Int J Adv Manuf Technol (2013) 68:2359–2369
DOI 10.1007/s00170-013-4844-x

ORIGINAL ARTICLE

Optimum tolerance design using component-amount and mixture-amount experiments

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Received: 7 May 2012 / Accepted: 21 January 2013 / Published online: 7 March 2013

Abstract The tolerance design problem involves optimizing component and assembly tolerances to minimize the total cost (sum of manufacturing cost and quality loss). Previous literature recommended using traditional response surface methodology (RSM) designs, models, and optimization techniques to solve the tolerance design problem for the worst-case scenario in which the assembly characteristic is the sum of the component characteristics. In this article, component-amount (CA) and mixture-amount (MA) experiment approaches are proposed as more appropriate for solving this class of tolerance design problems. The CA and MA approaches are typically used for product formulation problems, but can also be applied to this type of tolerance design problem. The advantages of the CA and MA approaches over the RSM approach and over the standard, worst-case tolerance-design method are explained. Reasons for choosing between the CA and MA approaches are also discussed. The CA and MA approaches (experimental design, response modeling, and optimization) are illustrated using real examples.

Keywords Assembly tolerance · Component-amount experiment · Component tolerances · Mixture-amount experiment · Tolerance design

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1 Introduction

Consider a product assembly that is composed of two or more component parts. In practice, tolerances on an assembly and its components must be specified because of inevitable variations in production processes, materials, environmental conditions, personnel, etc. Tolerance design is a procedure that determines the assembly tolerance and distributes it among the components. The design of assembly and component tolerances affects the manufacturing costs and performance of the product. Larger tolerances decrease manufacturing costs, but can cause a poor assembly and/or product performance. On the other hand, smaller tolerances ensure the performance of an assembly/product, but cause higher manufacturing costs.

In practice, the exact forms of relationships between assembly and component tolerances, costs, and functional performance are unknown. Kim and Chou [1] summarized several approximate models for manufacturing cost versus component tolerances. They also proposed using response surface methodology (RSM) with classical statistical experimental designs and models to (i) estimate the relationship between manufacturing cost and component tolerances and (ii) allocate component tolerances of an assembly to minimize manufacturing cost. Creveling [2] presented a comprehensive discussion of tolerance design, including using orthogonal arrays to design experiments. Şehirlioğlu and Özler [3] proposed using mixture experiment designs and models as more appropriate than classical RSM designs and models to address (i) and (ii).

Jeang [4, 5] emphasized that the functional performance of the product should be accounted for in the tolerance design problem, specifically by using the quality-loss function. Several tolerance design methods based on quality loss [4, 6–22] have been developed for the purpose of finding optimum tolerances of components and assemblies. Jeang

[5] also proposed using classical RSM to determine the optimum assembly and component tolerances by minimizing a total cost (TC) function, which is the sum of manufacturing cost (MC) and quality-loss cost (QLC). Classical RSM methods for experimental design, modeling, and optimization are discussed by Box and Draper [23], Khuri and Cornell [24], and Myers et al. [25].

In this article, component-amount (CA) and mixture-amount (MA) approaches are proposed as more appropriate than the classical RSM approach proposed by Jeang [5] for solving the tolerance design problem when using the worst-case method and the assembly characteristics are linear combinations of the component characteristics. The CA, MA, and RSM approaches use different kinds of experimental designs and models to solve the tolerance design problem, as discussed subsequently in the article.

In modern tolerance design, statistical methods that account for the probabilistic variation in component characteristics are often favored over the worst-case method because statistical methods lead to wider tolerances and hence lower MC [2]. However, the worst-case method may be preferred for tolerance design problems where safety, improved lifetime, or other issues are a major concern. Further, the CA and MA approaches discussed in this article provide for specifying constraints on the assembly tolerance, so that the optimal component tolerances are not as tight as would be the case when using the standard worst-case method. Hence, the CA and MA approaches for tolerance design may be thought of as improvements to the worst-case method because of (i) the ability to constrain the assembly tolerance and (ii) selecting assembly and component tolerances to minimize the TC.

The article is organized as follows. First, mixture, MA, and CA experiments are reviewed, including the forms of models and how to generate experimental designs. Next, the concepts and formulas for MC, QLC, and TC are discussed. Then, the tolerance design problems using CA and MA experimental approaches are presented and illustrated using real examples. Finally, the work is summarized and conclusions are made.

2 Review of mixture, mixture-amount, and component-amount experiments

In this section, we review mixture, MA, and CA experiments. The design, modeling, and optimization of tolerance design problems using the CA and MA approaches are illustrated with real examples in subsequent sections.

2.1 Mixture experiments

In a mixture experiment, the response (i) is a function of the proportions of $q \geq 2$ components present in the mixture, and (ii) is assumed to depend only on the proportions of the components, and not on the total amount of the mixture. If the proportion of the ith mixture component is denoted by x_i ($i=1, 2, \ldots, q$), then $0 \leq x_i \leq 1$ and $\sum_{i=1}^{q} x_i = 1$. In addition, there are often lower and upper constraints on component proportions $L_i \leq x_i \leq U_i$, and there may be multi-component constraints. Depending on the shape of the mixture experimental region, an appropriate mixture experiment design is generated, and the resulting data are used to develop a model for the response. Commonly used mixture models are the linear and quadratic canonical polynomials

$$\eta = \sum_{i=1}^{q} \beta_i x_i \tag{1}$$

and

$$\eta = \sum_{i=1}^{q} \beta_i x_i + \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \beta_{ij} x_i x_j \tag{2}$$

proposed by Scheffé [26]. When there are single-component and possibly multiple-component constraints on the component proportions, mixture experiment models are often fitted in terms of pseudocomponents ([27, 28]). Most commonly L-pseudocomponents are used, defined as

$$x_i' = \frac{x_i - L_i}{1 - \sum_{j=1}^{q} L_j} \tag{3}$$

L-pseudocomponent values can be expressed in terms of original component values via

$$x_i = L_i + \left(1 - \sum_{j=1}^{q} L_j\right) x_i' \tag{4}$$

For a comprehensive presentation of mixture experiment designs, models, and other data analysis methods, see Cornell [27] and Smith [28].

