Int J Adv Manuf Technol (2013) 65:343–353
DOI 10.1007/s00170-012-4173-5

ORIGINAL ARTICLE

A warpage optimization method for injection molding using artificial neural network with parametric sampling evaluation strategy

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Received: 13 December 2010 / Accepted: 13 April 2012 / Published online: 15 May 2012

Abstract A sequential optimization design method based on artificial neural network (ANN) surrogate model with parametric sampling evaluation (PSE) strategy is proposed in this paper. The quality index, such as warpage deformations, thickness uniformity, and so on, is a nonlinear, implicit function of the process conditions, which are typically evaluated by the solution of finite element (FE) equations, a complicated task which often involves huge computational effort. The ANN model can build an approximate function relationship between the design variables and quality index, replacing the expensive FE reanalysis of the quality index in the optimization. Moldflow Corporation's Plastics Insight software is used to analyze the quality index of the injection-molded parts. The optimization process is performed by a Parametric Sampling Evaluation (PSE) function. PSE is an infilling sampling criterion. Although the design of experiment size is small, this criterion can take the relatively unexpected space into consideration to improve the accuracy of the ANN model and quickly tend to the global optimization solution in the design space. As examples, a scanner, a TV cover, and a plastic lens are investigated. The results show that the sequential optimization method based on PSE sampling criterion can converge faster and effectively approach to the global optimization solution.

Keywords Injection molding · Warpage · Optimization · DOE · ANN · PSE function

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1 Introduction

Injection molding is widely applied to make plastic products with complex shape, highly precision, and low cost. Warpage is a serious defect in injection molded parts, especially for the thin-wall plastic products, so how to reduce warpage is becoming the key to improve the part quality. In general, injection processing can be divided into three stages: filling, packing, and cooling [1]. Warpage can be reduced by changing the geometry of parts, modifying the structure of molds, or adjusting the process parameters. The part design and mold design are usually determined in the initial stage of product development, which cannot be easily changed. So the optimization of process parameters should be more feasible and reasonable.

There have been many publications devoted to warpage optimization [2–13]. Kim and Lee [12] optimized the process conditions and the wall thickness to reduce warpage using the modified complex method and obtained significant warpage reduction (over 70 %). Sahu et al. [13] optimized the process conditions by a combined implementation of the modified complex method and design of experiments (DOE) to reduce warpage of injection molded parts. Their results show that these methods can reduce warpage effectively, but these methods are costly and time-consuming because they perform lots of expensive function evaluations. The Taguchi method [14, 15] has been employed over the years to improve products or manufacturing processes. It is a powerful and effective method to solve the quality problems of products. However, it can only find the best set of specified given process parameter level combinations, not the global optimal solution in the design

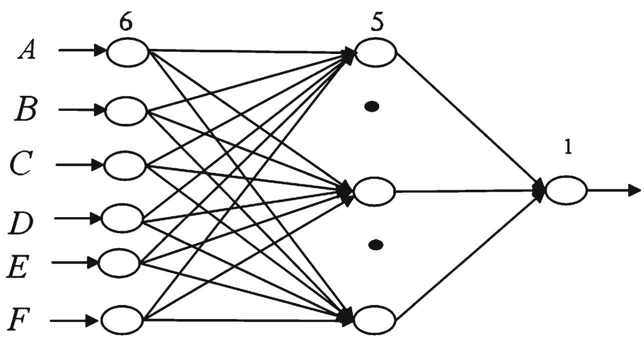

Fig. 1 Configuration of the ANN model

space. Hence, the efficient methods are necessary to efficiently and rapidly analyze process parameters and control the product quality.

To cope with these challenges, many researchers introduced some surrogate models, such as Kriging surrogate model, artificial neural network (ANN), response surface method (RSM), support vector regression (SVR), and so on. Gao and Wang [16] optimized the process parameters to reduce the warpage by combining Kriging surrogate model with expected improvement (EI) function method [17]. Kurtaran et al. [18] combined an ANN model with

