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Spatial-structure differences between urban and regional systems

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Abstract The total population and the overall average density are derived for a single city, in which the spatial structure conforms approximately to the negative exponential function. It is then shown that across the cities of an urban system, the areal extent of a city is positively related to its average density and also to its population. Consideration is next given to a system of regions and again to the manner in which areal extent is related to average density and population across regions. In this setting, however, the relationships are usually different from those found within an urban system.

JEL Classification R12 · R23 · R53

1 Introduction

The analysis of spatial structure is typically undertaken at the level of the individual city or region. An obvious drawback of this single-entity perspective is that it may divert attention away from important regularities that exist across cities or across regions, the emphasis of the following discussion. For cities, this will involve a national urban system, and for regions, the focus will be on a space-filling system of regions that collectively covers the entire nation. With the first case, the concern is with the urban population of a nation, while in the second case, it is with the total national population. The primary purpose of this paper is to impose a common framework on these two national systems, thus enabling certain aspects of their respective spatial structures to be compared. As a starting point, the density pattern of the individual city is examined.

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The approach is extended to examine the variation of spatial structure across a set of cities forming the urban system. Attention then moves to the regional system, where similar aspects of spatial structure are considered, though these display fundamentally different forms from those of the urban system. There follows a discussion as to why such spatial-structure differences between the two systems might exist.

2 The individual city

To place the discussion in context, the initial focus is on conditions within an individual city, above some minimum population. The term "city" refers throughout to a continuously built-up area, which in the case of a large city would represent the metropolitan area. Attention is confined to conditions within Western cities.

2.1 The marginal density function

Clark (1951) appears to have been the first to recognize that the negative exponential function provides a reasonably accurate description of the decline of population densities in a city with increasing distance from its center. As Clark was aware, the general phenomenon of urban density decay had been remarked upon considerably earlier by a number of writers, including Bleicher (1892) and Meuriot (1898). The negative exponential function has the form:

$$M(x) = C \exp(-bx) \qquad (b > 0;\ 0 \le x \le u) \tag{1}$$

or, in natural logarithms,

$$\ln M(x) = \ln C - bx$$

where $M(x)$ is the marginal density of population within a thin annular ring at distance x from the center, while C represents the density at the center, and $-b$ defines the slope of the function. The term "marginal density" is discussed shortly. Three cases of such a function are shown as curves M_1, M_2, and M_3 in Fig. 1. The term u is the radius of the city or the distance at which the function reaches some minimum urban density M', this density being the same for all cities.

Several important assumptions underpin the application of this function. The city is assumed to be monocentric, to have a circular shape, and to be radially symmetric, so that no physical barriers to urban development exist, although account can be taken of these to some extent (Weiss 1961). Also, the existence of a density crater very near the center of the city is ignored. The constant C thus represents an extrapolated value or "the point to which densities are tending" as the center is approached (Clark 1951, p. 491). An economic basis for the negative exponential function of urban densities has been suggested by such authors as Alonso (1964), Amson (1972); Muth (1969) and Winsborough (1961). However, Batty and Kim (1992) have questioned the validity of the negative exponential function in this connection. Since Eq. 1 is only concerned

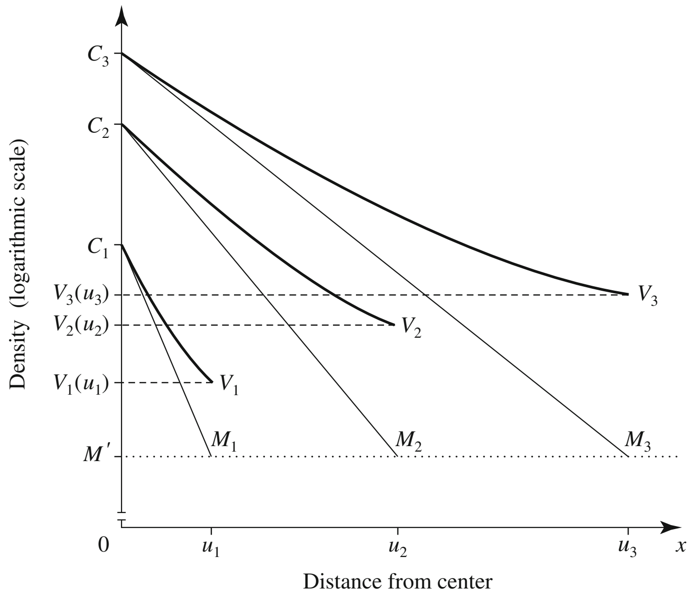

Fig. 1 Marginal and average density functions

with the level of density $M(x)$ at radial distance x, it may be termed the "marginal density function," although it has not been customary to refer to it in this way.