2.2 Mixture-amount experiments

A mixture-amount (MA) experiment is one in which mixture experiments are performed at two or more levels of total amount [29]. The response is assumed to depend on the x_i ($i=1, 2, \ldots, q$) and the total amount of the mixture (denoted A). An MA experiment explores the effects on the response of varying both the component proportions and the total amount of the mixture.

Piepel and Cornell [30] proposed developing MA models by writing the parameters of a mixture model as functions of the total amount variable. As an example, writing the coefficients of the quadratic mixture model (2) as polynomial functions of A yields

$$\eta = \sum_{i=1}^{q} \beta_i^0 x_i + \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \beta_{ij}^0 x_i x_j + \sum_{k=1}^{r-1} \left[\sum_{i=1}^{q} \beta_i^k x_i + \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \beta_{ij}^k x_i x_j \right] A^k, \quad (5)$$

where r represents the maximum polynomial degree of the effect of A. When A is coded to have mean zero, the β_i^0 and β_{ij}^0 represent the linear and quadratic blending effects of the components at the mean value of total amount, while the β_i^k and β_{ij}^k represent the kth-order effect of total amount on the linear and quadratic blending effects of the components. As a special case, when the amount of the mixture does not affect the blending properties of the mixture components but merely causes a constant change in the magnitude of the response, (5) reduces to

$$\eta = \sum_{i=1}^{q} \beta_i^0 x_i + \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \beta_{ij}^0 x_i x_j + \sum_{k=1}^{r-1} \beta_0^k A^k, \quad (6)$$

where the β_0^k (k=1, 2, …, $r-1$) represent the linear, quadratic, …, ($r-1$)th-degree effects of A on the response. Piepel and Cornell [30] recommended reducing full MA models such as (5) to avoid overfitting the data and reduce the uncertainty of model predictions. To select an appropriate reduced MA model, variable selection (e.g., stepwise or backward elimination) algorithms and the full- versus reduced-model F test can be used [25].

MA models may be fitted using L-pseudocomponents (x_i') instead of components (x_i) and a coded total amount (A') instead of the total amount (A). If there are lower and upper bounds on the total amount ($A_L \leq A \leq A_U$), then the coding

$$A' = \frac{A - \frac{A_L + A_U}{2}}{A_U - A_L} \quad (7)$$

yields $A'=-1$ when $A=A_L$ and $A'=+1$ when $A=A_U$. Piepel and Cornell [30] explain why this coding is useful. Coded values A' can be converted back to original values A via

$$A = \frac{A_L + A_U}{2} + A'(A_U - A_L) \quad (8)$$

MA designs appropriate for fitting MA models were considered by Piepel and Cornell [29]. They discussed ways to generate complete and fractional designs for both unconstrained and constrained MA experiments. Piepel and Cornell [29] proposed (i) setting up one or different mixture designs at the different levels of total amount, or (ii) using *optimal experimental design* (Atkinson et al. [31]) to develop designs for reduced MA models and unconstrained or constrained MA experiments. Optimal design procedures are very useful for fractionating designs and can generate designs of any number of experimental runs for any model form of interest. To efficiently estimate the elements of the parameter vector $\boldsymbol{\beta}$ in the mixture or MA model $\mathbf{y}=\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}$, a design criterion should be chosen. D-optimality, which seeks to minimize $\det(\mathbf{X'X})^{-1}$, is the criterion implemented in most commercial experimental design software. Other optimality criteria are discussed by Atkinson et al. [31]. Optimal MA designs can be constructed using several statistical packages, including Design-Expert (Stat-Ease [32]), JMP [33], SAS [34], and Minitab [35].

2.3 Component-amount experiments

Piepel and Cornell [30] also discussed the CA approach for experiments with two or more components and the total amount of the components varies. With the CA approach, the behavior of the response is studied in terms of the amounts of individual components. Piepel and Cornell [30] wrote the first- and second-degree polynomial CA models as

$$\eta = \alpha_0 + \sum_{i=1}^{q} \alpha_i a_i \quad (9)$$

$$\eta = \alpha_0 + \sum_{i=1}^{q} \alpha_i a_i + \sum_{i=1}^{q} \alpha_{ii} a_i^2 + \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \alpha_{ij} a_i a_j \quad (10)$$

where the a_i (i=1, 2, …, q) are the amounts of individual components. Piepel and Cornell [36] state that designs for fitting models (9) and (10) could be selected from the classical response surface designs (e.g., factorial, fractional factorial, Plackett–Burman, Box–Behnken, or central composite). In (10), each component amount can be written as $a_i=x_iA$, where A is the uncoded total amount and $a_1+a_2+\ldots+a_q=A$. Comparisons of MA and CA models and designs are given by Piepel and Cornell [30, 36]. They note that any MA model can be written in terms of the CA variables, with the resulting expression a nonpolynomial function of the a_i's being an expanded form of the second-degree CA model (10). Similarly, when MA models are generated by expanding coefficients of a mixture model using polynomials of different degrees, Piepel and Cornell [30] noted that the class of MA models contains the class of polynomial CA models as a special case. So, the class of MA models has the ability to approximate the true response surface better than the class of CA models.

2.4 Choosing between mixture-amount and component-amount approaches

Piepel and Cornell [30] pointed out that MA models provide information about the component blending properties at the average level of total amount and information on how the total amount of the mixture affects these blending properties. On the other hand, CA models do not separate these two types of information, but rather provide information on how the amounts of components affect the response.

