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ORIGINAL ARTICLE

Estimating multivariate linear profiles change point with a monotonic change in the mean of response variables

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Abstract In this paper, a maximum likelihood estimator (MLE) is developed to estimate change point when monotonic change occurs in the mean of response variables in multivariate linear profiles in Phase II. Performance of the proposed estimator is compared to the performance of step change and linear drift estimators under different shift types. To conduct comparisons, accuracy and precision of the estimators are considered as performance measures. Simulation results show that the average change point estimate of the proposed estimator is less biased than the one for the step and drift estimators in small shifts, because $\bar{\hat{\tau}}_{monotonic}$ is closer to the actual change point of 25 in small shifts. Also, the precision of the proposed estimator is approximately better than that of the step and drift estimators, because its precision values are higher. Hence, the proposed estimator has better performance in terms of both accuracy and precision in small shifts under any kinds of increasing changes. In single step and linear drift changes when the magnitude of shifts increases, the accuracy and precision of their corresponding estimators become better than the accuracy and precision of the proposed estimator. However, the proposed estimator has an advantage that it does not require assumptions about the change type, and its only assumption is that the mean of the response variables changes in an increasing manner. Additional evaluations on the effect of smoothing constant show that with smaller values of the smoothing constant, the proposed change point estimator has less biased estimates and smaller values of mean square error in small shifts rather than the step and drift estimators, leading to a better performance. Also, the larger values of smoothing constant lead to the better performance of the monotonic estimator in large shifts. Finally, the application of the proposed estimator is shown through a real case in the calibration process in the automotive industry.

Keywords Change point estimation · Monotonic change · Maximum likelihood estimator · Multivariate linear profile · Statistical process control

1 Introduction

In statistical process control (SPC), control charts are very effective tools to detect changes in a process. When a control chart signals, quality practitioners must search for assignable causes. Knowing the time at which the process began to vary, referred to as change point, can save the time to find and remove the special causes. Several approaches have been proposed to estimate change point related to an out-of-control signal. Page [1] and Nishina [2] proposed built-in change point estimators of cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts, respectively. Maximum likelihood estimator (MLE), clustering analysis, and artificial neural networks are the other approaches used in the change point estimation literature which are more capable than built-in estimators in estimating change point. Pignatiello and Samuel [3] compared the performance of the MLE approach with the CUSUM and EWMA built-in estimators' performance to estimate time of a single step change in a normal process mean. Comparison of MLE with CUSUM built-in estimator was carried out following an out-of-control signal from a CUSUM control chart. Results showed that in a small shift of δ=0.5, the

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average change point estimate using CUSUM built-in estimator is badly biased. But $\overline{\hat{\tau}}_{\text{MLE}}$ is much closer to the actual change point of 100 and it is less biased. When the magnitude of shift is $\delta=1$, both estimators are unbiased and the precision of the CUSUM built-in estimator was better than that of MLE. However, for other magnitudes of shifts, the MLE approach has better performance in terms of both accuracy and precision of the estimates. Comparison of MLE with EWMA built-in estimator was also performed after receiving an out-of-control signal from a EWMA control chart. Results showed that the EWMA built-in estimator performs better than the MLE approach only in small shift size of $\delta=0.5$, but for other shift sizes, the MLE has superior accuracy and precision. Hence, it can be concluded that the MLE is overally more capable than built-in estimators. Step change is one of the potential change types in a process. For example, step changes can arise from tool breakage. Hence, some researches have been performed to estimate the step change point. Samuel et al. [4] used the MLE approach to estimate time of a step change in $\bar{x}$ control chart. Samuel et al. [5] also considered estimating step change point in the variance of a normal process using MLE. Noorossana et al. [6] also used MLE to identify change time in high-yield processes. On the other hand, linear drift changes can also occur in processes. For example, tool wear leads to a linear trend in the process parameters. Therefore, the MLE approach has been developed to estimate drift changes. Perry and Pignatiello [7] introduced a maximum likelihood estimator to estimate linear trend disturbances in a normal process mean. Perry et al. [8] estimated drift change point in a Poisson rate parameter using MLE. They also compared the performance of the linear drift MLE to the one for the step MLE when linear trend disturbance is imposed to the process. Comparison results showed that to detect linear trend disturbances, the step MLE is not as capable as the linear drift MLE. In practice, sometimes the change type is not known a priori. The step and linear drift MLEs are derived under step and linear trend change assumptions. But monotonic MLE does not require any assumption about the change type. Isotonic and antitonic are two types of the monotonic change. The only assumption of the isotonic change is that the process parameter changes increasingly. Also, in antitonic changes, the only assumption is that the process parameter changes in a decreasing manner. Hence, in the monotonic MLE, assumption about the change type can be released. Perry et al. [9] and Noorossana and Shadman [10] used MLE to identify the time when isotonic change occurs in a process fraction with nonconforming and normal process mean, respectively. In this paper, estimating monotonic change point is considered and the MLE approach proposed by Perry et al. [9] is developed to estimate the isotonic change point in the mean of multivariate linear profiles.

The MLE approach can be used in Phase II, and it also requires knowing the underlying distribution. A clustering approach was also introduced by Ghazanfari et al. [11] to estimate a step change point which can be applicable to both Phases I and II; also it does not require any assumptions about an underlying distribution. Alaeddini et al. [12] proposed an approach based on hybrid fuzzy clustering to estimate change point of a process mean with capability of applying in fixed and variable sampling control charts and also normal and non-normal distributions. Finally, Ahmadzadeh [13] and Atashgar [14] proposed approaches for change point estimation in the mean of multivariate processes based on artificial neural network. Amiri and Allahyari [15] comprehensively reviewed the change point estimation literature.

In some situations, the quality of a process or product is defined as a regression relationship between a response variable and one or more explanatory variables entitled profile. This relationship can be characterized by a simple linear, multiple linear, polynomial, or nonlinear regressions. Many studies have been carried out on profile monitoring in both Phases I and II. Kang and Albin [16] proposed two approaches for monitoring simple linear profiles in both Phases I and II. The first approach is using the T^2 control chart, and the second method is applying combined EWMA and R control charts for the residuals. The T^2 control chart, which is a Shewhart-type control chart, can detect large shifts in the profile mean faster than EWMA-R approach. But the EWMA-R approach detects small to moderate mean shifts sooner. The R chart is also used to monitor the regression variation. Among the other approaches of Phase I monitoring of simple linear profile, the global F test and likelihood ratio test (LRT) control chart are two important methods. Mahmoud and Woodall [17] recommended the global F test which can test the similarity of the regression lines of all samples. They also suggested not using the EWMA-R approach in Phase I. They stated that the EWMA chart can detect small to moderate shifts quickly, but the goal of Phase I is not quick detection of process shifts. Also, in EWMA charts, several samples may cause an out-of-control signal. Thus, identification and elimination of out-of-control samples can be hard. Mahmoud et al. [18] used the LRT-based control chart which is more capable than the global F test in sustained shifts, because its probability of out-of-control signal is greater than that of the global F test. Finally, Yeh and Zerehsaz [19] recommended a control chart based on change point to monitor a simple linear profile in Phase I when only one observation in each sample is available.

Monitoring simple linear profile in Phase II has been also considered by several authors. As mentioned above, Kang and Albin [16] proposed T^2 and EWMA-R control charts. Kim et al. [20] suggested coding the x values to make estimated intercept and slope of the regression line independent. Hence, two separate control charts can be used to monitor the intercept and slope. They also applied three EWMA control charts for monitoring the intercept, slope, and variance of the regression line, separately. Gupta et al. [21] applied the coded method and three Shewhart-type control charts instead of three EWMA control charts to detect large shifts as soon as possible. Zou et al.

[22] proposed a control chart based on change point to monitor simple linear profiles in Phase II. Zou et al. [23] designed a self-starting control chart for Phase II monitoring of simple linear profiles when a sufficient number of samples to estimate Phase II control limits is unavailable. Zhang et al. [24] incorporated EWMA procedure to the construction of LRT which can detect shifts in the intercept, slope, variance, and simultaneous shifts by only one control chart. They also added the variable sampling interval (VSI) scheme to their proposed control chart. Mahmoud et al. [25] considered monitoring simple linear profiles in Phase II when only two observations are available for each subgroup. They suggested an approach to estimate the variance of the regression line with two observations. Li and Wang [26] integrated a VSI procedure to EWMA control chart. Using the VSI procedure improved the performance of the control chart to detect shifts. Zhu and Lin [27] proposed a Shewhart-type control chart to monitor the slope of linear profiles in both Phases I and II. Hosseonifard et al. [28] developed three monitoring methods based on artificial neural networks to monitor simple linear profiles. Noorossana and Ayoubi [29] have concentrated on monitoring simple linear profiles in Phase II when the underlying distribution of the observation is unknown. They proposed a nonparametric bootstrap T^2 control chart. They showed that when the control limits are estimated using sufficient data, the bootstrap T^2 control chart performs well in detecting changes.

Some authors also focused on monitoring multiple linear and polynomial profiles. Zou et al. [30] used a variance transformation and recommended using multivariate exponentially weighted moving average control chart to monitor the mean and variance of a general linear profile, simultaneously. They also added the VSI feature to their proposed control chart to improve its performance to detect all shifts. Mahmoud [31] proposed a data reduction approach for multiple linear profiles monitoring in Phase I in which, regardless of the number of explanatory variables, the profile response is monitored using only three parameters, an intercept, a slope, and a variance. The advantage of their method is improving detection of changes in the profile parameters in high-dimensional space. Kazemzadeh et al. [32] developed three methods including LRT, global F test, and T^2 control chart to monitor polynomial profiles in Phase I. Amiri et al. [33] considered monitoring an autocorrelated polynomial profile in an automotive industry.

