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ORIGINAL ARTICLE

Kinematics of 3-RPS parallel manipulators by means of screw theory

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Abstract In this work the forward position analysis of parallel manipulators with identical limbs, type revolute-prismatic-spherical (RPS), is carried out applying recursively the Sylvester dialytic elimination method. Afterwards, the velocity and acceleration analyses of the mechanisms at hand are addressed using the theory of screws. A numerical example is provided to prove the efficacy of the chosen methodology for the kinematic analyses of the mechanisms under study.

Keywords Parallel manipulator · Analytical form solution · Klein form · Screw theory · Forward kinematics

1 Introduction

According to the notation proposed by IFToMM [1], a parallel manipulator (PM for brevity) is a mechanism where the motion of the end-effector, namely the moving platform, is controlled by means of at least two kinematic chains. Unlike serial manipulators, not all the kinematic pairs of a PM are actuated and therefore the presence of passive kinematic pairs is a typical characteristic of these mechanisms. On the other hand, due to a compact topology, PMs are more precise and stiff than their serial counterparts, however suffer from a limited workspace, poor dexterity, and a recurrent problem of the so-called local singularities. These drawbacks have a direct connection with the nominal degrees of freedom assigned to the moving platform, the number of limbs, and the dimensions of the mechanism.

A 3-RPS parallel manipulator, see Fig. 1, is a mechanism where the moving platform is connected to the fixed platform by means of three limbs. Each limb is composed by a lower body and an upper body connected each other by means of a prismatic joint. The moving platform is connected at the upper bodies via three distinct spherical joints while the lower bodies are connected to the fixed platform by means of three distinct revolute joints. The prismatic joints are actuated independently providing three degrees of freedom over the moving platform.

The 3-RPS parallel manipulator was introduced in Hunt [2] and has been the motive of an exhaustive research field where a great number of contributions, encompassing a wide range of topics, such as kinematic and dynamic analyses, synthesis, singularity analysis, extensions to hyper-redundant manipulators, etc., see for instance [3–6]. In particular, screw theory has been proved to be an efficient mathematical resource for determining the kinematic characteristics of 3-RPS parallel manipulators [7–9], including the instantaneous motion analysis of the mechanism at the level of velocity analysis [10–12].

This paper addresses the kinematics, including the acceleration analysis, of 3-RPS parallel manipulators. The forward position analysis is carried out in analytical form solution using the Sylvester dialytic elimination method. The velocity and acceleration analyses are approached by means of the theory of screws. To this end, the velocity and reduced acceleration states of the moving platform, w.r.t.

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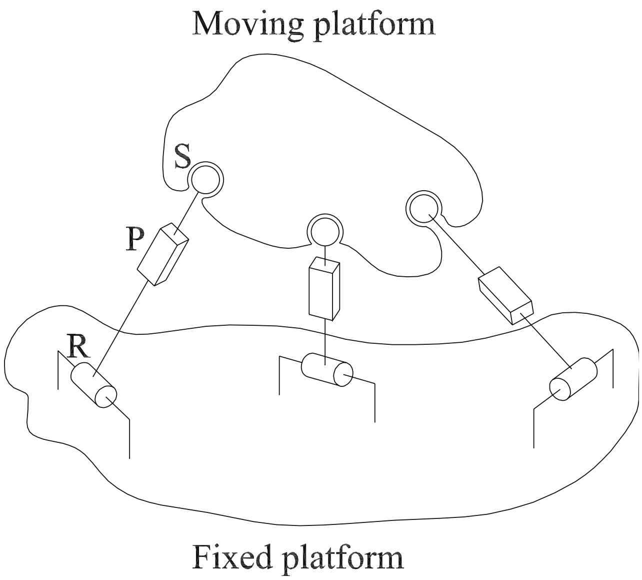

Fig. 1 Spatial 3-RPS parallel manipulator

fixed platform, are written in screw form through each one of the limbs of the mechanism. Finally, the systematic application of the Klein form to these expressions allows one to obtain simple and compact expressions for computing the velocity and acceleration analyses. A numerical example is provided.

2 Position analysis

Usually the forward position analysis of a 3-RPS parallel manipulator consists of finding the pose, position and orientation, of the moving platform w.r.t. the fixed platform given the limb lengths q_i $i \in \{1, 2, 3\}$. Clearly this problem is equivalent to the computation of the coordinates of the centers of the three distinct spherical joints, points $\mathrm{P}_i = (X_i, Y_i, Z_i) i \in \{1,2,3\}$, attached at the moving platform, see Fig. 2.

