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ORIGINAL ARTICLE

Capability indices and nonconforming proportion in univariate and multivariate processes

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Abstract The main usefulness of a capability index is to relate the actual variability of the process with the admissible one. This admissible variability is, in turn, related with the nonconforming proportion. Hence, the capability index should be closely related to the nonconforming proportion. In univariate and centered processes, the classical C_p index explicitly admits this interpretation. For instance, if $C_p = 0.5$, the standard deviation should be reduced to 50% to attain $C_p = 1$. However, for noncentered processes and multivariate processes, there is a lack of capability indices that admit such an interpretation. This article fills this gap in the literature and proposes univariate and multivariate capability indices that have a direct interpretation of how much the variability of the process should increase or decrease to attain a unitary index. Some numerical examples are used to compare the proposed indices with the existing ones, showing the advantages of the proposals.

Keywords Capability · Capability index · Multivariate processes · Nonforming proportion

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1 Introduction

One way of comparing the characteristics of the output of a manufacturing process with the engineering requirements is by using the concept of capability. In general, a process is capable if the probability of obtaining nonconforming items (outside some specifications) is small, typically 0.0027. Even though the interest is in the nonconforming proportion, it is customary to quantify the capability using a unitless index such that it can easily be computed and interpreted by the users. Typically, a process capability index (PCI) compares the natural variability of a stable process and the allowed variability. A general form of a PCI is

$$\frac{\textit{measure of allowable process spread}}{\textit{measure of actual process spread}}. \tag{1}$$

Therefore, a PCI is an index of the quality of the process that measures the risk of producing defective articles due to the natural variability of the process. It is used, for example, by quality engineers in their reports to their managers. It is also used when part of the production process is made by third parties, as it is customary, for instance, in the automotive industry. The report of a multivariate PCI is then included in the quality audit report.

A large number of publications related to PCI for univariate processes are available. An overview of different developments is provided in Kotz and Johnson [1]. In a univariate process, the quality is measured by one characteristic and the engineering requirements are usually represented by two specification limits: the upper and the lower specification limits. In the case of multivariate processes, some capability indices have also been proposed. In these processes, the quality is measured by the joint level of several characteristics, and the engineering specification is a tolerance region instead of an interval. In the case of independent characteristics, some proposals can be found in Yu et al. [2, 3]. In a more general case of correlated variables, alternative approaches of multivariate capability indices can be found in [4–13]. These proposals are summarized below.

In spite of the number of PCIs that can be found in the literature, there is a need for a PCI that clearly relates the actual variability of the process with the nonconforming proportion. In this article, we propose a PCI that fills this gap. This need is especially important in multivariate processes. For instance, some of the existing PCIs do not assure that values larger (smaller) than one imply that the nonconforming proportion is smaller (larger) than, say, 0.0027. Under the proposed approach, a unitary capability index is linked to a prespecified nonconforming proportion. Also, when the variability of the process changes linearly, the proposed PCI also changes linearly, irrespective of whether the process is centered or not. Apart from the original C_p index in unbiased processes, none of the existing univariate or multivariate PCI admits this clear interpretation. In this sense, the proposed PCI is the only one that retrieves the initial idea of relating natural variability with nonconforming proportion, applying it to noncentered univariate processes as well as to general multivariate processes.

The outline of the article is as follows: Section 2 describes briefly some univariate PCIs existing in the literature. Section 3 describes the methodology to estimate the newly proposed univariate capability index. A comparison with existing indices is also provided. Section 4 describes some multivariate PCIs existing in the literature. Section 5 extends the proposed univariate index to the multivariate case. Section 6 compares the multivariate indices using some real and simulated examples; some of them are extracted from existing literature. Section 7 concludes. The Matlab code used in the examples can be downloaded from the authors' website.

2 Capability indices for univariate processes

Let $X \sim N(\mu, \sigma^2)$ denote the univariate quality characteristic of interest. The basic capability index used in practice is

$$C_p = \frac{USL - LSL}{6\sigma}, \tag{2}$$

where USL and LSL are the upper and lower specification limits of X, respectively. In this index, it is assumed that X is centered within the tolerance region. Under these assumptions, there is a correspondence between C_p and the proportion of nonconforming items, which will be denoted as p. To see this point, let us denote

$$C_p^{p/2} = -\frac{1}{3}\Phi^{-1}(p/2), \tag{3}$$

with $\Phi(\cdot)$ the cumulative distribution function of the standard normal distribution. Then, it is straightforward to show that, for centered processes,

$$C_p = C_p^{p/2}. \tag{4}$$

We will define a process as capable if $p \leq \alpha$, where α is a small value, typically $\alpha = 0.0027$. Otherwise, the process is incapable. From Eq. 4, if $p \leq \alpha$, then $C_p \geq 1$. This index can be seen as the relative length of the tolerance interval to the length of the centered $100(1-\alpha)\%$ of the population.

In order to motivate our proposal for multivariate PCI, we will expose an alternative interpretation of C_p that will ease its extension to a multivariate process. Let us denote as σ_p to the maximum standard deviation of the process such that $C_p = 1$; that is,

$$\frac{USL - LSL}{6\sigma_p} = 1.$$

Hence, by Eq. 2,

$$\sigma_p = \frac{USL - LSL}{6} = C_p\sigma. \tag{5}$$

Then, C_p can be interpreted as the maximum factor that should be applied to σ to obtain a capable process; i.e., with $C_p = 1$. For instance, if $C_p = 2$, the standard deviation of the process could be multiplied by 2 and still have a capable process. Conversely, if $C_p = 0.5$, the standard deviation needs to be divided at least by 2 to have a capable process. Also, from Eq. 5, we obtain that

$$C_p = \frac{\sigma_p}{\sigma}, \tag{6}$$

which describes C_p as a ratio of standard deviations: the maximum allowable and the real one, which is in line with Eq. 1. Our proposal for a PCI presented in the following sections will be based on this interpretation.

As it is well known, a disadvantage of C_p is that it does not capture the location of the process. Then, if the process mean is far from the center of the tolerance zone, the process can have $C_p > 1$ with a high proportion of nonconforming items. To avoid this situation, the index

$$C_{pk} = \min\left(C_{pU}, C_{pL}\right), \; C_{pU} = \frac{USL - \mu}{3\sigma},$$

$$C_{pL} = \frac{\mu - LSL}{3\sigma}, \tag{7}$$

was proposed (see, e.g., Kane [14]). As in Eq. 4, and following Castagliola and García Castellanos [13], this index can be written in terms of the lower and upper proportions of nonconforming products as

$$C_{pk} = \frac{1}{3}\min\left\{-\Phi^{-1}(p_U), -\Phi^{-1}(p_L)\right\}, \tag{8}$$

where $p_U = P(X > USL)$ and $p_L = P(X < LSL)$. This index has also some drawbacks. In Kane [14] and Pearn and Lin [15], it is shown that processes with the same value of C_{pk} can have different percentage of nonconforming items, depending on the specification limits and σ. Under normality, it can easily be checked that the proportion of nonconforming items is bounded by

$$p \leq 2\Phi(-3C_{pk}), \tag{9}$$

and therefore,

$$C_{pk} \leq -\frac{1}{3}\Phi^{-1}\left(p/2\right) \equiv C_p^{p/2}, \tag{10}$$

where the equality holds for centered processes. Then, if $C_{pk} = 1$, the proportion of nonconforming items is $p \leq \alpha$. Also, for $p = \alpha$, we obtain $C_{pk} \leq 1$. Hence, this index is conservative. However, as pointed out in Wierda [6], if the process is sufficiently capable, a good approximation (still conservative) is

$$C_{pk} \approx -\frac{1}{3}\Phi^{-1}(p) \equiv C_{pk}^p. \tag{11}$$

Apart from this interpretation in terms of p, the index C_{pk} always admits an interpretation similar to Eqs. 5 and 6 in that C_{pk} is the maximum factor that should be applied to σ to obtain a process with $C_{pk} = 1$, and also

$$C_{pk} = \frac{\sigma_{pk}}{\sigma}, \tag{12}$$

where σ_{pk} is the standard deviation that corresponds to $C_{pk} = 1$. Note that if the process is not centered, $\sigma_{pk} < \sigma_p$. Although capability indices are usually used to evaluate the variability of the process, the index C_{pk} can also be used to evaluate the distance of μ to the specification limits. For instance, from Eq. 7, we can write

$$\frac{\left(USL - \mu\right)/C_{pU}}{3\sigma} = 1, \; \frac{\left(\mu - LSL\right)/C_{pL}}{3\sigma} = 1; \tag{13}$$

and, hence, the unilateral indices C_{pU} and C_{pL} can be interpreted as the maximum factor; we can reduce the distance to each specification limit to obtain a unilateral capability index equal to one. For instance, if the process has $C_{pL} = 2$, the distance between μ and LSL can be reduced as much as 50% and still have a capable process. In this article, and for the sake of conciseness, we are going to focus on using capability indices to evaluate only variability.

