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ORIGINAL ARTICLE

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Optimal control policy for dependent process steps with over-adjusted means and variances

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Abstract Over-adjustment to processes may result in shifts in process mean and variance, ultimately affecting the quality of products. An economic adjustment model is developed for the joint design of $\overline{X} - S^2$ control charts and $\overline{e} - S_e^2$ cause-selecting control charts to control both means and variances of two dependent process steps using the Markov chain approach. The objective is to determine the optimal control policy of the proposed control charts, which effectively detect and distinguish the shifts of means and variances on the dependent process steps and minimize the total quality control cost. Application of the proposed control charts is illustrated through a numerical example.

Keywords Markov chain · Over-adjusted control charts · Renewal reward processes · Special causes

1 Introduction

Control charts are an important tool of statistical quality control. These charts are used to monitor and maintain current control of a process. Deming [1] explains that a production worker can mistakenly over-adjust or under-adjust a process. He further explains that the control chart provides "a rational and economic guide to minimize loss from both mistakes". Duncan [2] first proposed the economic design of control charts. The pioneering work of Duncan is then extended by others. A review of the literature is available in Montgomery [3] and Vance [4]. Economic design optimizes the model by considering the cost of under-adjustment along with other costs, however it assumes that the search for a special cause is perfect.

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A common problem in statistical process control is process over-adjustment. Information about the state of the process is available only through sampling. When a control chart indicates that the process is out of control, it requires adjustment. Sometimes, the process may be adjusted unnecessarily, when a false alarm occurs. Saniga [5] describes that economic design of control charts does not consider statistical properties when selecting the design parameters for a control chart. Woodall [6] noted that the effect of over-adjustment is an increase in variability. This increases in variability and resultant loss of quality can be quite significant. He describes the probability of type I error in an economic design as being much higher than that in a statistical design. This results in greater false alarm frequency, which leads to over-adjustment, and ultimately an increase in the variability of the quality characteristic. Collani et al. [7] first solve this problem through an economic adjustment model for the $\overline{X}$ control chart with a single special cause that considers the effects of process over-adjustment and under-adjustment. Their model determines the design parameters of the $\overline{X}$ control chart which maximize the profitability of the process, or equivalently minimize the cost of over-adjustment and under-adjustment. Yang and Rahim [8] propose a Makovian chain approach to derive the statistically economic adjustment model for the $\overline{X}$ and S^2 control charts that considers the effects of process mean and variance over-adjustment and under-adjustment. Yang and Yang [9] study economic statistical process control for overadjusted process means under multiple assignable causes. Today, many industrial products are produced by several dependent processes not just one process. Consequently, it is not appropriate to monitor these processes with a control chart for each individual process. How to propose an appropriate method for controlling the dependent processes is worthy of study. Zhang [10] proposes the simple cause-selecting chart to monitor the second step of the two dependent steps with samples of size one. Wade and Woodall [11] review the basic principles of the cause-selecting chart for two dependent steps and suggest a modification to the use of simple cause-selecting chart. They also examine the relationship between the simple cause-selecting chart and the multivariate T^2 control chart. In their opinion, the simple cause-selecting control chart has some advantages over the T^2 control chart. Yang [12] extends the overadjusted process mean problem to two dependent processes by considering a single special cause. Yang and Yang [13] apply the approach of controlling over-adjusted process mean on two dependent processes to bank management. However, the effect of over-adjusted process mean and variance on dependent processes has not been addressed. From the viewpoint six sigma of GE company, the variance is important since feeling of quality is variance but not mean for any customer.

This paper considers that the dependent incoming quality of the first step and the out-going quality of the second step, which can be affected by two special causes resulting in shifts in the process mean and variance due to over-adjustment during operation. The control $\overline{X}-S^2$ charts and $\overline{e}-S_e^2$ cause-selecting control charts are proposed to signal the special causes on the first step and the second step, which result in a shift of the process mean and variance on the first step and the second step of the process. A Markovian chain approach is extended. The proposed approach allows easier derivation of the expected cycle time and the expected cycle cost than that of others. The paper is organized as follows. In the next section, the optimal adjustment model is derived using a Markov chain approach. An optimization technique is used to determine the optimal control policy for using the $\overline{X}-S^2$ control charts and $\overline{e}-S_e^2$ cause-selecting control charts which minimize the cost of the production process. An example is provided to illustrate an application of the $\overline{X}-S^2$ control charts and $\overline{e}-S_e^2$ cause-selecting control charts. A brief summary concludes the paper.