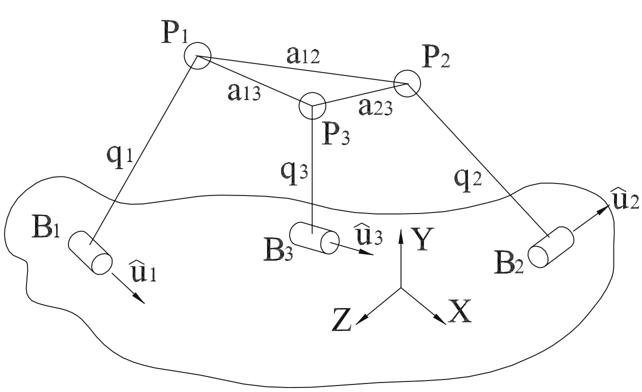

Fig. 2 The geometric scheme of a generic 3-RS structure

When the limbs of the parallel manipulator are locked $\dot{q}_i = 0\, i \in \{1,2,3\}$, the mechanism becomes into the 3-RS structure shown in Fig. 2. In order to simplify the analysis, the reference frame XYZ, attached at the fixed platform, is chosen in such a way that the points $\mathrm{B}_i = (b_{Xi}, 0, b_{Zi})$ $i \in \{1,2,3\}$, denoting the nominal positions of the three revolute joints, lie on the X-Z plane.

2.1 Closure equations

With reference to Fig. 2, the axes of the revolute joints are coplanar and three constraints are imposed by these kinematic pairs as follows

$$(\mathbf{P}_i - \mathbf{B}_i) \cdot \widehat{u}_i = 0 \quad i \in \{1,2,3\}, \tag{1}$$

where $\widehat{u}_i = (u_{Xi}, 0, u_{Zi})$ is the i-*th* unit vector along the screw axis of the i-*th* revolute joint.

Furthermore, clearly the limb lengths are restricted to

$$(\mathbf{P}_i - \mathbf{B}_i) \cdot (\mathbf{P}_i - \mathbf{B}_i) = q_i^2 \quad i \in \{1,2,3\}. \tag{2}$$

Finally, three compatibility constraints can be obtained as follows

$$(\mathbf{P}_2 - \mathbf{P}_3) \cdot (\mathbf{P}_2 - \mathbf{P}_3) = a_{23}^2, \tag{3}$$

$$(\mathbf{P}_1 - \mathbf{P}_3) \cdot (\mathbf{P}_1 - \mathbf{P}_3) = a_{13}^2, \tag{4}$$

and

$$(\mathbf{P}_1 - \mathbf{P}_2) \cdot (\mathbf{P}_1 - \mathbf{P}_2) = a_{12}^2. \tag{5}$$

Expressions (1–5); form a system of nine equations in nine unknowns given by $\{X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3\}$.

It is worth mentioning that expressions (1) were not considered, in the form derived here, by Tsai [13], and therefore the analysis reported in that contribution requires a particular arrangement of the positions of the revolute joints over the fixed platform accordingly to the reference frame XYZ. Furthermore, clearly expressions (1) are applicable not only to tangential 3-RPS parallel manipulators, like the mechanism of Fig. 1, but also to the so-called concurrent 3-RPS parallel manipulators.

2.2 Analytical form solution

In this subsection expressions (1)–(5) are systematically reduced into a non linear system of three equations in three unknowns. Afterwards, a 16th-order polynomial in one unknown is derived using the Sylvester dialytic elimination method.

Expressions (1) yields three linear equations given by

$$X_i = f(Z_i) \quad i \in \{1,2,3\}. \tag{6}$$

Afterwards, the substitution of (6) into expressions (2) leads to

$$Y_1^2 = \mathcal{P}i \quad i \in \{1, 2, 3\}, \tag{7}$$

where p_i are second-degree polynomials in Z_i.

Finally, the substitution of expressions (7) into expressions (3)–(5) results in

$$c_1Z_2^2 + c_2Z_3^2 + c_3Z_2^2Z_3 + c_4Z_2Z_3^2 + c_5Z_2Z_3 + c_6Z_2 + c_7Z_3 + c_8 = 0 \tag{8}$$

$$d_1Z_1^2 + d_2Z_3^2 + d_3Z_1^2Z_3 + d_4Z_1Z_3^2 + d_5Z_2Z_3 + d_6Z_1 + d_7Z_3 + d_8 = 0 \tag{9}$$

$$e_1Z_1^2 + e_2Z_2^2 + e_3Z_1^2Z_2 + e_3Z_1^2Z_2 + e_4Z_1Z_2^2 + e_5Z_1Z_2 + e_6Z_1 + e_7Z_2 + e_8 = 0 \tag{10}$$

where c, d, and e are coefficients that are calculated accordingly to the parameters and generalized coordinates of the parallel manipulator.

Expressions (8–10) represent a non-linear system of three equations in three unknowns given by $\{Z_1, Z_2, Z_3\}$. These equations are similar to those derived in Tsai [13], however their deduction is simpler due to the inclusion, in this contribution, of expressions (1).