Other capability indices have been proposed to consider the proximity of the process to a target value T, should it exist. These indices are denoted as C_{pm} and C_{pkm} and are analogous to C_p and C_{pk}, respectively. They are defined as

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{14}$$

$$C_{pkm} = \frac{\min(USL - \mu, \mu - LSL)}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{15}$$

Then, for the same process, $C_p \geq C_{pm}$ and $C_{pk} \geq C_{pmk}$ because the denominator includes the penalty term $(\mu - T)^2$. All these proposals are based on the assumption of normality. Departures from normality can result in erroneous conclusions when using these indices. In these situations, two main approaches can be applied. The first approach is to adjust a more appropriate parametric distribution. This can be attained by approximating the distribution with some other parametric model (e.g., Clements [16], Castagliola [17], Liu and Chen [18]). Alternatively, the process data can be transformed into normal by applying some transformation like the Box-Cox or the so-called root transformation of Hosseinifard et al. [19]. The second approach is the nonparametric one, like the kernel estimation proposed in Polansky [20] or the neural-network approach used in Abbasi [21].

3 A new proposal: the C_n index for univariate processes

We will define here a new capability index directly related to the proportion of nonconforming items p. The index is denoted as C_n, where n stands for nonconforming proportion. Let $X_{\max}$ be a process with density function similar to that of X but with variance $\sigma^2_{\max}$, where

σ^2_{max} is the value that verifies $P(LSL \leq X_{max} \leq USL) = 1-\alpha$. That is, σ_{max} is the maximum allowable standard deviation that should have X such that $p = \alpha$ without changing the mean. Then, C_n is defined as

$$C_n = \frac{\sigma_{max}}{\sigma}. \tag{16}$$

It can be verified that $C_p \geq C_n \geq C_{pk}$, where equality holds for a normal process centered in the tolerance zone. To illustrate the index C_n with an example, let us use $\alpha = 0.0027$ and let us assume that the process is normal but not centered, such that $p_U = 0.0020$ and $p_L = 0.0007$. Then, it is clear that $p = 0.0027$ and the process is capable. In this case, $\sigma_{max} = \sigma$, and, hence, $C_n = 1$. Therefore, the process cannot increase its variability; otherwise, it will become incapable. It can be checked, using Eq. 8, that $C_{pk} = 1/3 \min(2.87, 3.19) = 0.959$. Therefore, this capability index C_{pk} shows, erroneously, that the process is incapable. Also, we have that $USL - LSL = 6.0728\sigma$, and, hence, $C_p = 1.012$. Using C_p, the process is labeled as capable, but quality engineers would not care much if the standard deviation increases up to $\sigma_p = 1.012\sigma$ (assuming that $C_p = 1$ is satisfactory enough).

3.1 Computation of C_n

In the normal case with $X \sim N(\mu, \sigma^2)$, the computation of C_n is straightforward. The parameters μ and σ can be estimated from data, and then p can be computed as $p = 1 - P(LSL < X < USL)$. If $p = \alpha$, then $\sigma_{max} = \sigma$ and $C_n = 1$. If $p \neq \alpha$, σ_{max} should be computed. Since the relation between σ and p in a normal distribution is nonlinear, σ_{max} should be found using a searching procedure like, for instance, the bisection method. For instance, if $p < \alpha$, we compute the nonconforming proportion of the normal process $X^* \sim N(\mu, \sigma^{2*})$ and search for the value $\sigma^{2*} > \sigma^2$ that verifies $P(LSL < X^* < USL) = 1 - \alpha$.

Table 1 shows the results of some small experiments that compares the performance of C_n with C_p, $C_p^{p/2}$, C_{pk}, C_{pk}^p, C_{pm}, and C_{pkm}. In these experiments, it is assumed that the specification limits are $USL = 3$ and $LSL = -3$, and the target $T = 0$. The model used in the experiments is $X \sim N(\mu, \sigma^2)$ with alternative values of μ and σ^2. The nonconforming proportion p is then computed using the normality assumption and the specification limits. The computation of C_n has been made in Matlab, searching for σ_{max} using a combination of bisection and secant methods. The nonconforming probabilities of the normal distribution are based on the Matlab function normcdf.m. The Matlab code used in this table is available from the authors' website. In the first experiment (Panel A), the parameters of the model are $\mu = 0$, and alternative values of σ^2. In this experiment, it is clear that $\sigma_{max} = 1$ and, therefore, all the indices are equal except C_{pk}^p.

In the second experiment (Panel B), the process has $\mu = 1$ and, hence, is not centered in the specification interval. In this case, C_p is useless. It can be seen in Table 1 that, even with $p = 0.0228$, C_p erroneously suggests that the process is capable. On the other hand, C_{pk} is equal to C_{pk}^p (up to two digit precision) and both are rather conservative in this setting. For instance, when $p = 0.0027$, we have that $C_{pk} = C_{pk}^p = 0.93$, erroneously suggesting that the process is not capable. The indices C_{pm} and C_{pkm} are also very conservative, especially C_{pkm}. Conversely, C_n is defined to have values larger than 1 only if $p < 0.0027$, and its value is related to the admissible variation of σ. For

Table 1 Comparison of univariate capability indices for alternative models $X \sim N(\mu, \sigma^2)$

	μ	σ^2	p	C_p	$C_p^{p/2}$	C_{pk}	C_{pk}^p	C_{pm}	C_{pkm}	C_n
Panel A	0	0.4	0.0000	1.58	1.58	1.58	1.53	1.58	1.58	1.58
	0	0.8	0.0008	1.12	1.12	1.12	1.05	1.12	1.12	1.12
	0	1.0	0.0027	1.00	1.00	1.00	0.93	1.00	1.00	1.00
	0	1.2	0.0062	0.91	0.91	0.91	0.83	0.91	0.91	0.91
Panel B	1	0.4	0.0008	1.58	1.12	1.05	1.05	0.85	0.56	1.14
	1	0.51	0.0026	1.40	1.01	0.93	0.93	0.81	0.54	1.01
	1	0.517	0.0027	1.39	1.00	0.93	0.93	0.81	0.54	1.00
	1	0.53	0.0030	1.37	0.99	0.92	0.92	0.80	0.54	0.99
	1	0.6	0.0049	1.29	0.94	0.86	0.86	0.79	0.53	0.93
	1	1.0	0.0228	1.00	0.76	0.67	0.67	0.71	0.47	0.72
	1	1.2	0.0341	0.91	0.71	0.61	0.61	0.67	0.45	0.66
Panel C	−1.5	1	0.0668	1.00	0.61	0.50	0.50	0.55	0.28	0.54
	−0.5	1	0.0064	1.00	0.91	0.83	0.83	0.89	0.75	0.90
	0	1	0.0027	1.00	1.00	1.00	0.93	1.00	1.00	1.00
	0.9	1	0.0179	1.00	0.79	0.70	0.70	0.74	0.52	0.76
	1.7	1	0.0968	1.00	0.55	0.43	0.43	0.51	0.22	0.47

instance, when $\sigma^2 = 0.6$, we have that $C_n = 0.93$ and the process is not capable. However, the value of C_n shows us the path to attain a capable process: if the standard deviation of the process would be 93% of the actual one, that is, $\sqrt{0.6} \times 0.93 = 0.72 = \sqrt{0.517}$, we would get $p = 0.0027$. Note that, in this case, of an off-centered process, the capability index can also be improved by readjusting the process mean, instead of the variance. As mentioned above, and for the sake of conciseness, we will restric our exposition of capability indices in terms of process variability. The index $C_p^{p/2}$ is similar to C_n only in the neighborhood of $p = 0.0027$. As the process departs from this situation $C_p^{p/2}$ has a bias toward unity. Hence, its value is not related to nonconforming proportion as C_n do.

In the third experiment (panel C), the process has different values of μ. Again, none of the indices show the same profile as C_n. As before, C_p is unable to alert from the lack of capability; the indices C_{pk}, C_{pk}^p, and C_{pkm} show a pessimistic view of the process; and the index $C_p^{p/2}$ shows a tendency toward the unit value.