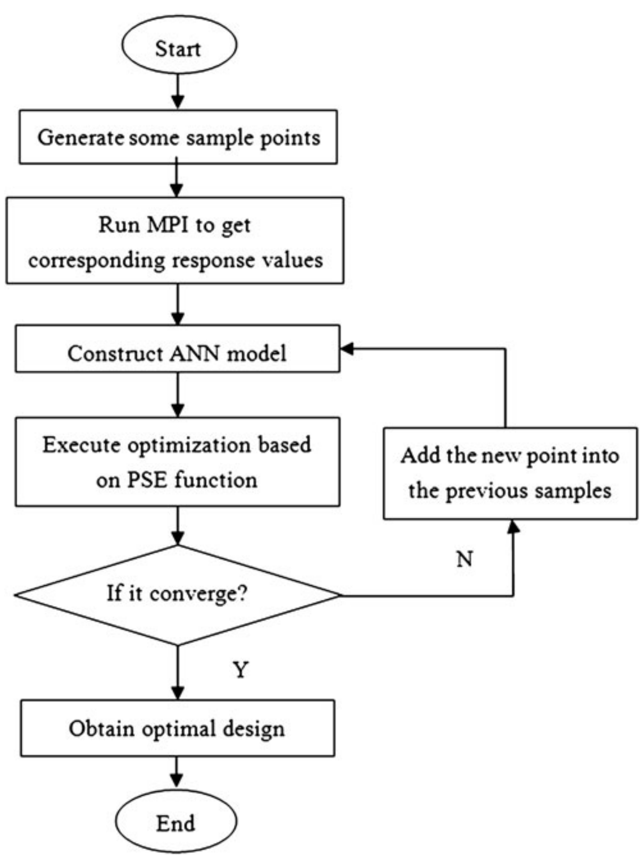

Fig. 2 Flow chart for warpage optimization method based on ANN with PSE

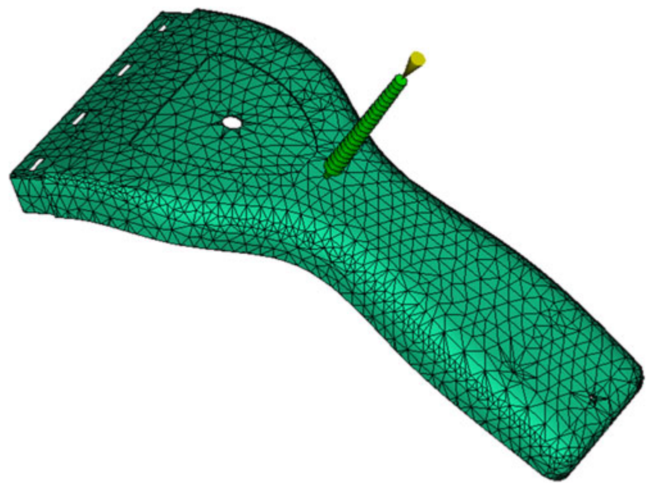

Fig. 3 Model of a scanner

genetic algorithm (GA) to optimize the process conditions for reducing the warpage. Shi et al. [19] optimized the process parameters to reduce warpage using combination of an ANN model with EI function. Ozcelik and Erzurumlu [20] optimized the process parameters to reduce warpage using integrated RSM with GA. Zhou et al. [21] used the SVR model and GA to optimize the process parameters. Their results show that these surrogate models can be considered as good way to reduce high computational cost in the warpage optimization. GA and EI function can be used to approach to the global optimal solution effectively.

In this study, the mold temperature, melt temperature, injection time, packing time, packing pressure, and cooling time are considered as process parameters. A surrogate model is then created by ANN, which is used to compute approximately values of quality index using the process parameters. A small-size design of experiment (DOE) is obtained by Latin hypercube design (LHD), and the quality index values are evaluated by MoldFlow Plastic Insight (MPI) software. The optimization process is performed by the parametric sampling evaluation (PSE) function. PSE function can adaptively select a better additional sampling point to improve the surrogate model and find the optimum value at every iteration. This method is similar to Res [17], which has been viewed as effective global optimization. However, the optimization by means of PSE function belongs to the smooth optimizations, which can make the convergence of optimization iterations rapid and steady.

Table 1 Ranges of the process parameters

Parameter	*A* (°C)	*B* (°C)	*C* (s)	*D* (s)	*E* (MPa)	*F* (s)
Lower limit	50	240	0.2	2	40	10
Upper limit	90	300	2	8	100	30

Table 2 Comparison of optimization results

Process parameters	A (°C)	B (°C)	C (s)	D (s)	E (MPa)	F (s)	Warpage (mm)	Iterations
Before optimization	84.09	289.17	0.43	6.214	67.45	16.20	0.217	
After optimization (EI)	74.25	296.05	1.44	5.26	100	30	0.1131	48
After optimization (PSE)	50.80	300	1.54	5.20	100	30	0.1072	21

2 Artificial neural network model

ANN has been widely used in many research fields, such as control, data compression, forecasting, optimization, pattern recognition, classification, regression, speech, vision, etc. [22–27]. One typical ANN is a back propagation network.

An ANN is a multilayered construction made up of one or more hidden layers placed between the input and output layers. The layers include several processing units called neurons. All of them are connected with variable weights that have to be determined. In the network, each neuron receives total input from all of the neurons in the preceding layer given as below:

$$\mathrm{net}_j = \sum_{j=0}^{N} w_{ij} x_i \quad (1)$$

where net_j is the total or net input, and N is the number of inputs to the ith neuron in the hidden layer. w_{ij} is the weight of the connection from the ith neuron in the forward layer to the jth neuron in the hidden layer and x_i is the input from the ith neuron in the preceding layer. A neuron in the network produces its output (out_j) by processing the net input through an activation (transfer) function f, such as logistic function chosen in this study given as follows:

$$\mathrm{out}_j = f\left(\mathrm{net}_j\right) = \frac{1}{1 + e^{-\mathrm{net}_j}} \quad (2)$$

An ANN is trained by repeatedly presenting a series of input/output pattern sets to the network. The neural network gradually "learns" the input/output relationship of interest by modifying the weights between its neurons to minimize the error between the actual and predicted output patterns of the training set. Then, a separate set of data called the test set to monitor network's performance. When the mean squared error (MSE) reaches a minimum value, network training is regarded complete and the weights are fixed.

In this study, a three-layer ANN model with one hidden layer was used. The mold temperature (A), melt temperature (B), injection time (C), packing time (D), packing pressure (E), and cooling time (F) are considered as input variables, and the quality index is regarded as output variable. Therefore, the neuron number of the input layer and output layer of ANN are determined. The neuron number of the hidden layer was determined through trials. The transfer function between the input layer and the hidden layer is "Logsig", and the transfer function between the hidden layer and the output layer is "Purelin". The train function is trainlm; performance function is MSE. The configuration of ANN used in this paper is shown in Fig. 1.

3 Parametric sampling evaluation method

ANN can be used as an arbitrary function approximation mechanism which "learns" from training data. For this study,