In planning an experiment where the amount of the mixture affects the response, an experimenter must choose between the MA and CA approaches. According to Piepel and Cornell [36], for problems where the experimenter wants to separately study the blending behavior of the components and how varying the total amount affects the component blending, an MA approach is natural. On the other hand, if the experimenter wants to understand how the component amounts interact and affect the response, a CA approach is natural.

3 Cost functions for optimum tolerance design

This section presents background information about MC, QLC, and TC. In the tolerance design problem addressed in this article, TC is the response variable in an empirical model developed using data from a designed experiment. The TC model is then used to optimally determine the tolerances.

The MC of a product is influenced by the tolerances on component characteristics. Tighter tolerances increase the MC. Larger tolerances lead to reduced MC, but lower product performance. A summary of MC functions is presented by Kim and Chou [1].

The quality-loss function proposed by Taguchi [37] gives a financial value for increasing costs as a product quality characteristic deviates from its target value. Taguchi considers quality loss as all the costs of reduced product quality (including cost of scrap, rework, downtime, warranty claims, and reduced market share). For the "target-is-best" case, the Taguchi loss function is $L(Y) = K(Y - m_Y)^2$, where Y is the quality characteristic, m_Y is the target value of the quality characteristic, and K is a constant depending on the cost at the specification limits and the width of the specification. The expected value of $L(Y)$, which is subsequently referred to as the *quality loss cost* (QLC), can be written as

$$QLC = \mathrm{E}[L(Y)] = K\left[(E_Y - m_Y)^2 + \sigma_Y^2\right], \tag{11}$$

where E_Y is the expected value of Y and σ_Y^2 is the variance of Y. In practice, $\widehat{E}_Y$ and $\widehat{\sigma}_Y^2$, which are estimates of E_Y and σ_Y^2, respectively, are used to estimate the QLC. Details and examples about finding the value of K are discussed by Ross [38].

To achieve a good tolerance design considering QLC and MC, Zhang et al. [39] recommended: (i) functional requirements of the product should be satisfied, (ii) variations of functional performance should be minimized, and (iii) MCs should be constrained. Violations in the first two requirements will influence the QLC. Manufacturing costs typically must satisfy some monetary constraints. Jeang [5, 10, 11] said that product-design tolerances should be determined to account for QLC and MC. The TC function is the sum of the QLC and MC functions, which for the CA approach we write as

$$\begin{aligned} &TC\left(t_1, t_2, \ldots, t_q, \widehat{E}_{Y(t_1,t_2,\ldots t_q)}, \widehat{\sigma}^2_{Y(t_1,t_2,\ldots t_q)}, K\right) \\ &\quad = QLC\left(\widehat{E}_{Y(t_1,t_2,\ldots t_q)}, \widehat{\sigma}^2_{Y(t_1,t_2,\ldots t_q)}, K\right) \\ &\quad + MC(t_1, t_2, \ldots, t_q) \end{aligned} \tag{12}$$

and for the MA approach we write as

$$\begin{aligned} &TC\left(x_1, x_2, \ldots, x_q, T, \widehat{E}_{Y(x_1,x_2,\ldots x_q,T)}, \widehat{\sigma}^2_{Y(x_1,x_2,\ldots x_q,T)}, K\right) \\ &\quad = QLC\left(\widehat{E}_{Y(x_1,x_2,\ldots x_q,T)}, \widehat{\sigma}^2_{Y(x_1,x_2,\ldots x_q,T)}, K\right) \\ &\quad + MC(x_1, x_2, \ldots, x_q, T). \end{aligned} \tag{13}$$

In these equations, q is the number of component tolerances, $(x_1,x_2,\ldots,x_q)$ denotes component proportional tolerances (defined in Section 4), QLC is a function of $\left(\widehat{E}_Y, \widehat{\sigma}_Y^2, K\right)$ and is calculated using Eq. (11), and other notation is as previously defined. Note that QLC and MC are functions of $(t_1,t_2,\ldots,t_q)$ or $(x_1,x_2,\ldots,x_q,T)$ to indicate their values may depend on the different points in a CA or MA experimental design, respectively. $\widehat{E}_Y$ and $\widehat{\sigma}_Y^2$ values could be calculated from characteristic measurements on multiple parts at each of the points $(t_1,t_2,\ldots,t_q)$ or $(x_1,x_2,\ldots,x_q,T)$ in a CA or MA experimental design, respectively.

Equations (12) and (13) can be used to calculate values of the response variable TC for the points in a CA or MA experimental design, respectively. Then, CA or MA models (as discussed in Section 2) can be fit to the TC values for the experimental design points.

4 Component-amount and mixture-amount approaches for optimum tolerance design to minimize total cost

The CA approach to solving a tolerance design problem in which (i) there is one assembly characteristic that is the sum of the component characteristics, and (ii) the goal is to minimize total cost is formulated as

$$\begin{aligned} &\text{Minimize} \quad TC\left(t_1, t_2, \ldots, t_q, \widehat{E}_{Y(t_1,t_2,\ldots t_q)}, \widehat{\sigma}^2_{Y(t_1,t_2,\ldots t_q)}, K\right) \\ &\text{subject to} \quad l_i \le t_i \le u_i,\ i = 1, 2, \ldots, q \text{ and} \\ &\qquad\qquad\quad T_L \le T = t_1 + t_2 + \ldots + t_q \le T_U, \end{aligned} \tag{14}$$

where l_i and u_i represent the lower and upper bounds of the ith component tolerance; t_i, T_L, and T_U represent the lower