If X follows a parametric distribution other than normal, the computation is similar. For instance, we can estimate the parameters using the method of moments. Those estimators will relate the parameters of the distribution with μ, σ, and perhaps with some other moments. Consequently p can be computed. If $p \neq \alpha$, $\sigma_{\max}$ should be searched by replacing the estimated value of σ by alternative values. If a parametric distribution for X cannot be assumed, p can be estimated using nonparametric techniques. These techniques are based on flexible smoothing methods that allow to build estimates of the density function of X, which in turn allow us to estimate p. In particular, Polansky [20] proposed a nonparametric procedure to estimate p specially designed to be applied to process capability problems. If the random sample leads to a value of $p \neq \alpha$, the data can be modified in such a way that the variance changes but the shape of the distribution remains the same. To this aim, the sample can be modified as $X^* = b(X - \hat{\mu})$, where $\hat{\mu}$ is the sample mean of X and $b > 0$ is some constant. Using alternative values of b, different nonparametric estimates of p will be obtained. Note that $\mathrm{var}(X^*) = b^2\sigma^2$. Hence, it can be checked that the value of b that leads to $p = \alpha$ is just the capability index C_n. The method can be summarized as follows:

(a) From the sample, we compute $\hat{\mu}$ and $USL^* = USL - \hat{\mu}$, $LSL^* = LSL - \hat{\mu}$; $k = 1$; $b_k = 1$, and $X^* = b_k(X - \hat{\mu})$.
(b) Compute the nonconforming proportion $p = 1 - P(LCL^* \leq X^* \leq UCL^*)$ using some nonparametric procedure. For example, the kernel estimation of the distribution function, as implemented in Polansky [20], can be used.
(c) If $p > \alpha$ (or $p < \alpha$), then set $k = k + 1$; $b_k < b_{k-1}$ (or $b_k > b_{k-1}$) and $X^* = b_k(X - \hat{\mu})$ and go to b). Otherwise, if $p = \alpha$, then $C_n = b_k$.

4 Capability indices for multivariate processes

Let $\mathbb{X} = (X_1, X_2, ..., X_m)'$ represent the vector of m quality characteristics of interest with mean vector $\mu = E(\mathbb{X}) = (\mu_1, \mu_2, ..., \mu_m)'$ and covariance matrix $\mathbf{\Sigma}$. Let us denote S to the tolerance region such that

$$S = \left\{\mathbb{X} \in R^m : (LSL_i \leq X_i \leq USL_i), i = 1, ..., m\right\}. \quad (17)$$

If the specifications of the i^{th} variable, USL_i and LSL_i, are independent of the remaining variables, then S will be a hyperrectangle. In some instances, the tolerances are interrelated and S can be a more complex region, as seen in the examples below. Several forms of capability indices for the multivariate processes have been proposed in the literature. Here, we will briefly describe some of them.

4.1 Capability indices using principal component analysis

In Wang and Chen [10], Wang and Du [11], and Wang [12], indices based on principal component analysis (PCA) are proposed. Under normality, the principal components (PCs) are independent. Capability indices are computed for each PC in a similar fashion as in the univariate case. Therefore, the computation of the capability is translated from a multivariate problem to a univariate one.

Following PCA, the original variables $\mathbb{X}$ are projected onto new m independent PCs. The PCs are linear combinations of $\mathbb{X}$ (defined by the eigenvectors). The variance of each PC_i is equal to its eigenvalue λ_i. For each PC_i, $i = 1, ..., m$, the authors set a specification interval for each component, $[LSL_{PC_i}, USL_{PC_i}]$, by projecting the multivariate specifications of $\mathbb{X}$ into each PC_i. Note that, by doing so, the new tolerance region is always a hyperrectangle with sides parallel to the PCs. It is also important to note that the new tolerance region is different from the original one in Eq. 17. For instance, if the specification limits USL_i and LSL_i of the i^{th} variable are independent, it is clear that the tolerance region is a hyperrectangle with sides parallel to the original axes and, hence, nonparallel to the PCs. Besides, it can be checked that the new tolerance region is larger than S in Eq. 17. Wang and Chen [10] and Wang and Du [11] propose an index obtained as the geometric mean of the capability indices of the principal components. This index, denoted here as C_{pk}^{PC}, can be written as

$$C_{pk}^{\text{PC}} = \left(\prod_{i=1}^{m} C_{pk;\,PC_i} \right)^{1/m}, \tag{18}$$

where $C_{pk;\,PC_i}$ is the univariate capability index of each component, obtained as in Eq. 2. This definition of capability index has the additional drawback that all the *PCs* are equally weighted. However, as it is well known, the first principal components will be more relevant than the last ones. A value of $C_{pk:PC_1} < 1$ should be more important than a value of $C_{pk:PC_m} < 1$. To solve this problem, in Wang [12], a weighted geometric mean is used instead of a simple geometric mean. The weights are based on the eigenvalues λ_i of each *PC*. The index, denoted here as C_{pk}^{WPC}, can be obtained as:

$$C_{pk}^{\text{WPC}} = \left[\prod_{i=1}^{m} \left(C_{pk;\,PC_i} \right)^{\lambda_i} \right]^{1 \Big/ \sum_{i=1}^{m} \lambda_i}. \tag{19}$$

The interpretation of these indices is analogous to the univariate case. Values higher than 1 are interpreted as that the process is capable. A variant of Eqs. 18 and 19 can be obtained using a number of principal components lower than m. In this case, the accounted variability of $\mathbb{X}$ does not reach 100%.

4.2 Capability indices as a ratio of volumes

We will expose two proposals based on a ratio of volumes. In the first one, proposed in Taam et al. [5], the volumes are based on the shape of the process region. Under multivariate normality of $\mathbb{X}$, the process can be represented as an elliptical region of dimension m. In the second one, proposed in Shahriari et al. [8], the volumes are based on the shape of the tolerance region.

The first proposal, due to Taam et al. [5], is a capability index built as a ratio of the volume of two ellipsoids (or hyperellipsoids), as analogous to the ratio of lengths used in the univariate C_p index. The index, denoted here as C_{pm}^{VPR}, is obtained as:

$$C_{pm}^{\text{VPR}} = \frac{\text{Volume of the modified tolerance region}}{\text{Volume of the } 100(1-\alpha)\% \text{ process region}} \equiv \frac{V_{MT}}{V_{PR}}. \tag{20}$$

Since the process region is an ellipsoid of dimension m, the modified tolerance region defined in [3] is the largest ellipsoid centered at the target $\mathbf{T}$. That is, the largest ellipsoid within the tolerance region of $\mathbb{X}$. It can be checked that the volume of the modified tolerance region is the product of the semiaxes length by $\pi^{m/2}\left[\Gamma(m/2+1)\right]^{-1}$. In the case of independent specifications for each variable, the semiaxes length is just half the specification interval. However, in a more general case, such computation can be very complex.

The elliptical process region with $100(1-\alpha)\%$ coverage is represented by the quadratic form $(\mathbb{X}-\mu)'\boldsymbol{\Sigma}_T^{-1}(\mathbb{X}-\mu) \leq \chi^2_{m,1-\alpha}$, where $\boldsymbol{\Sigma}_T = E\left[(\mathbb{X}-\mathbf{T})(\mathbb{X}-\mathbf{T})'\right]$; that is, the squared deviation matrix evaluated at target vector $\mathbf{T}$. The value $\chi^2_{m,1-\alpha}$ is the $100(1-\alpha)\%$ point of the chi-square distribution with m degrees of freedom. Based on the concept of ellipsoid of concentration, the volume of this process region can be obtained as:

$$V_{PR} = |\boldsymbol{\Sigma}_T|^{1/2} \left(\pi \chi^2_{m,1-\alpha}\right)^{m/2} \left[\Gamma(m/2+1)\right]^{-1}, \tag{21}$$

where $\Gamma(\cdot)$ is the gamma function. $\boldsymbol{\Sigma}_T$ can be written as $\boldsymbol{\Sigma}_T = \boldsymbol{\Sigma}\left[1+(\mu-\mathbf{T})'\boldsymbol{\Sigma}^{-1}(\mu-\mathbf{T})\right]$. This expression shows that C_{pm}^{VPR} resembles the index C_{pm} used in univariate processes. Using expressions (20) and (21), the index C_{pm}^{VPR} can be obtained as:

$$\begin{aligned} C_{pm}^{\text{VPR}} &= \frac{\text{Volume of the modified tolerance region}}{|\boldsymbol{\Sigma}|^{1/2}\left(\pi\chi^2_{m,1-\alpha}\right)^{m/2}\left[\Gamma(m/2+1)\right]^{-1}\left[1+(\mu-\mathbf{T})'\,\boldsymbol{\Sigma}^{-1}\,(\mu-\mathbf{T})\right]^{1/2}} \\ &= \frac{\text{Volume of the modified tolerance region}}{|\boldsymbol{\Sigma}|^{1/2}\left(\pi\chi^2_{m,1-\alpha}\right)^{m/2}\left[\Gamma(m/2+1)\right]^{-1}} \times \left[1+(\mu-\mathbf{T})'\,\boldsymbol{\Sigma}^{-1}(\mu-\mathbf{T})\right]^{-1/2} \\ &= C_p^{\text{VPR}}/\mathcal{D} \end{aligned} \tag{22}$$

The term C_p^{VPR} represents the process variability relative to the modified tolerance region. A value higher than 1 implies that the process has smaller variation than allowed by the specification limits. The term $\mathcal{D}$ reflects the process deviation from the target $\mathbf{T}$. Larger values of $\mathcal{D}^{-1}$ $(0 < \mathcal{D}^{-1} < 1)$ imply that the mean is closer to the target.