Please note that only two of the unknowns are present in Eqs. 8, 9, and 10, and therefore their solution appears to be an easy task. In fact, Z_2 and Z_3 can be obtained as functions of Z_1 from (10) and (9), afterwards the substitution of these variables into Eq.(8) yields a higher non-linear equation in Z_1. However, the handling of such a expression is a formidable an unpractical task. Thus, an appropriated strategy is required for solving the system of equations at hand. Some options are

1. A numerical technique such as the Newton-Raphson method. It is an effective option, however only one and imperfect solution can be computed, and there are not guarantee that all the solutions will be calculated.
2. Using computer algebra like Maple©. An absolutely viable option that guarantee the computation of all the possible solutions.
3. The application of the Sylvester dialytic elimination method. An elegant option that allows to compute all the possible solutions.

In this contribution the last option was selected and in what follows the results will be presented.

With the purpose to eliminate Z_3, expressions (8) and (9) are rewritten as follows

$$\mathcal{P}_1Z_3^2 + \mathcal{P}_2Z_3 + \mathcal{P}_3 = 0 \tag{11}$$

$$\mathcal{P}_4Z_3^2 + \mathcal{P}_5Z_3 + \mathcal{P}_6 = 0 \tag{12}$$

where $\mathcal{P}_i \in \{1, 2, 3\}$ are second-degree polynomials in Z_2 while $\mathcal{P}_i \in \{4, 5, 6\}$ are second-degree polynomials in Z_1.

After conducted mathematical manipulation, the term Z_3^2 is eliminated from (11) and (12). With this action, two linear equations in two unknowns, the variable Z_3 and the scalar 1, are obtained. Casting in matrix form such a expressions it follows that

$$M_1\begin{bmatrix} Z_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$

where

$$M_1 = \begin{bmatrix} \mathcal{P}_1\mathcal{P}_5 - \mathcal{P}_2\mathcal{P}_4 & \mathcal{P}_1\mathcal{P}_6 - \mathcal{P}_3\mathcal{P}_4 \\ \mathcal{P}_3\mathcal{P}_4 - \mathcal{P}_1\mathcal{P}_6 & \mathcal{P}_3\mathcal{P}_5 - \mathcal{P}_2\mathcal{P}_6 \end{bmatrix}$$

It is evident that expression (13) is valid if, and only if, det M_1=0. Thus

$$\det M_1 = \mathcal{P}_7Z_2^4 + \mathcal{P}_8Z_2^3 + \mathcal{P}_9Z_2^2 + \mathcal{P}_{10}Z_2 + \mathcal{P}_{11} = 0 \tag{14}$$

where $\mathcal{P}_i i \in \{7, 8, 9, 10, 11\}$ are fourth-degree polynomials in Z_1, and the first step of the Sylvester dialytic elimination method finishes with the computation of this eliminant.

In order to solve (14) it is necessary to take into proper account Eq. 5. Clearly, this equation can be rewritten as follows

$$\mathcal{P}_{12}Z_2^2 + \mathcal{P}_{13}Z_2 + \mathcal{P}_{14} = 0 \tag{15}$$

where $\mathcal{P}_i i \in \{12, 13, 14\}$ are second-degree polynomials in Z_1. It is very tempting to assume that the non-linear system of two equations formed by (14) and (15) in two unknowns, Z_1 and Z_2, can be easily solved obtaining first Z_2 in terms of Z_1 from Eq. 15 and later substituting it into Eq. 14. However, when one realize this apparent evident action with the aid of computer algebra, an excessively long expression is derived, and its handling is a hazardous task. Thus, the application of the Sylvester dialytic elimination method is a more viable option, however such a procedure requires of a little of experience for obtaining the corresponding coefficients.

As an initial step, in order to avoid extraneous roots, it is strongly advisable the deduction of a minimum of linear equations. For example, the term Z_2^4 is eliminated multiplying Eq. 14 by $\mathcal{P}_{12}$ and Eq. 15 by $\mathcal{P}_7 Z_2^2$. The subtraction of the obtained expressions leads to

$$(\mathcal{P}_{13}\mathcal{P}_7 - \mathcal{P}_{12}\mathcal{P}_8)Z_2^3 + (\mathcal{P}_{14}\mathcal{P}_7 - \mathcal{P}_{12}\mathcal{P}_9)Z_2^2 - \mathcal{P}_{12}\mathcal{P}_{10}Z_2 - \mathcal{P}_{12}\mathcal{P}_{11} = 0. \tag{16}$$

Thus, expressions (15) and (16) represent a linear system of two equations in four unknowns, $\{Z_2^3, Z_2^2, Z_2, 1\}$ and therefore it is necessary the search of two additional linear equations. An equation is easily obtained multiplying Eq. 15 by Z_2. In fact

$$\mathcal{P}_{12}Z_2^3 + \mathcal{P}_{13}Z_2^2 + \mathcal{P}_{14}Z_2 = 0. \tag{17}$$