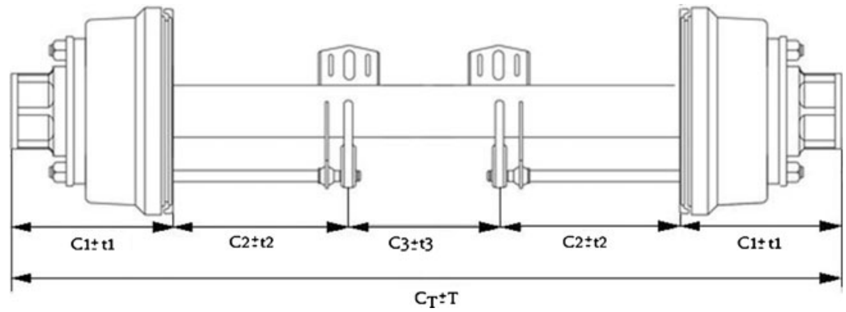

Fig. 1 A truck axle components and tolerances

and upper bounds of assembly tolerance $T=t_1+t_2+\ldots+t_q$, and *TC* is the total cost function in Eq. (12). This formulation of the tolerance design problem uses the CA approach, where the t_i take the roles of component amounts and there are lower and upper inequality constraints on the assembly tolerance (which can be written as the sum of the t_i for this class of problems). Hence, the experimental region for the component tolerances is polyhedral. This is why classical second-order response surface designs such as Box–Behnken designs (BBD) and central composite designs (CCD) proposed by Jeang [5] are not appropriate.

When the goal of a tolerance design problem is to minimize MC rather than TC, the optimum solution occurs when $T=T_U$. In that case, (14) can be re-expressed as

$$\begin{array}{ll}\text{Minimize} & MC\left(x_1, x_1, \ldots, x_q, T\right) \\ \text{subject to} & L_i \leq x_i \leq U_i,\ i=1,2,\ldots,q \text{ and} \\ & x_1+x_2+\ldots+x_q=1,\end{array} \tag{15}$$

where $x_i=t_i/T_U$, $L_i=l_i/T_U$, $U_i=u_i/T_U$, and $MC(x_1,x_2,\ldots x_q,T)$ is the manufacturing cost function represented in Eq. (13). In this approach, MC is regarded as the response variable. When the assembly tolerance is known to start, then it must be allocated among the components in some rational way. Such cases are called *tolerance allocation* problems, for which a mixture experiment approach is appropriate, as discussed by Şehirlioğlu and Özler [3].

The MA approach to solving a tolerance design problem when the goal is to minimize TC is formulated as

$$\begin{array}{ll}\text{Minimize} & TC\left(x_1, x_2, \ldots, x_q, T, \widehat{E}_{Y(x_1,x_2,\ldots x_q,T)}, \widehat{\sigma}^2_{Y(x_1,x_2,\ldots x_q,T)}, K\right) \\ \text{subject to} & L_i \leq x_i \leq U_i,\ i=1,2,\ldots,q \text{ and} \\ & T_L \leq T \leq T_U,\end{array} \tag{16}$$

where $x_i(=t_i/T)$ represents the proportion of the assembly tolerance (T) allocated to the ith component, L_i and U_i represent the lower and upper bounds of x_i, T_L and T_U represent the lower and upper bounds of assembly tolerance, and TC is the total cost function in Eq. (13). This formulation of the tolerance design problem uses the MA approach, where the x_i are the component proportions and T represents the total amount. Because of the lower and upper bounds on the component tolerance proportions and the lower and upper bounds on the assembly tolerance, the experimental region is polyhedral in shape. Hence, MA designs and models are required rather than classical response surface designs and models. After the optimal settings of the x_i and T variables are determined, they can be converted back to optimal tolerance settings using $t_i=x_iT$.

An important advantage of using mixture designs for (15) and CA or MA designs for (14) or (16) instead of BBDs or CCDs is that all design points will be (i) on the (q-1)-dimensional feasible region for (15) or (ii) inside the q-dimensional feasible region for (14) or (16). When a BBD or CCD is used, design points can fall outside of the feasible region. This situation may lead to less accurate TC models and/or larger variances of the predicted TC values, which in turn could lead to a sub-optimal solution to the tolerance design problem. In addition to this, assembly constraints could be exceeded at some design points, and this situation may result in functional problems on assemblies.

5 Example using the component-amount approach for a tolerance design problem to minimize total cost

In this section, we consider an example to demonstrate using the CA approach for the tolerance design problem to minimize TC. Figure 1 shows a truck axle, which consists of five components ($2\times C_1$, $2\times C_2$, and C_3) and their associated component tolerances (t_1, t_2, and t_3). It also shows the assembly characteristic (C_T) and its tolerance (T). There is only one assembly characteristic (length) in this example. The lower and upper limits for t_1, t_2, t_3, and T are given in Table 1. The objective is to find the optimal component

Table 1 Constraints on the components and assembly tolerances for the truck axle example

Component/ assembly name	Target value (mm)	Tolerance name	Lower (mm)	Upper (mm)
C1	315	t_1	1	4
C2	460	t_2	2	5
C3	490	t_3	2	7
C_T (assembly)	2040	T	8	20

Table 2 D-optimal component-amount design with values of manufacturing cost (MC), $\widehat{E}_Y$, $\widehat{\sigma}_Y^2$, quality-loss cost (QLC), and total cost (TC) for the truck axle example