A drawback of this index is that a value $C_{pm}^{\text{VPR}} = 1$ does not mean that the expected proportion of nonconforming items is necessarily $p = \alpha$, even if the process is centered at the midpoint of the tolerance region. The reason is that α is the probability of being outside the ellipsoid of coverage $100(1-\alpha)\%$, and not the probability of being outside the tolerance region (17). Therefore, the value of this index does not have a direct interpretation in terms of p. Besides, since Eq. 22 is a ratio of volumes, its scale depends on the dimension. For instance, if we multiply each dimension of the modified tolerance region by a factor k, the volume would increase by k^m. Consequently, C_{pm}^{VPR} will be altered by the factor k^m. Hence, small variations in both the tolerance region or the process region leads to very large variations in C_{pm}^{VPR}, making interpretation difficult. For the same reason, capability indices of processes of different dimensions are not comparable. This effect can be avoided if the root-m of the index is taken.

The second proposal, due to Shahriari et al. [8] is a vector of three components, defined also under multivariate normality of $\mathbb{X}$. We will denote this index as

$$C^{\mathrm{VTR}} = [C_{pM}, PV, LI]. \tag{23}$$

The first component of C^{VTR}, denoted as C_{pM}, is the following ratio of volumes:

$$C_{pM} = \left(\frac{\text{Volume of the tolerance region}}{\text{Volume of the modified process region}}\right)^{1/m}. \tag{24}$$

The so-called modified process region is defined as the smallest region similar in shape to the tolerance region, circumscribed about the $100(1-\alpha)\%$ elliptical process region. As opposed to Eq. 22, the root-m of the ratio of volumes in Eq. 24 is used as a way to linearize the index. Otherwise, and as mentioned above, as m grows, the index might take very high or low values even with small changes in the specifications, making the interpretation very complex. This index C_{pM} is only useful if the process is centered. For this reason, C^{VTR} has a second component, denoted as PV, defined as the following probability

$$PV = P\left[T^2 > \frac{m(n-1)}{n-m}F_{(m,n-m)}\right], \text{ with } T^2 = n\left(\bar{\mathbf{X}}-\mathbf{T}\right)'\hat{\boldsymbol{\Sigma}}^{-1}\left(\bar{\mathbf{X}}-\mathbf{T}\right), \tag{25}$$

where $F_{(m,n-m)}$ is the Snedecor's F distribution, $\bar{\mathbf{X}}$ is the vector of sample means with a sample of size n, and $\hat{\boldsymbol{\Sigma}}$ is the sampling covariance matrix. As the center of the process is closer to the target $\mathbf{T}$, PV goes to one. Therefore, low values of PV, say below 5%, will reveal a noncentered process. The third component of C^{VTR}, denoted as LI, is a dummy variable that takes the value zero if any part of the modified process region falls outside the specification limits. The drawback of C^{VTR} is similar to C_{pm}^{VPR} in that its value is not easily translated to a proportion of nonconforming items p.

4.3 Capability indices based on the proportion of nonconforming items

This family of capability indices search for a direct interpretation of the capability in terms of the proportion of nonconforming items. Some proposals compute the multivariate nonconforming proportion p and then translate this proportion into a capability index by using the cumulative distribution of the univariate standard normal distribution $\Phi(\cdot)$. For instance, Wierda [6] proposes to compute p by integrating the multivariate normal distribution of $\mathbb{X}$ using numerical integration. Once p is estimated, a capability index, denoted here as C_{pk}^{p}, is constructed by applying the approximation (11) as

$$C_{pk}^{p} = -\frac{1}{3}\Phi^{-1}(p), \tag{26}$$

where the approximation is based on the assumption that the process is sufficiently capable. Polansky [12] also uses this index (Eq. 26), where p is estimated using an optimal nonparametric procedure. This estimation procedure uses a kernel function with a bandwidth optimized for the estimation of p, whereas traditional bandwidths are optimized for the estimation of a density function. In line with this approach, we can extend the univariate index (Eq. 3) to the multivariate case and define

$$C_{p}^{p/2} = -\frac{1}{3}\Phi^{-1}(p/2), \tag{27}$$

where, in this definition, it is assumed that the process is centered.

A different approach can be found in Chen [7], which proposes an index that is a ratio of two radii as

$$C_{pk}^{r} = \frac{r_0}{r}, \tag{28}$$

where r_0 represents the tolerance region, and r represents the process region. The tolerance region is defined as $V = \{\mathbb{X} \in R^m : h(\mathbb{X}-\mathbf{T}) \leq r_0\}$, where r_0 is a positive number and $h(\mathbb{X}-\mu)$ is a specific positive function of $\mathbb{X}$. The radius r is $r = \min\{c : P(h(\mathbb{X}-\mathbf{T}) \leq c) \geq 1-\alpha\}$. This radius is calculated by solving the equation $P(h(\mathbb{X}-\mathbf{T}) \leq r) = 1-\alpha$ or also by Monte Carlo. In a general setting, the computation of both r_0 and r can be very complex.

Castagliola and García [13] propose a capability index for a bivariate normal processes, denoted as BC_{pk}. They uses the eigenvectors of $\mathbf{\Sigma}$ to divide the elliptical process region into four equal regions, A_i, $i = 1, ..., 4$. The proportion of observations at each region A_i is, therefore, 1/4. The proportion q_i, $i = 1, ..., 4$, is defined as the proportion of observations inside the tolerance region at region A_i. Then, the proportion $p_i = 1/4 - q_i$ is the nonconforming proportion at region A_i. The index BC_{pk} is obtained as:

$$BC_{pk} = \frac{1}{3}\min\left\{\Phi^{-1}(2\hat{p}_1), \Phi^{-1}(2\hat{p}_2), \Phi^{-1}(2\hat{p}_3), \Phi^{-1}(2\hat{p}_4)\right\}. \tag{29}$$

5 Multivariate extension of the proposed C_n index

In this section, we will describe the methodology to estimate a multivariate PCI based on the proposed C_n in Eq. 16. As in the univariate case, the index is built such that, if $C_n = 1$, then $P(\mathbb{X} \in S) = 1 - \alpha$, where S is the tolerance region (17). The index represents how much the square root of the covariance matrix of $\mathbb{X}$ can increases, or decrease, in order to obtain a desired value of α. This definition is in accordance with the original concept of a capability index (Eq. 1), as a ratio that compares the allowable variability (standard deviation) and the real one. The general form of the proposed capability index for multivariate processes can be written as

$$C_n = f\left(\frac{|\mathbf{\Sigma}^{\max}|^{1/2}}{|\mathbf{\Sigma}|^{1/2}}\right), \tag{30}$$

where alternative functions $f(\cdot)$ will be used below in Eqs. 33 and 34, related to specific uses of this index. In this general representation $\mathbf{\Sigma}$ is the covariance matrix of the process. $\mathbf{\Sigma}^{\max}$ is the maximum allowable covariance; that is, $\mathbf{\Sigma}^{\max}$ is the covariance of a process $\mathbb{X}^{\max}$ with multivariate density function similar to that of $\mathbb{X}$ but with a covariance matrix such that $P(\mathbb{X}^{\max} \notin S) = \alpha$. The problem now is to obtain $\mathbf{\Sigma}^{\max}$; that is, how to increase or decrease the determinant $|\mathbf{\Sigma}|$ to obtain $|\mathbf{\Sigma}^{\max}|$, since this variation should be compatible with the multivariate structure of the process.

5.1 Changing from Σ to $\Sigma^{\max}$

In general, the variables $\mathbb{X}$ will not be independent, but may be considered as a combination of independent factors $\mathbb{Y}$ that might not be directly measurable. In multivariate analysis, these independent factors are usually known as latent factors. These factors can be interpreted as the primary independent sources of variability of the process. Therefore, any change in $\mathbf{\Sigma}$ will necessarily be provoked by variations in the variance of the independent factors $\mathbb{Y}$. Reasoning in this way, we can move from the multivariate problem of analysing changes in $\mathbf{\Sigma}$ to the univariate problem of analysing changes in the variance of the independent factors $\mathbb{Y}$.

In order to ease the exposition, we will first assume that $\mathbb{X} \sim N(\mu, \Sigma)$ and we have a $n \times m$ matrix of observed data $\mathbf{X}$. Later, we will extend the procedure to the nonnormal case. Under the assumption of normality, it is well known that those independent factors can be obtained using principal component analysis (PCA). The capability analysis will be based on changes in the variance of these principal components. The PCA analysis is based on the singular value decomposition of $\mathbf{\Sigma}$, that can be written as $\mathbf{\Sigma} = \mathbf{CDC}'$, where $\mathbf{C}$ is the matrix of eigenvectors of $\mathbf{\Sigma}$, with columns $\mathbf{c}_i$, $i = 1, 2, ..., m$; and $\mathbf{D}$ is a diagonal matrix with the eigenvalues λ_i, $i = 1, ..., m$. The diagonal matrix $\mathbf{D}$ is the covariance matrix of the independent factors $\mathbb{Y}$.

