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ORIGINAL ARTICLE

Fuzzy multivariate exponentially weighted moving average control chart

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Abstract Traditional multivariate control charts such as Hotelling's χ^2 and T^2 control charts are designed to monitor vectors of variable quality characteristics. However, in certain situations, data are expressed in linguistic terms and, under these circumstances, variable or attribute multivariate control charts are not suitable choices for monitoring purposes. Fuzzy multivariate control charts such as fuzzy Hotelling's T^2 could be considered as efficient tools to overcome the problems of linguistic observations. The purpose of this paper is to develop a fuzzy multivariate exponentially weighted moving average (F-MEWMA) control chart. In this paper, multivariate statistical quality control and fuzzy set theory are combined to develop the proposed method. Fuzzy sets and fuzzy logic are powerful mathematical tools for modeling uncertain systems in industry, nature, and humanity. Through a numerical example, the performance of the proposed control chart was compared to the fuzzy Hotelling's T^2 control chart. Results indicate uniformly superior performance of the F-MEWMA control chart over Hotelling's T^2 control chart.

Keywords Multivariate control charts · Fuzzy set theory · Fuzzy multivariate control chart · Representative value · Hotelling's T^2 · Multivariate exponentially weighted moving average

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1 Introduction

Globalization and fierce competition in the economic arena have forced organizations to consider product or service quality improvement as a strategy for higher business performance. Quality improvement tools such as control charts, commonly considered as a featured tool of statistical process control, play a vital role in quality improvement initiatives. Although traditional control charting uses variable or attribute quality characteristics in univariate or multivariate environment to monitor the quality of a product or process, there are certain situations where linguistic terms could be used effectively for this purpose.

In multivariate environment, several related quality characteristics of a product or process are monitored using a multivariate control procedure. Several types of multivariate control charts, including Hotelling's χ^2 and T^2 control charts, have been developed in attempts to improve monitoring by using the correlation structure that exists between quality characteristics. If the correlation structure between the variables is ignored and a set of separate univariate control charts are used instead for monitoring purposes, then inefficient and potentially misleading results should be expected. Many authors including Mason et al. [2, 3] and Linna et al. [1] have reviewed developments in multivariate control charting.

Under certain situations where the quality of a product or process could be expressed effectively in linguistic terms, the use of control procedures based on fuzzy logic could be a realistic choice.

Fuzzy set theory supports subjective natural language descriptors of quality and provides a methodology for allowing them to enter into the modeling process. Measurement of quality in the context of fuzzy set theory where conventional modeling is not applicable provides an opportunity to monitor process appropriately. A major contribution of fuzzy set theory is its capability to represent and model linguistic data.

In this paper, a fuzzy multivariate exponentially weighted moving average (F-MEWMA) control chart is developed for monitoring linguistic observations. The performance of the proposed control chart is evaluated using the average run length (ARL) criterion.

Some basic concepts of fuzzy multivariate control procedures are presented in Section 2. The F-MEWMA procedure is discussed in Section 3. The performance of the proposed control chart is evaluated through a numerical example in Section 4. Our concluding remarks are presented in the final section.

2 Fuzzy multivariate control charts

The MEWMA control chart utilizes a series of observations rather than simply taking the current sample used in T^2 chart. The MEWMA control chart, similar to its univariate version, is more sensitive to small mean shifts and, therefore, could be an effective tool for identifying the change point.

Suppose that the p-related attribute quality characteristics (QCs) denoted as c_1, c_2, c_3, …, c_p are being monitored simultaneously. Each quality characteristic c_j for j=1, 2, 3, …, p is characterized by q_j categories or linguistic terms which are described by fuzzy subsets. Each of these fuzzy subsets is associated with a membership function. Using a fuzzy transformation method, each linguistic term is converted into a representative value.

Linguistic variable c_j is not expressed as a numerical value but rather as a certain word or phrase. In fuzzy set theory, this is represented by “term set” $T(c_j)$. In addition, each c_{jh} for h=1, …, q in term set $T(c_j)$ is characterized by a membership function $\mu_{jh}(x)$ where x is a measure of quality level. This measure is a basic variable standardized on an interval from zero to one with zero representing the best quality level and one representing the worst quality level. Several methods for developing and selecting membership functions have been proposed in the literature.

