Int J Adv Manuf Technol (2013) 67:261–267
DOI 10.1007/s00170-013-4772-9

ORIGINAL ARTICLE

Single machine scheduling jobs with a truncated sum-of-processing-times-based learning effect

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Received: 29 June 2012 / Accepted: 4 December 2012 / Published online: 17 February 2013

Abstract In this note, we consider a single-machine scheduling problem with truncated sum-of-processing-times-based learning considerations. We show that even with the introduction of the proposed model to job processing times, several single-machine problems remain polynomially solvable. For the following objective functions, the discounted total weighted completion time, the maximum lateness, we present heuristics according to the corresponding problems without learning effect. We also analyze the worst-case bound of our heuristics.

Keywords Scheduling · Single machine · Learning effect

1 Introduction

In classical scheduling problems, the processing time of a job is assumed to be constant. However, in many realistic problems of operations management, both machines and workers can improve their performance by repeating the production operations. Therefore, the actual processing time of a job is shorter if it is scheduled later in a sequence. This phenomenon is known as the "learning effect" in the literature [1]. Biskup [2] and Cheng and Wang [3] were among the pioneers that brought the concept of learning into the field of scheduling, although it has been widely employed in management science since its discovery by Wright [4]. An extensive survey of different scheduling models and problems with learning effects could be found in Biskup [5]. More recent papers which have considered scheduling jobs with learning effects include those of Cheng et al. [6, 7, 9, 22], Mosheiov [8], Lee and Wu [10, 11], Wu and Lee [12], Yang and Kuo [13, 16], Yin et al. [14], Wang et al. [15, 17, 27, 31], Wang and Wang [18, 19, 26], Wang and Li [20], Wu et al. [21, 25, 32], Yin and Wang [23], Huang et al. [24], Bai et al. [28], Shen et al. [29], and Wang and Feng [30].

However, the actual processing time of a given job drops to zero precipitously when the normal job processing times are large in the time-dependent learning model proposed by Yang and Kuo [33]. Motivated by this observation, Cheng et al. [22] and Wu et al. [25] proposed a learning model with truncated sum-of-processing-times-based learning considerations where the actual job processing time is a function which depends not only on the total normal processing times of the jobs already processed but also on a control parameter. The use of the truncated function can be justified on the grounds that learning, like other human activities, is limited. This paper extends the results of Cheng et al. [22] and Wu et al. [25] by considering some other single-machine scheduling with truncated sum-of-processing-times-based learning considerations. The remaining part of this note is organized as follows. In Section 2, we formulate the model. In Section 3, we consider several single-machine scheduling problems. The last section presents the conclusions.

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2 Problem description

There are given single-machine and n-independent and nonpreemptive jobs that are immediately available for processing. The machine can handle one job at a time, and preemption is not allowed. Associated with each job J_j ($j = 1, 2, ..., n$), there is a normal processing time p_j (the normal processing time of a job is incurred if the job is scheduled first in a sequence), a due date d_j and a weight w_j. In addition, let $p_{[k]}$ be the normal processing time of a job if it is scheduled in the kth position in a sequence, where $[k]$ is the index of the kth position of the sequence for $k = 1, 2, \ldots, n$. Let p_{jr}^A be the actual processing time of job J_j if it is scheduled in position r in a sequence. As in Cheng et al. [22] and Wu et al. [25], we consider a learning effect model, i.e.,

$$p_{jr}^A = p_j \max\left\{\left(1+\sum_{i=1}^{r-1} p_{[i]}\right)^a, \beta\right\}, r, j = 1, 2, \ldots, n, \tag{1}$$

where $a \leq 0$ is the learning index and β is a truncation parameter with $0 < \beta < 1$.