There are also some applications of more than one profile whose response variables are correlated, known as multivariate profiles. In this manner, separate monitoring of each profile can lead to misleading results. Hence, some approaches have been proposed to monitor multivariate profiles. For example, Noorossana et al. [34] considered Phase I monitoring of multivariate multiple linear profiles. They proposed four methods including T^2 control chart, the LRT approach, Wilk's lambda test, and principal component analysis (PCA). They presented that in sustained shifts, the LRT method has superior performance. Eyvazian et al. [35] considered monitoring multivariate multiple linear profiles in Phase II. They proposed four methods including the multivariate exponentially weighted moving average (MEWMA) method, MEWMA in low-dimensional space, LRT approach, and combined MEWMA and chi-square control charts. Zou et al. [36] also proposed a lasso-based approach to monitor multivariate linear profiles.

Several authors have considered applications of change point in profile monitoring. In Phase II, Zou et al. [22, 30] used the likelihood ratio method to monitor and estimate the time of a step shift in the mean of linear profiles. Eyvazian et al. [35] developed the likelihood ratio method for multivariate multiple linear profiles subject to a step change. In this paper, we develop the MLE to identify the time of a monotonic change in the mean of response variables of multivariate linear profiles in Phase II. Underlying profile model and MLE derivation are discussed in Section 2. The third section contains the description of MEWMA and chi-square control charts for monitoring multivariate linear profiles. In Section 4, simulation results and the performance comparison of the maximum likelihood estimators under single step, multiple step, and linear drift shifts are evaluated. Section 5 discusses about the effect of the smoothing coefficient on control chart's average run length (ARL) as well as the proposed estimator performance. A numerical example is provided in Section 6. Concluding remarks are presented in the final section.

2 Underlying model and MLE derivation

In this paper, the underlying model of multivariate multiple linear profiles considered by Eyvazian et al. [35] is used. The model is expressed as follows for sample j with fixed values of independent variables x and n observations of $(x_{i1},x_{i2},\ldots,x_{iq},y_{i1j},y_{i2j},\ldots,y_{ipj})$:

$$\mathbf{Y}_j = \mathbf{X}\mathbf{B}_j + \mathbf{E}_j \qquad (1)$$

For the jth sample, $\mathbf{Y}_j$ is $n\times p$ matrix of response variables in which p is the number of correlated profiles, $\mathbf{B}_j$ is $(q+1)\times p$ matrix of profile coefficients, and $\mathbf{E}_j$ is $n\times p$ matrix of correlated error terms containing n vectors of p-variate elements which have normal distribution with the mean vector $\underline{0}$ and $p\times p$ variance-covariance matrix of $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{np} \end{bmatrix}$.

Also, $\mathbf{X}$ is $n\times(q+1)$ matrix of independent variables.

When $p=1$, the model of multivariate multiple linear profiles in Eq. (1) collapses to a multiple linear profile. Also when both $p=1$ and $q=1$, the structure of Eq. (1) reduces to a simple linear profile. Hence, multiple and simple linear profiles are special cases of multivariate multiple linear profiles.

To derive the proposed change point estimator, it is assumed that there are no any changes in the covariance matrix of observations. Mean of response variables is in-control before an unknown time, τ; hence, $(\mathbf{XB})_j=(\mathbf{XB})$ for $j=1,2,\ldots,\tau$. The focus of this paper is on Phase II, hence the matrix of $\mathbf{B}$ is known; with the assumption of fixed x values, the matrix of $(\mathbf{XB})$ which is $n\times p$ matrix is known.

It is assumed that monotonic change occurs in the mean of response variables such that $(\mathbf{XB})_{\tau+1}\geq(\mathbf{XB})$, $(\mathbf{XB})_j\geq(\mathbf{XB})_{j-1}$ for $j=\tau+2,\ \tau+3,\ldots,T$, where $(\mathbf{XB})_j$, for $j=\tau+1,\tau+2,\tau+3,\ldots,T$, is obtained from the following equation:

$$(\mathbf{XB})_j=\mathbf{X}\widehat{\mathbf{B}}_j=\mathbf{X}\left[\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{Y}_j\right] \tag{2}$$

Each element of design matrix of $\mathbf{X}$ can be either positive or negative. If all elements of $\mathbf{X}$ are positive, monotonic change in each elements of $\mathbf{B}j$ can be concluded from the monotonic change in each element of $(\mathbf{XB})_j$, or vice versa. Otherwise, if at least one element of X is negative, a monotonic change in each element of $(\mathbf{XB})_j$ may result from an increase in some elements and decrease in some other elements of $\mathbf{B}_j$.

T is the time of the last subgroup at which a control chart shows an out-of-control (OC) signal. The likelihood function is given by the following equation:

$$L\left(\tau,(\mathbf{XB})_{\mathrm{T}-\tau}|\mathbf{Y},\mathbf{X}\right)=\prod_{j=1}^{\tau}\prod_{i=1}^{n}\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}e^{-\frac{1}{2}\left(\mathbf{y}_{ij}-\mathbf{x}_i\mathbf{B}\right)\boldsymbol{\Sigma}^{-1}\left(\mathbf{y}_{ij}-\mathbf{x}_i\mathbf{B}\right)^{\mathrm{T}}}$$
$$\times\prod_{j=\tau+1}^{T}\prod_{i=1}^{n}\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}e^{-\frac{1}{2}\left(\mathbf{y}_{ij}-\mathbf{x}_i\mathbf{B}_j\right)\boldsymbol{\Sigma}^{-1}\left(\mathbf{y}_{ij}-\mathbf{x}_i\mathbf{B}_j\right)^{\mathrm{T}}}, \tag{3}$$

where x_i and $\mathbf{y}_{ij}$ are the ith rows of the $\mathbf{X}$ and $\mathbf{Y}_i$ matrices, respectively. Taking a natural logarithm of the likelihood function leads to

$$Ln\left[L\left(\tau,(\mathbf{XB})_{\mathrm{T}-\tau}|\mathbf{Y},\mathbf{X}\right)\right]=U-\frac{1}{2}\sum_{j=1}^{\tau}\sum_{i=1}^{n}\left(\mathbf{y}_{ij}-\mathbf{x}_i\mathbf{B}\right)\sum^{-1}\left(\mathbf{y}_{ij}-\mathbf{x}_i\mathbf{B}\right)^{\mathrm{T}}-\frac{1}{2}\sum_{j=\tau+1}^{T}\sum_{i=1}^{n}\left[\mathbf{y}_{ij}-\mathbf{x}_i\mathbf{B}_j\right]\sum^{-1}\left[\mathbf{y}_{ij}-\mathbf{x}_i\mathbf{B}_j\right]^{\mathrm{T}}, \tag{4}$$

where U is constant. The following equation is always true in vector and matrix equations:

$$\sum_{i=1}^{n}\left(\mathbf{y}_i-\mathbf{x}_i\mathbf{B}\right)\sum^{-1}\left(\mathbf{y}_i-\mathbf{x}_i\mathbf{B}\right)^{\mathrm{T}}=tr\left[(\mathbf{Y}-\mathbf{XB})\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\mathbf{XB})^{\mathrm{T}}\right] \tag{5}$$

where $tr[(\mathbf{Y}-\mathbf{XB})\Sigma^{-1}(\mathbf{Y}-\mathbf{XB})^{\mathrm{T}}]$ is the trace of the matrix of $(\mathbf{Y}-\mathbf{XB})\sum^{-1}(\mathbf{Y}-\mathbf{XB})^{\mathrm{T}}$. Hence, Eq. (4) can be rewritten as follows:

$$Ln\left[L\left(\tau,(\mathbf{XB})_{\mathrm{T}-\tau}|\mathbf{Y},\mathbf{X}\right)\right]=U-\frac{1}{2}\sum_{j=1}^{\tau}tr\left[\left(\mathbf{Y}_j-(\mathbf{XB})\right)\boldsymbol{\Sigma}^{-1}\left(\mathbf{Y}_j-(\mathbf{XB})\right)^{\mathrm{T}}\right]-\frac{1}{2}\sum_{j=\tau+1}^{T}tr\left[\left(\mathbf{Y}_j-(\mathbf{XB})_j\right)\sum^{-1}\left(\mathbf{Y}_j-(\mathbf{XB})_j\right)^{\mathrm{T}}\right] \tag{6}$$

The proposed change point estimator is expressed as follows:

$$\widehat{\boldsymbol{\tau}}=\underset{0\leq t\leq T-1}{\arg\max}\left\{-\frac{1}{2}\sum_{j=1}^{t}tr\left[\left(\mathbf{Y}_j-(\mathbf{XB})\right)\sum^{-1}\left(\mathbf{Y}_j-(\mathbf{XB})\right)^{\mathrm{T}}\right]-\frac{1}{2}\sum_{j=t+1}^{T}tr\left[\left(\mathbf{Y}_j-(\widehat{\mathbf{XB}})_j\right)\sum^{-1}\left(\mathbf{Y}_j-(\widehat{XB})_j\right)^{\mathrm{T}}\right]\right\} \tag{7}$$

In order to evaluate $\widehat{\tau}$, $\left(\widehat{\mathbf{XB}}\right)_{\mathrm{j}}$ must be found. For this purpose, an initial estimate of each element of mean matrix is required.

$$\left[\left(\widetilde{\mathbf{XB}}\right)_j\right]_{iu}=\left\{\begin{array}{lll}\left[(\mathbf{X}B)_j\right]_{iu} & \text{if} & \left[(\mathbf{X}B)_j\right]_{iu}\geq(\mathbf{XB})_{iu}\\ (\mathbf{XB})_{iu} & \text{if} & \left[(\mathbf{X}B)_j\right]_{iu}<(\mathbf{XB})_{iu}\end{array}\right\}\text{ for } i=1,2,\ldots,n;\ \ u=1,\ldots,p\ \text{ and } j=\tau+1,\tau+2,\ldots,T \tag{8}$$