To plot sample quality characteristic values on multivariate control charts, fuzzy subsets associated with linguistic terms for each term set should be converted into scalar values. These values are called representative values. There are four transformation methods: fuzzy median, fuzzy mode, α-level fuzzy midrange, and fuzzy average. Since the distribution of fuzzy data may be asymmetrical, the fuzzy median method is used. The median is the midpoint of a distribution and is not affected by the extreme values. Hence, it is a suitable measure for evaluating the central tendency of an asymmetrical distribution. The median $m(A)$ of a fuzzy set A is defined as:

$$m(A) = \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(x)\mathrm{d}x.$$

Suppose each fuzzy number can be represented by a triangular fuzzy number, i.e., we can express each number as (a_{1j}, a_{2j}, a_{3j}). Using the fuzzy median method, the representative value is:

$$R_j = \begin{cases} a_{3j} - \left[\frac{(a_{3j}-a_{1j})(a_{3j}-a_{2j})}{2}\right]^{1/2}, \text{for } a_{2j} < \frac{a_{3j}+a_{1j}}{2} \\ a_{1j} - \left[\frac{(a_{3j}-a_{1j})(a_{2j}-a_{1j})}{2}\right]^{1/2}, \text{for } a_{2j} > \frac{a_{3j}+a_{1j}}{2} \end{cases} \quad (1)$$

Let there be q_j linguistic variables on the term set c_j denoted by c_{jh}, h=1, 2, …, q_j. Let the fuzzy set for each linguistic variable be F_{jh} which is characterized by the membership function μ_{jh}. Thus, a sample A with n observations is expressed as:

$$A = \left\{\{(F_{11}, n_{11}), ..., (F_{1q_1}, n_{1q_1})\}; ...; \{(F_{p1}, n_{p1}), ..., (F_{pq_p}, n_{pq_p})\}\right\}$$

where n_{jh} is the number of observations classified by linguistic variable c_{jh}. Using fuzzy arithmetic, each c_j is then associated with only one fuzzy subset in the following way:

$$F_j = \frac{1}{n}\sum_{h=1}^{q_j} n_{jh}F_{jh}. \quad (2)$$

The sample A from n observations is now expressed by:

$$A = \{F_1, F_2, F_3, ..., F_p\}.$$

Then, each fuzzy subset F_j using the fuzzy median method can be transformed to the representative value R_{ij}. This representative value of the fuzzy number F_j in Eq. 1 from m samples can be expressed as:

$$R = \begin{bmatrix} R_{11} & R_{12} & ... & R_{1p} \\ R_{21} & R_{22} & ... & R_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ R_{m1} & R_{m2} & ... & R_{mp} \end{bmatrix}.$$

With regards to the representative values in Eq. 1, R_{ij} is the representative value of fuzzy number F_j in sample i.

As it was stated earlier, MEWMA and multivariate cumulative sum (MCUSUM) control charts use the information available in current and past samples and, therefore, they are sensitive to small process variations compared to the T^2 control chart that uses only the information available in the current sample. According to Montgomery [4], the performance of the MEWMA control chart is almost similar to the performance of the MCUSUM control chart but its use in practice is relatively easier. Therefore, we consider the MEWMA control chart to design our proposed fuzzy multivariate control procedure.

3 The fuzzy MEWMA control chart

Suppose R_i, i=1, 2, …, p, is the representative value corresponding to the fuzzy number F_i such that $R_i \sim N_p(\mu_i, \Sigma_F)$ where μ_i is the mean vector and Σ_F is the known fuzzy covariance matrix. Without loss of generality, when process is in control, we assume the mean vector $\boldsymbol{\mu}_i$ to be equal to (0, 0, …, 0)′. The F-MEWMA vectors $\boldsymbol{Z}_i$ are obtained using:

$$Z_i = \lambda R_i + (I - \lambda) Z_{i-1}, \quad i = 1, 2, \ldots$$

where $\boldsymbol{\lambda}$ is a diagonal weight matrix with diagonal elements $\boldsymbol{\lambda}_i$ ($0<\boldsymbol{\lambda}_i\leq 1$) for i=1,2, …, p and Z_0=0. The F-MEWMA control chart is a plot of the following statistic:

$$F_i^2 = Z_i' \Sigma_{Z_i}^{-1} Z_i > h_F \tag{3}$$

where Σ_{Z_i} is the covariance matrix of F-MEWMA vectors. Consequently, an alarm indicating an out-of-control process is triggered when $F_i^2 \geq h_F$ where h_F is a predefined threshold yielding the desired in-control ARL. Suppose