For a given schedule $\pi = [J_1, J_2, \ldots, J_n]$, let $C_j = C_j(\pi)$ denote the completion time of job J_j. In this paper, we will consider the minimization of the following objective functions: the total lateness $\sum L_j$, the sum of the θth ($\theta \geq 0$) power of job completion times $\sum C_j^\theta$, the discounted total weighted completion time $\sum_{j=1}^n w_j(1 - e^{-\gamma C_j})$, where $\gamma \in (0, 1)$ is the discount factor (see Section 3.1 in Pinedo [34]), the maximum lateness $L_{\max} = \max\{L_j | j = 1, 2, \ldots, n\}$, where $L_j = C_j - d_j$. In the remaining part of the paper, all the problems considered will be denoted using the three-field notation scheme introduced by Grahamet al. [35].

3 Single-machine scheduling problems

Lemma 1 (Wu et al. [25]) *For the problem* $1|p_{jr}^A = p_j \max\left\{\left(1+\sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|C_{\max}$, *an optimal schedule can be obtained by sequencing the jobs in nondecreasing order of* p_j *(the shortest processing time (SPT) rule), where* $C_{\max} = \max\{C_j | j = 1, 2, \ldots, n\}$ *is the makespan of all jobs.*

Lemma 2 (Wu et al. [25]) *For the problem* $1|p_{jr}^A = p_j \max\left\{\left(1+\sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|\sum C_j$, *an optimal schedule can be obtained by sequencing the jobs in nondecreasing order of* p_j *(the SPT rule).*

Townsend [36] studied the single-machine scheduling with quadratic objective. He showed that the problem $1||\sum C_j^2$ can be solved optimally by the SPT rule. By the similar proof of Wu et al. [25], we can show that the solution of Townsend still holds for the problem $1|p_{jr}^A = p_j \max\left\{\left(1+\sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|\sum C_j^\theta$, where $\theta > 0$.

Theorem 1 *For the problem* $1|p_{jr}^A = p_j \max\left\{\left(1+\sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|\sum C_j^\theta$, *an optimal schedule can be obtained by sequencing the jobs in nondecreasing order of* p_j *(the SPT rule).*

Proof The proof follows directly from the pairwise interchange analysis. Let π and π' be two job schedules where the difference between π and π' is a pairwise interchange of two adjacent jobs J_j and J_k, that is, $\pi = [S_1, J_j, J_k, S_2], \pi' = [S_1, J_k, J_j, S_2]$, where S_1 and S_2 are partial sequences. Furthermore, we assume that there are $r-1$ jobs in S_1. Thus, J_j and J_k are the rth and the $(r+1)$th jobs with $p_j \leq p_k$, respectively, in π. Likewise, J_k and J_j are scheduled in the rth and the $(r+1)$th positions in π'. To further simplify the notation, let A denote the completion time of the last job in S_1 and J_h be the first job in S_2. In order to show that π dominates π', it suffices to show that (a) $C_k(\pi) \leq C_j(\pi')$ and (b) $C_j^\theta(\pi) + C_k^\theta(\pi) \leq C_k^\theta(\pi') + C_j^\theta(\pi')$. Under π, the completion times of jobs J_j and J_k are

$$C_j(\pi) = A + p_j \max\left\{\left(1+\sum_{i=1}^{r-1} p_{[i]}\right)^a, \beta\right\} \tag{2}$$

and

$$C_k(\pi) = A + p_j \max\left\{\left(1+\sum_{i=1}^{r-1} p_{[i]}\right)^a, \beta\right\} + p_k \max\left\{\left(1+\sum_{i=1}^{r-1} p_{[i]} + p_j\right)^a, \beta\right\}. \tag{3}$$

Under π', the completion times of jobs J_k and J_j are

$$C_k(\pi') = A + p_k \max\left\{\left(1+\sum_{i=1}^{r-1} p_{[i]}\right)^a, \beta\right\} \tag{4}$$

and

$$C_j(\pi') = A + p_k \max\left\{\left(1+\sum_{i=1}^{r-1} p_{[i]}\right)^a, \beta\right\} + p_j \max\left\{\left(1+\sum_{i=1}^{r-1} p_{[i]} + p_k\right)^a, \beta\right\}. \tag{5}$$

From Lemma 1, we have $C_k(\pi) \leq C_j(\pi')$; this completes the proof of part (a). In addition, from $p_j \leq p_k$, we have $C_j(\pi) \leq C_k(\pi')$, hence

$C_j^\theta(\pi) + C_k^\theta(\pi) \leq C_k^\theta(\pi') + C_j^\theta(\pi')$.

