_0 (=9,866) and r (=0.081) values. As shown in Table 3, the values of the average informative time T generated by the three chart systems are 1,104.92, 713.52, and 384.00 for the traditional $\overline{X}\&S$, suboptimal $\overline{X}\&S$, and optimal $\overline{X}\&S$ chart system,

Table 3 The three control chart systems in the example 2

System	Stage	Sample size n	Sampling interval h	Control limits			Type I error probability α	The average informative time T
				$LCL_{\bar{x},i}$	$UCL_{\bar{x}}$	UCL_s		
Traditional $\overline{X}\&S$ chart system	1	5	240.00	77.870	79.230	0.99556	0.006588	1,104.92
	2	5	240.00	92.868	93.332	0.33971	0.006588	
	3	10	120.00	102.073	102.927	0.75122	0.005575	
Suboptimal $\overline{X}\&S$ chart system	1	5	240.00	77.993	79.107	0.98058	0.018846	713.52
	2	5	240.00	92.858	93.342	0.37300	0.002692	
	3	10	120.00	102.028	102.972	0.81784	0.001394	
Optimal $\overline{X}\&S$ chart system	1	13	165.68	78.164	78.936	0.795483	0.009293	384.00
	2	10	203.11	92.920	93.280	0.30698	0.001859	
	3	7	235.18	102.036	102.964	0.83443	0.008494	

respectively. That means, the suboptimal $\overline{X}\&S$ chart system reduces T by 35.42 % and the optimal $\overline{X}\&S$ chart system reduces T by 65.25 %, compared to the traditional $\overline{X}\&S$ chart system. Similar to example 1, the optimal $\overline{X}\&S$ chart system provides considerably higher detection effectiveness.

5 Conclusions

This article proposes the $\overline{X}\&S$ control chart system for monitoring process shifts in mean and standard deviation in a multistage manufacturing system. The design algorithm considers all the charts within a system in an integrative and optimal manner. Consequently, the performance characteristics of the system as a whole can be significantly improved, and the product quality will be further guaranteed. Such an improvement is achieved without requiring additional cost and effort for on-line inspections.

The comparative studies show that the suboptimal $\overline{X}\&S$ chart system improves the effectiveness (in terms of reduction in detection time T) by about 27 %, and the optimal $\overline{X}\&S$ chart system improves the effectiveness by an even high percentage of 53 %, compared to the traditional 3-sigma $\overline{X}\&S$ chart system. The further improvement in effectiveness (26 % compared to the suboptimal $\overline{X}\&S$ chart system) achieved by the optimal $\overline{X}\&S$ chart system is attributable to the optimization of the sample sizes and sampling intervals in addition to the optimization of the control limits. In the optimal solution, majority of the detection power of the optimal $\overline{X}\&S$ chart system is allocated to a stage where sample size and type I error are large and the sampling interval is small.

The design of the $\overline{X}\&S$ chart system is more difficult than that of the traditional $\overline{X}\&S$ chart system. However, the design procedure can be easily computerized. Moreover, the general guidelines for adjusting the sample sizes, sampling intervals, and control limits of the $\overline{X}\&S$ charts in a system are particularly useful and widely applicable.

Finally, for model simplicity, the induced mean and standard deviation shifts ($\delta_{\mu,i}$ and $\delta_{\sigma,i}$) pertaining to the interdependency of the process stages have not been considered in the current article, and the focus is only on the process shifts resulting from the process stage itself. Therefore, the proposed chart system can be extended in the future for handling the interdependency between the process stages of a manufacturing system. Further studies can also be conducted considering machines with different operating characteristics for parallel streams.