2 Optimal adjustment model

2.1 Problem statement

In a production system, suppose that there are two dependent steps of a process, which may have two failure mechanisms. If the processes experience at least a failure mechanism, it goes out of control, otherwise it is in control. One failure mechanism may occur only in the first process and shifts the mean and variance of the quality variable (X) and another failure mechanism may occur only in the second process and shifts the mean and variance of the quality variable (Y). The in-control process becomes out-of-control if it is over-adjusted. The over-adjustment means the operator adjusted the process when adjustment was unnecessary. The out-of-control process keeps out-of-control if it is under-adjusted. The under-adjustment means the operator did not adjust the process when adjustment was necessary. The quality variable Y is influenced by the quality variable X since the two steps are dependent, and the variation of Y is increasing when the value of X is increasing. How to distinguish and detect the shifts of process mean and variance on the two dependent steps of process? In this analysis, $\overline{X}-S^2$ charts and $\overline{e}-S_e^2$ cause-selecting control charts are used to signal the need for adjustment of the first and the second steps, respectively. The problem here is what is the optimal dependent steps control policy? That is, what are the control charts, how long to take a sample, how large the sample size, how the over-adjustment affects the performance of the process control? Specifically, a sample of size n units of output is taken every h hours, and the process is adjusted if its sample mean and sample variance fall outside the control limits of their control charts. The objective is to derive the optimal adjustment model, and to determine the parameters n, h, k_1, k_2 (control limit coefficients of the $\overline{X}-S^2$ control charts) and k_3, k_4 (control limit coefficients of $\overline{e}-S_e^2$ cause-selecting control charts), so that the average long-term cost of the two-step process is minimized, and the optimal adjustment control $\overline{X}-S^2$ charts and $\overline{e}-S_e^2$ cause-selecting control charts are proposed.

2.2 Description of the production process

When m random samples of size n are taken from the second step of a process at every sampling time interval h, we get mn pairs of observations $(x_{11}, y_{11}), (x_{12}, y_{12}), (x_{13}, y_{13}), \ldots, (x_{1n}, y_{1n}), \ldots, (x_{m1}, y_{m1}), (x_{m2}, y_{m2}), (x_{m3}, y_{m3}), \ldots, (x_{mn}, y_{mn})$. Suppose that mn pairs of observations are taken to determine the design parameters of the proposed control charts. In reality, since some manufacturing problem may cause the variation of Y increasing as the value of X increasing. The model relating the two variables (X, Y) can take the general model:

$$Y_{ij}\left|X_{ij}\right. = f\left(X_{ij}\right) + \varepsilon_{ij}\,, \quad i = 1, 2, 3, \ldots, m, \quad j = 1, 2, 3, \ldots, n\,,$$

where ε_i is a random error, $\varepsilon_i \sim NID(0, \sigma_\varepsilon^2)$.

To monitor the two dependent steps of a process effectively, two control charts are constructed to control the mean and variance on the first step and the second step, respectively. To monitor the first step, the $\overline{X}-S^2$ control charts are set up based on the in-control distribution of statistics $\overline{X}$ and S^2. To monitor the second step, the specific quality of the second step can be specified by adjusting the effect of X on Y, that is the specific quality is presented by the average cause-selecting values (or average residuals), where $\overline{e}_i = \sum_{j=1}^{n} e_{ij}/n$ and $e_{ij} = Y_{ij}|X_{ij} - \hat{Y}_{ij}|X_{ij}$. Consequently, the $\overline{e}-S_e^2$ cause-selecting control chart is set up based on the in-control distributions of statistics average cause-selecting values and variance cause-selecting values.

Assume that when the first step and the second step are all in control, $\overline{X} \sim \mathrm{N}(\mu, \sigma_X^2/n)$ and $\overline{e}_i \sim \mathrm{N}(0, \sigma_e^2/n)$. When a special cause SC_1 occurs, there would be a shift in the in-control distribution of $\overline{X}$ to $\overline{X} \sim \mathrm{N}(\mu + \delta_1\sigma_X/\sqrt{n}, \delta_2^2\sigma_X^2/n)$, where $\delta_1 \neq 0, \delta_2 > 1$. When a special cause SC_2 occurs, there would be a shift in the in-control distribution of $\overline{e}$ to $\overline{e} \sim \mathrm{N}(\mu + \delta_3\sigma_e/\sqrt{n}, \delta_4^2\sigma_e^2/n)$, $\delta_3 \neq 0, \delta_4 > 1$. The time until occurrence of a special cause SC_i is assumed to be exponentially distributed with a mean of $1/\lambda_i$, $i = 1, 2$. It is also assumed that the process is not self-correcting, and the time to sampling and plot $(\overline{x}, s^2)$ and $(\overline{e}, s^2)$ is negligible.

An adjustment to the process is performed if the sampled values $(\overline{x}, s^2)$ and/or $(\overline{e}, s_e^2)$ fall outside the control limits of the $\overline{X}-S_e^2$ control charts and/or $\overline{e}-S_e^2$ cause-selecting control charts, respectively $\mathrm{LCL}_{\overline{X}}$, $\mathrm{UCL}_{\overline{X}}$, LCL_{S^2}, $\mathrm{UCL}_{\overline{e}}$, $\mathrm{LCL}_{\overline{e}}$, and $\mathrm{UCL}_{s_e^2}$, where

$$\mathrm{LCL}_{\overline{X}} = \mu - k_1\sigma_X/\sqrt{n}$$
$$\mathrm{UCL}_{\overline{X}} = \mu + k_1\sigma_X/\sqrt{n}$$
$$\mathrm{UCL}_{S^2} = k_2\sigma_X^2$$
$$\mathrm{LCL}_{\overline{e}} = k_3\sigma_e/\sqrt{n}$$
$$\mathrm{UCL}_{\overline{e}} = -k_3\sigma_e/\sqrt{n}$$
$$\mathrm{UCL}_{s_e^2} = k_4\sigma_e^2 .$$

The process correct-adjustment and over-adjustment can take one of the forms following the alarm from $\overline{X}-S^2$ charts or $\overline{e}-S_e^2$ cause-selecting charts:

- When the shift results in $\overline{X} \sim \mathrm{N}(\mu+\delta_1\sigma_X/\sqrt{n}, \delta_2^2\sigma_X^2/n)$ and $\overline{X}$ chart (or S^2 chart or $\overline{X}-S^2$ charts) has an alarm, the special cause SC_1 is adjusted to let the mean of $\overline{X}$ be μ and the variance of $\overline{X}$ be σ_X^2/n.
- When the shift results in $\overline{e} \sim \mathrm{N}(\mu+\delta_3\sigma_e/\sqrt{n}, \delta_4^2\sigma_e^2/n)$ and $\overline{e}$ cause-selecting chart (or S_e^2 chart or $\overline{e}-S_e^2$ charts) has an alarm, the special cause SC_2 is adjusted to let the mean of $\overline{e}$ be 0.
- When the process is in control but only $\overline{X}$ (or S^2 chart or $\overline{X}-S^2$ charts) chart has alarm, the first step is over-adjusted to let $\overline{X} \sim \mathrm{N}(\mu+\delta_1\sigma_X/\sqrt{n}, \delta_2^2\sigma_X^2/n)$.
- When the process is in control but only $\overline{e}$ cause-selecting chart (or S_e^2 chart or $\overline{e}-S_e^2$ charts) has alarm, the second step is over-adjusted to let $\overline{e} \sim \mathrm{N}(\mu+\delta_3\sigma_e/\sqrt{n}, \delta_4^2\sigma_e^2/n)$.
- When the process is in control but both $\overline{X}-S^2$ charts and $\overline{e}-S_e^2$ cause-selecting charts have alarms, the first step is over-adjusted to let $\overline{X} \sim \mathrm{N}(\mu+\delta_1\sigma_X/\sqrt{n}, \delta_2^2\sigma_X^2/n)$ and the second step is over-adjusted to let $\overline{e} \sim \mathrm{N}(\mu+\delta_3\sigma_e/\sqrt{n}, \delta_4^2\sigma_e^2/n)$.
- When only the first step is out of control but at least one of the $\overline{e}-S_e^2$ cause-selecting charts has an alarm, SC_2 is adjusted, the second step is over-adjusted to let $\overline{e} \sim \mathrm{N}(\mu+\delta_3\sigma_e/\sqrt{n}, \delta_4^2\sigma_e^2/n)$ and the first step is unchanged.
- When only the second step is out of control but at least one of the $\overline{X}-S^2$charts has alarm, the special cause SC_1 is adjusted, the first step is over-adjusted to let $\overline{X} \sim \mathrm{N}(\mu+\delta_1\sigma_X/\sqrt{n}, \delta_2^2\sigma_X^2/n)$ and the second step is unchanged.
- When only the first step is out of control but $\overline{X}-S^2$charts and $\overline{e}-S_e^2$ cause-selecting charts have alarms, the first step is correct-adjusted to let $\overline{X} \sim \mathrm{N}(\mu, \sigma_X^2/n)$ and the second step is over-adjusted to let $\overline{e} \sim \mathrm{N}(\mu+\delta_3\sigma_e/\sqrt{n}, \delta_4^2\sigma_e^2/n)$; similar to only the second step is out of control but $\overline{X}-S^2$ charts and $\overline{e}-S_e^2$ cause-selecting charts have alarms.

The decision rule can result in an over-adjustment following false alarm for either the first step or the second step, or for both together. It is assumed that a transition in the process from in control to out of control during sampling is impossible. The following notation is used.

2.3 Defining the probabilities of over-adjustment and under-adjustment

$\alpha_{\overline{X}}$: Probability that the first step is over-adjusted when the $\overline{X}$ control chart gives a false alarm, where

$$\alpha_{\overline{X}} = 1 - \mathrm{P}(\mathrm{LCL}_{\overline{X}} \le \overline{X} \le \mathrm{UCL}_{\overline{X}} | \overline{X} \sim \mathrm{N}(\mu, \sigma_X^2/n)) = 2\Phi(-k_1)\,,$$

where $\Phi(.)$ is the cumulative probability of a normal distribution.

α_{S^2}: Probability that the first step is over-adjusted when S^2 control chart gives a false alarm, where

$$\alpha_{S^2} = 1 - \mathrm{P}(\mathrm{LCL}_{S^2} \le S^2 \le \mathrm{UCL}_{S^2} | \overline{X} \sim \mathrm{N}(\mu, \sigma_X^2/n)) = 1 - F_{X^2}((n-1)k_2)\,,$$

where $F_{X^2}(\cdot)$ is the cumulative probability of a X^2 distribution.

$\alpha_{\overline{e}}$: Probability that the second step is over-adjusted when $\overline{e}$ cause-selecting control chart gives a false alarm, where

$$\alpha_{\overline{e}} = 1 - \mathrm{P}(\mathrm{LCL}_{\overline{e}} \le \overline{e} \le \mathrm{UCL}_{\overline{e}} | \overline{e} \sim \mathrm{N}(o, \sigma_e^2/n)) = 2\Phi(-k_3)\,.$$

$\alpha_{S_e^2}$: Probability that the second step is over-adjusted when S^2 control chart gives a false alarm, where

$$\alpha_{S_e^2} = 1 - \mathrm{P}(\mathrm{LCL}_{S_e^2} \le S_e^2 \le \mathrm{UCL}_{S_e^2} | \overline{e} \sim \mathrm{N}(o, \sigma_e^2/n)) = 1 - F_{X^2}((n-1)k_4)\,,$$

where $F_{X^2}(\cdot)$ is the cumulative probability of a x^2 distribution.