This completes the proof of part (b) and thus of the theorem. □

Theorem 2 *For the problem* $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}\left|\sum L_j\right.$, *an optimal schedule can be obtained by sequencing the jobs in nondecreasing order of* p_j *(the SPT rule).*

Proof The total lateness $\sum_{j=1}^n L_j$ can be calculated by

$$\sum_{j=1}^n L_j = \sum_{j=1}^n (C_j - d_j) = \sum_{j=1}^n C_j - \sum_{j=1}^n d_j.$$

The total lateness $\sum_{j=1}^n L_j$ is minimized if $\sum_{j=1}^n C_j$ is minimized as $\sum_{j=1}^n d_j$ is a constant. From Lemma 2, we can obtain that the problem $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|\sum C_j$ can be solved by the SPT rule. This completes the proof. □

It is well known that the weighted discounted smallest processing time first (WDSPT) rule yields an optimal schedule for the classical scheduling problem to minimize the total weighted discounted completion time, i.e., sequencing jobs in nondecreasing order of $\frac{1-e^{-\gamma p_j}}{w_j e^{-\gamma p_j}}$. However, this rule does not yield an optimal schedule under the proposed model as shown in the following example.

Example 1 $n = 2, p_1 = 1, p_2 = 2, w_1 = 10, w_2 = 31, \beta = 0.8, a = -0.5, \gamma = 0.1$. The schedule according to the WDSPT rule is $[J_2, J_1]$, yielding the value $\sum_{j=1}^n w_j(1 - e^{-\gamma C_j}) = 8.0615$. Obviously, the optimal sequence is $[J_1, J_2]$, yielding the optimal value $\sum_{j=1}^n w_j \left(1 - e^{-\gamma C_j}\right) = 8.0490$.

In order to solve the problem approximately, we will use the WDSPT rule as a heuristic for the problem $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|\sum w_j \left(1 - e^{-\gamma C_j}\right)$. The performance of the heuristic will be evaluated by its worst-case error bound.

Lemma 3 (Pinedo [34]) *For the problem* $1||\sum w_j \left(1 - e^{-\gamma C_j}\right)$, *an optimal schedule can be obtained by sequencing the jobs in nondecreasing order of* $\frac{1-e^{-\gamma p_j}}{w_j e^{-\gamma p_j}}$ *(the WDSPT rule).*

Lemma 4 (Wang et al. [17]) $1 - e^{-\gamma\alpha} \geq \alpha(1 - e^{-\gamma})$ *if* $0 \leq \gamma \leq 1$ *and* $0 \leq \alpha \leq 1$.

Theorem 3 *Let* π^* *be an optimal schedule and* π *be a WDSPT schedule for the problem* $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|\sum w_j(1 - e^{-\gamma C_j})$. *Then*

$$\rho_1 = \frac{\sum_{j=1}^n w_j \left(1 - e^{-\gamma C_j(\pi)}\right)}{\sum_{j=1}^n w_j \left(1 - e^{-\gamma C_j(\pi^*)}\right)} \leq \frac{1}{\max\left\{\left(1 + \sum_{j=1}^n p_j - p_{\min}\right)^a, \beta\right\}},$$

and this bound is tight, where $p_{\min} = \min\{p_1, p_2, \ldots, p_n\}$.