The subscript iu stands for the element of ith row and uth column of the matrix of interest. Now we can estimate $\left(\widehat{\mathbf{XB}}\right)_j$ by solving the following convex program:

$$\underset{\tilde{B}_{T-\tau}}{\text{Maximize}}\left\{-\frac{1}{2}\sum_{j=1}^{\tau} tr\left[\left(\mathbf{Y}_j-\mathbf{XB}\right)\sum^{-1}\left(\mathbf{Y}_j-\mathbf{XB}\right)^{\mathrm{T}}\right]-\frac{1}{2}\sum_{j=\tau+1}^{T} tr\left[\left(\mathbf{Y}_j-\left(\widetilde{\mathbf{XB}}\right)_j\right)\sum^{-1}\left(\mathbf{Y}_j-\left(\widetilde{\mathbf{XB}}\right)_j\right)^{\mathrm{T}}\right]\right\} \quad (9)$$

subject to $\left(\widetilde{\mathbf{XB}}\right)_j \geq \left(\widetilde{\mathbf{XB}}\right)_{j-1}$ for $j=\tau+1,\tau+2,\ldots,T$

To find $\left(\widehat{\mathbf{XB}}\right)_{T-\tau}$, we use fitting isotonic regression as Perry et al. [9] to each element of the matrix of $\left(\widetilde{\mathbf{XB}}\right)_{T-\tau}$.

$$\left[\left(\widehat{\mathbf{XB}}\right)_{iu}\right]_{T-\tau} = I\left(\left[\left(\widetilde{\mathbf{XB}}\right)_{iu}\right]_{T-\tau}\right) \quad \text{for } i=1,2,\ldots,n \text{ and } u=1,2,\ldots,p \quad (10)$$

Among the isotonic regression algorithms, the pool adjacent violators (PAV) algorithm described by Best and Chakravarti [37] is used in this paper. PAV algorithm must be applied for each element of $\widetilde{\mathbf{XB}}_{T-\tau}$ matrix, i.e., $n\times p$ times.

3 Monitoring method

In Phase II, Eyvazian et al. [35] proposed four methods to monitor multivariate multiple linear profiles, one of which is combined MEWMA and chi-square control charts. In this paper, the proposed maximum likelihood estimator of the change point is applied after getting an out-of-control signal from combined MEWMA and chi-square control charts. For the jth random sample, the MEWMA statistic is as follows:

$$\mathbf{z}_j = \lambda\bar{\boldsymbol{\varepsilon}}_j^{\mathrm{T}} + (1-\lambda)\mathbf{z}_{j-1}, \quad j=1,2,\ldots \quad (11)$$

where $\mathbf{z}_0$ is a $p\times 1$ vector of zeros. The average error vector $\bar{\varepsilon}_j = \left(\bar{\varepsilon}_{1j},\bar{\varepsilon}_{2j},\ldots,\bar{\varepsilon}_{pj}\right)$ has a p-variate normal distribution with mean vector zero and known covariance matrix of $\boldsymbol{\Sigma}_{\bar{\varepsilon}}= n^{-1}\boldsymbol{\Sigma}$ in which $\bar{\varepsilon}_{uj} = \frac{\sum_{i=1}^{n}\varepsilon_{iuj}}{n}$, for $u=1,2,..,p$. Equation (12) shows a covariance matrix of the MEWMA statistic.

$$\sum_{\mathbf{z}} = \frac{\lambda}{(2-\lambda)}\sum_{\bar{\varepsilon}} = \frac{\lambda}{n(2-\lambda)}\sum \quad (12)$$

Finally, the following statistic is used for MEWMA control chart:

$$T^2_{\mathbf{z}_j} = \mathbf{z}_j^T\sum\nolimits_z^{-1}\mathbf{z}_j, \quad j=1,2,\ldots \quad (13)$$

The upper control limit (UCL) of MEWMA control chart must be chosen to achieve the specified in-control (IC) ARL. Chi-square statistic is also expressed as follows:

$$\chi_j^2 = \sum_{i=1}^{n}\left(\boldsymbol{\varepsilon}_{ij}\sum^{-1}\boldsymbol{\varepsilon}_{ij}^T\right) \quad (14)$$

χ_j^2 has a chi-square distribution with np degrees of freedom. Hence with a given α, UCL of the chi-square control chart is $\chi^2_{np,\alpha}$.

4 Performance evaluation of the proposed estimator

In this section, Monte Carlo simulation with 5,000 replications is used to evaluate the performance of the proposed estimator. We also compare the performance of the proposed estimator to the step and linear drift change point estimators suggested by Kazemzadeh et al. [38] (see Appendixes 1 and 2) with the only difference that, in this paper, shifts occur in the mean matrix of response variables, i.e., **XB**, instead of the parameter matrix of **B**. Hence, the elements of **B** matrix do not have any restrictions of changing monotonically. If all elements of **X** are positive and monotonic changes take place in **XB** elements, the elements of **B** also change monotonically and vice versa. In simulation and case study sections of this paper, the elements of design matrix **X** are positive, so we consider three types of increasing changes, i.e., single step change, multiple step changes, and linear drift disturbance for the elements of **B** leading to increasing changes of **XB** elements. But if matrix of **X** has at least a negative element, increasing changes in the elements of **B** may cause a decrease in some elements of **XB**, which is not the case in this paper.

Comparisons are carried out after getting an out-of-control alarm from combined MEWMA (with λ=0.2) and chi-square control charts and when process mean, **XB**, is subject to a monotonic change. To conduct our simulation study, we deal with false alarms as mentioned by Perry et al. [8] and

Pignatiello and Samuel [5]. The MEWMA control chart incorporates previous data; hence, several observations may be combined together to produce a false alarm. So, all samples prior to false alarms are eliminated and the first sample after a false alarm is considered as the first sample. Subsequently, the out-of-control samples under different shift types in **XB** elements will be generated until the combined MEWMA and chi-square method signals an out-of-control condition. At this point, change point estimators are applied to estimate a time of change. For simulation studies, we consider the overall in-control ARL of 200. Hence, the UCL of the MEWMA control chart must be chosen to achieve the specified in-control ARL of 400. In this paper, upper control limit of MEWMA control chart is obtained using 50,000 simulation runs. We also consider α=0.0025 leading to in-control ARL of 400 for the chi-square control chart.

The accuracies and precisions of the proposed estimator are reported in Tables 1, 2, 3, 4, 5, 6, and 7. E(T) is the average time to observe a signal from the combined MEWMA and chi-square control charts, so estimated average run length ($\widehat{\mathrm{ARL}}$) would be equal to E(T)−τ. Each table consists of two sections: one for accuracy and the other for precision of the change point estimators. In the section of accuracy performances, standard errors of the estimators are shown in the parentheses. In precision performances section, $\widehat{P}_0=\widehat{p}(|\widehat{\tau}-\tau|=0)$, $\widehat{P}_1=\widehat{p}(|\widehat{\tau}-\tau|=1)$, $\widehat{P}_3=\widehat{p}(|\widehat{\tau}-\tau|=3)$, $\widehat{P}_5=\widehat{p}(|\widehat{\tau}-\tau|=5)$, $\widehat{P}_7=\widehat{p}(|\widehat{\tau}-\tau|=7)$, and $\widehat{P}_{10}=\widehat{p}(|\widehat{\tau}-\tau|=10)$ show precisions of the estimators. Precisions of step, linear drift, and monotonic maximum likelihood estimators are shown in the first, second, and third rows, respectively.

First, we consider the multivariate multiple linear profiles model. Accuracy and precision of the change point estimators are given in Tables 1, 2, and 3. The following in-control model considered by Eyvazian et al. [35] is used here:

$$Y_1=3+2x_1+x_2+\varepsilon_1,\quad Y_2=2+x_1+x_2+\varepsilon_2 \tag{15}$$

Independent variables (x_1 and x_2) have fixed values of (2, 1), (4, 2), (6, 3), and (8, 2). The vector of $(\varepsilon_1,\varepsilon_2)^T$ has bivariate normal distribution with mean vector zero. Covariance matrix of $\sum=\begin{bmatrix}1 & 0.5\\ 0.5 & 1\end{bmatrix}$ is also considered to conduct the simulations. Values of 11.1 and 23.7745 are set for the upper control limits of MEWMA and chi-square control charts, respectively.

If single increasing step change is imposed to the parameters, we consider a matrix, entitled **K**, with the same dimension of **B** as follows:

$$\mathbf{B}_j=\mathbf{B}+\mathbf{K},\quad \text{for } j=\tau+1,\tau+2,\ldots,T \tag{16}$$

$$\begin{bmatrix}\beta_{01j} & \beta_{02j} & \cdots & \beta_{0pj}\\ \beta_{11j} & \beta_{12j} & \cdots & \beta_{1pj}\\ \vdots & \vdots & \ddots & \vdots\\ \beta_{q1j} & \beta_{q2j} & \cdots & \beta_{qpj}\end{bmatrix}=\begin{bmatrix}\beta_{01} & \beta_{02} & \cdots & \beta_{0p}\\ \beta_{11} & \beta_{12} & \cdots & \beta_{1p}\\ \vdots & \vdots & \ddots & \vdots\\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp}\end{bmatrix}+\begin{bmatrix}k_{01} & k_{02} & \cdots & k_{0p}\\ k_{11} & k_{12} & \cdots & k_{1p}\\ \vdots & \vdots & \ddots & \vdots\\ k_{q1} & k_{q2} & \cdots & k_{qp}\end{bmatrix} \tag{17}$$

$k_{01}, k_{11},\ldots,k_{q1}$ are magnitudes of change in the parameters of $\beta_{01},\beta_{11},\ldots,\beta_{q1}$, respectively. Also, $k_{02},k_{12},\ldots,k_{q2}$ are magnitudes of change in the parameters of the second profile, and so on.