$\beta_{\overline{X}}$: Probability that the first step is under-adjusted since it is affected by the special cause SC_1, where

$$\beta_X = \mathrm{P}(\mathrm{LCL}_{\overline{X}} \le \overline{X} \le \mathrm{UCL}_{\overline{X}} | \overline{X} \sim \mathrm{N}(\mu+\delta_1\sigma_X/\sqrt{n}, \delta_2^2\sigma_X^2/n)) = \Phi(k_1/\delta_2 - \delta_1/\delta_2) - \Phi(-k_1/\delta_2 - \delta_1/\delta_2)\,.$$

β_{S^2}: Probability that the first step is under-adjusted since it is affected by the special cause SC_1, where

$$\beta_{S^2} = \mathrm{P}(S^2 \le \mathrm{UCL}_{S^2} | \overline{X} \sim \mathrm{N}(\mu+\delta_1\sigma_X/\sqrt{n}, \delta_2^2\sigma_X^2/n)) = F_{x^2}((n-1)k_2/\delta_2^2)\,.$$

$\beta_{\overline{e}}$: Probability that the second step is under-adjusted when it is affected by the special cause SC_2, where

$$\beta_{\overline{e}} = \mathrm{P}(\mathrm{LCL}_{\overline{e}} \le \overline{e} \le \mathrm{UCL}_{\overline{e}} | \overline{e} \sim \mathrm{N}(\delta_3\sigma_e/\sqrt{n}, \delta_4^2\sigma_e^2/n)) = \Phi(k_3/\delta_4 - \delta_3/\delta_4) - \Phi(-k_3/\delta_4 - \delta_3/\delta_4)\,.$$

$\beta_{S_e^2}$: Probability that the first step is under-adjusted since it is affected by the special cause SC_2, where

$$\beta_{S_e^2} = \mathrm{P}(S^2 \le \mathrm{UCL}_{S_e^2} | \overline{e} \sim \mathrm{N}(\delta_3\sigma_e/\sqrt{n}, \delta_4^2\sigma_e^2/n)) = F_{x^2}((n-1)k_4/\delta_4^2)\,.$$

2.4 Defining the terms associated with times and costs

T_f : Expected time of over-adjustment following a false alarm.
T_{sc_i} : Time before the special cause SC_i occurs in the process,

$$T_{\mathrm{sc}} \sim \exp(\lambda_i),\ i = 1, 2\,.$$

T_{sr} : Expected time to search and repair any special cause.
C_f : Expected cost of over-adjustment.
C_0 : Production cost per unit time when the process is in control.
C_1 : Production cost per unit time when the process is affected by the special cause SC_1.
C_2 : Production cost per unit time when the process is affected by the special cause SC_2.
C_{12} : Production cost per unit time when the process is affected by the special cause SC_1 and SC_2.
C_{sr} : Expected cost to search and repair any special cause.
a: Fixed cost per sample and test.
b: Cost per unit sampled and tested.
τ_i : Expected arrival time of the special cause, given that it occurred in the first sampling interval, where

$$\tau_i = \frac{1-(1+\lambda_i h)e^{-\lambda_i h}}{\lambda_i - \lambda_i e^{-\lambda_i h}}$$

(see Lorenzen and Vance [14]).

2.5 Description of Markov chain

In order to derive the cost function, we use the Markov chain approach to derive the expected cycle time (ET) and the expected cycle cost (EC) first. All possible states at the end of each sampling and testing time must be examined. Depending on the state of the system, the transition probabilities and transition costs can be computed. There are 64 possible states at the end of every sampling and testing time, and these states are defined as follows (Table 1).

The 64 states can be classified into two types of states: transient states and absorbing states. States 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, and 64 are absorbing states, others are transient states. Transition probability from state i to state j in time interval h is described in Appendix 1.

The transition probability matrix is denoted as $P_{11} = [P_{i,j}]$, $i, j \neq 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64$; $P_{12} = [P_{i,j}]$, $j = 18, 19, 22, 36,37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64$, $i \neq j$; zero matrix $0 = [P_{i,j}]$, $P_{i,j} = 0$ for $i = 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64$, $j \neq i$. Identity matrix $I = [P_{i,j}]$, $P_{i,j} = 1$ for $i, j = 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64$, and matrix P is the combination of submatrices P_{11}, P_{12}, I, and 0. That is

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & I \end{bmatrix}.$$

The cycle time is the time from the start of the process in control until an alarm is detected, repaired, and the process is restarted or equivalently it is the time from transient state 1 to reach any absorbing state 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, or 64. The state variable $Y_t (t = 0, h, 2h, \ldots)$ is a Markov chain on the state 1, 2, ... , 64 and so the Markov property can be effectively used to find the expected cycle time.