In order to compute the capability index C_n, we should transform the process $\mathbb{X}$ to $\mathbb{X}^{\max}$, where only realistic changes from $\mathbf{\Sigma}$ to $\mathbf{\Sigma}^{\max}$ should be considered. A realistic change in the process will be due to changes in the variance of the independent factors $\mathbb{Y}$. That is, the factor causing the variability in $\mathbb{X}$ remains the same but with larger variance. Therefore, realistic changes in $\mathbf{\Sigma}$ with practical meaning should be restricted to those maintaining the eigenvectors $\mathbf{C}$. A change in $\mathbf{C}$ would only be possible if a radical and drastic change has happened in the process, such that the independent sources of variability become dependent. It does not seem reasonable to design a capability index based on those exceptional situations. Therefore, changes in $\mathbf{\Sigma}$ will be due to changes in the eigenvalues matrix $\mathbf{D}$ (see González and Sánchez [22] for further applications of this concept).

Many alternative changes in $\mathbf{D}$ can be considered. Since the factors $\mathbb{Y}$ are independent, it is reasonable to analyze the case where only one factor changes its variance at a time. Under this assumption, a capability index defined for each factor will be useful. Alternatively, it is also of interest to analyze the case where all the factors change simultaneously and proportionally. This simultaneous change can reflect the situation of a poor quality management of the whole process, whereas the changes in only one factor reflects the case of specific quality problems associated with such factor. Although both cases can lead to different values of capability index, it is of practical interest to compute both of them.

5.2 Definition of C_n in a multivariate case

Paralleling the univariate case, the index C_n should be based on the constant that should be multiplied by the standard deviation to attain $p = \alpha$. Let Y_i, be the i^{th} independent factor obtained with PCA. We can alter the variance of this factor by multiplying the corresponding eigenvalue λ_i by a factor b_i^2. The new matrix of eigenvalues, denoted as $\mathbf{D}^*_{(i)}$, can be expressed as:

$$\mathbf{D}^*_{(i)} = \text{diag}\left(\lambda_1, \lambda_2, ..., b_i^2\lambda_i, ..., \lambda_m\right). \tag{31}$$

This change in the variance of the i-th component will provoke a change in $\mathbb{X}$, denoted as $\mathbb{X}^*_{(i)}$ such that $\mathbb{X}^*_{(i)} \sim N\left(\mu, \mathbf{\Sigma}^*_{(i)}\right)$, where

$$\mathbf{\Sigma}^*_{(i)} = \mathbf{CD}^*_{(i)}\mathbf{C}'. \tag{32}$$

This new distribution will imply a proportion of nonconforming items $p^*_{(i)} = P(\mathbb{X}^*_{(i)} \notin S)$. Alternative methods can be used to compute $p^*_{(i)}$. For instance, if m is low, numerical integration can be a feasible method. Also, Monte Carlo simulation can be used, especially if m is large. Since the value of $p^*_{(i)}$ can be very low, a large number of replications should be used. The goal then is to look for a value b_i, denoted as $b_i^{\max}$, such that $p^*_{(i)} = \alpha$. This $b_i^{\max}$ leads to the covariance matrix $\mathbf{\Sigma}^{\max}_{(i)}$ where the subindex i denotes the factor that has been altered. The capability index associated to this factor should represent how much the variance of the i^{th} component can be increased (or decreased) to obtain a unitary capability index (that leads to a nonconforming proportion α). Therefore, the general expression (Eq. 30) leads to define

$$C_{n,i} = \left(\frac{\left|\mathbf{\Sigma}^{\max}_{(i)}\right|}{|\mathbf{\Sigma}|}\right)^{1/2} = \left(\frac{\left(b_i^{\max}\right)^2 \prod_{i=1}^{m}\lambda_i}{\prod_{i=1}^{m}\lambda_i}\right)^{1/2} = b_i^{\max}. \tag{33}$$

A process will be capable if all $C_{n,i} \geq 1$, $i = 1, ..., m$. However, if the process is not capable, the indices $C_{n,i}$ can show us which factor, or set of factors, should be improved. Note that this index, contrary to $C_{pk;PC_i}$ in Eqs. 18 and 19, is built computing the nonconforming proportion outside the tolerance region S, instead of using univariate tolerance limits for each principal component.

A simultaneous change in all the components can be modeled by multiplying all the eigenvalues by b^2 such that $\mathbf{D}^* = b^2\mathbf{D}$. Hence, the change in $\mathbb{X}$ is $\mathbb{X}^* \sim N(\mu, \mathbf{\Sigma}^*)$, with $\mathbf{\Sigma}^* = \mathbf{CD}^*\mathbf{C}'$. As before, we look for a value $b^{\max}$ such that $p^* = P(\mathbb{X}^* \notin S) = \alpha$. This $b^{\max}$ leads to the covariance matrix $\mathbf{\Sigma}^{\max}$, and hence, the general expression Eq. 30 leads us to define

$$C_n^S = \left(\frac{|\mathbf{\Sigma}^{\max}|}{|\mathbf{\Sigma}|}\right)^{\frac{1}{2m}} = \left\{\frac{\left(b^{\max}\right)^{2m} \prod_{i=1}^{m}\lambda_i}{\prod_{i=1}^{m}\lambda_i}\right\}^{\frac{1}{2m}} = b^{\max}, \tag{34}$$

where the superscript index S stands for simultaneous change. As opposed to Eq. 33, the root-2m is used in Eq. 34, so that C_n^S represents how much the standard deviation of each factor can be increased (or decreased) simultaneously to obtain a nonconforming proportion of α. By doing so, the values of the index (34) for processes of different dimensions m can be compared.

It is straightforward to see that, in this simultaneous change, $\mathbf{\Sigma}^* = \mathbf{CD}^*\mathbf{C}' = b^2\mathbf{CDC}' = b^2\mathbf{\Sigma}$. Hence, the simultaneous capability index C_n^S can be obtained without using PCA. The estimation of C_n^S is made by computing $p^* = P(\mathbb{X}^* \notin S)$ by Monte Carlo using the normal distribution of mean μ and covariance $\mathbf{\Sigma}^* = b^2\mathbf{\Sigma}$, and searching the value $b^{\max}$ so that $P(\mathbb{X}^{\max} \notin S) = \alpha$.

5.3 Extension to the nonnormal case

The proposed C_n^S and $C_{n,i}$ can be extended to the nonnormal case. Some aspects are, however, different. First, the set of independent factors $\mathbb{Y} = (Y_1, ..., Y_m)$ can not be obtained using PCA as we used in Eqs. 31 and 32. In this case, the appropriate technique to obtain independent factors is independent component analysis (ICA) (see, for instance Hyvärinen et al. [23] and González and Sánchez [24]). Second, in order to change the variance of the ith independent component Y_i we can not do it as in Eq. 31 because in ICA there is not such thing as the eigenvalues. This change might be obtained by multiplying the independent component Y_i by a constant b_i. This change in Y_i will provoke a new matrix $\mathbb{X}^*_{(i)}$, which will allow to estimate the new variance $\mathbf{\Sigma}^*_{(i)}$. The goal is then similar to the normal case: to look for a value b_i, denoted as $b_i^{\max}$, such that $p^*_{(i)} = P(\mathbb{X}^*_{(i)} \notin S) = \alpha$. Third, the computation of $p^*_{(i)}$ can not rely on the normal distribution. This proportion could be computed using a nonparametric procedure as in Polansky [12]. The capability index $C_{n,i}$ associated to this independent component is just $C_{n,i} = b_i^{\max}$. To obtain C_n^S, and as in the normal case, it is not necessary to compute the independent factors. In this case, the original variables $\mathbb{X}$ are multiplied by the same factor b obtaining as a result a new matrix $\mathbb{X}^*$, that allows to