Proof Without loss of generality, we can suppose that $\frac{1-e^{-\gamma p_1}}{w_1 e^{-\gamma p_1}} \leq \frac{1-e^{-\gamma p_2}}{w_2 e^{-\gamma p_2}} \leq \ldots \leq \frac{1-e^{-\gamma p_n}}{w_n e^{-\gamma p_n}}$, then

$$\sum_{j=1}^n w_j\left(1 - e^{-\gamma C_j(\pi)}\right) = w_1(1 - e^{-\gamma p_1}) + w_2\left(1 - e^{-\gamma\left[p_1 + p_2 \max\{(1+p_1)^a, \beta\}\right]}\right) + \ldots + w_n\left(1 - e^{-\gamma\left[p_1 + p_2 \max\{(1+p_1)^a, \beta\} + \ldots + p_n \max\left\{\left(1 + \sum_{i=1}^{n-1} p_i\right)^a, \beta\right\}\right]}\right) \leq \sum_{j=1}^n w_j\left(1 - e^{-\gamma \sum_{k=1}^j p_k}\right).$$

$$\begin{aligned}\sum_{j=1}^n w_j \left(1 - e^{-\gamma C_j(\pi^*)}\right) &= w_{[1]}\left(1 - e^{-\gamma p_{[1]}}\right) + w_{[2]}\left(1 - e^{-\gamma\left[p_{[1]} + p_{[2]} \max\{(1+p_{[1]})^a, \beta\}\right]}\right) \\ &\quad + \ldots + w[n]\left(1 - e^{-\gamma\left[p_{[1]} + p_{[2]} \max\{(1+p_{[1]})^a, \beta\} + \ldots + p_{[n]} \max\left\{\left(1 + \sum_{i=1}^{n-1} p_{[i]}\right)^a, \beta\right\}\right]}\right) \\ &\geq \sum_{j=1}^n w_{[j]}\left(1 - e^{-\gamma \max\left\{\left(1 + \sum_{j=1}^n p_j - p_{\min}\right)^a, \beta\right\} \sum_{k=1}^j p_{[k]}}\right) \\ &\geq \max\left\{\left(1 + \sum_{j=1}^n p_j - p_{\min}\right)^a, \beta\right\} \sum_{j=1}^n w_{[j]}\left(1 - e^{-\gamma \sum_{k=1}^j p_{[k]}}\right) \\ &\geq \max\left\{\left(1 + \sum_{j=1}^n p_j - p_{\min}\right)^a, \beta\right\} \sum_{j=1}^n w_j\left(1 - e^{-\gamma \sum_{k=1}^j p_k}\right).\end{aligned}$$

The first inequality derived from $1 - e^{-\gamma C_j}$ is an increasing function on C_j and $1 \geq (1 + p_{[1]})^a \geq \ldots \geq \left(1 + \sum_{i=1}^{n-1} p_{[i]}\right)^a \geq \left(1 + \sum_{j=1}^{n} p_j - p_{\min}\right)^a$; the second inequality is obtained from Lemma 4, i.e., $\left(1 - e^{-\gamma \max\left\{\left(1+\sum_{j=1}^{n} p_j - p_{\min}\right)^a, \beta\right\} \sum_{k=1}^{j} p_{[k]}}\right) \geq \max\left\{\left(1 + \sum_{j=1}^{n} p_j - p_{\min}\right)^a, \beta\right\}\left(1 - e^{-\gamma \sum_{k=1}^{j} p_{[k]}}\right)$; while the third inequality is obtained by Lemma 3, i.e., $\sum_{j=1}^{n} w_{[j]}\left(1 - e^{-\gamma \sum_{k=1}^{j} p_{[k]}}\right) \geq \sum_{j=1}^{n} w_j\left(1 - e^{-\gamma \sum_{k=1}^{j} p_k}\right)$ for $\frac{1-e^{-\gamma p_1}}{w_1 e^{-\gamma p_1}} \leq \frac{1-e^{-\gamma p_2}}{w_2 e^{-\gamma p_2}} \leq \ldots \leq \frac{1-e^{-\gamma p_n}}{w_n e^{-\gamma p_n}}$. Hence,

$$\rho_1 = \frac{\sum_{j=1}^{n} w_j\left(1 - e^{-\gamma C_j(\pi)}\right)}{\sum_{j=1}^{n} w_j\left(1 - e^{-\gamma C_j(\pi^*)}\right)} \leq \frac{1}{\max\left\{\left(1 + \sum_{j=1}^{n} p_j - p_{\min}\right)^a, \beta\right\}}.$$

It is not difficult to see that the bound is tight, since if $a = 0$, we have $\rho_1 = \frac{\sum_{j=1}^{n} w_j(1-e^{-\gamma C_j(\pi)})}{\sum_{j=1}^{n} w_j(1-e^{-\gamma C_j(\pi^*)})} = \frac{1}{\max\{1,\beta\}} = 1$. This result is intuitive as when a=0, the WDSPT schedule is optimal. □

Although the WDSPT sequence does not provide the optimal schedule for the problem $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|\sum w_j(1 - e^{-\gamma C_j})$, it is still optimal for some special conditions.