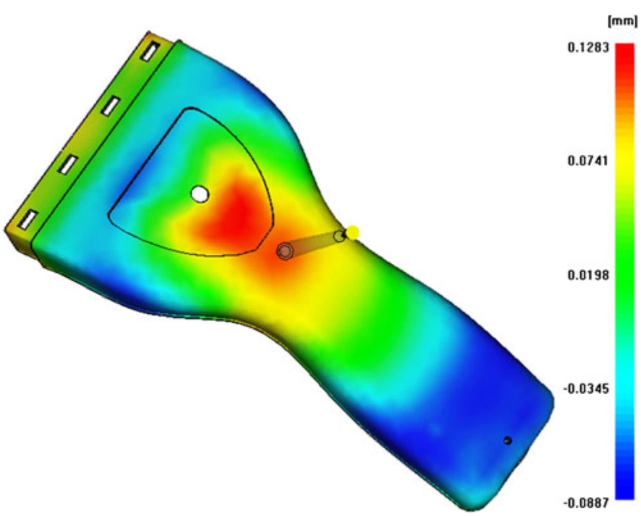

Fig. 4 Warpage of the scanner before optimization

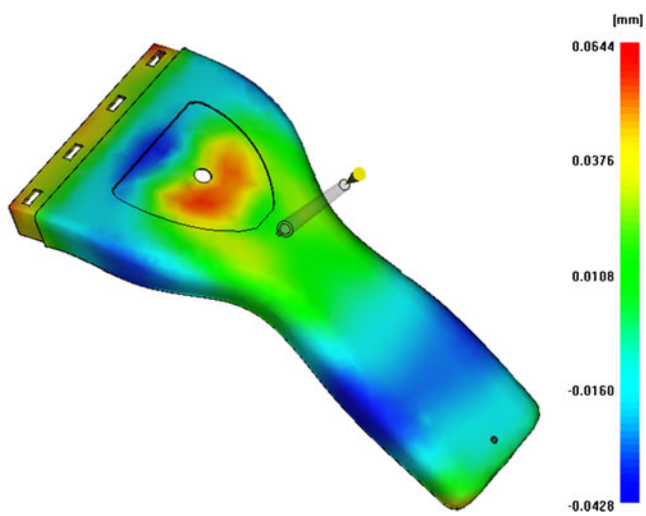

Fig. 5 Warpage of the scanner after optimization

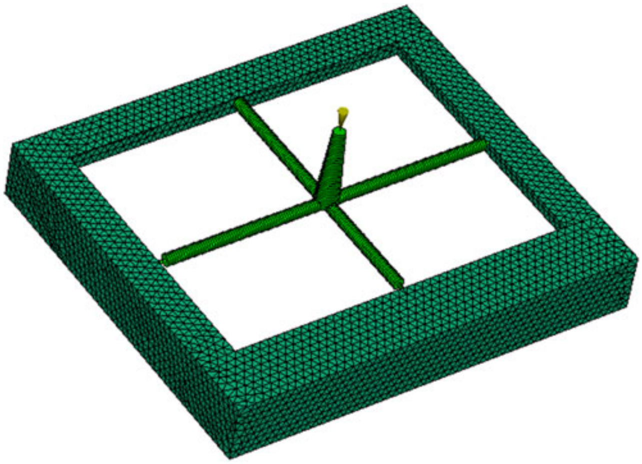

Fig. 6 Model of a TV cover

ANN is used to construct an approximate function relationship between the quality index value and the process parameters, replacing the expensive analysis and reanalysis of simulation programs in the optimization process. In general, this function may have many extremum points, making the optimization algorithms employing such functions converge to a local minimum. PSE algorithm is here introduced to approach to the global optimization solution.

3.1 Expected improvement combined with ANN model

EI combined with ANN model includes computing the possible improvement at a given point. It is an efficient global optimization algorithm, which utilizes an ANN model to select sample points [17]. The new points, or "infill samples", are selected based on a criterion called EI function. It can balance local and global search. The basic formulation of the EI method is briefly described as follows:

Before sampling at some point x, the response value $y(x)$ is unpredictable. $y(x)$ at a candidate point x is normally distributed with mean $\widehat{y}(x)$ and variance σ^2calculated by using the ANN predictor. If the current best response value is $Y_{\min}$, then an improvement $I=Y_{\min}-y(x)$ by the ANN predictor can be achieved. The likelihood of this improvement is given by the normal density

$$\frac{1}{\sqrt{2\pi}\sigma(x)}\exp\left[-\frac{\left(Y_{\min}-I-\widehat{y}(x)\right)^2}{2\sigma^2(x)}\right] \tag{3}$$

Table 3 Ranges of the process parameters

Parameter	A (°C)	B (°C)	C (s)	D (s)	E (%)	F (s)
Lower limit	40	220	2	2	60	10
Upper limit	80	280	6	10	90	30

Then the expected value of the improvement is found by integrating over this density:

$$e(I)=\int_{I=0}^{I=\infty}\left\{\frac{1}{\sqrt{2\pi}\sigma(x)}\exp\left[-\frac{\left(Y_{\min}-I-\widehat{y}(x)\right)^2}{2\sigma^2(x)}\right]\right\}dI \tag{4}$$

Using integration by parts, Eq. (4) can be written as

$$e(I)=\sigma(x)[u\Phi(u)+\phi(u)] \tag{5}$$

Where Φ and ϕ are the normal cumulative distribution function and density function, respectively, and

$$u=\frac{Y_{\min}-\widehat{y}(x)}{\sigma(x)} \tag{6}$$

The first term of Eq. (5) is the difference between the current minimum response value $Y_{\min}$ and the predicted value $\widehat{y}(x)$ at x, penalized by the probability of improvement. Hence the first term is large when $\widehat{y}(x)$ is small. The second term is a product of predicted error $\sigma(x)$ and normal density function $\phi(u)$. The normal density function value is large when the error $\sigma(x)$ is large and $\widehat{y}(x)$ is close to $Y_{\min}$. Thus, the expected improvement will tend to be large at a point with the predicted value smaller than $Y_{\min}$ and/or with much predicted uncertainty. The framework of this method may be described as follows:

- Step 1 Generate m sample points using chosen DOE method, and run the simulation program to get the output (response) values of the corresponding sample points. Every sample point presents a group of processing combination.
- Step 2 Perform ANN simulation based on sample points and corresponding response values.
- Step 3 Select optimization algorithms to implement the optimization design based on EI function and obtain the modified processing design.
- Step 4 Check convergence: if the maximal relative EI satisfies its convergence accuracy, then the optimization solution is obtained; else add the modified design (as a new sample point) into the set of samples, and go to step 2.