The following equation is concluded from Eq. (16):

$$(\mathbf{XB})_j=\mathbf{XB}+\mathbf{XK}\,,\quad \text{for } j=\tau+1,\tau+2,\ldots,T \tag{18}$$

For three increasing step changes, three change points are τ_1=25,τ_2=35, and τ_3=45, respectively. Matrixes of K_1, K_2, and K_3 ($\mathbf{K}_3$>$\mathbf{K}_2$>$\mathbf{K}_1$) are with the same dimension of **B**. Changes are defined as follows:

$$\begin{aligned}&\mathbf{B}_j=\mathbf{B}+\mathbf{K}_1,\ \text{so}\ (\mathbf{XB})_j=\mathbf{XB}+\mathbf{XK}_1,\ \text{for } j=26,27,\ldots,35\\ &\mathbf{B}_j=\mathbf{B}+\mathbf{K}_2,\ \text{so}\ (\mathbf{XB})_j=\mathbf{XB}+\mathbf{XK}_2,\ \text{for } j=36,37,\ldots,45\\ &\mathbf{B}_j=\mathbf{B}+\mathbf{K}_3,\ \text{so}\ (\mathbf{XB})_j=\mathbf{XB}+\mathbf{XK}_3,\ \text{for } j=46,47,\ldots,T\end{aligned} \tag{19}$$

We also investigate performance of the estimators under increasing drift changes in the parameters. The change is defined using following equation:

$$\begin{aligned}&\mathbf{B}_j=\mathbf{B}+(j-\tau)\mathbf{K},\ \text{so}\ (\mathbf{XB})_j\\ &=\mathbf{XB}+(j-\tau)\mathbf{XK},\ \text{for } j=\tau+1,\tau+2,\ldots,T\end{aligned} \tag{20}$$

Results of Table 1 show that when single step shift is exposed to the elements of **XB**, the proposed monotonic estimator has superior performance in small to moderate shift sizes. But in moderate to large shifts, the step estimator has better performance, because it obtains under the assumption of step change. The drift estimator has an acceptable uniform performance from small to large shifts.

From Table 2, it is clear that the performance of monotonic estimator is much better than step and drift estimators under three step changes in small to moderate changes. The performance of the step estimator improves as the magnitude of shifts increases. But the performance of the drift estimator is

Table 1 Accuracy and precision performances of the step, linear drift, and monotonic MLEs for the multivariate linear profiles under a single step change in the parameters. τ=25, N=5,000 replications

$\mathbf{K}=\begin{bmatrix} k_{01} & k_{02} \\ k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$			$\begin{bmatrix} 0 & 0 \\ 0.05 & 0.0375 \\ 0 & 0.05 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0.05 & 0.05 \\ 0 & 0.05 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0.075 & 0.05 \\ 0 & 0.075 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0.075 & 0.075 \\ 0 & 0.075 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0.1 & 0.1 \\ 0 & 0.075 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix}$
$\widehat{ARL}=E(T)-\tau$			28.8158	21.3224	14.1478	9.9804	6.9056	3.8618
Accuracy performances	$\overline{\widehat{\tau}}_{SC}$		31.0964 (0.2105)	29.2492 (0.172)	27.0598 (0.1199)	26.3366 (0.1004)	25.9006 (0.0782)	25.558 (0.0503)
	$\overline{\widehat{\tau}}_{LD}$		23.6302 (0.245)	22.4408 (0.1876)	21.8208 (0.1373)	21.8062 (0.1094)	22.1344 (0.0839)	22.9976 (0.0518)
	$\overline{\widehat{\tau}}_{Monotonic}$		28.1386 (0.2279)	23.865 (0.143)	20.4758 (0.0939)	19.3704 (0.0835)	18.496 (0.0792)	17.8056 (0.0781)
Precision performances	$\widehat{P}_0$	Step	0.084	0.1194	0.1736	0.2218	0.2832	0.4422
		Drift	0.0446	0.066	0.084	0.1086	0.1632	0.2822
		Monotonic	0.05	0.0596	0.0612	0.0554	0.045	0.0222
	$\widehat{P}_1$	Step	0.2066	0.2466	0.3316	0.4046	0.473	0.6212
		Drift	0.1232	0.1552	0.202	0.2666	0.364	0.5684
		Monotonic	0.1534	0.1842	0.1952	0.1814	0.1534	0.1044
	$\widehat{P}_3$	Step	0.3654	0.4252	0.53	0.5906	0.655	0.781
		Drift	0.2402	0.3026	0.4002	0.5008	0.624	0.831
		Monotonic	0.3448	0.3968	0.4214	0.4082	0.3614	0.302
	$\widehat{P}_5$	Step	0.4766	0.5406	0.6412	0.7038	0.7668	0.894
		Drift	0.3366	0.4156	0.5394	0.645	0.7696	0.9136
		Monotonic	0.4776	0.5528	0.5872	0.579	0.5304	0.4824
	$\widehat{P}_7$	Step	0.5654	0.623	0.724	0.7818	0.8468	0.9626
		Drift	0.4152	0.5082	0.6378	0.7398	0.8464	0.9428
		Monotonic	0.593	0.6726	0.709	0.6994	0.6638	0.6188
	$\widehat{P}_{10}$	Step	0.655	0.7222	0.8134	0.8656	0.9228	0.9826
		Drift	0.5238	0.6286	0.7456	0.8322	0.9062	0.9644
		Monotonic	0.7102	0.7964	0.8296	0.8178	0.797	0.7622

Table 2 Accuracy and precision performances of the step, linear drift, and monotonic MLEs for the multivariate linear profiles under three step changes. τ_1=25, τ_2=35, τ_3=45 and N=5,000 replications

$\mathbf{K}_1 = \begin{bmatrix} k_{01} & k_{02} \\ k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$			$\begin{bmatrix} 0.2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.8 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\mathbf{K}_2 = \begin{bmatrix} k_{01} & k_{02} \\ k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$			$\begin{bmatrix} 0.3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.45 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.6 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.6 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\mathbf{K}_3 = \begin{bmatrix} k_{01} & k_{02} \\ k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$			$\begin{bmatrix} 0.4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.8 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.8 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\widehat{\mathrm{ARL}} = \mathrm{E}(T) - \tau$			29.7568	25.1752	22.0106	21.1468	21.0488	21.0038
Accuracy performances	$\bar{\hat{\tau}}_{SC}$		35.4028 (0.201)	29.3526 (0.1719)	31.3798 (0.1677)	30.3864 (0.1617)	27.6918 (0.1355)	29.2958 (0.1436)
	$\bar{\hat{\tau}}_{LD}$		25.9134 (0.2351)	17.35 (0.1742)	17.4258 (0.1463)	13.9206 (0.1019)	13.7822 (0.1043)	14.5428 (0.1137)
	$\bar{\hat{\tau}}_{Monotonic}$		34.5806 (0.1587)	26.1114 (0.1082)	24.3958 (0.0884)	21.835 (0.0598)	21.5904 (0.0604)	21.7778 (0.0552)
Precision performances	$\widehat{P}_0$	Step	0.0432	0.1544	0.1454	0.3102	0.4088	0.4394
		Drift	0.025	0.0212	0.0268	0.0062	0.0048	0.0032
		Monotonic	0.0372	0.1062	0.1162	0.2064	0.219	0.2458
	$\widehat{P}_1$	Step	0.1122	0.3026	0.2808	0.4712	0.5622	0.5652
		Drift	0.0704	0.0604	0.0792	0.013	0.011	0.009
		Monotonic	0.1	0.273	0.2892	0.419	0.4158	0.4432
	$\widehat{P}_3$	Step	0.2164	0.4718	0.4276	0.5812	0.677	0.6524
		Drift	0.1662	0.1558	0.1952	0.0664	0.0416	0.0268
		Monotonic	0.2132	0.4936	0.526	0.661	0.6536	0.672
	$\widehat{P}_5$	Step	0.2938	0.5642	0.5094	0.6288	0.727	0.694
		Drift	0.2424	0.2614	0.3226	0.1824	0.1394	0.1146
		Monotonic	0.308	0.6376	0.6802	0.7962	0.7846	0.8006
	$\widehat{P}_7$	Step	0.3702	0.6314	0.568	0.6548	0.7548	0.7118
		Drift	0.3236	0.3604	0.4412	0.3228	0.287	0.2764
		Monotonic	0.3962	0.7424	0.7918	0.8704	0.86	0.8832
	$\widehat{P}_{10}$	Step	0.4656	0.6942	0.6332	0.6762	0.7776	0.7256
		Drift	0.4342	0.4954	0.583	0.5312	0.5122	0.5148
		Monotonic	0.54	0.846	0.9046	0.9316	0.9288	0.9392

Table 3 Accuracy and precision performances of the step, linear drift, and monotonic MLEs for the multivariate linear profiles under drift changes in the parameters. τ=25, N=5,000 replications