Table 2 Multivariate process capability indices for Example 1

	c_1	c_2	c_3	p	$C_{n,1}$	$C_{n,2}$	C_n^S	C_{pk}^{PC}	C_{pk}^{WPC}	C_{pm}^{VPR*}	C^{VTR}	C_{pk}^r	C_{pk}^p	$C_p^{p/2}$	BC_{pk}
Panel A	1.00	1.20	1.20	5.1×10^{-5}	1.37	2.13	1.33	2.56	1.46	1.62	[1.22; 0.54; 1]	1.32	1.29	1.35	1.27
	1.00	0.906	0.906	0.0027	1.00	1.00	1.00	1.93	1.10	1.23	[0.92; 0.54; 0]	1.00	0.93	1.00	0.94
	1.00	0.80	0.80	0.0087	0.86	<0.01	0.88	1.70	0.97	1.09	[0.81; 0.54; 0]	0.88	0.79	0.87	0.81
	1.00	0.60	0.60	0.0563	0.53	<0.01	0.66	1.27	0.73	0.81	[0.61; 0.54; 0]	0.66	0.53	0.64	0.57
Panel B	1.000	1.00	1.00	0.0009	1.12	1.47	1.10	2.13	1.22	1.35	[1.02; 0.54; 0]	1.10	1.04	1.11	1.06
	0.937	1.00	1.00	0.0027	1.00	0.99	1.00	1.95	1.01	1.26	[1.02; 0.02; 0]	1.00	0.93	1.00	0.91
	0.920	1.00	1.00	0.0044	0.94	<0.01	0.95	1.89	0.96	1.22	[1.02; 0.003; 0]	0.96	0.87	0.95	0.86
	0.850	1.00	1.00	0.0258	0.67	<0.01	0.72	1.65	0.73	1.03	[1.02; 1.5×10^{-6}; 0]	0.81	0.65	0.74	0.63
Panel C	0.850	1.00	2.00	1.1×10^{-6}	1.70	2.81	1.61	3.09	1.72	1.54	[1.53; 1.5×10^{-6}; 1]	1.55	1.58	1.63	1.32
	0.850	1.00	1.235	0.0027	1.00	1.00	1.00	2.06	1.01	1.15	[1.14; 1.5×10^{-6}; 0]	1.00	0.93	1.00	0.92
	0.850	1.00	1.10	0.0106	0.82	<0.01	0.84	1.83	0.85	1.08	[1.07; 1.5×10^{-6}; 0]	0.89	0.77	0.85	0.75
	0.850	1.00	0.80	0.1089	0.25	<0.01	0.48	1.27	0.48	0.93	[0.92; 1.5×10^{-6}; 0]	0.67	0.41	0.53	0.39

estimate the new variance $\boldsymbol{\Sigma}^*$. Then, the nonconforming proportion p^* can be computed using, for instance, Polansky [12]. We search the value $b^{\max}$ so that $p^* = \alpha$, and then $C_n^S = b^{\max}$.

6 Illustrative examples

In this section we illustrate the usefulness of the proposed indices with two examples from the literature in the context of multivariate processes. Also, we include a simulated example with a tenth-dimensional process.

The search of $b_i^{\max}$ in Eq. 33, and $b^{\max}$ in Eq. 34 has been made using a combination of bisection and secant methods. In this search, the nonconforming proportion $p^*_{(i)}$ for the computation of $C_{n,i}$, and p^* for the computation of C_n^S are obtained by Monte Carlo using the normality assumption. In each case, the nonconforming proportion is the empirical proportion of times the simulated multidimensional observation is outside the multidimensional tolerance region. The Matlab code used in these examples are available from the authors' website.

6.1 Example 1

This example was initially proposed in Sultan [25] and has been used by other authors. The data comes from a bivariate process where the quality characteristics are: the brinell hardness H, and the tensile strength E of steel sections. The process is assumed normal. The data set consists on 25 observations from the in control process and can be found in [25]. From the data set we obtain the following estimations

$$\widehat{\mu} = (177.2, 52.32)' \,;\, \hat{\boldsymbol{\Sigma}} = \begin{bmatrix} 338.000 & 88.8925 \\ 88.8925 & 33.6247 \end{bmatrix}. \quad (35)$$

The process is analyzed under different conditions, using different targets and tolerance regions. In all cases the capability indices are obtained for $\alpha = 0.0027$. Table 2 summarizes the results. The target is set at $\mathbf{T} \equiv (T_1, T_2) = (177c_1, 53c_1)$, with alternative values of c_1 representing alternative noncentered situations. The

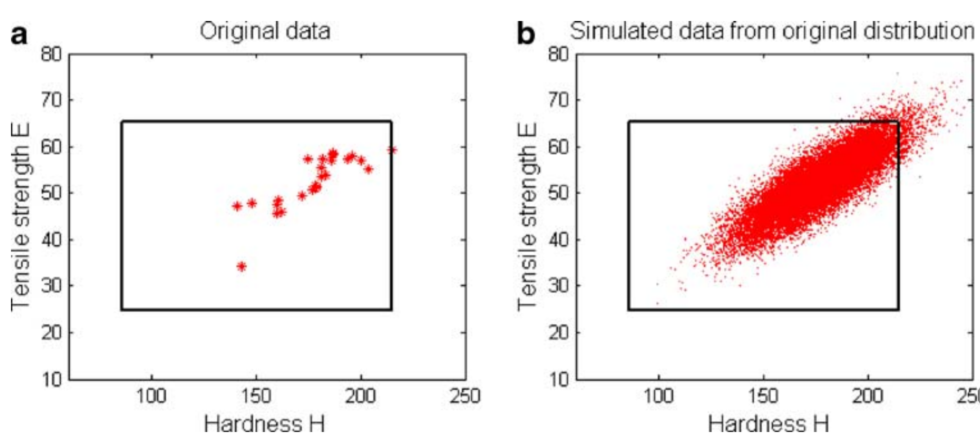

Fig. 1 Data from Example 1 with tolerance region. **a** Original data. **b** Monte Carlo simulations from a bivariate normal estimated from original data

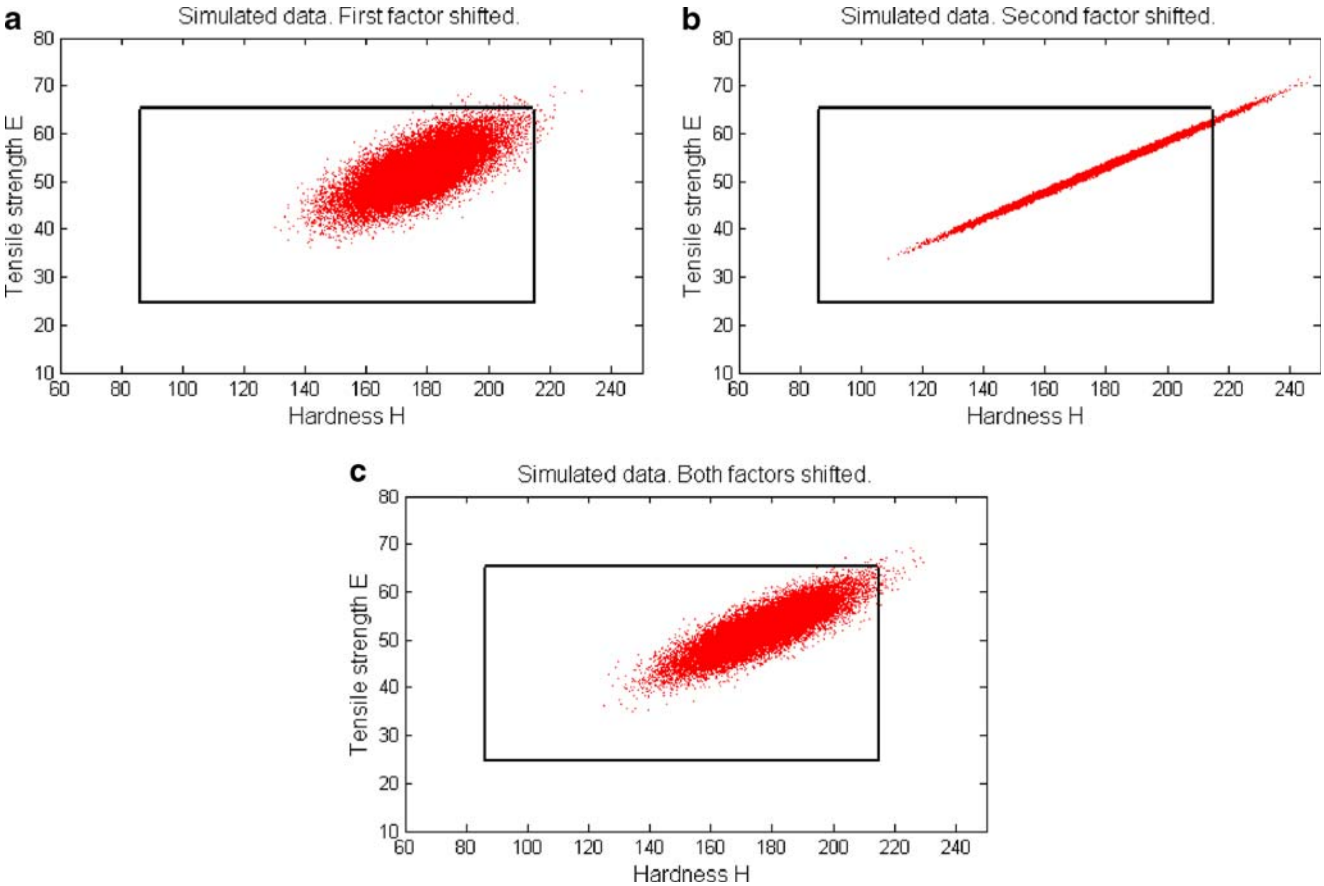

Fig. 2 Monte Carlo simulations from a bivariate normal distribution with tolerance region of Example 1. **a** Same distribution as in Fig. 1 where first factor has a reduced variance such that $p = \alpha$. **b** Same distribution as in Fig. 1 where second factor has a reduced variance. **c** Same distribution as in Fig. 1 where both factors have a reduced variance such that $p = \alpha$