3.2 Parametric sampling evaluation function

As mentioned above, EI function is used to find a better sampling point, the EI method combined with ANN model is essentially a sequential minimization method, which can approximate to the optimal solution by minimizing the maximum value of EI function step by step. It is proved that the EI method can find the global minimum [17]. However, disadvantages of the traditional method arise from the fact that the

Table 4 Comparison of optimization results

Process parameters	A (°C)	B (°C)	C (s)	D (s)	E (%)	F (s)	Warpage (mm)	Iterations
Before optimization	40.52	236.21	4.56	7.33	70.09	18.59	1.362	
After optimization (EI)	65.74	220.11	6	10	90	30	0.84631	50
After optimization (PSE)	57.20	220.58	6	10	90	30	0.84	22

maximum value of EI function jumps from one set of sampling points to another as the optimization proceeds, making the convergence of algorithms employing such method slow and unsteady. A PSE function $S(E)$ is proposed to improve the traditional EI method (see Appendix 1). A new sampling point can be obtained by minimizing PSE function

$$S(E) = \frac{1}{p} \ln \sum_{j=1}^{m} \exp\left(pe_j(I)\right) \tag{7}$$

where p is a positive real variable, which can be chosen in the range 10^1–10^5, and the sampling evaluation function at sampling point j

$$e_j(I) = \sigma(x)[u_j\Phi(u_j) + \phi(u_j)] \; u_j = \frac{Y_j - \widehat{y}(x)}{\sigma(x)} \tag{8}$$

where Y_j is the predicted value correspond to the sampling point j.

This infilling sampling method has some advantages: (1) it can intelligently add sample points to improve the ANN model, so it allows "learns" from a small DOE size; (2) it can avoid searching the areas with relative large function values and decrease the computational cost; (3) it can also avoid to add some points close to each other in the design space and keep the stability of ANN prediction.

4 The optimization procedure based on PSE function

A optimization design problem can be described as follows:

$$\begin{aligned} &\text{Find } x_1, x_2, \cdots, x_m \\ &\text{Minimize } y(x) \\ &\text{Subjectto } \underline{x}_j \le x_j \le \overline{x}_j \quad j = 1, 2, \cdots, m \end{aligned} \tag{9}$$

where $x_1, x_2, \cdots, x_m$ are the design variables, $y(x)$ is the quality index, which will be replaced by an approximate function based on the ANN model in optimization iterations, and $\underline{x}_j$ and $\overline{x}_j$ are the lower and upper limits of the jth design variable.

ANN model used here is to approximate $y(x)$ as $\widehat{y}(x)$, and the objective of optimization problem based on PSE function should be transformed into minimize $S(E)$, in which Y_j is the predicted value correspond to the sampling point j, and the function value $\widehat{y}(x)$ and $\sigma^2(x)$ at every new point x predicted by ANN model. Optimization procedure can be illustrated in Fig. 2.

The convergence criterion is here to satisfy:

$$\frac{S(E)}{Y_{\min}} \le \varepsilon \tag{10}$$

where ε is a given convergence tolerance, and $Y_{\min}$ is the optimal response value in all samples. The left hand side is a ratio between the maximum $S(E)$ and $Y_{\min}$, so ε can be given