Theorem 4 *For the problem* $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|\sum w_j\left(1 - e^{-\gamma C_j}\right)$, *if the jobs have agreeable weights, i.e.,* $p_j \leq p_k$ *implies* $w_j \geq w_k$ *for all the jobs* J_i *and* J_j, *an optimal schedule can be obtained by sequencing the jobs in nondecreasing order of* $\frac{1-e^{-\gamma p_j}}{w_j e^{-\gamma p_j}}$ *(the WDSPT rule).*

Proof Here, we still use the same notations as in the proof of Theorem 1. From Theorem 1, we have $C_k(\pi) \leq C_j(\pi')$, $C_j(\pi) \leq C_k(\pi')$ and $C_k(\pi') \leq C_j(\pi')$. Suppose that $\frac{1-e^{-\gamma p_j}}{w_j e^{-\gamma p_j}} \leq \frac{1-e^{-\gamma p_k}}{w_k e^{-\gamma p_k}}$, implies that $p_j \leq p_k$ and $w_j \geq w_k$, hence from Theorem 1, we have $C_j(\pi) \leq C_k(\pi')$, $C_k(\pi) \leq C_j(\pi')$ and $C_j(\pi') \geq C_k(\pi')$. In order to show that π dominates π', it suffices to show that $\sum_{j=1}^{n} w_j(1 - e^{-\gamma C_j(\pi)}) \leq \sum_{j=1}^{n} w_j\left(1 - e^{-\gamma C_j(\pi')}\right)$.

It should be clear that the completion times of J_i in sequences π and π' are equal if it is in S_1 because both sequences have the same jobs in these positions, i.e., $C_{[i]}(\pi) = C_{[i]}(\pi')$ for positions $i = 1, 2, \ldots, r-1$.

From Eqs. 2–5, we have

$$\begin{aligned} & w_k\left(1 - e^{-\gamma C_k(\pi')}\right) + w_j\left(1 - e^{-\gamma C_j(\pi')}\right) \\ & \quad - w_j\left(1 - e^{-\gamma C_j(\pi)}\right) - w_k\left(1 - e^{-\gamma C_k(\pi)}\right) \\ & \quad = w_j e^{-\gamma C_j(\pi)} + w_k e^{-\gamma C_k(\pi)} \\ & \qquad - w_k e^{-\gamma C_k(\pi')} - w_j e^{-\gamma C_j(\pi')} \\ & \quad \geq w_j e^{-\gamma C_k(\pi')} + w_k e^{-\gamma C_j(\pi')} \\ & \qquad - w_k e^{-\gamma C_k(\pi')} - w_j e^{-\gamma C_j(\pi')} \\ & \quad = (w_j - w_k)\left(e^{-\gamma C_k(\pi')} - e^{-\gamma C_j(\pi')}\right) \\ & \quad \geq 0. \end{aligned}$$

From Lemma 1, we have $C_u(\pi') \leq C_u(\pi)$ for any J_u in S_2. Hence, we have $\sum_{j=1}^{n} w_j\left(1 - e^{-\gamma C_j(\pi)}\right) \leq \sum_{j=1}^{n} w_j\left(1 - e^{-\gamma C_j(\pi')}\right)$. This completes the proof of the theorem. □

Corollary 1 *For the problem* $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}, p_j = p|\sum w_j\left(1 - e^{-\gamma C_j}\right)$, *an optimal schedule can be obtained by sequencing the jobs in nonincreasing order of* w_j.