$\mathbf{K} = \begin{bmatrix} k_{01} & k_{02} \\ k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$			$\begin{bmatrix} 0 & 0.001 \\ 0.003 & 0 \\ 0 & 0.001 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.001 \\ 0.01 & 0 \\ 0 & 0.001 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.01 \\ 0.02 & 0 \\ 0 & 0.0015 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.01 \\ 0.025 & 0 \\ 0 & 0.005 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.01 \\ 0.025 & 0 \\ 0 & 0.01 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.05 \\ 0.03 & 0 \\ 0 & 0.05 \end{bmatrix}$
$\widehat{ARL} = E(T) - \tau$			27.5162	12.4238	8.2334	7.3032	7.5036	6.2884
Accuracy performances	$\bar{\bar{\tau}}_{SC}$		40.0858 (0.1839)	31.2714 (0.1056)	28.7422 (0.0795)	28.2976 (0.0732)	28.2482 (0.0715)	27.8138 (0.0653)
	$\bar{\bar{\tau}}_{LD}$		32.49 (0.1952)	26.1686 (0.1094)	25.005 (0.0811)	24.74 (0.0751)	24.9284 (0.0751)	24.6564 (0.068)
	$\bar{\bar{\tau}}_{\text{Monotonic}}$		34.9584 (0.134)	24.7488 (0.0863)	21.8618 (0.081)	21.2452 (0.0791)	21.1282 (0.0809)	19.4434 (0.079)
Precision performances	$\widehat{P}_0$	Step	0.014	0.0298	0.0544	0.0614	0.0558	0.0644
		Drift	0.0256	0.0736	0.119	0.1312	0.1348	0.156
		Monotonic	0.0282	0.0768	0.094	0.1024	0.1	0.0814
	$\widehat{P}_1$	Step	0.0416	0.0864	0.1518	0.1734	0.178	0.2004
		Drift	0.0746	0.1942	0.3154	0.3486	0.3508	0.422
		Monotonic	0.0752	0.2222	0.291	0.2948	0.2896	0.2264
	$\widehat{P}_3$	Step	0.0974	0.2192	0.3812	0.4146	0.4202	0.4788
		Drift	0.173	0.4068	0.6012	0.6654	0.6612	0.7522
		Monotonic	0.1666	0.4782	0.567	0.5708	0.5538	0.4486
	$\widehat{P}_5$	Step	0.1556	0.3764	0.5696	0.6154	0.6246	0.6692
		Drift	0.2546	0.5762	0.7882	0.8388	0.8368	0.8942
		Monotonic	0.2502	0.6826	0.7346	0.7116	0.6982	0.597
	$\widehat{P}_7$	Step	0.2168	0.5184	0.7098	0.76	0.7858	0.866
		Drift	0.342	0.7182	0.8926	0.9178	0.92	0.9394
		Monotonic	0.3368	0.8192	0.8164	0.7948	0.7916	0.7118
	$\widehat{P}_{10}$	Step	0.3138	0.6852	0.9032	0.9622	0.9606	0.9764
		Drift	0.4578	0.8542	0.944	0.9554	0.9516	0.9576
		Monotonic	0.4784	0.923	0.8924	0.883	0.8706	0.826

Table 4 Accuracy and precision performances of the step, linear drift, and monotonic MLEs for the multiple linear profiles under a single step change in the parameters. τ=25, N=5,000 replications

$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$			$\begin{bmatrix} 0.15 \\ 0.02 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.2 \\ 0.025 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.4 \\ 0.025 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.6 \\ 0.025 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.6 \\ 0.05 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.15 \\ 0 \end{bmatrix}$
$\widehat{\mathrm{ARL}} = \mathrm{E}(T) - \tau$			20.5238	12.6628	5.6122	3.599	3.0642	1.5856
Accuracy performances	$\overline{\overline{\tau}}_{SC}$		30.81 (0.1985)	27.5728 (0.1381)	25.714 (0.0822)	25.289 (0.063)	25.3958 (0.0555)	25.0534 (0.0173)
	$\overline{\overline{\tau}}_{LD}$		25.7592 (0.2155)	23.6888 (0.1467)	22.4608 (0.0873)	22.8234 (0.0652)	23.0572 (0.0595)	24.4762 (0.0225)
	$\overline{\overline{\tau}}_{\text{Monotonic}}$		26.6924 (0.1694)	22.4954 (0.0928)	20.229 (0.0704)	19.8654 (0.0696)	19.7726 (0.0687)	19.7312 (0.0677)
Precision performances	$\widehat{P}_0$	Step	0.0878	0.131	0.2484	0.3634	0.4108	0.8106
		Drift	0.0598	0.0898	0.185	0.294	0.3544	0.7006
		Monotonic	0.0734	0.0892	0.0826	0.0644	0.0634	0.0568
	$\widehat{P}_1$	Step	0.1958	0.2676	0.436	0.5374	0.5768	0.903
		Drift	0.1438	0.2012	0.3972	0.5738	0.6474	0.9308
		Monotonic	0.2084	0.2572	0.2598	0.2228	0.2102	0.2026
	$\widehat{P}_3$	Step	0.354	0.4498	0.6106	0.729	0.7788	0.9872
		Drift	0.2854	0.3848	0.6482	0.807	0.8548	0.9792
		Monotonic	0.4312	0.5172	0.5174	0.4874	0.4772	0.4644
	$\widehat{P}_5$	Step	0.4622	0.563	0.7458	0.876	0.923	0.9914
		Drift	0.3902	0.5216	0.777	0.8828	0.9122	0.9856
		Monotonic	0.5728	0.6702	0.6754	0.6602	0.654	0.6452
	$\widehat{P}_7$	Step	0.5484	0.6534	0.8336	0.937	0.9628	0.9948
		Drift	0.483	0.6268	0.8466	0.9178	0.9372	0.991
		Monotonic	0.675	0.78	0.7854	0.7712	0.7668	0.7604
	$\widehat{P}_{10}$	Step	0.649	0.7564	0.9176	0.965	0.9748	0.9974
		Drift	0.6018	0.7428	0.904	0.9456	0.9558	0.9948
		Monotonic	0.7786	0.876	0.8784	0.872	0.8678	0.8728

Table 5 Accuracy and precision performances of the step, linear drift, and monotonic MLEs for the multiple linear profiles under three step changes. τ_1=25, τ_2=35, τ_3=45 and N=5,000 replications

$\mathbf{k}_1 = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$			$\begin{bmatrix} 0 \\ 0 \\ 0.085 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.095 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.52 \end{bmatrix}$
$\mathbf{k}_2 = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$			$\begin{bmatrix} 0 \\ 0 \\ 0.095 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.52 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.54 \end{bmatrix}$
$\mathbf{k}_3 = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$			$\begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.15 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.54 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0.56 \end{bmatrix}$
$\widehat{\mathrm{ARL}} = \mathrm{E}(T) - \tau$			33.2766	25.787	21.5282	21.019	21.0038	21.004
Accuracy performances	$\overline{\hat{\tau}}_{SC}$		32.4376 (0.2272)	32.4428 (0.1895)	32.3494 (0.1528)	31.8636 (0.1518)	24.7308 (0.0304)	24.8178 (0.0316)
	$\overline{\hat{\tau}}_{LD}$		21.7368 (0.2533)	22.0764 (0.2105)	20.0606 (0.1428)	15.7596 (0.0751)	14.5896 (0.0653)	14.7184 (0.0615)
	$\overline{\hat{\tau}}_{\text{Monotonic}}$		28.3138 (0.162)	25.094 (0.1106)	23.122 (0.0841)	21.1918 (0.0608)	22.1784 (0.0501)	22.2656 (0.0488)
Precision performances	$\widehat{P}_0$	Step	0.0896	0.097	0.102	0.389	0.7556	0.7626
		Drift	0.026	0.0232	0.0448	0.0066	0.0044	0.0054
		Monotonic	0.069	0.0806	0.093	0.1464	0.232	0.2518
	$\widehat{P}_1$	Step	0.1958	0.2082	0.2084	0.531	0.8944	0.9016
		Drift	0.0604	0.067	0.1186	0.0206	0.0094	0.012
		Monotonic	0.201	0.2328	0.2644	0.355	0.4718	0.4822
	$\widehat{P}_3$	Step	0.3312	0.3396	0.3388	0.6112	0.9612	0.9682
		Drift	0.133	0.1598	0.2794	0.092	0.0278	0.0286
		Monotonic	0.4004	0.4712	0.5176	0.6252	0.7308	0.7352
	$\widehat{P}_5$	Step	0.4284	0.42	0.4316	0.641	0.9822	0.984
		Drift	0.211	0.2476	0.4276	0.241	0.106	0.0986
		Monotonic	0.5406	0.6178	0.6912	0.7666	0.849	0.856
	$\widehat{P}_7$	Step	0.5012	0.4792	0.505	0.6552	0.9898	0.9888
		Drift	0.2842	0.3314	0.5458	0.4272	0.2814	0.2658
		Monotonic	0.639	0.7182	0.8094	0.8488	0.9074	0.9136
	$\widehat{P}_{10}$	Step	0.5812	0.5514	0.622	0.6648	0.9918	0.9906
		Drift	0.3828	0.4498	0.6794	0.675	0.5848	0.5846
		Monotonic	0.7498	0.8288	0.9158	0.919	0.955	0.9572

Table 6 Accuracy and precision performances of the step, linear drift, and monotonic MLEs for the simple linear profile under a single step change in the parameters. τ=25, N=5,000 replications.