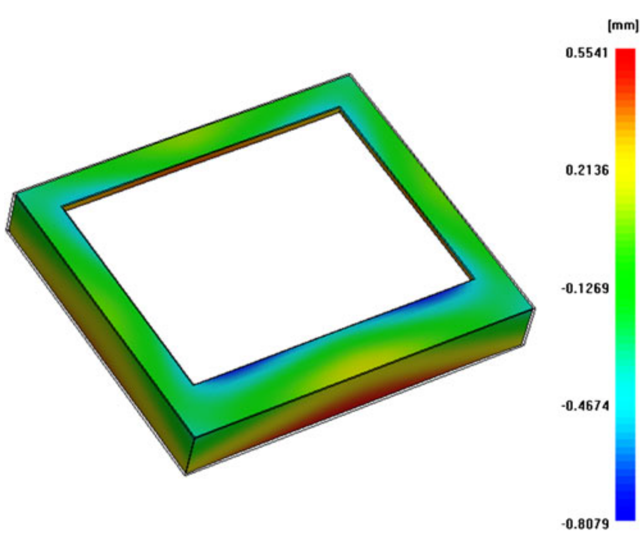

Fig. 7 Warpage of the TV cover before optimization

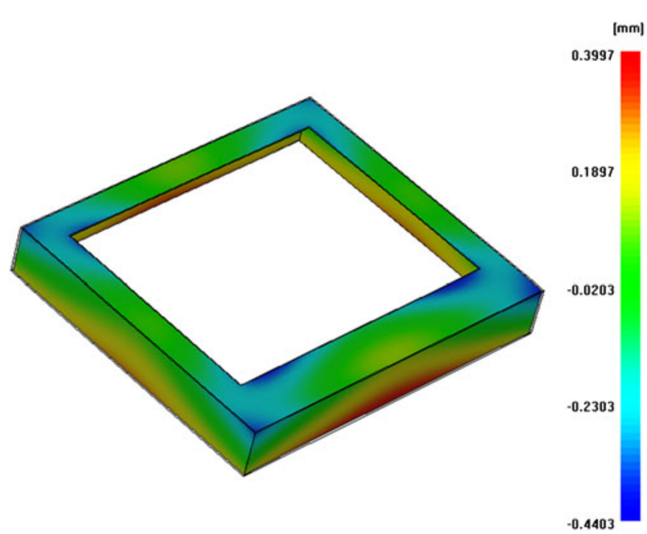

Fig. 8 Warpage of the TV cover after optimization

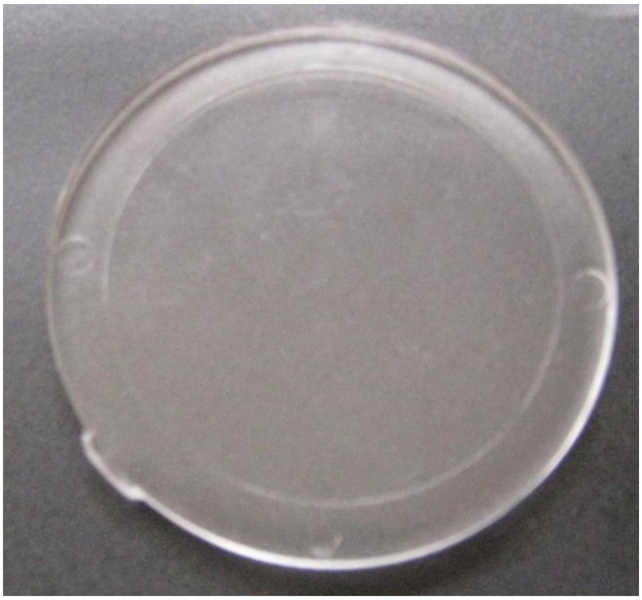

Fig. 9 The geometry model of a plastic lens

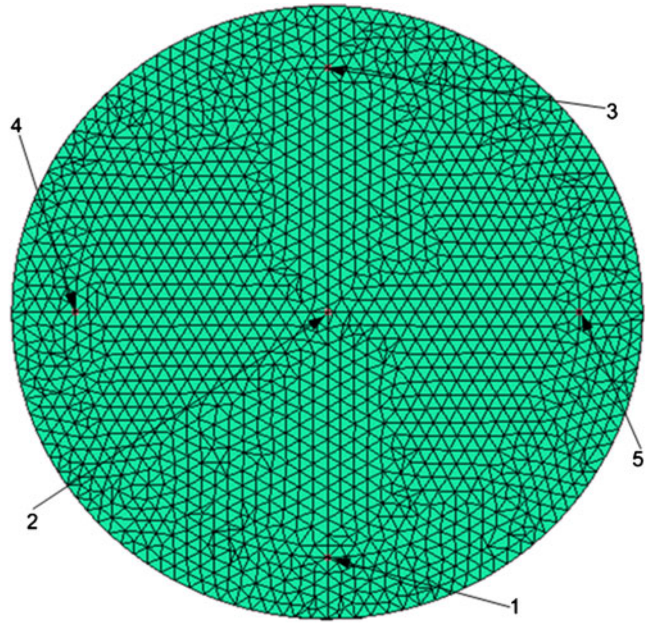

Fig. 11 FE model of a plastic lens with five measuring points

without consideration of the magnitudes and taking ε=0.01 % here.

5 The optimization procedure applications

5.1 The application examples

The first example is a scanner. The cover is discretized by 8,046 triangle elements shown in Fig. 3. It is made of PC/ABS.

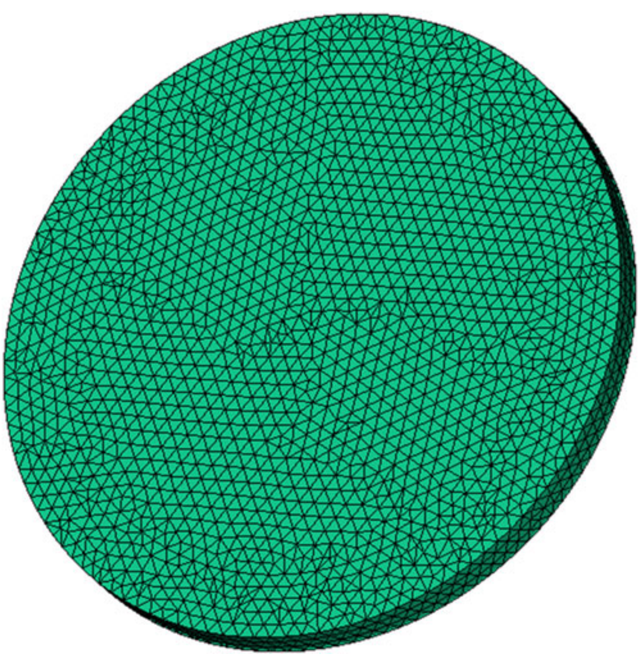

Fig. 10 FE model of a plastic lens

The mold temperature (*A*), melt temperature (*B*), injection time (*C*), packing time (*D*), packing pressure (*E*), and cooling time (*F*) are considered as design variables. The warpage value is as a quality index, which is quantified by the out-of-plane displacement, which is the sum of the maximum upward deformation and the maximum downward deformation with reference to the default plane in MPI software. The ranges of mold temperature and melt temperature are based on the recommended values in MPI. The ranges of injection time, packing time, packing pressure, and cooling time are determined by the experience of the mold analyzer. The ranges of the six variables are given in Table 1.

First, 20 initial samples are selected by LHD, then the corresponding warpage are obtained by running MPI software, finally, an approximate function relationship between the warpage and the process parameters is built by ANN model.

The optimization process based on PSE function is here solved using the sequential quadratic programming (SQP) [28]. First, the point with minimum warpage value in all sample points is selected as initial design, then the minimize value of PSE function is calculated by using SQP, if the minimal value of PSE satisfies its convergence accuracy, the optimization solution is obtained; else add the modified design (as a new sample point) into the set of samples, and go to next optimization iteration. The optimal solution was obtained after

Table 5 Ranges of the process parameters

Parameter	*A* (°C)	*B* (°C)	*C* (s)	*D* (s)	*E* (MPa)	*F* (s)
Lower limit	50	220	2.6	2	20	15
Upper limit	80	280	4.6	10	80	25