2.2 The average density function

The concern here, however, will be with average density rather than marginal density. Average density $V(x)$ is the overall density of population within a circular city of radius x. This aspect of spatial structure does not appear to have been employed in studies of urban density. Yet information on the overall density of a city out to different distances from the center is of obvious relevance in analyses concerned with housing availability, private and public transportation, service provision, environmental management, etc. The measurement of average density is initially approached in terms of population. The value $p(x)$, the population within the thin annular ring at distance x, is

$$p\,(x) = 2\pi x\ C\ \exp\,(-bx) \qquad (b > \ 0;\ \ 0 \leq x \leq u) \tag{2}$$

Function 2 reaches its maximum at distance $1/b$ (Bussière 1972). This is the most populous, as opposed to the most densely populated, part (ring) of the city. The distance $2/b$ represents the approximate average distance of an inhabitant to the center of the city (Ashenfelter 1976). Other measures of central tendency in the negative exponential function are discussed by Edmonston and Davies (1978).

Important for our purposes, however, is the value $P(x)$, the population located within a circle of radius x. This is obtained by integrating Eq. 2 over the interval 0 to x to give

$$P\,(x) = \frac{2\pi C}{b^2}\left[1 - \exp\,(-bx)\,(1\ + bx)\right] \qquad (0 \leq x \leq u) \tag{3}$$

The right side of Eq. 3 has two components: the first refers to the level of population where $x = \infty$; the second component is the term in square parentheses, which is the proportion of the first component that is located within a perimeter of radius x(this component assumes a value of unity for $x = \infty$). When $x = u$, Eq. 3 gives the total population of the city.

The value $V(x)$, the average density within a circle of radius x, is obtained by dividing Eq. 3 by the area of the circle so that

$$V(x) = \frac{2C}{(bx)^2}\left[1 - \exp(-bx)(1 + bx)\right] \tag{4}$$

when $x = u$, Eq. 4 indicates the average density of the entire city. Three average density or V_i functions are indicated in Fig. 1, along with their respective marginal density or M_i functions ($i = 1, 2, 3$). Each V_i function decreases at a decreasing rate out to a distance u_i, the boundary of city i, where the corresponding marginal density function M_i reaches level M', the minimum urban density. The M_i function refers to the marginal density at radial distance x, whereas the corresponding V_i function refers to the average density within a circle of radius x. The average density of the entire city i is thus expressed in Fig. 1 as $V_i(u_i)$.

Though common enough in other contexts, the notion of average density appears not to have been used in relation to density functions, and it is unclear why this should have been the case. Perhaps one explanation for this is that Clark, himself, used the term "average density" in a particular context. In outlining his construction of a density function, Clark (1951, p. 491) stated: "…it was found convenient to calculate total population, and hence *average density*, in a series of concentric rings about the center of the city, generally drawn at each mile radius" [emphasis added]. This was, of course, a perfectly valid use of the term, which drew attention to the fact that density within a given ring is necessarily of an average nature. However, Clark's employment of the term "average density" effectively precluded its use in any other context. This may explain why Clark (and subsequent commentators) never considered the concept of average density as defined earlier and simply used the term "average density" to describe the density within a given ring, or what is referred to here as "marginal density."

3 Spatial structure of the urban system

Building on this discussion, we now take a cross-sectional view of cities in an urban system, in which the internal structure of individual cities conforms to the negative exponential function. For a contemporary national urban system, more than a little evidence has accumulated to suggest that for a given point in time, the greater the city population, the higher tends to be the value of C (Muth 1969; Stewart and Warntz 1958) and the lower the value of b (Berry et al. 1963; Mills 1972; Weiss 1961). These cross-sectional relationships are, of course, oblivious to the fact that cities may have developed at different periods of time and against backgrounds of differing transportation technologies, income levels, and residential preferences. Such relationships, which are therefore no more than approximations, are reflected in Fig. 1 by the positioning of the marginal density functions (and thus the average density functions), where the subscripts 1, 2, and 3 refer, respectively, to a city of a small, medium, and large population.

If Fig. 1 is regarded as representing a rudimentary urban system, two features of its spatial structure begin to emerge, the first of which is fairly obvious. It can be seen in Fig. 1 that u_i, the radius (and therefore the area) of city i, is positively related to its population ($i = 1, 2, 3$). A second feature of the spatial structure is less apparent. Figure 1 indicates that the greater the population of city i, the higher and less steep is its average density or V_i function, and therefore, the greater the value of u_i and the greater the value of $V_i(u_i)$, the average density of the entire city i. It follows that the area of a city is positively related to its overall average density.

The general forms of these two spatial-structure characteristics of the urban system (i.e., the positive relationships between area and both population and average density) are not yet established. Important light is cast on this question, however, by the empirical work of Stewart and Warntz (1958). This focused on cities as well as metropolitan districts in the United States for 1940 and also on cities in England and Wales for 1951. Stewart and Warntz were able to specify the manner in which area increased with city population (now denoted as B) across the urban system, although few details of the statistical analysis were provided. Expressed in terms of common logarithms, it was found that

$$\log A_i = \log K + k\,(\log B_i) \qquad (0 < k < 1) \tag{5}$$

where A_i is the area of city i (with $i = 1, 2, \ldots, I$), and B_i is the population of city i. The terms K and k are constants, k being the slope, which was estimated to be around 0.75). Similar findings were reported by Vining and Louw (1978) for "built-up" areas in the United States, Sweden, Norway, and Japan at various years from 1950 to 1975.