The search of the fourth equation is more elusive, for details the reader is referred to Tsai [13]. To this end, multiply (14) by $\mathcal{P}_{12}Z_2 + \mathcal{P}_{13}$ and (15) by $\mathcal{P}_7 Z_2^3 + \mathcal{P}_8 Z_2^2$ and substract the resulting expressions to finally obtain

$$(\mathcal{P}_{12}\mathcal{P}_9 - \mathcal{P}_7\mathcal{P}_{14})Z_2^3 + (\mathcal{P}_{12}\mathcal{P}_{10} + \mathcal{P}_{13}\mathcal{P}_9 - \mathcal{P}_9\mathcal{P}_{14})Z_2^2 + (\mathcal{P}_{12}\mathcal{P}_{11} + \mathcal{P}_{13}\mathcal{P}_{10})Z_2 + \mathcal{P}_{13}\mathcal{P}_{11} = 0. \tag{18}$$

Casting in matrix form expressions (15)–(18) it follows that

$$M_2 \begin{bmatrix} Z_2^3 \\ Z_2^2 \\ Z_2^1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

where

$$M_2 = \begin{bmatrix} 0 & \mathcal{P}_{12} & \mathcal{P}_{13} & \mathcal{P}_{14} \\ \mathcal{P}_{13}\mathcal{P}_7 - \mathcal{P}_{12}\mathcal{P}_8 & \mathcal{P}_{14}\mathcal{P}_7 - \mathcal{P}_{12}\mathcal{P}_9 & -\mathcal{P}_{12}\mathcal{P}_{10} & -\mathcal{P}_{12}\mathcal{P}_{11} \\ \mathcal{P}_{12} & \mathcal{P}_{13} & \mathcal{P}_{14} & 0 \\ \mathcal{P}_{12}\mathcal{P}_9 - \mathcal{P}_7\mathcal{P}_{14} & \mathcal{P}_{12}\mathcal{P}_{10} + \mathcal{P}_{13}\mathcal{P}_9 - \mathcal{P}_8\mathcal{P}_{14} & \mathcal{P}_{12}\mathcal{P}_{11} + \mathcal{P}_{13}\mathcal{P}_{10} & \mathcal{P}_{13}\mathcal{P}_{11} \end{bmatrix}.$$

Once again, expression (19) is valid if, and only if, det M_2=0. This eliminant yields a sixteenth-order polynomial in the unknown Z_1. It is worth mentioning that expressions (14) and (15) have the same structure of those derived by Innocenti and Parenti-Castelli [14] for solving the forward position analysis of the Stewart platform mechanism. However, this work differs from that contribution in that, while in this contribution the application of the Sylvester dialytic elimination method finishes with the computation of the determinant of a 4×4 matrix, the contribution of Innocenti and Parenti-Castelli [14] finishes with the computation of the determinant of a 6×6 matrix.

Once Z_1 is calculated, Z_2 and Z_3 are calculated, respectively, from expressions (15) and (12). Afterwards, the remaining components of the coordinates, Y_i and X_i, are computed directly from expressions (7) and (6). It is important to mention that in order to determine the feasible values of the coordinates of the points P_i, the signs of the corresponding discriminants of Z_2, Z_3 and Y_i must be taken into proper account. Of course, We can ignore this last recommendation if the non-linear system equations formed by expressions (8–10), is solved by means of computer algebra like Maple©.

Finally, once the coordinates of the points P_1, P_2, and P_3 are calculated, the geometric center of the moving platform expressed in the reference frame XYZ, vector $\mathbf{r}_{C/O}$, results in

$$\mathbf{r}_{C/O} = (P_1 + P_2 + P_3)/3. \tag{20}$$

3 Velocity analysis

In this section the velocity analysis of the 3-RPS parallel manipulator is carried out using the theory of screws which is isomorphic to the Lie algebra *e*(3). This section applies well known screw theory; however, for readers unfamiliar with this mathematical resource, some appropriated reference are provided at the end of this work.

The mechanism under study is a spatial mechanism; thus, the Lie algebra involved requires that *dime*(3)=6. In order to satisfy the dimension of the subspace spanned by the screw system for each limb, the 3-RPS parallel manipulator can be modelled as a 3-R*RPS parallel manipulator, see Huang [11], in which the revolute joints R* are fictitious kinematic pairs where the corresponding joint rates are all equal to zero.

Let ω=(ω_X, ω_Y, ω_Z) the angular velocity of the moving platform, w.r.t. fixed platform, and v_O=(v_{OX}, v_{OY}, v_{OZ}) the translational velocity of the point *O*, see Fig. 3, where both three-dimensional vectors are expressed in the reference frame XYZ, see Fig. 3. Then, the velocity state $\boldsymbol{V}_O$=(ω, $\boldsymbol{v}_O$), also known as the twist about a screw, of the moving platform w.r.t. fixed platform, can be written, see Sugimoto [15], through each one of the limbs as follows

$$_0\omega_1^{i0}\$_i^1 +_1 \omega_2^{i1}\$_i^2 +_2 \omega_3^{i2}\$_i^3 +_4 \omega_5^{i4}\$_i^5 +_5 \omega_6^{i5}\$_i^6 = V_O \quad i \in \{1,2,3\} \tag{21}$$