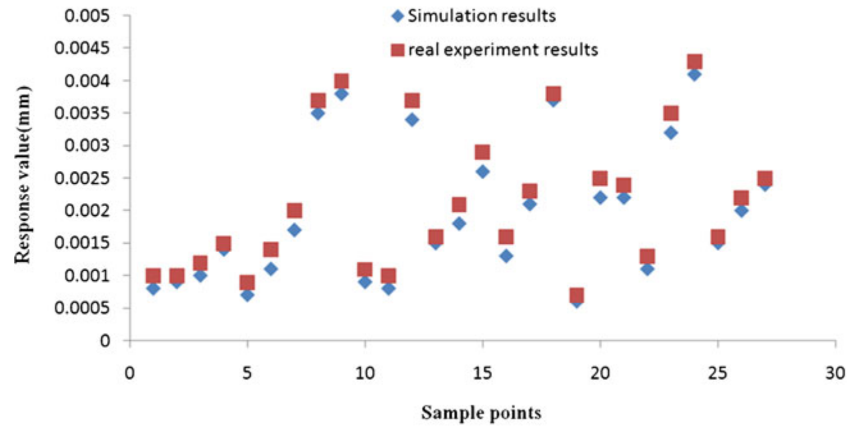

Fig. 12 The comparison of results

21 iterations. The optimization results are given in Table 2. Figures 4 and 5 show the warpage values before and after optimization, respectively. Table 2 also gives the results obtained by the traditional EI method for comparison.

The second example is a TV cover. It is discretized by 5,483 triangle elements, as shown in Fig. 6. Its length, width, height, and thickness are 350, 300, 50, and 2.5 mm, respectively. It is made of ABS.