Appendix 1: Calculation of the in-control ATS_0 [15]

The in-control ATS_0 of a chart system is determined by

$$\mathrm{ATS}_0 = \frac{1}{1-\prod_{i=1}^{s}(1-\alpha_i/h_i)^{g_i}} \tag{13}$$

On the other hand, for a specified value τ for ATS_0, any individual α_i can be expressed in terms of other α_j (j=1, 2, …, s, $j\neq i$).

$$\alpha_i = h_i\left[1-\left(\frac{1-1/\tau}{\prod_{j=1,j\neq i}^{s}\left(1-\alpha_j/h_j\right)^{g_j}}\right)^{1/g_i}\right] \tag{14}$$

where α_i is the type I error probability of a stream in the ith stage. In the mid-level search, a dynamic search algorithm is employed to determine the optimal set of (α_1, α_2, …, α_s). In this algorithm, the optimal values of α_i of the first ($s-1$) control charts are searched step by step in ($s-1$) levels, using the same step size $d\alpha$. The last α_i is finally determined so that the resultant ATS_0 is exactly equal to the specified τ (Eq. (4)). Equation (14) is used to determine the range of possible α_i values in the ith level search. The mid-level search is carried out by using a total number of search points of 1,000 (corresponding to step size $d\alpha$ of 0.0004159).

Appendix 2: Search algorithm for the high-level optimization [16]

The objective of the high-level optimization is, for a given inspection capacity R, to determine an optimum set of (n_i, h_i) (i=1, 2, …, s) that will minimize the average informative time T and ensure that the actual or required capacity r is equal to the specified value R (i.e., constraint (6) is satisfied). A gradient-based search algorithm is employed to approach the optimal set of n_i and h_i step by step until the reduction in T is negligible. The actual inspection capacity r required by an integrated $\overline{X}\&S$ chart system is

$$r = \sum_{i=1}^{s}\frac{n_i g_i t_i}{h_i} \tag{15}$$

The minimum increment of sample size n_i is 1. Therefore, the step size (Δn_i) of n_i for all stages is taken as 1. Then, for an increment (Δn_i=1) of every sample size, the corresponding inspection capacity increment is

$$\overline{\Delta r_n} = \sum_{i=1}^{s} \frac{\partial r}{\partial n_i} \Delta n_i = \sum_{i=1}^{s} \frac{g_i t_i}{h_i} \tag{16}$$

It is rational to make the step size Δh_i of h_i proportional to h_i itself, that is

$$\Delta h_i \alpha h_i, \quad \text{or}, \Delta h_i = b h_i \tag{17}$$

where b (b>0) is a proportionality constant. Then, for a decrement of ($-\Delta h_i$) of every sampling interval, the increment of inspection capacity is

$$\begin{aligned}\overline{\Delta r_h} &= \sum_{i=1}^{s} \frac{\partial r}{\partial h_i}(-\Delta h_i) = \sum_{i=1}^{s} \left(-\frac{n_i g_i t_i}{h_i^2}\right)(-b h_i) \\ &= b \sum_{i=1}^{s} \frac{n_i g_i t_i}{h_i}\end{aligned} \tag{18}$$

By equating $\overline{\Delta r_n}$ and $\overline{\Delta r_h}$,

$$b = \left(\sum_{i=1}^{s} \frac{g_i t_i}{h_i}\right) \Bigg/ \left(\sum_{i=1}^{s} \frac{n_i g_i t_i}{h_i}\right) \tag{19}$$

Generally, this b value will make the increment of r due to a decrease of h_i by one Δh_i similar to the increment of r because of an increase of n_i by one Δn_i.

Now, the sample sizes n_i and the sampling intervals h_i can be increased or decreased step by step with the step size Δn_i and Δh_i. It is well known that, increasing n_i by Δn_i results in the decrease of T (or gaining detection power) and decreasing n_i by Δn_i results in the increase of T (or losing detection power). Conversely, decreasing h_i by Δh_i means moving the sampling interval in the gaining direction and increasing h_i by Δh_i means moving the sampling interval in the losing direction.