Corollary 2 *For the problem* $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}, w_j = w|\sum w_j\left(1 - e^{-\gamma C_j}\right)$, *an optimal schedule can be obtained by sequencing the jobs in nondecreasing order of* p_j *(the SPT rule).*

Corollary 3 *For the problem* $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}, w_j p_j = \mu|\sum w_j\left(1 - e^{-\gamma C_j}\right)$, *an optimal schedule can be obtained by sequencing the jobs in nondecreasing order of* p_j *(the SPT rule).*

The earliest due date (EDD) rule provides an optimal schedule for the classical problem to minimize the maximum lateness. However, it does not yield an optimal schedule under the proposed model as shown in the following example.

Example 2 $n = 2, p_1 = 1, p_2 = 100, d_1 = 1, d_2 = 0, \beta = 0.8, a = -0.5$. The schedule according to the EDD rule is $[J_2, J_1]$, yielding the value $L_{\max} = 100$. Obviously, the optimal sequence is $[J_1, J_2]$, yielding the optimal value $L_{\max} = 81$.

In order to solve the problem approximately, we use the EDD rule as a heuristic for the problem $1|p_{jr}^A = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|L_{\max}$. To develop a worst-case performance ratio for a heuristic, we have to avoid cases involving nonpositive $L_{\max}$. Similar to that of Cheng and Wang [3], the worst-case error bound is defined as follows:

$$\rho_2 = \frac{L_{\max}(\pi) + d_{\max}}{L_{\max}(\pi^*) + d_{\max}},$$

where π and $L_{\max}(\pi)$ denote the heuristic schedule and the corresponding maximum lateness, respectively, while π^* and $L_{\max}(\pi^*)$ denote the optimal schedule and the minimum maximum lateness value, respectively, and $d_{\max} = \max\{d_j | j = 1, 2, \ldots, n\}$.

Theorem 5 *Let π^* be an optimal schedule and π be an EDD schedule for the problem $1|p^A_{jr} = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|L_{\max}$. Then*

$$\rho_2 = \frac{L_{\max}(\pi) + d_{\max}}{L_{\max}(\pi^*) + d_{\max}} \leq \frac{\sum_{i=1}^n p_i}{C^*_{\max}},$$

and the bound is tight, where $C^*_{\max}$ is the optimal makespan of $1|p^A_{jr} = p_j \max\left\{\left(1 + \sum_{l=1}^{r-1} p_{[l]}\right)^a, \beta\right\}|C_{\max}$.

Proof Without loss of generality, supposing that $d_1 \leq d_2 \leq \ldots \leq d_n$, we have

$$\begin{aligned} L_{\max}(\pi) &= \max\{p_1 + p_2 \max\{(1 + p_1)^a, \beta\} \\ &\quad + \ldots + p_j \max\left\{\left(1 + \sum_{i=1}^{j-1} p_i\right)^a, \beta\right\} \\ &\quad - d_j | j = 1, 2, \ldots, n\} \\ &\leq \max\{p_1 + p_2 + \ldots + p_j - d_j | j = 1, 2, \ldots, n\} \\ &= L'_{\max}(\pi), \end{aligned}$$

where $L'_{\max}(\pi)$ is the optimal value of the classical version of the problem, i.e., $p_{j,r} = p_j$.