Table 1 Food process data

m	c_{11}	c_{12}	c_{13}	c_{21}	c_{22}	c_{23}	c_{31}	c_{32}	c_{33}	c_{34}
1	210	7	3	206	9	5	167	48	3	2
2	211	6	3	207	8	5	176	42	2	0
3	206	9	5	202	12	6	163	55	2	0
4	211	5	4	207	8	5	163	51	5	1
5	203	16	1	194	18	8	175	45	0	0
6	210	6	4	206	9	5	174	44	1	1
7	208	7	5	204	9	7	174	40	5	1
8	207	7	6	204	9	7	169	46	3	2
9	206	7	7	202	9	9	169	48	2	1
10	186	25	9	200	12	8	169	48	3	0
11	196	13	11	196	13	11	163	46	10	1
12	203	12	5	200	13	7	167	44	9	0
13	203	9	8	198	11	11	174	42	3	1
14	202	9	9	198	11	11	174	40	6	0
15	209	6	5	207	9	4	172	42	5	1
16	210	3	7	205	5	10	172	44	4	0
17	205	11	4	201	13	6	172	45	2	1
18	210	6	4	206	8	6	169	48	2	1
19	206	10	4	203	13	4	172	46	0	2
20	206	12	2	202	14	4	169	46	5	0

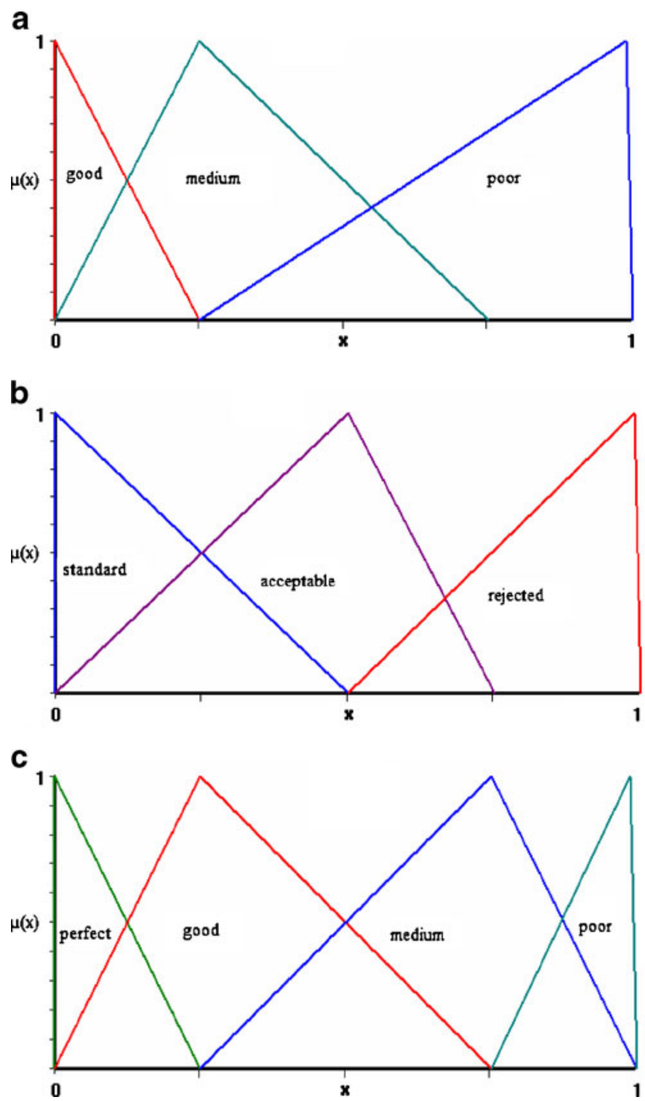

Fig. 1 Sets of membership functions related to **a** appearance, **b** color, and **c** taste

that equal weights are being applied to the p components of the F-MEWMA vectors, i.e., $\boldsymbol{\lambda}_i=\boldsymbol{\lambda}$ and i=1, 2, …, p, then the F-MEWMA vectors can be written as:

$$Z_i = \lambda R_i + (1 - \lambda) Z_{i-1} \tag{4}$$

and:

$$\Sigma_{Z_i} = \frac{\lambda}{2 - \lambda}\left[1 - (1 - \lambda)^{2i}\right]\Sigma_F \tag{5}$$

with the following asymptotic form:

$$\Sigma_{Z_i} = \frac{\lambda}{2 - \lambda}\Sigma_F. \tag{6}$$

The parameter $0<\boldsymbol{\lambda}\leq 1$ is the smoothing parameter which controls the weight assigned to the new observation vector. The parameter Σ_F should be estimated from the analysis of preliminary samples of size n taken when the process is