$\mathbf{k}=\begin{bmatrix}k_1\\k_2\end{bmatrix}$			$\begin{bmatrix}0.2\\0.01\end{bmatrix}$	$\begin{bmatrix}0.2\\0.025\end{bmatrix}$	$\begin{bmatrix}0.25\\0.05\end{bmatrix}$	$\begin{bmatrix}0.4\\0.075\end{bmatrix}$	$\begin{bmatrix}0.6\\0.1\end{bmatrix}$	$\begin{bmatrix}1\\0.2\end{bmatrix}$
$\widehat{\mathrm{ARL}}=\mathrm{E}(T)-\tau$			33.1546	20.0914	9.4656	4.8666	3.1716	1.7484
Accuracy performances	$\overline{\overline{\hat{\tau}}}_{SC}$		36.7964 (0.2919)	30.8874 (0.1898)	27.3596 (0.1058)	26.1218 (0.0666)	25.919 (0.0519)	25.0572 (0.0209)
	$\overline{\overline{\hat{\tau}}}_{LD}$		29.9492 (0.3179)	25.1086 (0.1972)	22.6396 (0.1083)	22.6766 (0.0695)	23.3208 (0.0486)	24.4192 (0.0207)
	$\overline{\overline{\hat{\tau}}}_{\text{Monotonic}}$		33.2872 (0.319)	24.529 (0.1492)	20.4004 (0.0853)	19.6138 (0.0774)	19.3138 (0.0786)	19.1782 (0.079)
Precision performances	$\widehat{P}_0$	Step	0.0596	0.1012	0.1882	0.3018	0.4318	0.7932
		Drift	0.0362	0.0618	0.1228	0.2334	0.3514	0.6492
		Monotonic	0.049	0.0658	0.0852	0.085	0.0654	0.0604
	$\widehat{P}_1$	Step	0.1408	0.2202	0.3554	0.4896	0.576	0.8886
		Drift	0.0972	0.1496	0.282	0.4868	0.6616	0.9078
		Monotonic	0.1494	0.2042	0.2566	0.2358	0.217	0.2084
	$\widehat{P}_3$	Step	0.2624	0.3774	0.5264	0.6668	0.7422	0.9842
		Drift	0.1936	0.3036	0.514	0.7498	0.8772	0.9782
		Monotonic	0.3086	0.4228	0.5034	0.481	0.4658	0.4602
	$\widehat{P}_5$	Step	0.3544	0.4892	0.648	0.7928	0.8822	0.991
		Drift	0.2844	0.426	0.6602	0.8542	0.9344	0.9892
		Monotonic	0.4274	0.5676	0.6576	0.6472	0.631	0.6224
	$\widehat{P}_7$	Step	0.4334	0.5844	0.7364	0.8878	0.9496	0.9942
		Drift	0.3636	0.5276	0.7608	0.8994	0.9552	0.9924
		Monotonic	0.525	0.6746	0.7614	0.7482	0.7428	0.7246
	$\widehat{P}_{10}$	Step	0.5262	0.6838	0.8332	0.9568	0.9822	0.9964
		Drift	0.4602	0.6344	0.8422	0.9388	0.9706	0.996
		Monotonic	0.6198	0.787	0.8522	0.8444	0.838	0.8296

Table 7 Accuracy and precision performances of the step, linear drift, and monotonic MLEs for the simple linear profile under three step changes. τ_1=25,τ_2=35,τ_3=45 and N=5,000 replications

$\mathbf{k}_1 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$			$\begin{bmatrix} 0 \\ 0.025 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.05 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.125 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.175 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$
$\mathbf{k}_2 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$			$\begin{bmatrix} 0 \\ 0.035 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.075 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.125 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.15 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.4 \end{bmatrix}$
$\mathbf{k}_3 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$			$\begin{bmatrix} 0 \\ 0.05 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.15 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.175 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.6 \end{bmatrix}$
$\widehat{\text{ARL}} = \text{E}(T)-\tau$			50.063	26.1698	22.4144	21.835	21.199	21
Accuracy performances	$\overline{\widehat{\tau}}_{SC}$		44.7834 (0.3323)	34.1652 (0.1896)	29.2466 (0.1542)	27.9928 (0.1343)	27.9574 (0.1289)	25.241 (0.0417)
	$\overline{\widehat{\tau}}_{LD}$		33.356 (0.367)	22.3576 (0.1929)	15.6786 (0.1167)	14.9052 (0.0989)	14.4098 (0.0854)	18.393 (0.06)
	$\overline{\widehat{\tau}}_{Monotonic}$		43.2356 (0.3133)	25.2644 (0.1155)	20.838 (0.0819)	20.5428 (0.0762)	20.4912 (0.0738)	22.1054 (0.0559)
Precision performances	$\widehat{P}_0$	Step	0.0212	0.0774	0.225	0.3142	0.4282	0.668
		Drift	0.0148	0.0334	0.0188	0.0112	0.0058	0.007
		Monotonic	0.018	0.0694	0.1042	0.1144	0.1358	0.254
	$\widehat{P}_1$	Step	0.055	0.165	0.4084	0.5136	0.608	0.8424
		Drift	0.0374	0.0906	0.051	0.0296	0.0168	0.0254
		Monotonic	0.0558	0.2032	0.2964	0.304	0.3298	0.4832
	$\widehat{P}_3$	Step	0.1212	0.2872	0.567	0.6702	0.7288	0.9272
		Drift	0.0942	0.2072	0.152	0.1032	0.0658	0.1566
		Monotonic	0.1424	0.4116	0.555	0.564	0.5792	0.7274
	$\widehat{P}_5$	Step	0.179	0.3692	0.647	0.7354	0.7744	0.952
		Drift	0.142	0.3164	0.2882	0.237	0.1854	0.414
		Monotonic	0.2106	0.548	0.6984	0.7022	0.7192	0.8388
	$\widehat{P}_7$	Step	0.2264	0.4428	0.6936	0.7708	0.7972	0.968
		Drift	0.1978	0.4156	0.4186	0.3974	0.3432	0.6468
		Monotonic	0.276	0.6696	0.7888	0.799	0.8008	0.8978
	$\widehat{P}_{10}$	Step	0.301	0.5486	0.7406	0.7986	0.814	0.985
		Drift	0.267	0.5422	0.5946	0.5968	0.5808	0.8614
		Monotonic	0.3704	0.8138	0.8728	0.8782	0.8806	0.9434

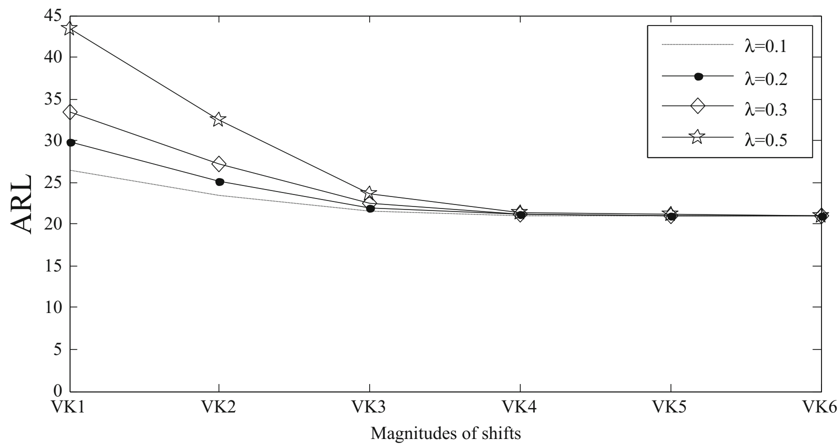

Fig. 1 Effect of smoothing constant on ARL of the combined MEWMA-chi[2] control charts under three increasing step changes in the mean of multivariate linear profiles

not acceptable, and it deteriorates when shift sizes become larger.

When linear drift changes occur in the elements of **XB**, results of Table 3 show that the drift estimator has superior performance, because it is derived under the assumption of the linear drift change. Monotonic estimator performs well in small to moderate shifts. Also, step estimator performs well and has better performance compared to monotonic estimator in moderate to large shifts.

The performance of the estimators is also investigated for multiple and simple linear profiles which are the special cases of multivariate multiple linear profiles. Results of the increasing drift changes are only reported for the multivariate profiles model and not reported here for the special cases because of identical consequences.

The following model is considered for the in-control multiple linear profile using (2, 1), (4, 3), (6, 2), (8, 4), (4, 4), (3, 2), (1, 3), and (4, 1) as the fixed values of (x_1 and x_2). The error term is a normal random variable with mean 0 and variance 1.

$$Y = 3 + 2x_1 + x_2 + \varepsilon \tag{21}$$

Values of 8.3 and 23.7745 are considered for the upper control limits of MEWMA and chi-square control charts, respectively. Simulation results are presented in Tables 4 and 5.

Tables 4 and 5 have predictable results that a step estimator has better performance with the existence of single step shifts; also, monotonic estimator performs well when three step shifts occur. Results are the same as multivariate profiles in Tables 1, 2, and 3.

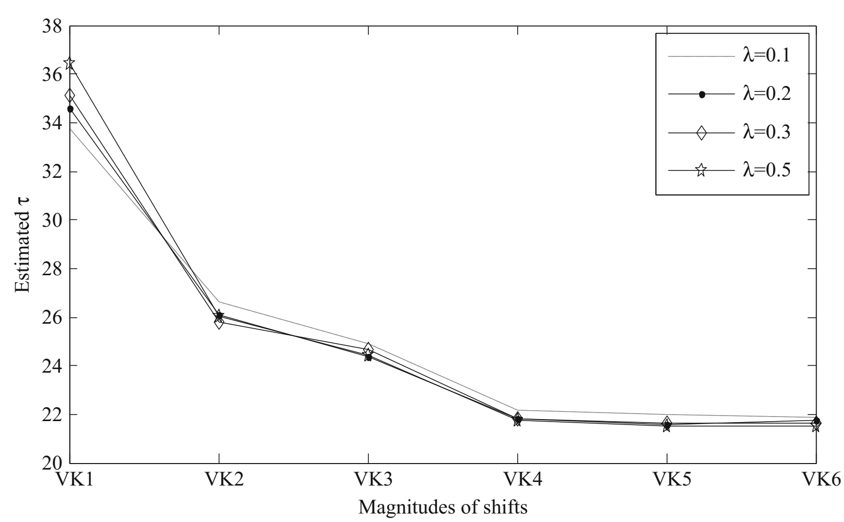

Fig. 2 Accuracy of the proposed change point estimator under three increasing step changes in the mean of multivariate linear profiles for different values of smoothing constant

Finally, the underlying in-control model for a simple linear profile discussed by Kang and Albin [16] is considered for simulation study. The model is as follows:

$$Y = 3 + 2x + \varepsilon \tag{22}$$

Fixed values considered for variable x are 2, 4, 6, and 8, and ε follows a normal distribution with mean of 0 and variance of 1. Values of 8.2 and 16.4239 are also considered for the upper control limits of MEWMA and chi-square control charts, respectively. Results are shown in Tables 6 and 7 which report identical consequences to the multivariate and multiple linear profiles.