rectangular tolerance region S is set at $LSL_i = T_i - 3.5c_2\widehat{\sigma_i}$, $USL_i = T_i + 3.5c_3\widehat{\sigma_i}$, $i = 1, 2$, where alternative values of c_2 and c_3 can be used to obtain symmetric or asymmetric specification limits with respect to the target $\mathbf{T}$. The estimator $\widehat{\sigma_i}$ is the estimated standard deviation of each variable; that is, $\hat{\sigma}_1 = 18.38$ and $\hat{\sigma}_2 = 5.8$. For each value of (c_1, c_2, c_3) the nonconforming proportion p is computed with the empirical nonconforming proportions in a Monte Carlo simulation under normality. In all cases, 10^9 Monte Carlo replications were used. The cases $(c_1, c_2, c_3) = (1, 1, 1)$ and $(c_1, c_2, c_3) = (0.85, 1, 1)$ have been considered in Chen [7], and Wang and Chen [10]. It should be noticed that in [10] C_{pk}^{PC} is computed using only the first principal component, whereas the value of C_{pk}^{PC} in Table 2 uses all the principal components. Indices are computed as shown in Section 2, with the exception of C_{pm}^{VPR}. In this case, the root-m of the index is used, that is $C_{pm}^{\mathrm{VPR}*} = \left(C_{pm}^{\mathrm{VPR}}\right)^{1/m}$. The index C_{pk}^{p} in Eq. 26 and $C_{p}^{p/2}$ in Eq. 27 are computed using the value p obtained in the Monte Carlo simulation.The index BC_{pk} is computed estimation the proportion p_i in Eq. 29 with 10^9 Monte Carlo replications.

In Panel A of Table 2, the process mean is very close to the target $\mathbf{T}$. The tolerance region is symmetric ($c_2 = c_3$), but of different sizes. As a first conclusion, it can be seen that only the proposed $C_{n,1}$, $C_{n,2,}$ C_n^S, and

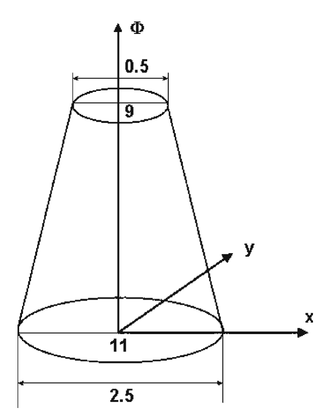

Fig. 3 Tolerance region of pin

Table 3 Multivariate process capability indices for Example 2, Case 1 and Case 2

Case	p	$C_{n,1}$	$C_{n,2}$	$C_{n,3}$	C_n^S	C_{pk}^{PC}	C_{pk}^{WPC}	C_{pm}^{VPR*}	C^{VTR}	C_{pk}^r	C_{pk}^p	$C_p^{p/2}$
1	7.6×10^{-4}	1.16	1.52	1.75	1.12	1.78	1.66	1.19	[1.18; 0.24; 1]	1.31	1.06	1.12
2	2.5×10^{-7}	2.02	2.87	5.24	1.80	4.08	3.92	1.52	[1.95; 0.00; 1]	1.81	1.68	1.72

the indices $C_p^{p/2}$ and C_{pk}^r have a value of 1.00 for $p = 0.0027$. The remaining indices fail to detect whether the process is capable or not. In this Panel A, and since this process is almost centered, the values of C_n^S, $C_p^{p/2}$ and C_{pk}^r are almost identical. The cases with $C_{n,2} < 0.01$ mean that it is not possible to attain a capable process by reducing the variance of that component. Hence, the lack of capability is due to the other component. In Panel B of Table 2, the tolerance region has always the same size and shape, but its location changes with c_1. As before, only the proposed $C_{n,1}$, $C_{n,2}$, C_n^S, as well as $C_p^{p/2}$ and C_{pk}^r have a value of 1.00 for $p = 0.0027$, with C_{pk}^{WPC} having also a value very close to unity. Contrary to Panel A, the index C_{pk}^r is not similar to C_n^S for values lower than one. Now, the index C_{pk}^{WPC} is similar to C_n^S for values lower than one but it is different for values larger than one. These results confirm that the existing indices do not have a clear interpretation in terms of p.

Figures 1 and 2 illustrate the behavior of the proposed $C_{n,i}$ and C_n^S in the case $c_1 = 0.85$ and $c_2 = c_3 = 1$, which corresponds to the last row of Panel B in Table 2. Figure 1a displays the original data with the tolerance region. This figure shows that the process is not centered in the tolerance region. Figure 1b shows Monte Carlo replications from bivariate normal distribution using the estimated parameters (35). From this simulation, it can be computed that the nonconforming proportion is $p = 0.0258 > \alpha$. The process is not capable. Figure 2 shows Monte Carlo replications from a bivariate normal distribution using the same mean vector as in Eq. 35 but a modified covariance matrix with the goal of attaining a capable process. In Fig. 2a the covariance matrix is obtained as in Eqs. 31 and 32 by multiplying the first eigenvalue by $\left(b_1^{\max}\right)^2 = 0.67^2$. With this transformation, it is obtained $p = \alpha$. Hence $C_{n,1} = 0.67$. In Fig. 2b the covariance matrix is obtained by multiplying the second eigenvalue by a very small value. However, as it can be seen from this figure, it is not feasible to obtain a capable process by only reducing the variability of this component. In Fig. 2c both components are multiplied by $(b^{\max})^2 = 0.72^2$. With this transformation, it is obtained that $p = \alpha$. Hence, $C_n^S = 0.72$.

In Panel C of Table 2 the target $\mathbf{T}$ remains constant and it is the same as in the last row of Panel B (see Fig. 1). The tolerance region has different sizes and shapes. The LSL remains constant, but the USL increases with c_3. In this case, the index C_{pk}^r is relatively different from C_n^S, apart from the case $p = 0.0027$. It is lower than C_n^S for $p < 0.0027$, but larger than C_n^S for $p > 0.0027$. The index $C_p^{p/2}$ is also similar to C_n^S in the vicinity of one, but they are different at large values of p. In this case, the index C_{pk}^{WPC} is similar to C_n^S. It seems that C_{pk}^r, $C_p^{p/2}$, and C_{pk}^{WPC} can sometimes have similar values to C_n^S, but that they do not always admit an interpretation like the proposed C_n^S. As a conclusion, only the proposed C_n^S and $C_{n,i}$ can guarantee an interpretation in terms of how much the variance of the process should change to attain a desired nonconforming proportion.

6.2 Example 2

This is an example of geometric dimensioning and tolerancing (GD&T). GD&T is an engineering standard

Table 4 Multivariate process capability indices for Example 3

	c_1	c_2	c_3	p	C_n^S	C_{pk}^{PC}	C_{pk}^{WPC}	C_{pm}^{VPR*}	C^{VTR}	C_{pk}^r	C_{pk}^p	$C_p^{p/2}$
Panel A	0.00	1.00	1.00	2.9×10^{-6}	1.46	11.16	2.06	3.39	[0.96; 1; 0]	1.47	1.51	1.56
	1.50	1.00	1.00	0.0010	1.08	10.53	1.52	3.11	[0.96; 0; 0]	1.06	1.03	1.10
	1.78	1.00	1.00	0.0027	1.00	10.39	1.41	3.07	[0.96; 0; 0]	1.00	0.93	1.00
	2.00	1.00	1.00	0.0055	0.93	10.28	1.33	3.04	[0.96; 0; 0]	0.96	0.85	0.93
	4.00	1.00	1.00	0.4000	0.31	8.89	0.53	2.85	[0.96; 0; 0]	0.69	0.08	0.28
Panel B	0.00	1.00	0.5	0.0233	0.78	7.84	1.09	2.54	[0.72; 1; 0]	0.84	0.66	0.76
	0.00	1.00	0.644	0.0027	1.00	8.81	1.37	2.79	[0.79; 1; 0]	1.00	0.93	1.00
	0.00	1.00	0.8	1.43×10^{-4}	1.24	9.85	1.67	3.05	[0.87; 1; 0]	1.21	1.21	1.27
	0.00	1.00	0.9	1.71×10^{-5}	1.37	10.51	1.87	3.22	[0.92; 1; 0]	1.35	1.38	1.43
	0.00	1.00	3.0	1.40×10^{-6}	1.55	20.02	2.29	6.78	[1.93; 1; 0]	1.22	1.56	1.61

that provides a unified terminology to describe the geometry tolerances of the product features (shape, orientation, profile, etc). This example was proposed in Karl and Taam [26]. In Wang et al. [27] this example is also used to illustrate the performance of C_{pm}^{VPR}, C^{VTR}, and C_{pk}^{r} under two alternative cases.