2.6 Expected cycle time and cost

Let random variable T_i be the time until absorption from transient state i. Then, using the Markov property and conditioning on the first step:

$$\begin{aligned}
&P(T_i = h + T_{\mathrm{sr}}) = P_{i,j} \\
&\quad \text{where } j = 18, 19, 22, 36, 37, 43, \quad i \neq j\,. \\
&P(T_i = h + 2T_{\mathrm{sr}}) = P_{i,j} \\
&\quad \text{where } j = 55, 56, 57, 58, 60, 61, 62, 63, 64, \quad i \neq j. \\
&P(T_i = h + T_f + T_j) = P_{i,j} \\
&\quad \text{where } i = 2, 3, 4, 5, 6, 7, 8, 11, 20, 21, 34, 35, 38, \\
&\qquad j = 17, 33, 49. \\
&P(T_i = h + 2T_f + T_j) = P_{i,j} \\
&\quad \text{where } i = 9, 10, 12, 13, 14, 15, 16, \quad j = 49. \\
&P(T_i = h + T_f + T_{\mathrm{sr}} + T_j) = P_{i,j} \\
&\quad \text{where } i = 23, 24, 25, 26, 28, 29, 30, 31, 32, 39, 40, 41, 42, \\
&\qquad 44, 45, 46, 47, 48, \quad j = 17, 33. \\
&P(T_i = h + T_j) = P_{i,j} \\
&\quad \text{where } i = 1, 17, 33, 49, \quad j \neq 18, 19, 22, 36, 37, 43, 55, \\
&\qquad 56, 57, 58, 60, 61, 62, 63, 64\,.
\end{aligned} \tag{1}$$

Equation 1 can be expressed in matrix form

$$\boldsymbol{M} = h\boldsymbol{1} + \boldsymbol{P}_{11}\boldsymbol{M}_{\mathrm{sr1}} + \boldsymbol{P}_{11}\boldsymbol{M} + \boldsymbol{P}_{12}\boldsymbol{M}_{\mathrm{sr2}}$$

So

$$\boldsymbol{M} = h(\boldsymbol{I}-\boldsymbol{P}_{11})^{-1}\boldsymbol{1} + (\boldsymbol{I}-\boldsymbol{P}_{11})^{-1}\boldsymbol{P}_{11}\boldsymbol{M}_{\mathrm{sr1}} + (\boldsymbol{I}-\boldsymbol{P}_{11})^{-1}\boldsymbol{P}_{12}\boldsymbol{M}_{\mathrm{sr2}}\,,$$

where $\boldsymbol{M}$ is a (49×1) vector, with the expected time up to absorption from transient state i, $i \neq 18, 19, 22, 36,37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64$.

$\boldsymbol{1}$ is a (49×1) vector, with elements 1,
$\boldsymbol{M}_{\mathrm{sr1}}$ is a (49×1) vector, $\boldsymbol{M}_{\mathrm{sr1}}^T = [0\ T_f\ T_f\ T_f\ T_f\ T_f\ T_f\ T_f\ 2T_f\ 2T_f\ T_f\ 2T_f\ 2T_f\ 2T_f\ 2T_f\ 2T_f\ 0\ T_f\ T_f\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ 0\ T_f\ T_f\ T_f\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ T_f+T_{\mathrm{sr}}\ 0\ T_{\mathrm{sr}}\ T_{\mathrm{sr}}\ T_{\mathrm{sr}}\ T_{\mathrm{sr}}\ T_{\mathrm{sr}}\ T_{\mathrm{sr}}]$,
$\boldsymbol{M}_{\mathrm{sr2}}$ is a (15×1) vector, $\boldsymbol{M}_{\mathrm{sr2}}^T = [T_{\mathrm{sr}}\ T_{\mathrm{sr}}\ T_{\mathrm{sr}}\ T_{\mathrm{sr}}\ T_{\mathrm{sr}}\ T_{\mathrm{sr}}\ 2T_{\mathrm{sr}}\ 2T_{\mathrm{sr}}\ 2T_{\mathrm{sr}}\ 2T_{\mathrm{sr}}\ 2T_{\mathrm{sr}}\ 2T_{\mathrm{sr}}\ 2T_{\mathrm{sr}}\ 2T_{\mathrm{sr}}\ 2T_{\mathrm{sr}}]$, P_{11} is defined as above.

The expected cycle time is the first element of vector $\boldsymbol{M}$, i.e., $\boldsymbol{M}_1$ or $E(T_1)$.