If all n_i and h_i are substituted by the general variable X_j (j= 1, 2, …, 2s) (i.e., an X_j may be a sample size or a sampling interval), the gaining factor G_j pertaining to an X_j for an increment of inspection capacity r is given by,

$$G_j = \frac{|\Delta T_j|}{\Delta r_j} = \frac{-\Delta T_j}{\dfrac{\partial r}{\partial X_j} \Delta X_j} \tag{20}$$

where ΔT_j is the change in the average informative time T when an X_j moves one step in the gaining direction (i.e., 1 for n_i or $-\Delta h_i$ for h_i) and all other X_js are kept unchanged. The quantity ΔT_j is always smaller than zero. Specifically, if an X_j is a sample size,

$$G_j = \frac{-\Delta T_j}{\dfrac{\partial r}{\partial n_j} \Delta n_j} = \frac{-\Delta T_j}{\dfrac{g_i t_i}{h_i}(1)} = \frac{(-\Delta T_j) \cdot h_i}{g_i t_i} \tag{21}$$

And if an X_j is a sampling interval,

$$G_j = \frac{-\Delta T_j}{\dfrac{\partial r}{\partial h_j} \Delta h_j} = \frac{-\Delta T_j}{\left(-\dfrac{n_i g_i t_i}{h_i^2}\right)(-b h_i)} = \frac{(-\Delta T_j) \cdot h_i}{n_i g_i t_i b} \tag{22}$$

If the gaining factor G_j of an X_j is larger, then, when this X_j moves in the gaining direction, greater reduction in T will be obtained with relatively smaller increase in r. Conversely, a lower G_j means that if the corresponding X_j moves in the losing direction, greater reduction in r will be obtained with relatively smaller increase in T. Therefore, the general variables X_j are first ranked according to the descending order of G_j. Then, at each design point, N general variables X_j with the largest G_j values will be moved one step in their gaining directions (that is, if an X_j is a sample size, increase it by Δn_i; if an X_j is a sampling interval, decrease it by Δh_i). This will usually lead to a substantial reduction in T with an insignificant increase of r. In order to balance out the increase in r, the rest (2s−N) general variables X_j with smaller G_j will be moved one step in their losing directions. This second movement may adversely increase T, but this increase in T must be smaller than the decrease of T in the first movement. A searching mechanism is employed to identify the maximum possible value for N, which ensures that, in each search step, the maximum reduction in T is achieved, and meanwhile, the actual or required inspection capacity r is equal to the specified value of R.

Appendix 3: Calculation of the informative power p_i of a process in stage i [21]

For a pair of mean shift $\delta_{\mu,i}$ and standard deviation shift $\delta_{\sigma,i}$ (expressed in terms of $\sigma_{0,i}$)

$$\mu_i = \mu_{0,i} + \delta_{\mu,i}\sigma_{0,i}, \quad \sigma_i = \delta_{\sigma,i}\sigma_{0,i} \tag{23}$$

When the process is in control, $\delta_{\mu,i}$=0 and $\delta_{\sigma,i}$=1.

There are infinite number of pairs of ($\delta_{\mu,i}$, $\delta_{\sigma,i}$) that jointly reduce the process capability index $C_{pk,i}$ from the in-control value to the out-of-control value $c_{\min,i}$. Or in other words, a given $c_{\min,i}$ may result from infinite number of different combinations of $\delta_{\mu,i}$ and $\delta_{\sigma,i}$ (e.g., large $\delta_{\mu,i}$ and small $\delta_{\sigma,i}$, small $\delta_{\mu,i}$ and large $\delta_{\sigma,i}$, or moderate $\delta_{\mu,i}$ and $\delta_{\sigma,i}$). A factor λ ($0\le\lambda\le1$) is used to take all of these combinations into account. When λ=0, $\delta_{\sigma,i}$ is equal to 1 and $\delta_{\mu,i}$ gets its maximum value $\delta_{\mu,\max,i}$.

$$\delta_{\mu,\max,i} = \frac{\mathrm{USL}_i - \mu_{0,i} - 3c_{\min,i}\sigma_{0,i}}{\sigma_{0,i}} \tag{24}$$

On the other hand, when λ=1, $\delta_{\mu,i}$ is equal to zero and $\delta_{\sigma,i}$ gets its maximum value $\delta_{\sigma,\max,i}$.