Table 2 The values of the sample statistics

m	R_{j1}	R_{j2}	R_{j3}	Z_{j1}	Z_{j2}	Z_{j3}	F_i^2
1	0.0900	0.1720	0.1410	0.0450	0.0860	0.0705	930
2	0.0890	0.1710	0.1240	0.0670	0.1285	0.0973	1,551
3	0.0980	0.1790	0.1380	0.0825	0.1538	0.1176	2,109
4	0.0910	0.1710	0.1460	0.0868	0.1624	0.1318	2,453
5	0.0930	0.1920	0.1220	0.0899	0.1772	0.1269	2,677
6	0.0920	0.1720	0.1270	0.0935	0.1746	0.1270	2,599
7	0.0960	0.1780	0.1340	0.0962	0.1763	0.1305	2,668
8	0.0990	0.1780	0.1390	0.0991	0.1771	0.1347	2,727
9	0.0102	0.1850	0.1340	0.1136	0.1811	0.1344	2,782
10	0.1280	0.1850	0.1330	0.1173	0.1830	0.1337	2,655
11	0.1210	0.1960	0.1540	0.1091	0.1895	0.1438	2,945
12	0.1010	0.1830	0.1450	0.1081	0.1863	0.1444	2,968
13	0.1070	0.1930	0.1310	0.1090	0.1896	0.1377	2,945
14	0.1100	0.1930	0.1330	0.1020	0.1913	0.1354	2,936
15	0.0950	0.1690	0.1360	0.1020	0.1802	0.1357	2,748
16	0.0980	0.1830	0.1320	0.1000	0.1816	0.1338	2,785
17	0.0970	0.1800	0.1310	0.0985	0.1808	0.1324	2,749
18	0.0920	0.1740	0.1340	0.0953	0.1774	0.1332	2,719
19	0.0960	0.1740	0.1310	0.0956	0.1757	0.1321	2,655
20	0.0920	0.1750	0.1360	0.0938	0.1753	0.1341	2,701

assumed to be in control. Suppose that m samples are available. The sample representative values are obtained using Eq. 1 and the $p\times 1$ mean vector is the estimated using:

$$\overline{R} = \left(\overline{R}_1, \overline{R}_2, \ldots, \overline{R}_p\right)'$$

where $\overline{R}_j = \frac{1}{m}\sum_{i=1}^{m} R_{ij}$ and R_{ij} is the representative value of the fuzzy subset associated with the ith sample in the jth quality characteristic.

The variances of these representative values in Eq. 1 are given by:

$$S_j^2 = \frac{1}{m-1}\sum_{i=1}^{m}\left(R_{ij} - \overline{R}_j\right)^2 \quad (7)$$

and the covariance between the jth and kth quality characteristics is given by:

$$S_{ij} = \frac{1}{m-1}\sum_{i=1}^{m}\left(R_{ij} - \overline{R}_j\right)\left(R_{ik} - \overline{R}_k\right), \quad i \neq k. \quad (8)$$

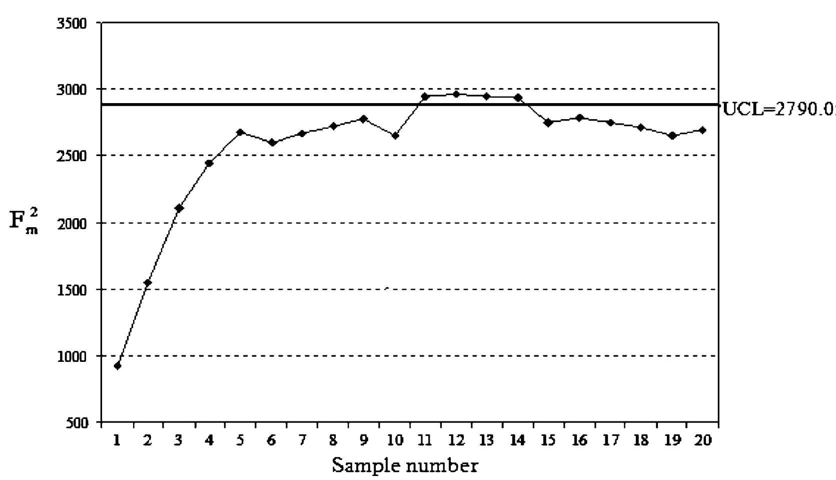

Fig. 2 Fuzzy multivariate control chart (F-MEWMA)