Table 1. Definition for each state

State	SC1 occurs?	SC2 occurs?	$\overline{X}$ chart signal?	S^2 chart signal?	$\overline{e}$ chart signal?	S_e^2 chart signal?	Process over-adjustment and which step?
1	No	No	No	No	No	No	No
2	No	No	Yes	No	No	No	First
3	No	No	No	Yes	No	No	First
4	No	No	No	No	Yes	No	Second
5	No	No	No	No	No	Yes	Second
6	No	No	Yes	Yes	No	No	First
7	No	No	Yes	No	Yes	No	Second
8	No	No	Yes	No	No	Yes	Second
9	No	No	No	Yes	Yes	No	First+Second
10	No	No	No	Yes	No	Yes	First+Second
11	No	No	No	No	Yes	Yes	Second
12	No	No	Yes	Yes	Yes	No	First+Second
13	No	No	Yes	Yes	No	Yes	First+Second
14	No	No	Yes	No	Yes	Yes	First+Second
15	No	No	No	Yes	Yes	Yes	First+Second
16	No	No	Yes	Yes	Yes	Yes	First+Second
17	Yes	No	No	No	No	No	No
18	Yes	No	Yes	No	No	No	No
19	Yes	No	No	Yes	No	No	No
20	Yes	No	No	No	Yes	No	Second
21	Yes	No	No	No	No	Yes	Second
22	Yes	No	Yes	Yes	No	No	No
23	Yes	No	Yes	No	Yes	No	Second
24	Yes	No	Yes	No	No	Yes	Second
25	Yes	No	No	Yes	Yes	No	Second
26	Yes	No	No	Yes	No	Yes	Second
27	Yes	No	No	No	Yes	Yes	Second
28	Yes	No	Yes	Yes	Yes	No	Second
29	Yes	No	Yes	Yes	No	Yes	Second
30	Yes	No	Yes	No	Yes	Yes	Second
31	Yes	No	No	Yes	Yes	Yes	Second
32	Yes	No	Yes	Yes	Yes	Yes	Second
33	No	Yes	No	No	No	No	No
34	No	Yes	Yes	No	No	No	First
35	No	Yes	No	Yes	No	No	First
36	No	Yes	No	No	Yes	No	No
37	No	Yes	No	No	No	Yes	No
38	No	Yes	Yes	Yes	No	No	First
39	No	Yes	Yes	No	Yes	No	First
40	No	Yes	Yes	No	No	Yes	First
41	No	Yes	No	Yes	Yes	No	First
42	No	Yes	No	Yes	No	Yes	First
43	No	Yes	No	No	Yes	Yes	No
44	No	Yes	Yes	Yes	Yes	No	First
45	No	Yes	Yes	Yes	No	Yes	First
46	No	Yes	Yes	No	Yes	Yes	First
47	No	Yes	No	Yes	Yes	Yes	First
48	No	Yes	Yes	Yes	Yes	Yes	First
49	Yes	Yes	No	No	No	No	No
50	Yes	Yes	Yes	No	No	No	No
51	Yes	Yes	No	Yes	No	No	No
52	Yes	Yes	No	No	Yes	No	No
53	Yes	Yes	No	No	No	Yes	No
54	Yes	Yes	Yes	Yes	No	No	No
55	Yes	Yes	Yes	No	Yes	No	No
56	Yes	Yes	Yes	No	No	Yes	No
57	Yes	Yes	No	Yes	Yes	No	No
58	Yes	Yes	No	Yes	No	Yes	No
59	Yes	Yes	No	No	Yes	Yes	No
60	Yes	Yes	Yes	Yes	Yes	No	No
61	Yes	Yes	Yes	Yes	No	Yes	No
62	Yes	Yes	Yes	No	Yes	Yes	No
63	Yes	Yes	No	Yes	Yes	Yes	No
64	Yes	Yes	Yes	Yes	Yes	Yes	No

Once the expected cycle time is obtained, the expected cycle cost must be calculated, and the economic adjustment model can be derived by taking the ratio of the expected cycle cost to the expected cycle time.

The derivation of the expected cycle cost uses the Markov property in a similar manner to that used for the expected cycle time. Let $C_{i,j}$ be the expected cumulative cost that is associated with transition from state i to j in time interval h; $i, j \neq 18$, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64. The calculation of $C_{i,j}$ is illustrated in Appendix 2.

The transition cost matrices are denoted as: $\boldsymbol{C}_{11} = [C_{i,j}]$, $i, j \neq 18$, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64; $\boldsymbol{C}_{12} = [C_{i,j}]$, $j = 18$, 19, 22, 36,37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64, $i \neq j$; zero matrix $0 = [C_{i,j}]$, $C_{i,j} = 0$ for $i = 18$, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64, $j \neq i$; $\boldsymbol{C}_{22} = C_{\mathrm{sr}}\boldsymbol{I}$, $\boldsymbol{I}$ is identity matrix for $i, j = 18$, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64, and matrix $\boldsymbol{C}$ is the combination of submatrices $\boldsymbol{C}_{11}, \boldsymbol{C}_{12}, \boldsymbol{C}_{22}$, and $\boldsymbol{0}$. That is

$$C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22}\,. \end{bmatrix}$$

The cycle cost is the cumulative cost from the start of the process, in control, until an alarm is detected, the process is repaired and re-started, or equivalently, it is the cost from transient state 1 until it reaches an absorbing state.

Let random variable C_i be the cumulative cost up to absorption from transient state $i, i \neq 18$, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64. Then using the Markov property and conditioning on the first step:

$$\begin{aligned} &P(C_i = C_{i,j}) = P_{i,j} \qquad \text{where} \\ &j = 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64, \\ &i \neq j \qquad (2) \\ &P(C_i = C_{i,j} + C_j) = P_{i,j} \qquad \text{where} \\ &i, j \neq 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64. \end{aligned}$$

Equation 2 can be expressed in matrix form

$$\boldsymbol{U} = \boldsymbol{P}_{11} * \boldsymbol{C}_{11} + \boldsymbol{P}_{11}\boldsymbol{U} + \boldsymbol{P}_{12} * \boldsymbol{C}_{12}$$

where $*$ denotes the Hadamard product of the two matrices, and $\boldsymbol{U}$ is a (49×1) vector with the expected cost up to absorption from transient state $i, i \neq 18$, 19, 22, 36,37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64. So $\boldsymbol{U} = (\boldsymbol{I} - \boldsymbol{P}_{11})^{-1}\boldsymbol{W}\boldsymbol{I}$, where $\boldsymbol{W} = [\boldsymbol{P}_{11} * \boldsymbol{C}_{11} \boldsymbol{P}_{12} * \boldsymbol{C}_{12}]$, and the first element of the vector, U_1, is the expected cycle cost.