$$\delta_{\sigma,\max,i} = \frac{\mathrm{USL}_i - \mu_{0,i}}{3c_{\min,i}\sigma_{0,i}} \tag{25}$$

Apart from above two extreme cases, $\delta_{\mu,i}$ and $\delta_{\sigma,i}$ are determined by a linear interpolation when $0<\lambda<1$.

$$\begin{aligned} \delta^2_{\mu,i}(\lambda) &= (1-\lambda) b \delta^2_{\mu,\max,i} \\ \delta^2_{\sigma,i}(\lambda) &= \lambda b \delta^2_{\sigma,\max,i} \end{aligned} \tag{26}$$

$$b = \left[\frac{USL_i - \mu_{0,i}}{\sigma_{0,i}\left(3c_{\min,i}\sqrt{\lambda}\delta_{\sigma,\max,i} + \sqrt{1-\lambda}\delta_{\mu,\max,i}\right)}\right]^2 \tag{27}$$

where the common parameter b ensures that $\delta_{\mu,i}$ and $\delta_{\sigma,i}$ jointly make $C_{pk,i}$ exactly equal to $c_{\min,i}$. The interpolation is applied to the squares of $\delta_{\mu,i}$ and $\delta_{\sigma,i}$ (rather than $\delta_{\mu,i}$ and $\delta_{\sigma,i}$ themselves), because the sum of $\delta_{\mu,i}{}^2$ and $\delta_{\sigma,i}{}^2$ is equal to the average value of the loss function, except for a constant [28]. For a given value of λ, the power $p_{\bar{x},i}(\lambda)$ of the $\overline{X}$ chart in the ith stage is

$$p_{\bar{x},i}(\lambda) = 1 - \left\{\Phi\left[\frac{\mathrm{UCL}_{\bar{x},i} - \left(\mu_{0,i} + \delta_{\mu,i}(\lambda)\sigma_{0,i}\right)}{\delta_{\sigma,i}(\lambda)\sigma_{0,i}/\sqrt{n_i}}\right] - \Phi\left[\frac{\mathrm{LCL}_{\bar{x},i} - \left(\mu_{0,i} + \delta_{\mu,i}(\lambda)\sigma_{0,i}\right)}{\delta_{\sigma,i}(\lambda)\sigma_{0,i}/\sqrt{n_i}}\right]\right\} \tag{28}$$

The power $p_{s,i}(\lambda)$ of the corresponding S chart is

$$p_{s,i}(\lambda) = 1 - \chi^2_{n_i-1}\left[\frac{n_i - 1}{\left(\delta_{\sigma,i}(\lambda)\sigma_{0,i}\right)^2} \times \mathrm{UCL}^2_{s,i}\right] \tag{29}$$

The power $p_{\bar{x}\&s,i}(\lambda)$ of the joint $\overline{X}$ and S charts is

$$p_{\bar{x}\&s,i}(\lambda) = p_{\bar{x},i}(\lambda) + p_{s,i}(\lambda) - p_{\bar{x},i}(\lambda) p_{s,i}(\lambda) \tag{30}$$

Finally, the informative power of stage i is equal to the following integration over the whole range of λ.

$$p_i = \int_0^1 p_{\bar{x}\&s,i}(\lambda) f(\lambda) d\lambda \tag{31}$$

where $f(\lambda)$ is the probability density function of λ and is equal to 1 (assuming a uniform distribution). Equation (31) takes into account of all the possible combinations of $\delta_{\mu,i}$ and $\delta_{\sigma,i}$ that correspond to the out-of-control condition ($C_{pk,i}$=$c_{\min,i}$). The integration (31) can be evaluated by using the Gauss-Legendre translation rule on a few integration points.

All of the formulae developed in the appendices have been verified by Monte Carlo simulation.