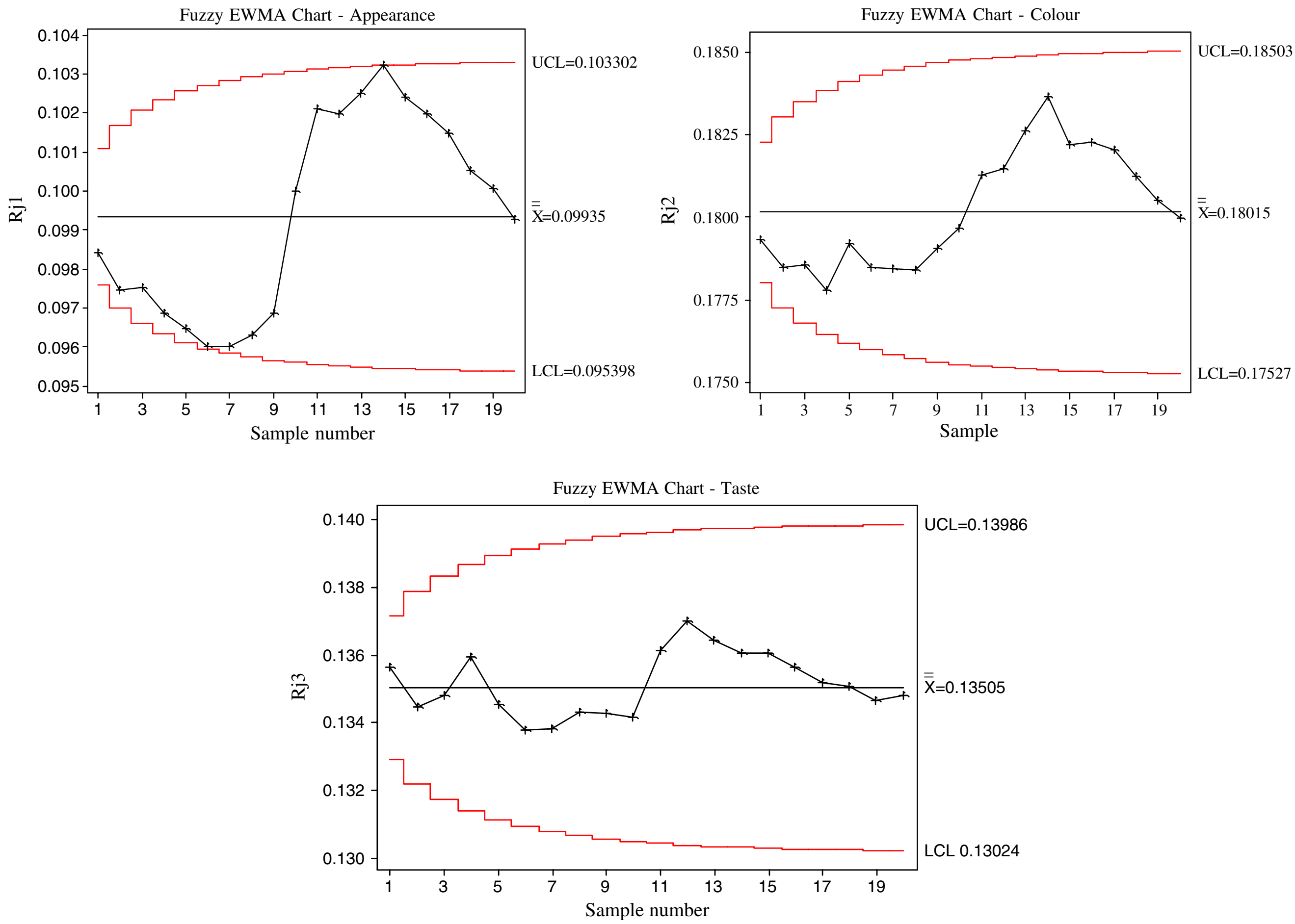

Fig. 3 Separate fuzzy univariate EWMA control charts

The sample covariance matrix $\boldsymbol{S}$ based on in-control observations is expressed as:

$$S = \begin{bmatrix} S_1^2 & S_{12} & \dots & S_{1p} \\ S_{21} & S_2^2 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ S_{p1} & \dots & \dots & S_p^2 \end{bmatrix}.$$

To determine the threshold value h_F, the bootstrap resampling method with over 1,000 samples is used. Detailed computations for the parameters of the proposed method are presented in the next section using a numerical example.