Design point	Run order	t_1	t_2	t_3	T	MC	$\widehat{E}_Y$	$\widehat{\sigma}_Y^2$	QLC	TC
1	14	1.00	2.00	2.00	8.00	15.75	2,040.01	1.89	0.47	16.23
2	19	1.00	2.00	2.00	8.00	16.25	2,040.08	1.87	0.47	16.72
3	10	4.00	2.00	2.00	14.00	11.81	2,041.69	0.44	0.82	12.64
4	1	4.00	2.00	2.00	14.00	11.92	2,039.63	5.07	1.30	13.23
5	7	2.50	3.50	2.00	14.00	13.42	2,040.74	3.12	0.92	14.34
6	17	1.00	5.00	2.00	14.00	12.60	2,039.32	5.76	1.56	14.15
7	9	4.00	5.00	2.00	20.00	10.17	2,037.27	7.84	3.82	13.99
8	11	4.00	5.00	2.00	20.00	10.39	2,039.35	7.99	2.10	12.49
9	2	1.00	3.50	2.13	11.13	14.26	2,039.61	2.33	0.62	14.88
10	13	1.95	5.00	4.28	18.17	10.17	2,040.02	9.56	2.39	12.56
11	16	4.00	2.92	4.35	18.18	9.09	2,040.18	9.45	2.37	11.46
12	4	2.65	2.00	4.65	13.95	11.25	2,040.73	2.39	0.73	11.98
13	3	1.00	3.65	4.70	14.00	11.20	2,040.45	4.57	1.19	12.39
14	8	3.01	4.12	5.55	19.80	8.04	2,040.09	4.02	1.01	9.05
15	5	1.00	2.00	7.00	13.00	7.99	2,039.04	3.23	1.04	9.03
16	20	1.00	2.00	7.00	13.00	7.84	2,039.83	1.75	0.44	8.29
17	18	4.00	2.50	7.00	20.00	3.37	2,039.39	1.47	0.46	3.83
18	12	2.37	2.93	7.00	17.59	5.97	2,040.37	4.52	1.16	7.13
19	15	1.50	5.00	7.00	20.00	5.66	2,041.62	1.68	1.07	6.74
20	6	1.50	5.00	7.00	20.00	5.54	2,037.99	0.80	1.21	6.75

tolerances (t_1,t_2, and t_3) and assembly tolerance (T) that give the minimum TC. Because the lower and upper values given in the Table 1 are on "amounts" of the component tolerances, the CA approach can naturally be selected to solve this tolerance design problem.

From Fig. 1, the relationship between the component and assembly characteristics can be written as $2C_1+2C_2+C_3=C_T$. So, the relationship between the component tolerances and assembly tolerance can be written as $2t_1+2t_2+t_3=T$. To construct a CA design using optimal experimental design methods, we first specify the constrained experimental region using the component and assembly tolerance constraints in Table 1, namely $1\le t_1\le 4$, $2\le t_2\le 5$, $2\le t_3\le 7$, and $8\le 2t_1+2t_2+t_3\le 20$.

Design-Expert (Stat-Ease [32]) was used to generate a 20-point D-optimal CA design assuming the CA model (10), using t_i's instead of a_i's. The design replicates five points and includes five points for assessing model lack-of-fit. The experimental runs were executed in a random order, with five parts produced for each design point. Table 2 presents, for each design point, the settings of t_1, t_2, and t_3, as well as the values of MC, QLC, and TC. The TC values were obtained by adding the MC and QLC values, as given in (12). The MC values were obtained by measuring the times for operation, inspection, assembly, and set-up per unit, and converting these times to costs. The $\widehat{E}_Y$ and $\widehat{\sigma}_Y^2$ values were calculated from measurements on the five parts produced at each point (t_1,t_2,…,t_q) in the CA experimental

Table 3 ANOVA for the truck axle component-amount model (17)

Source	Sum of Squares	df	Mean Square	F Value	p value Prob>F	
Model	231.26	6	38.54	130.96	< 0.0001	Significant
t_1	18.53	1	18.53	62.97	< 0.0001	
t_2	2.05	1	2.05	6.98	0.0204	
t_3	197.15	1	197.15	669.86	< 0.0001	
t_1t_2	2.92	1	2.92	9.91	0.0077	
t_1t_3	1.01	1	1.01	3.44	0.0866	
t_3^2	10.70	1	10.70	36.35	< 0.0001	
Residual	3.83	13	0.29			
Lack of fit	2.13	8	0.27	0.79	0.6370	Not significant
Pure error	1.69	5	0.34			
Cor total	2325.08	19				

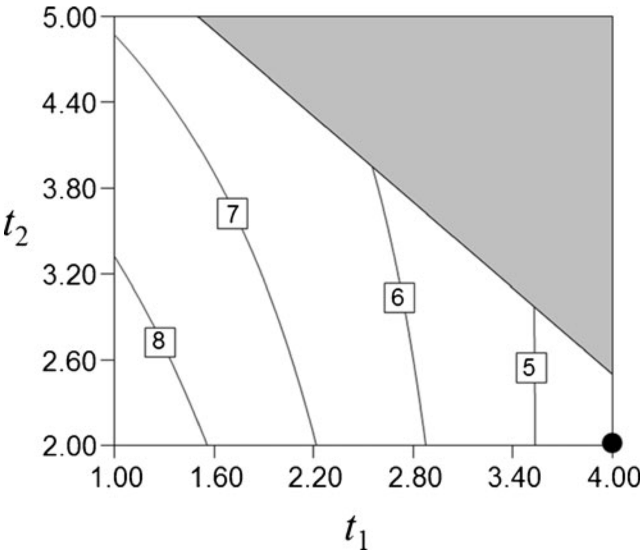

Fig. 2 Contour plot of total cost versus t_1 and t_2 with t_3=7 for the truck axle example, with the optimum solution shown as a *dot*

design, and are listed in Table 2. The QLC values were calculated with Eq. (11) using the $\widehat{E}_Y$ and $\widehat{\sigma}_Y^2$ values in Table 2 and K=0.25.

The second-order CA model (10) for TC was estimated using Design-Expert (Stat-Ease [32]), with the t_1^2, t_2^2, and t_2t_3 terms being statistically non-significant (p>0.10). Dropping those terms and refitting the model yielded

$$\widehat{TC} = 17.6844 - 1.3770t_1 - 0.8932t_2 + 1.2610t_3 + 0.2512t_1t_2 - 0.0918t_1t_3 - 0.2931t_3^2 \quad (17)$$

with R^2=0.9837, $R_A^2 = 0.9762$, and $R_P^2 = 0.9474$. The analysis of variance table for (17) is given in Table 3 and shows that the model has a non-significant lack of fit.