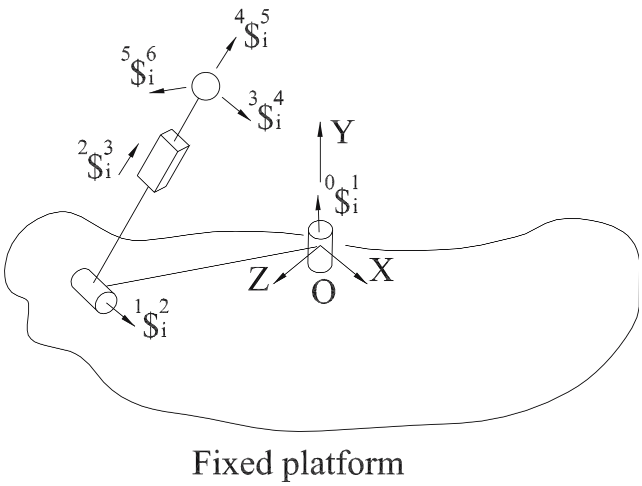

Fig. 3 A limb with its infinitesimal screws

where, in particular, ${}_1\omega_2^i = \dot{q}_i$ is the joint rate velocity of the i-*th* actuated prismatic joint, while ${}_0\omega_1^i = 0$ is the joint rate velocity of the i-*th* imaginary revolute joint, in the same limb. Furthermore, in order to simplify the computation of the Plücker coordinates of the fictitious kinematic pairs, these are placed at the origin O of the reference frame XYZ with their axes along the Y axis.

With these considerations in mind, in what follows the inverse and forward velocity analyses of the mechanism under study are easily obtained using the theory of screws.

The inverse velocity analysis consists of finding the joint rate velocities of the parallel manipulator, given the velocity state of the moving platform w.r.t. fixed platform. Accordingly to expression (21), it follows that

$$\Omega_i = J_i^{-1} V_O \tag{22}$$

therein $J_i = \left[{}^0\$_i^1 \quad {}^1\$_i^2 \quad {}^2\$_i^3 \quad {}^4\$_i^5 \quad {}^5\$_i^6\right]$ is the i-*th* Jacobian matrix of the corresponding limb and $\Omega_i = \left[{}_0\omega_1^i \quad {}_1\omega_2^i \quad {}_2\omega_3^i \quad {}_3\omega_4^i \quad {}_4\omega_5^i \quad {}_5\omega_6^i\right]^T$ is the i-*th* matrix of joint rate velocities.

On the other hand, the forward velocity analysis consists of finding the velocity state of the moving platform, w.r.t. the fixed platform, given the joint rate velocities $\left\{\dot{q}_1, \dot{q}_2, \dot{q}_3\right\}$. In this analysis the Klein form of the Lie algebra $e(3)$ plays a central role.

Given two elements $\$_1 = (\widehat{s}_1, \mathbf{s}_{O1})$ and $\$_2 = (\widehat{s}_2, \mathbf{s}_{O2})$ of the Lie algebra $e(3)$, the Klein form, {*;*}, is defined as follows

$$\{\$_1; \$_2\} = \widehat{s}_1 \cdot s_{O2} + \widehat{s}_2 \cdot s_{O1} \tag{23}$$

where the dot denotes the usual inner product operation of the three dimensional vectorial algebra.

Applying the Klein form of the screw ${}^4\$_i^5$ to both sides of expression (21), and taking into account that ${}_0\omega_1^i = 0$, the reduction of terms leads to

$$\left\{V_O; {}^4\$_i^5\right\} = \dot{q}_i \quad i \in \{1, 2, 3\}. \tag{24}$$

A similar result is obtained chosen the screw ${}^3\$_i^4$ as the *cancellator screw*. Indeed

$$\left\{V_O; {}^3\$_i^4\right\} = 0 \quad i \in \{1, 2, 3\}. \tag{25}$$

Casting in matrix form expressions (24) and (25), the velocity state $\boldsymbol{V}_O$ can be calculated from the expression

$$(J\Delta)^T V_O = \dot{Q} \tag{26}$$

wherein $J = \left[{}^4\$_1^5 \quad {}^4\$_2^5 \quad {}^4\$_3^5 \quad {}^3\$_1^4 \quad {}^3\$_2^4 \quad {}^3\$_3^4\right]$ is the Jacobian matrix of the manipulator $\dot{Q} = \left[\dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3 \quad 0 \quad 0 \quad 0\right]^T$ is the matrix of generalized joint rate velocities $\Delta = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}$ is an operator of polarity defined by the identity matrix I_3.

Finally, once the angular velocity of the moving platform and the translational velocity of the point O fixed at it are calculated, the translational velocity of the center of the moving platform, $\mathbf{v}_C$, is calculated using classical kinematics. Indeed

$$\mathbf{v}_C = \mathbf{v}_O + \omega \times \mathbf{r}_{C/O} \tag{27}$$

4 Acceleration analysis

In this section the acceleration analysis of the parallel manipulator is carried out by means of the theory of screws.