4 A numerical example

In this section, we present a numerical example based on the data used by Taleb et al. [5] to evaluate the performance of the proposed F-MEWMA control chart. The example discussed here considers a case in food industry. According to Taleb et al. [5], the appearance, color, and taste of a frozen food are three important quality characteristics that have to be monitored simultaneously. The product appearance could be classified as good, medium, or poor. Its color can also be classified as standard, acceptable, or rejected. In addition, the taste of a product can be classified as perfect, good, medium, or poor. Table 1 shows the

Table 3 Resampled data for the food process

m	c_{11}	c_{12}	c_{13}	c_{21}	c_{22}	c_{23}	c_{31}	c_{32}	c_{33}	c_{34}	R_{j1}	R_{j2}	R_{j3}	F_i^2
21	202	10	8	204	11	5	169	44	5	2	0.108	0.175	0.142	2,169
22	208	7	5	196	13	11	174	44	1	1	0.096	0.196	0.127	2,222
23	206	6	8	196	13	11	174	40	5	1	0.104	0.196	0.134	2,297
24	210	2	8	198	12	10	165	44	1	10	0.100	0.191	0.162	2,455

linguistic data related to this example. Hence, we have three term sets of linguistic variables:

Term set 1 related to the appearance and defined as:

$$T(c_1) = \{c_{11}, c_{12}, c_{13}\} = \{\text{good, medium, poor}\}.$$

Term set 2 related to the color and defined as:

$$T(c_2) = \{c_{21}, c_{22}, c_{23}\} = \{\text{standard, acceptable, rejected}\}.$$

Term set 3 related to the taste and defined as:

$$T(c_3) = \{c_{31}, c_{32}, c_{33}\} = \{\text{perfect, good, medium, poor}\}.$$

In this section, we use m=20 generated samples of size n=220 to illustrate our approach for monitoring the vector of linguistic quality characteristics.

Using the fuzzy set theory, linguistic terms, c_{jh}, j=1, 2, 3; h=1, …, q_j can be characterized by membership functions μ_{jh}. Membership functions associated with these three term sets are shown in Fig. 1. Fuzzy subsets F_j associated with the quality characteristics in a given sample are determined by Eq. 1. The membership functions associated with F_j are calculated now using fuzzy arithmetic. Thus, the representative values for F_j are computed using Eq. 2.

The representative values, components of the F-MEWMA vectors, and values of the F_i^2 statistic were calculated for each sample and summarized in Table 2. For example, for sample 1, R_{11} is computed by transforming F_{11} into its representative value. The triangular membership function μ_{11} associated with F_{11} is obtained using:

$$F_{11} = \frac{1}{220}[210, 7, 3]\begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0.25 & 0.75 \\ 0.25 & 1 & 1 \end{bmatrix}$$
$$= [0.00341, 0.02159, 0.27614],$$