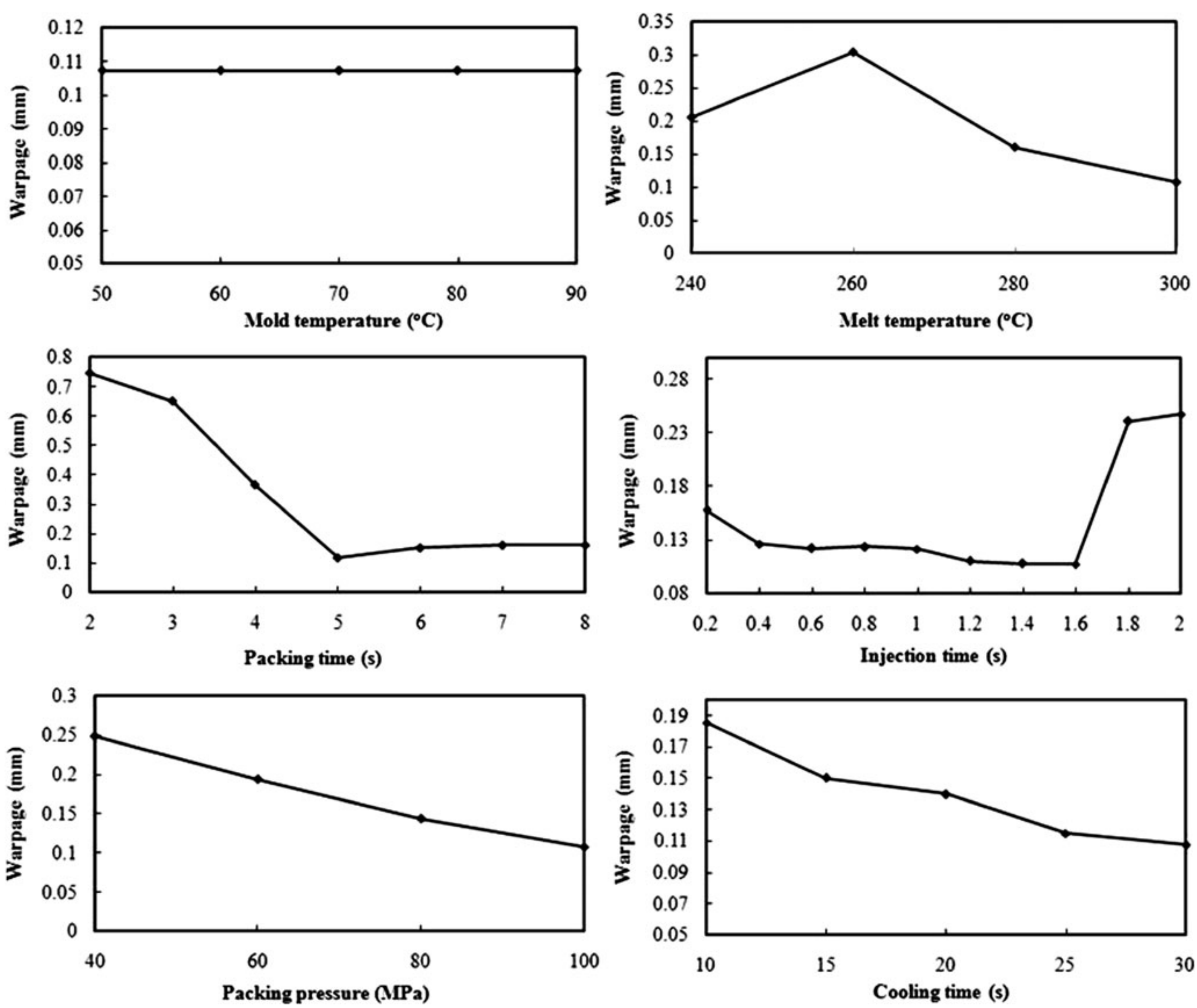

Fig. 13 The effect of each factor on the warpage of the scanner

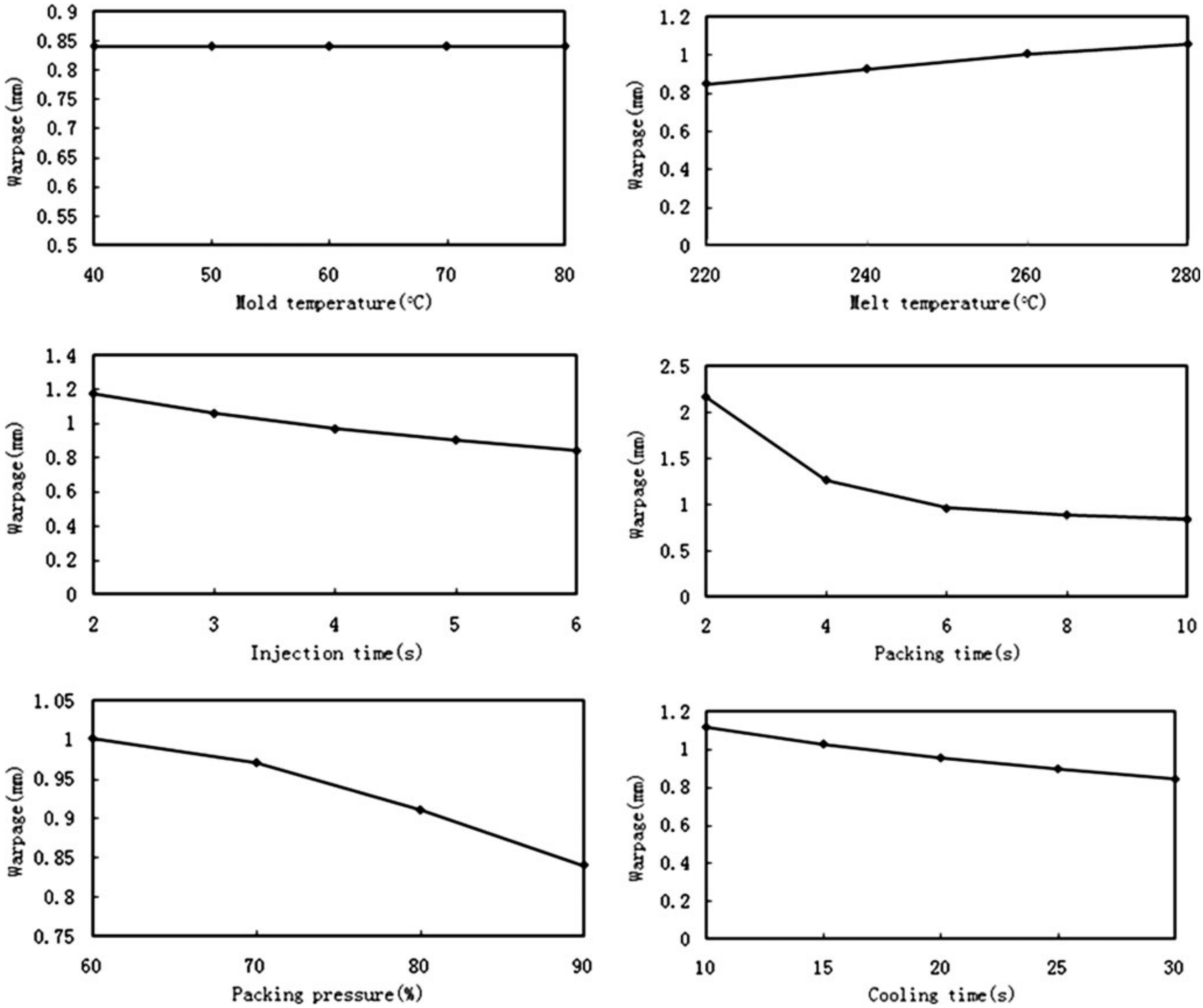

Fig. 14 The effect of each factor on the warpage of the TV cover

The mold temperature (*A*), melt temperature (*B*), injection time (*C*), packing time (*D*), packing pressure (*E*), and cooling time (*F*) are considered as design variables. The warpage value is also as a quality index, which is quantified by the out-of- plane displacement, which is the sum of the maximum upward deformation and the maximum downward deformation with reference to the default plane in MoldFlow Plastic Insight (MPI) software. The ranges of mold temperature and melt temperature are based on the recommended values in MPI. The ranges of injection time, packing time, packing pressure, and cooling time are determined by the experience of the mold analyzer. The ranges of the six variables are given in Table 3.

Twenty initial samples are selected by LHD; the optimal solution was obtained after 22 iterations. The results are given in Table 4. Figures 7 and 8 show the warpage before and after optimization, respectively. Table 4 also gives the results obtained by the traditional EI method for comparison.

The third example is a plastic lens. Its radius and thickness are 25 and 3 mm, respectively. The geometry model is shown

Table 6 Comparison of optimization results

Parameter	*A* (°C)	*B* (°C)	*C* (s)	*D* (s)	*E* (MPa)	*F* (s)	Td (mm)
Before optimization	80	220	4.6	6	20	25	0.0006
After optimization	50	243	3.6	5.5	60	20	0.0003
Experiment	50	243	3.6	5.5	60	20	0.00035

in Fig. 9. It is discretized by 64,691 three-dimensional tetrahedral elements, as shown in Fig. 10. It is made of PMMA.

The mold temperature (*A*), melt temperature (*B*), injection time (*C*), packing time (*D*), packing pressure (*E*), and cooling time (*F*) are considered as design variables. The uniform thickness variation is an important quality index of the plastic lens. Therefore, five points on the plastic lens surface are selected, as shown in Fig. 11. The difference between the maximum thickness and minimum thickness (Td) among these points is characterized a quality index of plastic lens. The ranges of mold temperature and melt temperature are based on the recommended values in MPI. The ranges of injection time, packing time, packing pressure, and cooling time are determined by the experience of the mold analyzer. The ranges of the six variables are given in Table 5.

First, 27 initial samples are selected by Taguchi method [14], then the real experiment were run by using SUMITOMO injection molding machine and the numerical simulation were run by the MPI software. The real experiment results and simulation results are shown in Fig. 12.

The optimal solution was obtained after 35 iterations. The results are given in Table 4. Table 4 also gives the results obtained by experiment for comparison.

6 Discussions

Tables 2 and 4 show that most process parameters are lying in the boundaries of the limits. The effect of each factor on the warpage when all other factors are kept at their optimal level is shown in Figs. 13 and 14, respectively.

Figures 13 and 14 show that the packing time has the most significant effect on warpage. The injection time, cooling time, melt temperature, packing pressure are more effective parameters in minimizing warpage. Compared with other process parameters, the mold temperature had no significant effect on the warpage for these examples.