For this truck axle example illustrating the CA approach, the optimization problem (14) is

$$\begin{array}{ll} \text{Minimize} & \widehat{TC}\text{ given in (17)} \\ \text{subject to} & 1 \le t_1 \le 4, 2 \le t_2 \le 5, 2 \le t_3 \le 7, \\ & \text{and } 8 \le T \le 20. \end{array}$$

Using Solver in Excel, the optimal values that minimize the TC were determined to be (t_1,t_2,t_3)=(4, 2, 7). The assembly tolerance and predicted TC at the optimum tolerances are 13 and 4.29, respectively. The contour plot of TC at t_3=7 is given in Fig. 2, with the optimal solution shown to be on a vertex of the constrained experimental region.

6 Example using the mixture-amount approach for a tolerance design problem to minimize total cost

In this section, we modify an example used by Şehirlioğlu and Özler [3] to illustrate using a mixture experiment approach for a tolerance allocation problem to minimize MC. The modified example illustrates using an MA approach for the tolerance design problem to minimize TC.

Figure 3 shows the four components of a portion of a steering mechanism (C_1, C_2, C_3, and C_4) along with their associated component tolerances (t_1, t_2, t_3, and t_4). There is only one assembly characteristic (length) in this example. With the MA approach, we are interested in studying how the component tolerance proportions and the assembly tolerance influence TC. In this example, the lower and upper limits of the assembly tolerance are equal to 3.8 and 5.0 mm, respectively, and the tolerance of component 3 is constant at 0.2 mm (the same as Şehirlioğlu and Özler [3]). Hence, the lower and upper limits of the assembly tolerance for the remaining three components (denoted T) are 3.6 and 4.8 mm. The proportional tolerances of the remaining three components are denoted as x_1 (for C_1), x_2 (for C_2), and x_3 (for C_4). The lower and upper constraints on the x_i and T, as well as the L-pseudocomponents (x_i') and coded total amount (A'), are given in Table 4. The objective is to find the optimal proportions (x_1,x_2,x_3) and assembly tolerance (T) that give the minimum TC. After the optimal settings of these variables are determined in the x_i' and A', they can be converted back to x_i and A (=T) values [using (4) and (8)], and then the x_i can be converted back to optimal tolerance settings using $t_i = x_iT$.

To construct an MA design using optimal experimental design methods, an MA model adequate to represent the relationship of TC as a function of the x_i (i=1, 2, 3) and T must be selected. An 18-term, quadratic-by-quadratic MA model given by (5) with q=3 and r=3 was chosen. With r=3, this MA model provides for the assembly tolerance (T) to have a quadratic effect on TC. In (5), L-pseudocomponent proportions x_i' per (3) and the coded total amount A' per (7) were used instead of the x_i and A.

Design-Expert (Stat-Ease [32]) was used to generate a 28-point D-optimal MA design for the 18-term quadratic-by-quadratic MA model. The design replicates five points and includes five points for assessing model lack-of-fit when the full 18-term MA model is fit. The experimental runs were executed in a random order, with five parts

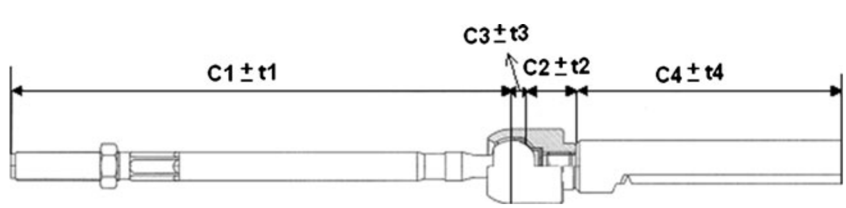

Fig. 3 Components and tolerances for a portion of a steering mechanism

Table 4 Constraints on the proportional component and assembly tolerances for the steering mechanism example

Variable name	Original components and T			L-Pseudocomponents and coded T		
	Variable	Minimum	Maximum	Variable	Minimum	Maximum
C_1	x_1	0.500	0.667	x_1'	0	0.60
C_2	x_2	0.111	0.250	x_2'	0	0.50
C_3	NA	NA	NA	NA	NA	NA
C_4	x_3	0.111	0.250	x_3'	0	0.50
C_T (assembly of 1, 2, and 4)	T	3.6	4.8	A'	−1	+1

produced for each design point. Table 5 presents, for each design point, the settings of x_1, x_2, x_3, and A, as well as the MC, QLC, and TC values. The TC values were obtained by adding the MC and QLC values, as given in (13). The MC values were obtained by measuring the times for operation, inspection, assembly, and set-up per unit, and converting these times to costs. The $\widehat{E}_Y$ and $\widehat{\sigma}_Y^2$ values were calculated from measurements on the five parts produced at each point (x_1, x_2, …,x_q, T) in the MA experimental design, and are listed in Table 5. The QLC values were calculated with Eq. (11) using the $\widehat{E}_Y$ and $\widehat{\sigma}_Y^2$ values in Table 5 and K=10.

The stepwise regression method in Design-Expert (Stat-Ease [32]) was used to select a reduced form of the full 18-term quadratic×quadratic MA model for TC:

Table 5 D-Optimal mixture-amount design and values of manufacturing cost (MC), $\widehat{E}_Y$, $\widehat{\sigma}_Y^2$, quality-loss cost (QLC), and total cost (TC) for the steering mechanism example