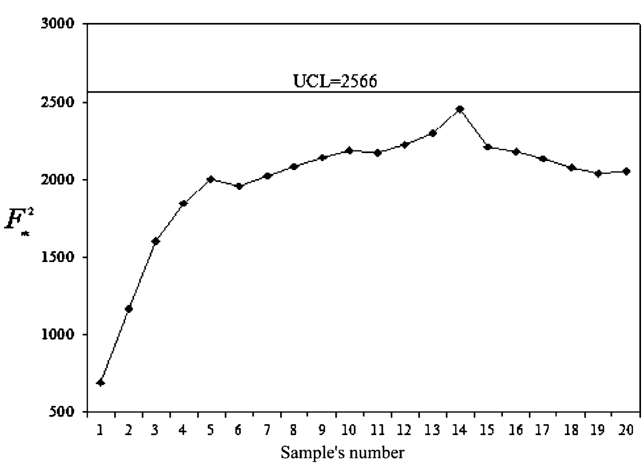

Fig. 4 F-MEWMA control chart for revised data

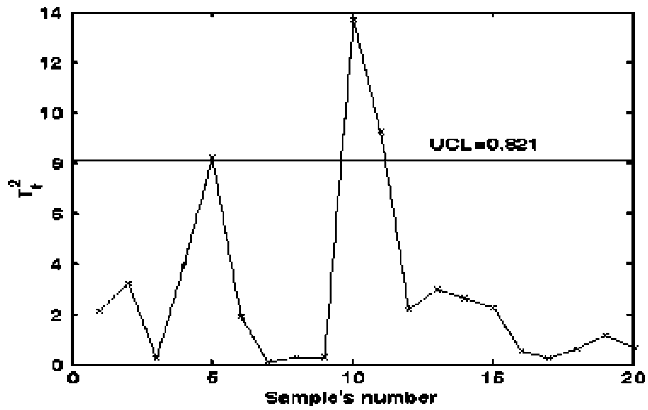

Fig. 5 T^2 control chart for the revised data

which leads to R_{11}=0.090. Thus, the vector of means is given by:

$$\overline{R} = \left(\overline{R}_1, \overline{R}_2, \overline{R}_3\right) = (0.0994, 0.1802, 0.1351).$$

The covariance matrix $\boldsymbol{S}$ can be obtained using Eqs. 7 and 8. For the given example, the sample covariance matrix is given by:

$$S = \begin{bmatrix} 0.0001046605 & 0.0000592079 & 0.0000233500 \\ 0.0000592079 & 0.0000696079 & 0.0000050974 \\ 0.0000233500 & 0.0000050974 & 0.0000561553 \end{bmatrix}.$$

The sample covariance matrix $\boldsymbol{S}$ is the estimate of Σ_F when the process is in control.

The sample representative values R_{j1}, R_{j2}, R_{j3} along with the MEWMA vector calculated using Eq. 4 with λ=0.5 are presented. The F_i^2 values in Eq. 3 are also presented using the covariance matrix in Eq. 5. The upper control limit or

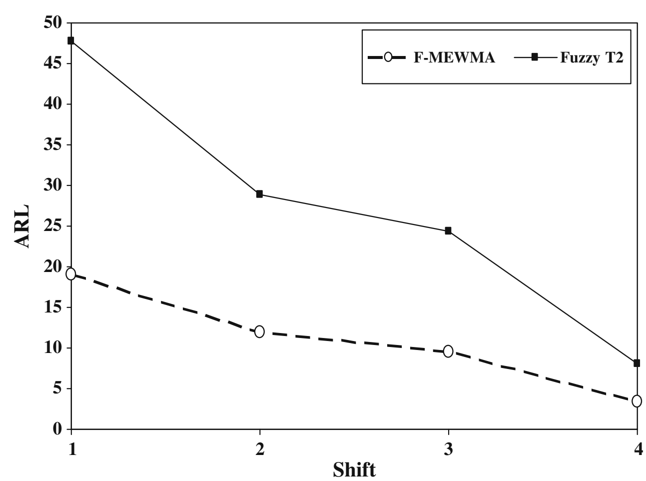

Fig. 6 ARL of F-MEWMA and T^2 charts

the threshold value is computed using the bootstrap resampling method. The F-MEWMA control chart for the sample data is shown in Fig. 2, indicating out-of-control conditions for samples 11 through 14. According to Fig. 3, if one has mistakenly used three separate fuzzy univariate EWMA control charts instead, then the process would have been announced under statistical control.

Following Taleb et al. [5], four out-of-control samples were replaced using the bootstrap resampling method. The data and their corresponding statistics are shown in Table 3. Based on a computer simulation, the threshold value is set to h_4=2566. The revised F-MEWMA control chart is shown in Fig. 4. According to the revised F-MEWMA control chart, the process is in control, whereas the fuzzy T^2 control chart proposed by Taleb et al. [5] indicates an out-of-control condition (see Fig. 5).

The ARL performance of the two control charts using 10,000 simulation runs is shown in Fig. 6. This figure indicates that F-MEWMA uniformly outperforms the fuzzy T^2 control chart.

5 Conclusions

The fuzzy multivariate control chart is an alternative control chart for handling linguistic observations. Since this chart uses cumulative information of the current and past samples, one should expect a better performance when compared to a control chart that uses only the information in the most recent sample. The multivariate control chart along with fuzzy logic was considered simultaneously to develop the F-MEWMA control chart. The performance of the proposed control chart was compared to the fuzzy T^2 control chart using the ARL criterion computed based on the bootstrap resampling data. The numerical results indicate a superior performance of the F-MEWMA control chart over the fuzzy T^2 control chart.

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