$$\begin{aligned} L_{\max}(\pi^*) &= \max\{p_{[1]} + p_{[2]} \max\{(1 + p_{[1]})^a, \beta\} + \ldots + p_{[j]} \max\left\{\left(1 + \sum_{i=1}^{j-1} p_{[i]}\right)^a, \beta\right\} - d_{[j]} | j = 1, 2, \ldots, n\} \\ &= \max\left\{\sum_{i=1}^j p_{[i]} - d_{[j]} - \sum_{i=1}^j p_{[i]} + \sum_{i=1}^j p_{[i]} \max\left\{\left(1 + \sum_{l=1}^{i-1} p_{[l]}\right)^a, \beta\right\} | j = 1, 2, \ldots, n\right\} \\ &= \max\left\{\sum_{i=1}^j p_{[i]} - d_{[j]} - \sum_{i=1}^j p_{[i]} \left(1 - \max\left\{\left(1 + \sum_{l=1}^{i-1} p_{[l]}\right)^a, \beta\right\}\right) | j = 1, 2, \ldots, n\right\} \\ &\geq \max\left\{\sum_{i=1}^j p_{[i]} - d_{[j]} - \sum_{i=1}^n p_{[i]} \left(1 - \max\left\{\left(1 + \sum_{l=1}^{i-1} p_{[l]}\right)^a, \beta\right\}\right) | j = 1, 2, \ldots, n\right\} \\ &= \max\left\{\sum_{i=1}^j p_{[i]} - d_{[j]} | j = 1, 2, \ldots, n\right\} - \sum_{i=1}^n p_{[i]} + \sum_{i=1}^n p_{[i]} \max\left\{\left(1 + \sum_{l=1}^{i-1} p_{[l]}\right)^a, \beta\right\} \\ &\geq L'_{\max}(\pi) - \sum_{i=1}^n p_i + C^*_{\max}, \end{aligned}$$

hence,

$$L_{\max}(\pi) \leq L'_{\max}(\pi) \leq L_{\max}(\pi^*) + \sum_{i=1}^n p_i - C^*_{\max},$$

and so

$$\begin{aligned} \rho_2 &= \frac{L_{\max}(\pi) + d_{\max}}{L_{\max}(\pi^*) + d_{\max}} \\ &\leq \frac{L_{\max}(\pi^*) + \sum_{i=1}^n p_i - C^*_{\max} + d_{\max}}{L_{\max}(\pi^*) + d_{\max}} \\ &\leq 1 + \frac{\sum_{i=1}^n p_i - C^*_{\max}}{L_{\max}(\pi^*) + d_{\max}} \\ &\leq 1 + \frac{\sum_{i=1}^n p_i - C^*_{\max}}{C^*_{\max}} \\ &= \frac{\sum_{i=1}^n p_i}{C^*_{\max}}, \end{aligned}$$

where $C^*_{\max}$ can be obtained by the SPT rule (see Lemma 1).

It is not difficult to see that the bound is tight, since if $a = 0$, we have $C_{\max} = \sum_{i=1}^n p_i$ and $\rho_2 = \frac{L_{\max}(\pi) + d_{\max}}{L_{\max}(\pi^*) + d_{\max}} = \frac{\sum_{i=1}^n p_i}{\sum_{i=1}^n p_i} = 1$. This result is intuitive because when $a = 0$, the EDD schedule is optimal. □

4 Conclusions

In this note, we considered single-machine scheduling problems with truncated sum-of-processing-times-based learning considerations. We proved that the total lateness minimization problem and the sum of the θth power of job completion times minimization problem can be solved by the SPT rule, respectively. For the discounted total weighted completion time and the maximum lateness minimization problems, we studied popular heuristics used in the corresponding problems without learning effects. We also analyze the worst-case bound of heuristics WDSPT and EDD. We further showed that under certain conditions, these heuristic rules do provide optimal schedules. The computational complexity of the problems $1|p_{jr}^{A} = p_j \max\left\{\left(1+\sum_{l=1}^{r-1} p_{[l]}\right)^{a}, \beta\right\}|\sum w_j C_j$, $1|p_{jr}^{A} = p_j \max\left\{\left(1+\sum_{l=1}^{r-1} p_{[l]}\right)^{a}, \beta\right\}|\sum w_j\left(1-e^{-\gamma C_j}\right)$ and $1|p_{jr}^{A} = p_j \max\left\{\left(1+\sum_{l=1}^{r-1} p_{[l]}\right)^{a}, \beta\right\}|L_{\max}$ remains unsolved. These unsolved problems seem to be an interesting topic for future research.

Acknowledgments We are grateful to the editor and two anonymous referees for their helpful comments on earlier version of this paper. This research was supported by the National Natural Science Foundation of China under grant number 11001181 and The Hong Kong Polytechnic University (PolyU's project G-UA71).

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