The data corresponds to a three dimensional process. One of the variables is the diameter of a pin, denoted as ϕ. The engineering specifications require a diameter pin between 9 and 11 tenths of an inch. The other variables are related to the perpendicularity of the centerline of the pin. The tolerance of the perpendicularity depends on ϕ. At maximum material condition (MMC), that is, when the $\phi = 11$, the specifications requiere that the centerline is within a cylinder of diameter 0.5 tenths of an inch. However, at least material condition (LMC), ($\phi = 9$), the centerline should be within a cylinder of diameter 2.5 tenths of an inch. Therefore, the specifications lead to a three dimensional tolerance region in the shape of a frustrum or 'lamp shade' as illustrated in Fig. 3, where x and y are the coordinates of the center point of the top surface of the pin with respect to the centerpoint of the bottom one, and the vertical axis is the diameter ϕ.This tolerance region implies that an item is conforming if $9 \leq \phi \leq 11$ and $x^2 + y^2 \leq \left[(11 - \phi)/2 + 0.25\right]^2$.

The two cases used in [27] are used here to compare the performance of the proposed C_n^S and $C_{n,i}$. In both cases, normality is assumed, and alternative targets, mean vectors, and covariance matrices are proposed. In the first case, the target for the variables (x, y, ϕ) is $\mathbf{T} = (0, 0, 10)$. The parameters of the process are

$\mu_1 = (-0.0124, -0.0062, 10.0586)'$ and

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 0.01313 & -0.00371 & 0.0084 \\ -0.00371 & 0.01618 & -0.01031 \\ 0.0084 & -0.01031 & 0.06473 \end{bmatrix}.$$

In the second case, the nominal value is $\mathbf{T} = (0, 0, 9.5)$. The mean vector and covariance matrix are

$\mu_2 = (-0.1, -0.1, 9.5)'$;

$$\mathbf{\Sigma}_2 = \begin{bmatrix} 0.00640 & 0.00256 & 0.00160 \\ 0.00256 & 0.00640 & 0.00160 \\ 0.00160 & 0.00160 & 0.01000 \end{bmatrix}.$$

Table 3 shows the values of the competing capability indices for $\alpha = 0.0027$. As before, the computation of p is based on 10^9 Monte Carlo simulations. In this table, the values of C^{VTR}, and C_{pk}^{r} are the ones reported in [27]. The index $C_{pm}^{\mathrm{VPR}*}$ is obtained from the index C_{pm}^{VPR} reported in [27]. In this example, the indices C_{pk}^{PC} and C_{pk}^{WPC} have a poor performance, that can be explained from the complexity of the tolerance region. It should be reminded that theses indices are based on building a rectangular tolerance regions for the principal components that is different from the real one. This new tolerance region tends to be larger than the original one, specially if the original one is not rectangular. This effect explains the high values of C_{pk}^{PC} and C_{pk}^{WPC}. The index C_{pk}^{r} is similar to C_n^S only in Case 2, whereas the index $C_p^{p/2}$ is similar to C_n^S only in Case 1.

6.3 Example 3

In this example, a correlated tenth-dimensional process is simulated. The process has a zero mean an covariance matrix,

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0.91 & 0.44 & 0.99 & 0.99 & 0.99 & 0.97 & 0.98 & 0.99 & 0.34 \\ 0.91 & 1 & 0.41 & 0.91 & 0.91 & 0.91 & 0.88 & 0.90 & 0.91 & 0.31 \\ 0.44 & 0.41 & 1 & 0.44 & 0.44 & 0.45 & 0.42 & 0.43 & 0.45 & 0.15 \\ 0.99 & 0.91 & 0.44 & 1 & 0.98 & 0.99 & 0.96 & 0.97 & 0.99 & 0.35 \\ 0.99 & 0.91 & 0.44 & 0.98 & 1 & 0.98 & 0.96 & 0.97 & 0.98 & 0.34 \\ 0.99 & 0.91 & 0.45 & 0.99 & 0.98 & 1 & 0.96 & 0.98 & 0.99 & 0.34 \\ 0.97 & 0.88 & 0.42 & 0.96 & 0.96 & 0.96 & 1 & 0.95 & 0.96 & 0.33 \\ 0.98 & 0.90 & 0.43 & 0.97 & 0.97 & 0.98 & 0.95 & 1 & 0.98 & 0.34 \\ 0.99 & 0.91 & 0.45 & 0.99 & 0.98 & 0.99 & 0.96 & 0.98 & 1 & 0.35 \\ 0.34 & 0.31 & 0.15 & 0.35 & 0.34 & 0.34 & 0.33 & 0.34 & 0.35 & 1 \end{bmatrix}$$

This covariance matrix has been obtained as follows. First, a random vector $\mathbf{x}_1$ of size 1000 has been obtained from a standard normal distribution. Then, a set of vectors $\mathbf{x}_i$, $i = 2, ..., 9$ have been obtained as $\mathbf{x}_i = v_i \mathbf{x}_1 + d\mathbf{a}_i$, where v_i is a random number from a uniform distribution in the [0,1] interval and $\mathbf{a}_i$ is also a random vector from the standard normal distribution. By using alternative values of d the degree of correlation among the variables can be changed. The smaller the value of d, the larger the correlations. In this example, $d = 0.1$. The correlation matrix of $\mathbf{x}_i$, $i = 2, ..., 9$ is used as covariance matrix of the process. Hence, we are assuming standardized variables. The nominal value of the process is set to $\mathbf{T} = \mathbf{0} + c_1$, where $\mathbf{0}$ is a vector of zeros. The rectangular tolerance region S is set at $LSL_i = T_i - 5c_2$, $USL_i = T_i + 5c_3$, $i = 1, ..., 10$, where alternative values c_2, c_3 can be used to obtain symmetric or asymmetric specification limits, with respect to the target $\mathbf{T}$.

Table 4 summarizes the results. In Panel A, the rectangular tolerance region remains of constant volume. As a first result, it can be seen that the index $C_{pm}^{\text{VPR}*}$ has high values. The first element of C^{VTR} is always $0.96 < 1$ suggesting that the process is not capable. However, the process is capable for $c_1 \leq 1.78$. The index C_{pk}^{PC} also shows very high values that can be explained by the high dimensionality of the process and the high correlations. In these cases, there will be several principal components with very low eigenvalues. Hence, they will have high values of $C_{pk;PC_i}$ in Eq. 18. Since all the principal components are equally weighted in C_{pk}^{PC}, the index will tend to reach very high values even if the process is not capable. The index C_{pk}^{WPC} solves this problem, but its value is not related to the capability of the process. For instance, for $p = 0.0027$ we obtain $C_{pk}^{\text{WPC}} = 1.33$ suggesting that the process is capable. Similarly, the value of the index C_{pk}^{p} fails to detect when the process is capable. For instance, when $p = 0.0027$, $C_{pk}^{p} = 0.93$, erroneously suggesting that the process is not capable. Only the indices $C_p^{p/2}$ and C_{pk}^{r}, as well as the proposed C_n^S, obtain the unit value when $p = 0.0027$. Besides, they have similar values for p around 0.0027. As the process departures from $p = 0.0027$ these indices are different. Hence, C_{pk}^{r} and $C_p^{p/2}$ can not be interpreted, as the proposed C_n^S, as how to modify the variability of the process to obtain $p = 0.0027$.

In Panel B of Table 4 the value of $\mathbf{T}$ is constant, but USL_i changes with the alternative values of c_3. It can be seen in this panel that the conclusions are similar as in Panel A.

7 Concluding remarks

Capability indices are a useful tool that allows to communicate the quality level of the process to interested parts. For instance, alternative manufacturing processes of an industrial plant can report the quality of their processes to the quality department using such indices. Also, suppliers evaluation are customary made using these capability indices. An accurate capability measure provides insights into the process situation of competing suppliers who may enter into a long-term partnership with a firm. Therefore, it is necessary a capability index with a direct interpretation in terms of nonconforming proportion that can be used in all circumstances: univariate or multivariate processes, centered or noncentered processes, normal or nonnormal processes, rectangular or nonrectangular tolerance regions, and so on.

In univariate and centered processes, the classical C_p index explicitly admits this interpretation. For instance, if $C_p = 1.5$ the standard deviation can be increased by a 50% and obtain $C_p = 1$. However, for some other cases there is a lack of capability indices that admit such an interpretation. This article proposes univariate and multivariate capability indices that have a direct interpretation of how much the variability of the process should be decreased to attain a unitary index. The examples shown in the article clearly demonstrate the advantages of the proposed indices.

In the multivariate case, a capability index can be built for each independent source of variability of the process, denoted as $C_{n,i}$ indices. These indices allows to identify the most critical parts of the process and how to modify their variability in order to attain a desired nonconforming proportion. In this respect, the proposed capability indices can be used as a tool to improve the performance of the process, and not only a descriptive measure.

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