Let $\dot{\omega} = (\dot{\omega}_X, \dot{\omega}_Y, \dot{\omega}_Z)$ the angular acceleration of the moving platform, w.r.t. fixed platform, and $a_O = (a_{OX}, a_{OY}, a_{OZ})$ the translational acceleration of the point O, where both three-dimensional vectors are expressed in the reference frame XYZ, see Fig. 3. The reduced acceleration state $\mathbf{A_O} = (\dot{\omega}, \mathbf{a_O} - \omega \times \mathbf{v_O})$, or accelerator for brevity, of the moving platform w.r.t. fixed platform can be written, for details see Rico and Duffy [16], through each one of the limbs as follows

$$\mathbf{A}_0 = {}_0\dot{\omega}_1^0\$^1 + {}_1\dot{\omega}_2^1\$^2 + {}_2\dot{\omega}_3^2\$^3 + {}_3\dot{\omega}_4^3\$^4 + {}_4\dot{\omega}_5^4\$^5 + {}_5\dot{\omega}_6^5\$^6 + \$_{Lie_i} \tag{28}$$
$$i \in \{1, 2, 3\}$$

where $\$_{Lie_i}$ is the i-*thLie screw*, which is calculated as follows

$$\begin{aligned}\$_{Lie_i} =& \left[{}_0\omega_1^0\$^1 \quad {}_1\omega_2^1\$^2 + \ldots +_5 \omega_6^5\$^6\right] \\ &+ \left[{}_1\omega_2^1\$^2 \quad {}_2\omega_3^2\$^3 + \ldots +_5 \omega_6^5\$^6\right] \\ &+ \ldots + \left[{}_4\omega_5^4\$^5 \quad {}_5\omega_6^5\$^6\right]\end{aligned}$$

and the brackets $[* \quad *]$ denote the Lie product.

Following the trend of Sect. 3, the inverse acceleration analysis, or in other words the computation of the joint rate accelerations of the parallel manipulator given the accelerator of the moving platform w.r.t. fixed platform, can be calculated, accordingly to expression (28), as follows

$$\dot{\Omega}_i = J_i^{-1}(\mathbf{A}_O - \$_{Lie_i}), \tag{29}$$

where $\dot{\Omega}_i = \left[{}_0\dot{\omega}_1^i \quad {}_1\dot{\omega}_2^i \quad {}_2\dot{\omega}_3^i \quad {}_4\dot{\omega}_5^i \quad {}_5\dot{\omega}_6^i\right]^T$.

Whereas the forward acceleration analysis, or in other words the computation of the accelerator of the moving platform w.r.t. fixed platform given the joint rate accelerations $\left\{\ddot{q}_1, \ddot{q}_2, \ddot{q}_3\right\}$ of the actuated prismatic joints, are calculated by means of the expression

$$(J\Delta)^T\mathbf{A}_O = \ddot{Q} \tag{30}$$

where

$$\ddot{Q} = \begin{bmatrix} \ddot{q}_1 + \left\{{}^4\$_1^5; \$_{Lie_1}\right\} \\ \ddot{q}_2 + \left\{{}^4\$_2^5; \$_{Lie_2}\right\} \\ \ddot{q}_3 + \left\{{}^4\$_3^5; \$_{Lie_3}\right\} \\ \left\{{}^3\$_1^4; \$_{Lie_1}\right\} \\ \left\{{}^3\$_2^4; \$_{Lie_2}\right\} \\ \left\{{}^3\$_3^4; \$_{Lie_3}\right\} \end{bmatrix}$$

It is interesting to mention that Eq. 30 does not require the values of the passive joint rate accelerations of the parallel manipulator.

Finally, the translational acceleration of the moving platform, vector $\mathbf{a}_C$, expressed in the reference frame XYZ results in

$$\mathbf{a}_C = \mathbf{a}_O + \omega \times \left(\omega \times \mathbf{r}_{C/O}\right) + \dot{\omega} \times \mathbf{r}_{C/O} \tag{31}$$

where the translational acceleration $\mathbf{a}_O$ is calculated from the dual part of the accelerator, $\mathbf{A}_O$, as follows

$$\mathbf{a}_O = D(\mathbf{A}_O) + \omega \times \mathbf{v}_O.$$

5 Case study. Numerical example

In this section a numerical example, using SI units, is solved with the aid of computer codes.