Design point	Run order	x_1	x_2	x_3	A	t_1	t_2	t_4	T	MC	$\widehat{E}_Y$	$\widehat{\sigma}_Y^2$	QLC	TC
1	24	0.6390	0.2500	0.1110	−1	2.30	0.90	0.40	3.6	19.44	400.07	0.38	3.85	23.29
2	12	0.6670	0.1665	0.1665	−1	2.40	0.60	0.60	3.6	20.23	400.20	0.32	3.57	23.80
3	16	0.5695	0.2500	0.1805	−1	2.05	0.90	0.65	3.6	21.57	400.14	0.43	4.48	26.05
4	23	0.6390	0.1110	0.2500	−1	2.30	0.40	0.90	3.6	19.33	400.04	0.34	3.42	22.75
5	7	0.6390	0.1110	0.2500	−1	2.30	0.40	0.90	3.6	20.74	400.52	0.17	4.39	25.13
6	6	0.5695	0.1805	0.2500	−1	2.05	0.65	0.90	3.6	23.60	399.96	0.43	4.36	27.96
7	17	0.5000	0.2500	0.2500	−1	1.80	0.90	0.90	3.6	23.43	400.08	0.38	3.81	27.24
8	1	0.5000	0.2500	0.2500	−1	1.80	0.90	0.90	3.6	24.86	399.98	0.45	4.53	29.39
9	26	0.6670	0.2220	0.1110	−0.5	2.60	0.87	0.43	3.9	18.68	400.03	0.41	4.08	22.76
10	20	0.5695	0.1805	0.2500	−0.5	2.22	0.70	0.98	3.9	19.54	400.27	0.37	4.39	23.93
11	15	0.6390	0.2500	0.1110	0	2.68	1.05	0.47	4.2	17.52	400.62	0.17	5.58	23.10
12	2	0.6670	0.1665	0.1665	0	2.80	0.70	0.70	4.2	14.57	399.87	0.51	5.29	19.86
13	18	0.5695	0.2500	0.1805	0	2.39	1.05	0.76	4.2	16.50	399.60	0.37	5.29	21.79
14	28	0.6390	0.1110	0.2500	0	2.68	0.47	1.05	4.2	15.76	400.43	0.42	5.99	21.75
15	22	0.5695	0.1805	0.2500	0	2.39	0.76	1.05	4.2	20.06	400.12	0.52	5.35	25.41
16	14	0.5000	0.2500	0.2500	0	2.10	1.05	1.05	4.2	19.47	400.50	0.32	5.65	25.12
17	9	0.6390	0.2500	0.1110	0.5	2.88	1.13	0.50	4.5	11.40	399.61	0.64	7.90	19.30
18	4	0.6390	0.1110	0.2500	0.5	2.88	0.50	1.13	4.5	12.95	400.01	0.71	7.08	20.03
19	27	0.5000	0.2500	0.2500	0.5	2.25	1.13	1.13	4.5	19.76	400.33	0.67	7.77	27.53
20	13	0.6390	0.2500	0.1110	1	3.07	1.20	0.53	4.8	12.34	399.82	0.81	8.38	20.72
21	3	0.6670	0.1665	0.1665	1	3.20	0.80	0.80	4.8	10.42	399.95	0.79	7.89	18.31
22	8	0.6670	0.1665	0.1665	1	3.20	0.80	0.80	4.8	10.42	400.37	0.65	7.89	18.31
23	10	0.5695	0.2500	0.1805	1	2.73	1.20	0.87	4.8	11.84	400.53	0.56	8.45	20.29
24	11	0.5695	0.2500	0.1805	1	2.73	1.20	0.87	4.8	14.01	399.76	0.71	7.69	21.70
25	5	0.6390	0.1110	0.2500	1	3.07	0.53	1.20	4.8	7.02	399.66	0.83	9.47	16.49
26	19	0.5695	0.1805	0.2500	1	2.73	0.87	1.20	4.8	11.25	399.93	0.93	9.31	20.56
27	21	0.5695	0.1805	0.2500	1	2.73	0.87	1.20	4.8	13.07	399.61	0.64	7.90	20.97
28	25	0.5000	0.2500	0.2500	1	2.40	1.20	1.20	4.8	14.01	400.08	0.89	8.96	22.97

Table 6 ANOVA for the steering mechanism mixture-amount model (18)

Source	Sum of squares	*df*	Mean square	*F* value	*P* value Prob>*F*	
Model	240.99	5	48.20	32.25	< 0.0001	Significant
Linear mixture	106.28	2	53.14	35.56	< 0.0001	
$x_1'x_3'$	5.71	1	5.71	3.82	0.0635	
$x_1'A'$	18.20	1	18.20	12.18	0.0021	
$x_3'A'$	30.21	1	30.21	20.21	0.0002	
Residual	32.88	22	1.49			
Lack of fit	26.66	17	1.57	1.26	0.4310	Not significant
Pure error	6.22	5	1.24			
Cor total	273.87	27				

$$\widehat{TC} = 20.89x_1' + 22.63x_2' + 30.21x_3' - 18.29x_1'x_3' - 3.19x_1'A' - 4.24x_3'A' \tag{18}$$

For this estimated model, R^2=0.880, $R_A^2 = 0.853$, and $R_P^2 = 0.809$. The analysis of variance table for (18) is given in Table 6 and shows that the model has a non-significant lack of fit. Even though we started out considering the quadratic×quadratic MA model with 18 terms, the reduced MA model (18) contains only 6 terms. This model shows that (i) the three tolerance proportions have linear blending effects on TC, (ii) the first and third tolerance proportions have a nonlinear blending effect on TC, and (iii) the linear effects of the first and third tolerance proprotions on TC depend on the total amount (assembly tolerance) in a linear way. This type of interpretation is possible with the MA approach but is not possible with the CA approach.