References

1. Woodal WH, Montgomery DC (1999) Research issues and ideas in statistical process control. J Qual Technol 31(4):376–386
2. Ohta H, Kimuar A, Rahim A (2002) An economic model for $\overline{X}$ and R charts with time-varying parameters. Qual Reliab Eng Int 18:131–139
3. Shamsuzzaman M, Wu Z, Elias MRS (2009) Designs of $\overline{X}$ &S control charts with optimal manpower deployment. Comput Ind Eng 56:1589–1596
4. Montgomery DC (2009) Introduction to statistical quality control. John Wiley & Sons, New York
5. Runger GC, Alt FB, Montgomery DC (1996) Controlling multiple stream processes with principal components. Int J Prod Res 34(11): 2991–2999
6. Nelson LS (1986) Control chart for multiple stream processes. J Qual Technol 18:255–256
7. Mortell RR, Runger GC (1995) Statistical process control of multiple stream processes. J Qual Technol 27:1–12
8. Williams WW, Peters MR (1989) Economic design of an attributes control system for a multistage serial production process. Int J Prod Res 27:1269–1286
9. Hawkins DM (1993) Regression adjustment for variables in multivariate quality control. J Qual Technol 25(3):170–182
10. Wade MR, Woodall WH (1993) A review and analysis of cause-selecting control charts. J Qual Technol 25:161–169
11. Ding Y, Ceglarek D, Shi J (2002) Fault diagnosis of multistage manufacturing processes by using state space approach. ASME Trans J Manuf Sci Eng 124:313–322
12. Zantek PF, Wright GP, Plante RD (2002) Process and product improvement in manufacturing systems with correlated stages. Manag Sci 48(5):591–606
13. Zhou SY, Chen Y, Shi JJ (2004) Statistical estimation and testing for variation root-cause identification of multistage manufacturing processes. IEEE Trans Autom Sci Eng 1(1):73–83

14. Zantek PF, Wright GP, Plante RD (2006) A self-starting procedure for monitoring process quality in multistage manufacturing systems. IIE Trans 38(4):293–308
15. Wu Z, Lam YC, Zhang S, Shamsuzzaman M (2004) Optimization designs of control chart systems. IIE Trans 36(5):447–455
16. Lam YC, Shamsuzzaman M, Zhang S, Wu Z (2005) Integrated control chart system—optimization of sample sizes, sampling intervals and control limits. Int J Prod Res 43(3):563–582
17. Shamsuzzaman M, Lam YC, Wu Z (2005) Control chart systems with independent quality characteristics. Int J Adv Manuf Technol 26:1298–1305
18. Wu Z, Shamsuzzaman M, Wang Q (2007) The cost minimization and manpower deployment to SPC in a multistage manufacturing system. Int J Prod Econ 106:275–287
19. Shamsuzzaman M, Xie M, Goh TN, Zhang HY (2009) Integrated control chart system for time-between-events monitoring in a multistage manufacturing system. Int J Adv Manuf Technol 40(3–4):373–381
20. Zhang HY, Xie M, Goh TN, Shamsuzzaman M (2011) Economic design of time-between-events control chart system. Comput Ind Eng 60(4):485–492
21. Wu Z, Shamsuzzaman M (2005) The design and application of the integrated control charts for monitoring process mean and variance. J Manu Syst 24(4):302–314
22. Duncan AJ (1986) Quality control and industrial statistics. Irwin, Homewood
23. Wu Z, Xie M, Tian Y (2002) Optimization design of the $\overline{X}$ &S charts for monitoring process capability. J Manu Syst 21: 83–92
24. Reynolds MR, Glosh BK (1981) Designing control charts for means and variances. ASQC Qual Cong Trans, American Soci for Qual Con, San Francisco 400–407
25. Carlyle WM, Montgomery DC, Runger GC (2000) Optimization problems and methods in quality control and improvement. J Qual Technol 32:1–17
26. Duncan AJ (1956) The economic design of $\overline{X}$ charts used to maintain current control of a process. J Am Stat Assoc 51:228–242
27. Zhang G (1990) A new diagnosis theory with two kinds of quality. Total Qual Manag 1(2):249–258
28. Cyrus D (1997) Statistical aspects of quality control. Academic Press Ltd., London