2.7 Determination of optimal design parameters

Applying the property of renewal reward processes [15], the objective function (L), the expected cost per unit time is derived by taking the ratio of the expected cycle cost (U_1) to the expected cycle time ($\boldsymbol{M_I}$); $L = U_1/\boldsymbol{M_I}$. The expected long term loss is the function of design parameters n, h, k_1, k_2, k_3, k_4; $L(n, h, k_1, k_2, k_3, k_4)$. Hence, the optimal design parameters of the optimal adjustment design of the $\overline{X} - S^2$ control charts and $\overline{e} - S_e^2$ cause-selecting control charts can be determined by minimization of the objective function or cost model, that is minimize $L(n, h, k_1, k_2, k_3, k_4)$.

3 A numerical example

In this section, we give an example to illustrate how the proposed method is used to solve a real process control problem.

A quality engineer found that there is a large variability for the thickness of the thin golden films. From the quality data analysis, he found that the thickness of the thin golden films (Y) in the second process step was primarily affected by gold concentration (X) in the first process, and the variation of thickness increases as concentration increases. Two independent filling machines, say machine 1 and machine 2, may fail and influence the means and variances of the gold concentration and thickness respectively. Since the unacceptable means and variance of the thickness may be influenced by filling machine 1 or gold concentration. To effectively maintain the variability of the gold concentration and thickness and distinguish which process step is out of control, four control charts are constructed as described before.

The thickness of the thin golden films can be obtained from knowledge of gold concentration, so their relationship can be found by analysis of history data. In the history data, 40 paired in-control data is collected. The following represents the computed regression: $\hat{Y}|X = 141.6$–$0.124X$. Consequently, the in-control distributions of X and e (also called cause-selecting value) are illustrated as follows.

$$X \sim \mathrm{N}(52, 3015)$$
$$e \sim \mathrm{N}(0, 1.5)$$

The control limits of the optimal $\overline{X} - S^2$ charts and $\overline{e} - S_e^2$ control charts are constructed as follows.

$$\begin{aligned} &\mathrm{LCL}_{\overline{X}} = 52 - 54.91k_1/\sqrt{n} \\ &\mathrm{UCL}_{\overline{X}} = 52 + 54.91k_1/\sqrt{n} \\ &\mathrm{UCL}_{S^2} = 3015k_2 \\ &\mathrm{LCL}_{\overline{e}} = k_3/\sqrt{n} \\ &\mathrm{UCL}_{\overline{e}} = -k_3/\sqrt{n} \\ &\mathrm{UCL}_{s_e^2} = 1.5k_4 \end{aligned}$$

In the production process, machine 1 could be out-of-control in the first step, and machine 2 could be out-of-control in the second step. Since the machines do not tend to deteriorate with time. It is of prime concern in process control to be able to distinguish in which one of the process steps the out-of-control situation occurs. An out-of-control situation occurring in the first step would cause the mean and variance of the $\overline{X}$ distribution to change or result in shifts in the process mean and variance of $\overline{X}$ distribution due to over-adjustment during operation. An out-of-control situation in the second step would cause the mean and variance

Table 2. Decision rules

Combinations	$\overline{X}-S^2$ charts signal?	$\overline{e}-S_e^2$ cause-selecting charts signal?	Which step stop?
1	No	No	No
2	Yes	No	First, adjust machine 1
3	No	Yes	Second, adjust machine 2
4	Yes	Yes	First and second, adjust Machine1 and machine 2

of the $\overline{e}$ distribution to change or result in shifts in the process mean and variance of $\overline{e}$ distribution due to over-adjustment during operation.

The $\overline{X}-S^2$ charts and $\overline{e}-S_e^2$ cause-selecting charts which minimize cost of over-adjustment and under-adjustment are constructed to monitor the two process steps effectively. To determine the optimal design parameters of the $\overline{X}-S^2$ charts and $\overline{e}-S_e^2$ cause-selecting charts, the process and cost parameters are estimated as follows.

$\delta_1=2$, $\delta_2=2.5$, $\delta_3=2.5$, $\delta_4=2$, $\lambda_1=0.05$, $\lambda_2=0.01$, $a=\$\,0.5$, $b=\$\,0.1$, $C_f=\$\,10$, $C_{sr}=\$\,35$, $T_f=0.1$ (hours), $T_{sr}=0.5$ (hours), $C_0=\$\,25$, $C_1=\$\,40$, $C_2=\$\,50$.

The algorithm used to obtain the approximate optimum values $(n^*, h^*, k_1^*, k_2^*, k_3^*)$ of the design values (n, h, k_1, k_2, k_3), with constraints $0<k_1, k_2, k_3, k_4<6$, $1<n\le 25$, $0<h\le 8$, is a simple grid search method yielding the following result: $n^*=6$, $h^*=1.5$, $k_1^*=2.5$, $k_2^*=2.8$, $k_3^*=2.3$, $k_4^*=2.6$.

That is, the upper and lower control limits of the economic $\overline{X}$ charts should be set at 108.5 and −4.05, respectively. The upper control limit of the economic S^2 chart should be set at 8442. The upper control limit of $\overline{e}$ cause-selecting chart should be set at 0.939; the lower control limit of $\overline{e}$ chart should be set at −0.939. The upper control limit of S_e^2 cause-selecting charts should be set at 3.9. To monitor the process states, every 1.5 hours a sample of size six is taken and tested.