For this steering mechanism example illustrating the MA approach, the optimization problem (16) is

$$\begin{array}{ll} \text{Minimize} & \widehat{TC} \text{ given in (18)} \\ \textit{subject to} & 0 \le x_1' \le 0.60,\ 0 \le x_2' \le 0.50,\ 0 \le x_3' \le 0.50 \\ & (\text{equivalent to } 0.500 \le x_1 \le 0.667,\ 0.111 \le x_2 \le 0.250, \\ & 0.111 \le x_3 \le 0.250), \\ & \text{and } -1 \le A' \le 1 (\text{equivalent to } 3.6 \le T \le 4.8). \end{array}$$

Using Solver in Excel, the optimal values that minimize the TC were determined to be (x_1',x_2',x_3')=(0.60, 0, 0.40) and A'=1. The same solution was obtained using the optimization capability of Design-Expert [32]. Using Eq. (4) and $T = 0.6A' + 4.2$ yielded (x_1,x_2,x_3)=(0.667, 0.111, 0.222) and T=4.8. Finally, using t_i=x_iT gave the optimal tolerances as t_1=3.20, t_2=0.53, t_4=1.07 with T=4.8. The predicted TC at the optimum tolerances is 16.61. The contour plot of TC as a function of (x_1,x_2, x_3) for T=4.8 is given in Fig. 4, with the optimal solution shown to be on a vertex of the constrained experimental region.

7 Summary and conclusions

This article discusses using CA or MA approaches for tolerance design problems in which (i) there is one assembly characteristic, (ii) the assembly tolerance is the sum of the component tolerances, and (iii) the goal is to determine the values of the tolerances that minimize the total cost (manufacturing cost plus quality-loss cost). These are non-standard applications of CA and MA experiments because they do not involve mixtures of components (ingredients) in the usual sense. However, this kind of tolerance design problem has a structure to which CA and MA experimental

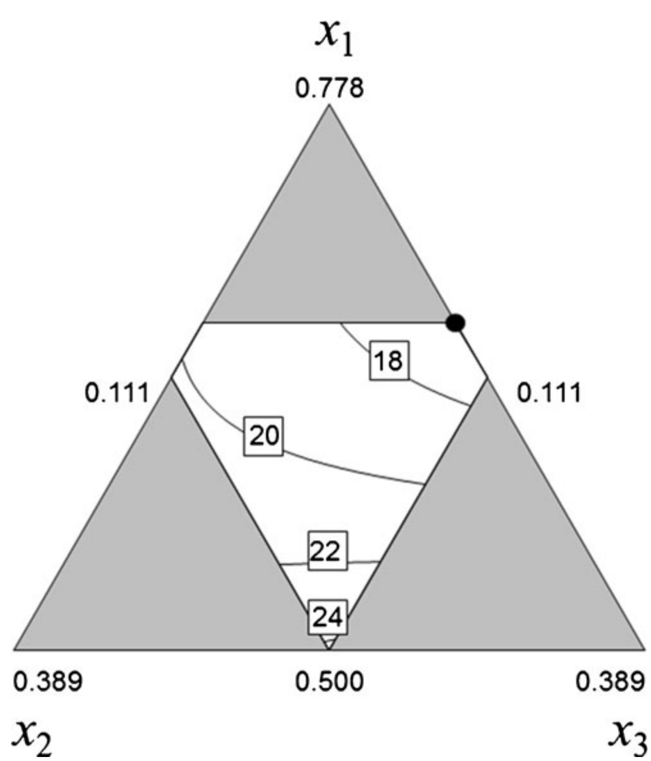

Fig. 4 Contour plot of total cost versus component tolerance proportions (x_1,x_2,x_3) with T=4.8 for the steering mechanism example, with the optimum solution shown as a *dot*

designs and models can be adapted. In addition, CA and MA designs avoid difficulties associated with using classical response surface designs such as a CCD or BBD (as recommended by Jeang [5]). All design points in a CA or MA design will be within the relevant experimental region, whereas some design points in a CCD or BBD may fall outside the relevant experimental region. This may result in functional problems of the components and/or the assembly. It also may affect the cost-tolerance model, and hence result in a sub-optimum solution obtained using the model. Ultimately, because a designer does not want to explore such infeasible regions, using an MA or a CA design is more reasonable than using a CCD or BBD. Further, the CA and MA approaches to solving the tolerance design problem provide less extreme solutions compared to the worst-case method, so that the advantages of that method can be realized while limiting the disadvantages.

In the planning stage of an experiment to address a tolerance design problem for which the CA and MA approaches are appropriate, a designer must make a choice between two approaches. The CA approach is probably most natural for many tolerance design problems because component tolerances may naturally be thought of as component amounts. However, there are two main advantages of the MA approach compared to the CA approach. First, the MA approach separately investigates the effects on the total cost of (i) the assembly tolerance, and (ii) the proportional tolerances of the assembly components. Hence, it is possible to learn how the proportional allocation of component tolerances should change as the assembly tolerance changes. In the CA approach, the effects of (i) and (ii) on total cost are confounded and cannot be studied and understood separately. Second, as pointed out by Piepel and Cornell [30, 36], CA models are special cases of corresponding MA models. Hence, MA models have the potential to fit experimental data from a tolerance design problem better than CA models. Ultimately, the choice between the CA and MA approaches for a tolerance design problem to minimize total cost depends on the naturalness of the CA approach versus the possible interpretative and fitting advantages of the MA approach. However, either the CA or MA approach is preferred to the inappropriate RSM approach using classical designs such as the CCD and BBD.

Both the CA and MA approaches are easily implemented in many software packages that (i) provide for single- and multi-variable constraints to define the experimental region, and (ii) apply optimal experimental design to construct a design over the experimental region that is generally polyhedral in shape. Then, classical response surface models for the CA approach, or MA models for the MA approach, can be fit to the resulting experimental data to obtain a model for total cost. Finally, the resulting model for total cost is used with constrained optimization software to identify the optimum tolerance design in terms of the CA or MA variables and their constraints, as shown in (14) and (16), respectively. The optimum settings in terms of MA variables can easily be converted back to optimum tolerance values.

Acknowledgments The work of the first author was completed on his personal time and was not part of any project at Pacific Northwest National Laboratory (PNNL). We thank Ahmet Kurt (general manager of SISMAK Inc. Izmir, Turkey) for providing the presented examples from his shop. We also acknowledge Scott Cooley (PNNL) for reviewing a draft of the manuscript. Finally, we thank an anonymous referee for comments that led to several improvements in the article.

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