5 Effect of smoothing coefficient on ARL and the proposed change point estimator performance

In Section 4, the value of 0.2 was used for smoothing constant, λ, which is consistent with the approach of Kim et al. [20]. In general, smaller λ is used for quick detection of small shifts and larger λ leads to a rapid detection of large shifts (see references [39] and [40]). In this section, the effect of smoothing constant on the control chart ARL and performance of the proposed change point estimator is investigated.

Upper control limits of the MEWMA control charts for different values of λ are chosen by 50,000 simulation runs to give approximately ARLs of 400, leading to an overall in-control ARL of 200 with combination of chi-square control chart. Hence, the values of 10.12, 11.1, 11.4, and 11.65 are set for the upper control limits of MEWMA control charts with λ=0.1, 0.2, 0.3, and 0.5, respectively. To conduct simulations, three increasing step changes in the mean of response variables in multivariate multiple linear profiles are considered. For this purpose, six values are used for K_1, K_2, and K_3, the same as Table 2 shown by VK1, VK2, VK3, VK4, VK5, and VK6 in Table 2. For example, VK1 contains $\mathbf{K}_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{K}_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $\mathbf{K}_3 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, and also VK6 consists of $\mathbf{K}_1 = \begin{bmatrix} 1.4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{K}_2 = \begin{bmatrix} 1.6 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $\mathbf{K}_3 = \begin{bmatrix} 1.8 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. In this section, 5,000 simulation runs with τ_1=25, τ_2=35, and τ_3=45 are used. Results are reported in Figs. 1, 2, and 3.

Since three increasing step changes are exposed to response variables mean based on Eq. (19), the combined control charts cannot issue a signal before sample 46; hence, ARLs in Fig. 1 are equal or greater than 21. Figure 1 confirms that the smaller λ, the better detection of small shifts. Finally, Figs. 2 and 3 show the effect of λ values on the performance of the proposed monotonic change point estimator. From Fig. 2, it can be concluded that when λ increases, the performance of monotonic estimator in estimating the change point becomes worse in small shifts and better in large shifts. Effect of λ values on mean square error of the proposed estimator is also the same, i.e., smaller values of λ perform better in small shifts and larger values of λ perform better in large shifts.

6 A real case

A calibration case of an electrical torqometer measuring torque required to fasten twins at Irankhodro Corporation is

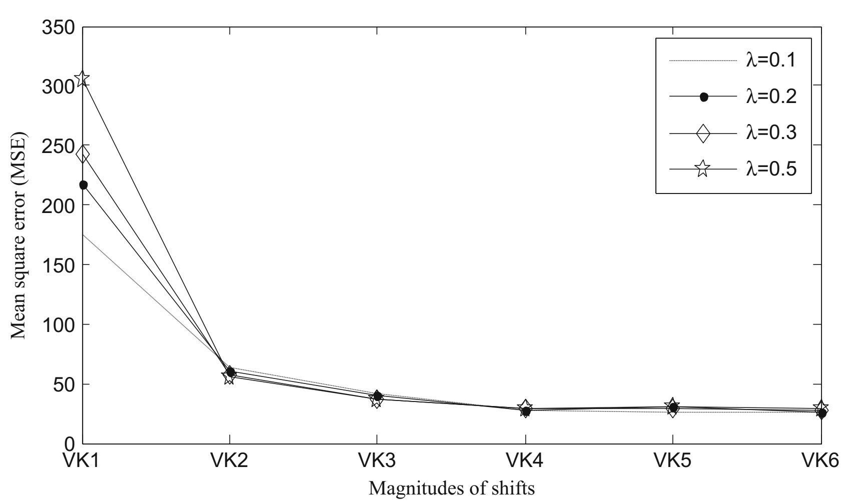

Fig. 3 Mean square error of the proposed change point estimator under three increasing step changes in the mean of multivariate linear profiles for different values of smoothing constant

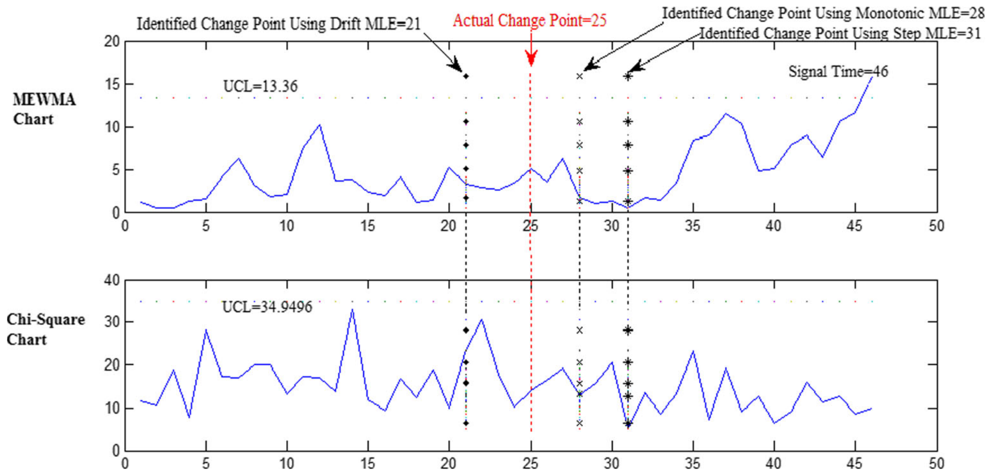

Fig. 4 Comparison of the change point estimators for torqometer calibration case study at Irankhodro Corporation

considered here to show the application of the proposed change point estimator. Three different connection types of fastening twins which are hard, semihard, and soft were used. Fixed values of torque are set to be measured by torqometer on the three connection types. Magnitudes of torque measured on the three types of connection are correlated because of measuring using the same torqometer. Hence, it can be modeled using three-variate simple linear profiles.

At first, normality assumption of each profile per sample was tested using Jarque-Bera test confirming there is no violations in the normality assumption. There is also a high correlation between measurements on three connections. Ten samples were obtained from the process, all of which are in-control. (Data are available in Table 9 in Appendix 4). An in-control model fitted on stable data with fixed x values of 20, 25, 30, 35, and 40 is as follows:

$$y_1 = 1.0696 + 0.9881x + \varepsilon_1$$
$$y_2 = -0.3758 + 0.9534x + \varepsilon_2$$
$$y_3 = -3.0574 + 1.0340x + \varepsilon_3$$

Also, the vector of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ has three-variate normal distribution with mean vector zero and covariance matrix of $\hat{\Sigma} = \begin{bmatrix} 0.8514 & -0.5728 & -0.4667 \\ -0.5728 & 4.0003 & 3.6758 \\ -0.4667 & 3.6758 & 3.6971 \end{bmatrix}$, calculated using the following formulas with the assumption of having m stable samples:

$$\hat{\Sigma} = \frac{\sum_{j=1}^{m} \hat{\Sigma}_j}{m},$$

$$\hat{\Sigma}_j = \frac{\left(\mathbf{Y}_j - \mathbf{X}\hat{\mathbf{B}}_j\right)^T \cdot \left(\mathbf{Y}_j - \mathbf{X}\hat{\mathbf{B}}_j\right)}{n-q-1}, j = 1, 2, \ldots, m$$

Upper control limits of MEWMA and chi-square control charts are 13.36 and 34.9496, respectively. Twenty-five samples from underlying in-control model is generated and then three step shifts are exposed to the elements of **XB** as follows:

$$\mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 0.005 \\ 0 & 0.005 & 0 \end{bmatrix}, \ \mathbf{K}_2 = \begin{bmatrix} 0 & 0 & 0.0075 \\ 0 & 0.0075 & 0 \end{bmatrix}, \ \mathbf{K}_3 = \begin{bmatrix} 0 & 0 & 0.01 \\ 0 & 0.01 & 0 \end{bmatrix}$$

Combined MEWMA and chi-square control chart method alarms an OC condition at sample 46. Hence, the change point estimators can be applied. Figure 4 and Table 8 (see Appendix 3) show that monotonic, step, and drift estimators have estimated the first out-of-control 29th, 32nd, and 22nd samples, respectively. The result also emphasizes on the effective performance of monotonic estimator with the existence of multiple step changes which is a case of monotonic change.

7 Conclusion

In this paper, we used the MLE method to estimate a monotonic change point in the mean of response variables in multivariate linear profiles. Simulation results showed that the proposed monotonic MLE performs well in small to moderate shifts over all kinds of increasing changes. Comparisons of the proposed estimator to the step change estimator described that under a single step change, the proposed estimator has superior performance in small to moderate shifts, but when the magnitude of shift increases, the performance of the step estimator becomes better. Also, under linear drift change, the proposed estimator has acceptable performance in small to moderate shifts. The performance of the drift estimator is acceptable when step and linear drift changes exposed to response variable mean, but the drift estimator performance deteriorates in three increasing step changes. The effect of smoothing coefficient was also evaluated on the performance of the proposed estimator which led to the results with better performance in small shifts under a small smoothing parameter. Also, the performance of the proposed estimator becomes better in large shifts when the smoothing parameter is large.