There are four possible results for the process. These outcomes with the associated actions are displayed in Table 2. Combination 1 means that the process is in control, so the process continues and the next sample is taken after 1.5 hours. Combination 2 means that the first step should be stopped and machine 1 is adjusted. Combination 3 means that the second step should be stopped and machine 2 is adjusted. Combination 4 means that both the first step and the second step should be stopped and machine 1 and 2 are adjusted.

4 Summary

A model of two dependent production process steps is proposed, whose quality can be affected by the occurrence of two special causes, which result in a shift in the means and variances of the first step and the second step. A shift in either may also result from over-adjustment of the process when the process is in control. Deming [1] discusses this common situation for a single process in practice. The proposed model is an improvement to the economic design with a single over-adjusted process and a two-step over-adjusted process, since it considers the effects of over-adjusted process means and variances on two dependent process steps with two special causes. Using the proposed design, the two steps may be distinguished and adjusted with minimum cost, since the only information about process state available is from sampling.

A Markov chain approach is extended to derive double special-cause economic adjustment model of the two dependent process steps used to determine the design parameters of the $\overline{X}-S^2$ charts and $\overline{e}-S_e^2$ cause-selecting control charts, which together minimize the long term cost resulting from processes over-adjustment or under-adjustment. It is demonstrated that the expression for the economic adjustment model is easier to obtain through the proposed approach rather than by others. Several important extensions of the developed model can be developed. It is straightforward to extend the proposed model to study autocorrelated observations or other control charts, like time series model or attributes charts or multivariate control charts. One particularly interesting research area for future research involves the economic modeling of production processes subject to adaptive sampling time and/or sampling size, etc.

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Appendix 1

The transition probability $P_{i,j}$ is calculated based on the following formula.

$P_{i,j} = P$ (process reaches state j| process is at state i)
$= P$ (SC_1 occurs in next time interval h?| step 1 is in control?)

- P (SC_2 occurs in next time interval h?| step 2 is in control?)
- P ($\overline{X}$ chart signals?| SC_1 occurs in next time interval h?)
- P (S^2 chart signals?| SC_1 occurs in next time interval h?)
- P ($\overline{e}$ chart signals?| SC_2 occurs in next time interval h?)
- P (S_e^2 chart signals?| SC_2 occurs in next time interval h?) , for state i, $j \neq 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64$.

$P_{i,i} = 1$, $P_{i,j} = 0$ for state $i = 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64$. $j \neq i$

Some examples for calculating the transition probabilities $P_{1,3}$, $P_{2,23}$, and $P_{7,64}$ are illustarted as follows.

$P_{1,3} = P$ (SC_1 does not occur in next time interval h| process step 1 is in control)

- P (SC_2 does not occur in next time interval h| proess step 2 is in control)
- P ($\overline{X}$ chart has no false signal)
- P (S^2 chart has false signal)
- P ($\overline{e}$ chart has no false signal)
- P (S_e^2 chart has no false signal)

$= \exp(-\lambda_1 h)\exp(-\lambda_2 h)(1-\alpha_{\overline{X}})(\alpha_{S^2})(1-\alpha_{\overline{e}})(1-\alpha_{S_e^2})$.

$P_{2,23} = P$ (SC_2 does not occur in next time interval h| process step 1 is out of control)

- P ($\overline{X}$ chart has true signal)
- P(S^2 chart has no true signal)
- P ($\overline{e}$ chart has false signal)
- P (S_e^2 chart has no false signal)

$= \exp(-\lambda_2 h)(1-\beta_{\overline{X}})(\beta_{S^2})(\alpha_{\overline{e}})(1-\alpha_{S_e^2})$.

$P_{7,64} = P$ ($\overline{X}$ chart has true signal and S^2 chart has true signal | process step 1 is out of control since over-adjusted process step 1)

- P ($\overline{e}$ chart has true signal and chart has true signal | process step 2 is out of control since over-adjusted process step 2)

$= (1-\beta_{\overline{X}})(1-\beta_{S^2})(1-\beta_{\overline{e}})(1-\beta_{S_e^2})$.

Appendix 2

$C_{i,j}$ = expected cumulative cost that is assoiated with transition from state i to j in time interval h,

= (expected cost ocurs in next time interval h, given the process is at state i) + (sampling cost) + (cost of false signal and/or cost of searhing special cause) for i, $j \neq 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64$.

$$C_{i,i} = C_{\mathrm{sr}} \text{ for state } i = 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63 .$$

$$C_{i,i} = 2C_{\mathrm{sr}} \text{ for state } i = 64 ,$$

$$C_{i,j} = 0 \text{ for state } i = 18, 19, 22, 36, 37, 43, 55, 56, 57, 58, 60, 61, 62, 63, 64, \ j \neq i .$$

Some examples for calculating the expected cumulative cost for $C_{1,3}$, $C_{2,23}$, and $C_{7,64}$ are illustarted as follows.

$C_{1,3}$ = expected cost in next time interval h with in-control process step 1 and 2, given the process is in control + sampling cost + cost of false signal

$= (C_0 h) + (a + bn) + C_f$.

$C_{2,23}$ = expected cost in next time interval h with in-control process step 2 , given the process step 1 is out of control + sampling cost

+ cost of false signal and cost of searhing special cause 1

$= (C_1 h) + (a + bn) + C_f + C_{\mathrm{sr}}$.

$C_{7,64}$ = expected cost in next time interval h, given the process step 1 and 2 are out of control + sampling cost

+ cost of searhing special cause 1 and 2

$= (C_{12} h) + (a + bn) + 2C_{\mathrm{sr}}$.