The unitary vectors $\widehat{u}_i$ are chosen as follows

$$\begin{aligned}\widehat{u}_1 &= (1, 0, 0) \\ \widehat{u}_2 &= (-0.5, 0, -0.866) \\ \widehat{u}_3 &= (-0.5, 0, 0.866)\end{aligned}$$

while the vectors $\mathbf{B}_i$ are given by

$$\begin{aligned}B_1 &= (0, 0, 2.0) \\ B_2 &= (1.5, 0, -1.0) \\ B_3 &= (-1.75, 0, -1.0)\end{aligned}$$

on the other hand the moving platform has the following dimensions

$$a_{12} = 2.0 \quad a_{13} = 1.5 \quad a_{23} = 2.0$$

whereas the generalized coordinates are chosen as follows

Table 1 The feasible coordinates of the spherical joints

P_1, P_2, P_3
Sol. 1 (0., 1.138655536 i, −3.128014862) (15.08245784, 14.86543493 i, −8.842065726) (−17.85236191, 18.04078203 i, −10.29697570)
Sol. 2 (0., 17.75590589 i, 20.44646833) (16.20218815, 16.22374830 i, −9.488561286) (−16.45136447, 16.36849769 i, −9.488085723]
Sol. 3 (0., 21.89006684 i, 24.45384213) (−2.934387619, 1.103981869 i, 1.560269988) −14.27560916, 13.74557801 i, −8.231875959)
Sol. 4 (0., 3.579698670 i, −4.149328628) (−3.690328142, 3.304522427 i, 1.996725255) −6.188370085, 2.452731251 i, −3.562569333
Sol. 5 (0., 4.562–0.479 i, −0.304–0.948 i) (−1.643+0.144 i, 3.447+0.175 i,0.814–0.083 i) (−1.660–0.312 i, 4.513+0.008 i, −0.948–0.180 i)
Sol. 6 (0., 4.562+0.479 i, −0.304+0.948 i) (−1.643–0.144 i, 3.447–0.175 i,0.814+0.083 i) (−1.660+0.312 i, 4.513–0.008 i,−0.948+0.180 i)
Sol. 7 (0., 4.861291225, 0.8304498164) (1.152600116, 4.983882229, −0.7994226997) (−0.6930235387, 4.331325336, −0.3897364542)
Sol. 8 (0., 3.565519210, −1.505292108) (1.240471996, 4.991011150, −0.8501570417) (−0.5482428734, 4.280694434, −0.3061448461)
Sol. 9 (0., 3.688434531, −1.375714844) (1.455855068, 4.999740152, −0.9745121637) (0.1504572482, 3.928648499, 0.09726169066)
Sol. 10 (0., 4.909208763, 1.051491000) (−1.490603781, 3.615919956, 0.7266765480) (0.4549873986, 3.710428415, 0.2730874126)
Sol. 11 (0., 4.114835463, −0.8404452309) (−1.319598476, 3.794688888, 0.6279436931) (0.6499629272, 3.545437155, 0.3856598887)
Sol. 12 (0., 4.833152888, 0.7191279669) (1.234355126, 4.990582032, −0.8466253617) (0.6676262098, 3.529399906, 0.3958580888)
Sol. 13 (0., 19.77956134 i, 22.40174127) (15.04836408, 14.82389339 i, −8.822381107) (2.318048019, 1.347470430 i, 1.348757517)
Sol. 14 (0., 3.349063202 i, −4.017991719) (−3.805785544, 3.540578701 i, 2.063386573) (2.848851993, 2.819515293 i, 1.655226324)
Sol. 15 (0., 3.012146316 i, −3.837210415) (6.760025091, 3.448336756 i, −4.036965988) (3.006854961, 3.149709196 i, 1.746452056)
Sol. 16 (0., 3.913964557 i, 8.349733739) (−3.620476000, 3.155874741 i, 1.956394919) 3.023486763, 3.183082382 i, 1.756054713)

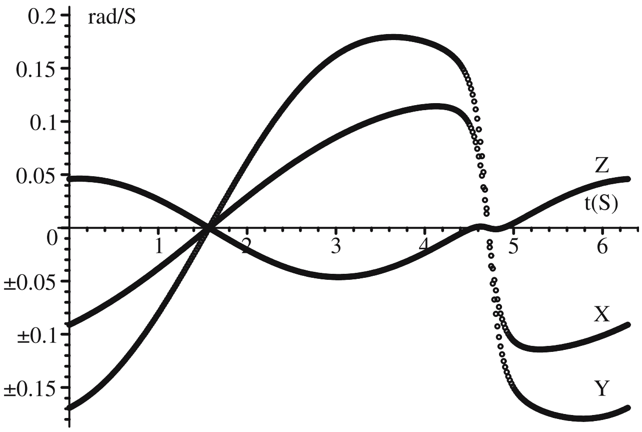

Fig. 4 Time history of the angular velocity of the moving platform

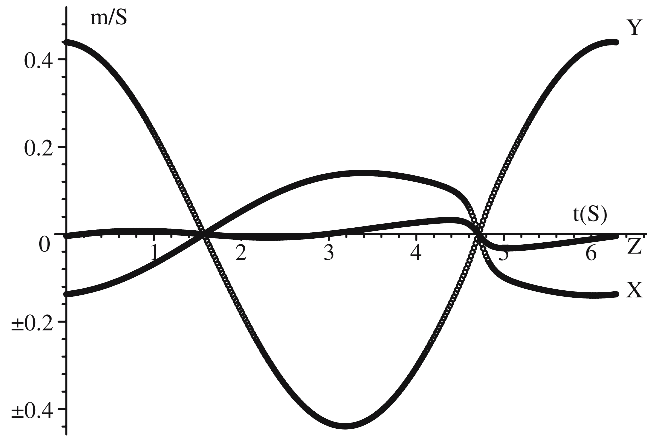

Fig. 6 Time history of the translational velocity of the center of the moving platform

$$
\begin{aligned}
q_1 &= 5.0 + 0.25 \sin t, \\
q_2 &= 5.0 + 0.50 \sin t, \\
q_3 &= 4.5 + 0.35 \sin t. \\
&\quad 0 \le t \le 2\pi
\end{aligned}
$$

Thus, the moving platform begins its motion at the time t=0 and 2π seconds later returns to its original pose.