Tables 2 and 4 also show that the optimization process based on PSE function is faster than that based on EI function.

The numerical results show that the PSE function can balance maximum and other EI value effect in the optimization iterations, making the convergence quick and steady.

Tables 6 show the comparison with the real experiment results, showing that the sequential optimization method based on PSE sampling criterion is effective and practical.

7 Conclusions

In this study, a sequential optimization method using ANN with PSE strategy is developed for improving the injection-molded part quality. The mold temperature, melt temperature, injection time, packing time, packing pressure, and cooling time are considered as the design variables. A predictive model for the quality index is built in terms of the process parameters using ANN to decrease the computational time of the optimization process. An ANN is integrated with PSE function to perform the optimization process. The PSE function cannot only overcome disadvantage of the traditional EI methods that the maximum value of EI function jumps from one set of sampling points to another as the optimization proceeds, but also take the relatively unexpected space into consideration to improve the accuracy of the ANN model and quickly tend to the global optimization solution in the design space with a few sample points. A scanner, a TV cover, and a plastic lens are considered as the application examples. The results show that this sequential optimization method can effectively optimize the part quality and quickly approach to the global optimization solution.

Acknowledgments The authors gratefully acknowledge financial support for this work from the Natural Science Foundation (no. 11072048), the National Basic Research Program of China (no. 2012CB025905) and the National High-tech Research and Development Program (no. 2012AA040912).

Appendix 1

Definition 1 If p is a positive real variable, and $E=\{e_j(I), j=1, 2, \ldots, m\}$ is a set of sampling evaluation functions, then

$$S(E) = \frac{1}{p} \ln \sum_{j=1}^{m} \exp\left(pe_j(I)\right) \tag{1}$$

is a parametric sampling evaluation function, where p is a positive real variable and the sampling evaluation function at sampling point j is

$$e_j(I) = \sigma(x)[u_j\Phi(u_j) + \phi(u_j)] \; u_j = \frac{Y_j - \widehat{y}(x)}{\sigma(x)} \tag{2}$$

Definition 2 If, for any $E(I) = \{e_1(I), e_2(I), \cdots, e_m(I)\}$, and $\overline{E}(I) = \{\overline{e}_1(I), \overline{e}_2(I), \cdots, \overline{e}_m(I)\}$, $E(I), \overline{E}(I) \in E^m$ with $e_j(I) \leq \overline{e}_j(I)$ $(1 \leq j \leq m)$ and there exists at least one $j_0 (1 \leq j_0 \leq m)$ such that $e_{j_0}(I) < \overline{e}_{j_0}(I)$, then $E(I) \leq \overline{E}(I)$ or simply $E \leq \overline{E}$.

Definition 3 If, for any $E, \overline{E} \in E^m$ with $E \leq \overline{E}, S(E) < S(\overline{E})$, then $S(E)$ is a strictly monotone increasing function of E.

Theorem 1 *The PSE function S(E) is a strictly monotone increasing function of E, and if* $p \to \infty$ *then*

$$S(E) = \frac{1}{p} \ln \sum_{j=1}^{m} \exp\left(pe_j(I)\right) = \max e_j(I) \tag{3}$$

Proof. Let

$$E = \left\{e_j(I)\right\} \leq \overline{E} = \left\{\overline{E}_j(I)\right\}, j = 1, 2, \cdots, m \tag{4}$$

by Definition 2

$$e_j(I) \leq \overline{e}_j(I), j = 1, 2, \cdots, m \tag{5}$$

and there exists at least one $j_0 (1 \leq j_0 \leq m)$ such that

$$e_{j_0}(I) < \overline{e}_{j_0}(I) \tag{6}$$

Then for $p>0$,

$$pe_{j_0}(I) < p\overline{e}_{j_0}(I) \tag{7}$$

$$\exp\left(pe_{j_0}(I)\right) < \exp\left(p\overline{e}_{j_0}(I)\right) \tag{8}$$

Hence

$$\sum_{j=1}^{m} \exp(pe_j(I)) < \sum_{j=1}^{m} \exp(p\overline{e}_j(I)) \tag{9}$$

Taking logarithms on both sides and dividing by p

$$\begin{aligned} S(E) &= (1/p) \ln \sum_{j=1}^{m} \exp(pe_j(I)) \\ &< (1/p) \ln \sum_{j=1}^{m} \exp(p\overline{e}_j(I)) \end{aligned} \tag{10}$$

i.e., $S(E)$ is a strictly monotone increasing function of increasing function of E.

The p norm of the q-dimensional vector

$$E_e = \left\{e^{e_1(I)}, e^{e_2(I)}, \cdots, e^{e_m(I)}\right\}^T \tag{11}$$

is given by

$$\|E_e\|_p = \left(\sum_{j=1}^{m} e^{pe_j(I)}\right)^{(1/p)} \tag{12}$$

The uniform norm, also called the maximum norm, is defined by

$$\|E_e\|_\infty = \lim_{p \to \infty} \left[\sum_{j=1}^{m} e^{pe_j(I)}\right]^{(1/p)} \tag{13}$$

Since $e^{e_j(I)} > 0$ by Jensen's inequality, the norm is a strictly monotone decreasing function of its order, i.e.

$$S_p < S_r \quad \text{for } r < p \tag{14}$$

The importance of this inequality is that it holds also in the limit as $p \to \infty$. Thus, we have

$$\begin{aligned} S_p(E_e) &= \lim_{p \to \infty} \left(\sum_{j=1}^{m} e^{pe_j(I)}\right)^{\frac{1}{p}} = S_\infty(E_e) \\ &= \max\left(e^{e_j(I)}\right) \end{aligned} \tag{15}$$

Taking logarithms on both side of equation gives

$$\lim_{p \to \infty} (1/p) \ln \sum_{j=1}^{m} \exp(pe_j(I)) = \max\left(e_j(I)\right) \tag{16}$$

and the proof is completed.

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