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Appendix 1: Derivation of the linear drift estimator

The logarithm of likelihood function for drift estimator is as follows (see Kazemzadeh et al. [38] for more details):

$$Ln\left(L\left(\tau,\mathbf{K}\middle|\mathbf{Y},\mathbf{X}\right)\right)=U-\frac{1}{2}\sum_{j=1}^{\tau}tr\left[\left(\mathbf{Y}_{\mathrm{j}}-\mathbf{XB}\right)\sum\nolimits^{-1}\left(\mathbf{Y}_{\mathrm{j}}-\mathbf{XB}\right)^{\mathrm{T}}\right]-\frac{1}{2}\sum_{j=\tau+1}^{T}tr\left[\left[\mathbf{Y}_{\mathrm{j}}-\mathbf{XB}-(j-\tau)\mathbf{XK}\right]\sum\nolimits^{-1}\left[\mathbf{Y}_{\mathrm{j}}-\mathbf{XB}-(j-\tau)\mathbf{XK}\right]^{\mathrm{T}}\right] \tag{23}$$

where U is a constant value and $tr[\mathbf{W}]$ is the trace of the matrix of $\mathbf{W}$. Taking a derivative of the aforementioned function with respect to the matrix of $\mathbf{XK}$ to estimate the slope of the changes and solving for $\mathbf{XK}$ leads to

$$\frac{\partial Ln\left(L\left(\tau,\mathbf{K}\middle|\mathbf{Y},\mathbf{X}\right)\right)}{\partial\mathbf{XK}}=\sum_{j=\tau+1}^{T}\left[(j-\tau)\left(\mathbf{Y}_{j}-\mathbf{XB}-(j-\tau)\mathbf{XK}\right)\sum\nolimits^{-1}\right] \tag{24}$$

$$\frac{\partial Ln\left(L\left(\tau,\mathbf{K}\middle|\mathbf{Y},\mathbf{X}\right)\right)}{\partial\mathbf{XK}}=0,\ \ so\ \ \widehat{\mathbf{XK}}=\frac{\left[\sum_{j=\tau+1}^{T}(j-\tau)\left(\mathbf{Y}_{j}-\mathbf{XB}\right)\right]}{\sum_{j=\tau+1}^{T}(j-\tau)^{2}} \tag{25}$$

Finally, the change point estimator is as follows:

$$\widehat{\tau}_{\boldsymbol{LD}}=\underset{0\le t\le T-1}{\arg\max}\left\{-\frac{1}{2}\sum_{j=1}^{t}tr\left[\left(\mathbf{Y}_{j}-\mathbf{XB}\right)\sum\nolimits^{-1}\left(\mathbf{Y}_{j}-\mathbf{XB}\right)^{\mathrm{T}}\right]-\frac{1}{2}\sum_{j=t+1}^{T}tr\left[\left[\mathbf{Y}_{j}-\mathbf{XB}-(j-\tau)\widehat{\mathbf{XK}}\right]\sum\nolimits^{-1}\left[\mathbf{Y}_{j}-\mathbf{XB}-(j-\tau)\widehat{\mathbf{XK}}\right]^{\mathrm{T}}\right]\right\} \tag{26}$$

For the special cases of multivariate linear profiles, i.e., multiple and simple linear profiles, the only difference is that in the above equations, matrixes of $\mathbf{Y}_j$, $\mathbf{B}$, and $\mathbf{K}$ reduce to vectors of $\mathbf{y}_j$, $\boldsymbol{\beta}$, and $\mathbf{k}$, respectively.

Appendix 2: Derivation of the step change estimator

The logarithm of the likelihood function with the assumption of step change yields

$$Ln\left(L\left(\tau,\mathbf{B}_1\middle|\mathbf{Y},\mathbf{X}\right)\right)=U-\frac{1}{2}\sum_{j=1}^{\tau}tr\left[\left(\mathbf{Y}_{j}-\mathbf{XB}\right)\boldsymbol{\Sigma}^{-1}\left(\mathbf{Y}_{j}-\mathbf{XB}\right)^{\mathrm{T}}\right]-\frac{1}{2}\sum_{j=\tau+1}^{T}tr\left[\left[\mathbf{Y}_{j}-(\mathbf{XB})_{1}\right]\boldsymbol{\Sigma}^{-1}\left[\mathbf{Y}_{j}-(\mathbf{XB})_{1}\right]^{\mathrm{T}}\right] \tag{27}$$

U is constant, so the derivative of the logarithm of the likelihood function with respect to $(\mathbf{XB})_1$ is

$$\frac{\partial Ln\left(L\left(\tau,\mathbf{B}_1\middle|\mathbf{Y},\mathbf{X}\right)\right)}{\partial(\mathbf{XB})_1}=\sum_{j=\tau+1}^{T}\left[\left(\mathbf{Y}_j-(\mathbf{XB})_1\right)\mathbf{\Sigma}^{-1}\right] \tag{28}$$

Also, the maximum likelihood estimator of $(\mathbf{XB})_1$ is

$$\left(\widehat{\mathbf{XB}}\right)_1=\frac{\left[\sum_{j=\tau+1}^{T}\mathbf{Y}_j\right]}{(T-\tau)} \tag{29}$$

$$\widehat{\tau}_{SC}=\arg\max_{0\le t\le T-1}\left\{-\frac{1}{2}\sum_{j=1}^{t}tr\left[\left(\mathbf{Y}_j-\mathbf{XB}\right)\sum^{-1}\left(\mathbf{Y}_j-\mathbf{XB}\right)^{\mathrm{T}}\right]-\frac{1}{2}\sum_{j=t+1}^{T}tr\left[\left[\mathbf{Y}_j-\left(\widehat{\mathbf{XB}}\right)_1\right]\sum^{-1}\left[\mathbf{Y}_j-\left(\widehat{\mathbf{XB}}\right)_1\right]^{\mathrm{T}}\right]\right\} \tag{30}$$

For multiple and simple linear profiles, the matrixes of $\mathbf{Y}_j$ and $\mathbf{B}$ reduce to vectors of $\mathbf{y}_j$ and $\boldsymbol{\beta}$, respectively.

Appendix 3: Computations of the proposed change point estimators

Table 8 Computations of change point estimators for torqometer calibration case study at Irankhodro Corporation. The first out-of-control sample occurs on the 26th sample (τ_1=25)

Sample number	Ln $L_{\mathrm{Monotonic}}$	Ln L_{SC}	Ln L_{LD}	Sample number	Ln $L_{\mathrm{Monotonic}}$	Ln L_{SC}	Ln L_{LD}
1	−358.586	−345.898	−341.912	24	−349.001	−341.178	−339.519
2	−359.222	−345.436	−341.751	25	−348.573	−339.841	−339.558
3	−358.87	−345.351	−341.585	26	−348.208	−341.066	−339.748
4	−357.498	−345.147	−341.392	27	−347.888	−341.509	−339.849
5	−357.184	−344.157	−341.185	28	−346.181	−341.168	−339.939
6	−355.759	−344.836	−341.013	29	***−345.317***	−340.561	−340.004
7	−354.384	−344.127	−340.811	30	−345.9	−339.266	−340.136
8	−355.32	−342.995	−340.656	31	−345.875	−338.609	−340.424
9	−354.231	−342.904	−340.56	32	−345.768	***−338.218***	−340.862
10	−351.394	−342.756	−340.433	33	−345.91	−339.393	−341.291
11	−351.347	−342.276	−340.313	34	−345.906	−339.411	−341.686
12	−350.302	−341.273	−340.222	35	−345.788	−341.322	−342.034
13	−351.042	−341.601	−340.196	36	−346.306	−344.216	−342.155
14	−351.991	−341.444	−340.177	37	−347.046	−344.904	−341.927
15	−352.617	−342.699	−340.171	38	−347.379	−344.894	−341.419
16	−352.044	−342.15	−340.125	39	−347.213	−344.672	−340.91
17	−352.589	−343.224	−340.094	40	−346.373	−342.125	−340.388
18	−352.763	−344.439	−340.004	41	−346.095	−340.881	−340.468
19	−352.323	−343.703	−339.799	42	−345.894	−340.5	−340.93
20	−351.899	−342.423	−339.63	43	−345.998	−340.762	−341.853
21	−349.631	−342.408	−339.541	44	−346.829	−342.612	−343.363
22	−349.526	−340.366	***−339.401***	45	−347.873	−344.618	−344.504
23	−348.307	−340.122	−339.436	46	−349.288	−345.624	−345.624

Appendix 4

Table 9 Data of torqometer calibration case study at Irankhodro Corporation

	Measured torque			Measured torque		
	Hard	Semihard	Soft	Hard	Semihard	Soft
Actual torque	First sample			Sixth sample		
20	20.83	19.77	19.56	20.66	20.49	19.25
25	25.09	22.046	21.89	25.43	23.04	22.22
30	30.9	25.64	26.954	32.73	25.73	26.65
35	34.65	32.23	32.58	34.3	32	32.65
40	40.38	39.78	39.35	40.2	39.33	40.2
	Second sample			Seventh sample		
20	20.25	20.416	19.41	21.4	20.76	19.33
25	24.28	23.02	21.44	25.38	22.93	22.33
30	32.5	26.72	25.616	31.8	26.47	26.58
35	35.1	32.22	32.51	35	32.2	32.33
40	40.13	39.58	40.04	40.93	39.98	39.25
	Third sample			Eighth sample		
20	20.89	20.599	19.4	20.99	20.66	19.12
25	25.47	22.92	21.623	25.4	22.59	22.49
30	31.7	25.86	26.057	30.43	26.5	26.91
35	36.71	32.11	31.84	36.7	32.3	31.9
40	40.5	39.95	39.91	40.84	39.15	39.87
	Fourth sample			Ninth sample		
20	21.16	20.4	19.42	20.73	20.58	19.85
25	25.34	22.835	21.38	26.17	23.1	21.95
30	30.5	25.9	26.09	31.32	25.88	25.73
35	33.8	32.3	32.6	35.8	32.44	32.6
40	40.2	39.16	40.8	40.33	39.29	39.99
	Fifth sample			Tenth sample		
20	20.07	20.36	19.3	20.3	20.49	19.48
25	26.11	22.35	21.56	26.39	22.7	21.42
30	29.99	26.93	25.51	32.58	25.93	25.79
35	36.9	32.1	32.74	35.1	32.1	32.23
40	41.1	39.68	40.34	40.2	39.73	40.02

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