With these data the sixteenth polynomial in Z_1 results in

$$
\begin{aligned}
&- 1.359431245 - .631756980 Z_1 + 3.486893135 Z_1^2 + 3.695332610 Z_1^3 \\
&- .187738864 Z_1^4 - 4.832016789 Z_1^5 - 4.948034544 Z_1^6 + .200209266 Z_1^7 \\
&+ 2.781561234 Z_1^8 + 1.44465504 Z_1^9 + .209159657 Z_1^{10} - .259999949e - 1 Z_1^{11} \\
&+ .6979845384e - 2 Z_1^{12} + .2060075678e - 3 Z_1^{13} + .604272936e - 4 Z_1^{14} \\
&- .393584570e - 5 Z_1^{15} + .6694771747e - 7 Z_1^{16} = 0.
\end{aligned}
$$

Thus, the solution of this polynomial, in combination with expressions (1)–(5), yields the 16 solutions of the forward position analysis, which are listed in Table 1.

Finally, taking the solution 8 of Table 1 as the initial configuration of the parallel manipulator, the most representative numerical results obtained for the forward velocity and acceleration analyses are shown in Figs. 4, 5, 6 and 7.

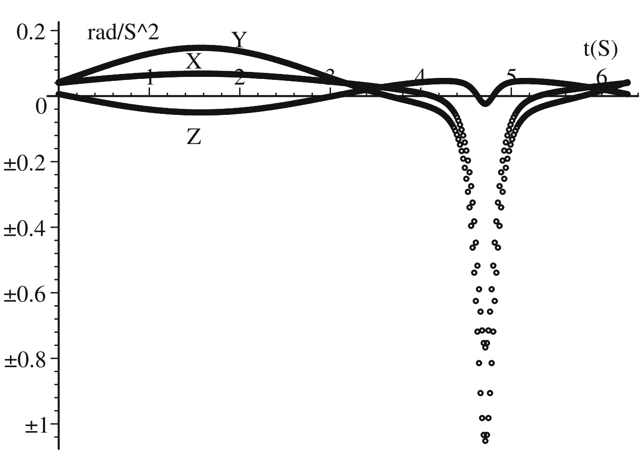

Fig. 5 Time history of the angular acceleration of the moving platform

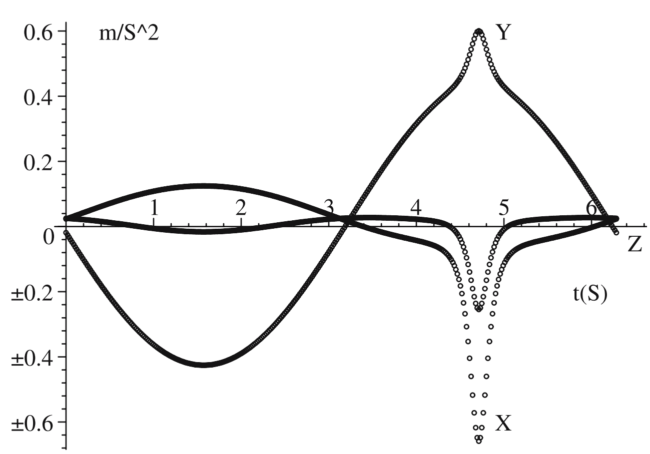

Fig. 7 Time history of the translational acceleration of the center of the moving platform

6 Conclusions

In this work the kinematics, including the acceleration analysis, of 3-RPS parallel manipulators has been successfully approached by means of screw theory. Firstly, the forward position analysis was carried out using recursively the Sylvester dialytic elimination method, such a procedure yields a 16-th polynomial expression in one unknown, thus all the possible solutions of this initial analysis are systematically calculated. Afterwards, the velocity and acceleration analyses are addressed using screw theory. To this end, the velocity and reduced acceleration states of the moving platform, w.r.t. fixed platform are written in screw form through each one of the three limbs of the manipulator. In particular, simple and compact expressions were derived in this contribution for solving the forward kinematics of the spatial mechanism by taking advantage of the concept of reciprocal screws via the Klein form of the Lie algebra $e(3)$. The obtained expressions are simple, compact and can be easily translated into computer codes. Finally, in order to exemplify the versatility of the chosen methodology, a case study was included in this work.

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