Since city population is the product of its area and average density, the relationship between area and average density (now denoted as G) may be expressed as

$$\log A_i = \log H + h\,(\log G_i) \qquad (h > 0) \tag{6}$$

where G_i is the average density of the entire city i, and the terms H and h are constants, with the value of h equal to $k/(1-k)$. Equation 6 can also be derived from the Bussière and Stovall (1977) study, which was initially concerned with the size distribution of cities within the urban system. To facilitate subsequent comparison with conditions in the regional system, Eq. 5 is rewritten in terms of h as follows:

$$\log A_i = \log K + \frac{h}{1+h}\,(\log B_i) \tag{7}$$

Expressions 6 and 7 represent power functions with slopes of h and $h/(1+h)$, respectively, the slope for average density being greater than the slope for population. Thus, for $h > 0$, area increases with average density across cities at $1 + h$ times the rate that it increases with population. Alternatively phrased, area increases with population at only $1/(1 + h)$ times the rate that it increases with average density.

4 The regional scale

We now wish to explore conditions at the level of the region. This is not a straightforward task, primarily because the region, in contrast to the city, is less amenable to unambiguous definition. Regions can be of substantially different types, not all of which lend themselves to the approach used previously. Attention is restricted to the nodal region, one of the three broad types of economic region considered by Meyer (1963). The nodal region, which can vary widely in terms of scale, consists of two zones: a node or core, and a surrounding hinterland containing rural population and an urban population (sometimes much larger), located in a network of centers of varying size. Such a region is characterized by substantial interactions between the two zones, particularly involving trade in goods and services and well as factor movements.

In principle at least, it is possible to apply the approach of the previous sections to regions, i.e., to examine conditions within the individual region and then to explore the way in which area is related to density and population across regions. Certain modifications would be necessary, of course. Thus, the negative exponential function would have to be replaced by some alternative function such as the linear gamma (Amson 1972) or lognormal (Parr 1985). Apart from being able to incorporate the density crater near the center of the city or node, both alternative functions reflect the fact that beyond the boundary of the city, marginal density declines at a slower rate than in the negative exponential function. However, there exists a serious impediment to replicating the previous approach. This involves the virtual absence of systematic studies of nodal regions within individual nations, the work of Bogue (1950) on the United States possibly being the only example. As a consequence, data on the parameters of the relevant marginal density functions for individual regions, and therefore data on how these parameters vary across regions within a nation, are unavailable, thus precluding the derivation of expressions equivalent to Eq. 6 for average density and Eq. 7 for population.

5 Spatial structure of the regional system

Given this paucity of cross-sectional data on nodal regions, we disregard the internal structure of the individual region and move directly to the spatial structure of a regional system of a nation. Here, we rely on the regression analysis of Stephan (1972). This was based on the official units (political or administrative divisions) for 98 nations, and in each nation, the relevant set of units, referred to here as “regions,” exhausted the national space. The Stephan study is not an entirely satisfactory source, since the regional boundaries may reflect the population distribution at some previous period of history rather than around the time of the study. This difficulty notwithstanding, in the majority of nations, the official regions possessed a sufficiently pronounced nodal aspect as to correspond to regions of the type described earlier. Note that in the emergence of official regions, there has often been the tendency for these to coincide with nodal regions, reflecting the advantages stemming from such correspondence (Parr 2007).

Table 1 Slopes in Eqs. 8 and 9 for given values of n

Value of n	Slope of Eq. 8	Slope of Eq. 9
0	0	0
$0 < n < 1$	$\in (-1, 0)$	$\in (-\infty, 0)$
1	-1	$-\infty$
$1 < n < \infty$	$\in (-\infty, -1)$	$\in (1, \infty)$
∞	$-\infty$	1

5.1 Relationship of area to average density and to population

The central finding of the Stephan (1972) analysis was that for the regions of a nation, area was negatively related to average density in the following manner:

$$\log A_j = \log N - n \left(\log G_j\right) \qquad (n \geq 0) \tag{8}$$

where A_j is the area of region j (with $j = 1, 2, \ldots, J$), and G_j is the average density of region j. The terms N and n represent constants, $-n$ being the slope of the function. This negative relationship was observed in 94 of the 98 cases, and in 78 of these, the relationship was significant beyond the 5 percent level. Equation 8 can be transformed into the following expression which relates area to population, the population of region j:

$$\log A_j = \log L - \frac{n}{1-n} \left(\log B_j\right) \qquad (n \geq 0) \tag{9}$$

where B_j is the population of region j, and the terms L and n are constants[1]. Equations 8 and 9 represent inverse power functions, having respective slopes of $-n$ and $-n/(1-n)$. It is evident from their negative form that Eqs. 8 and 9 for the regional system differ fundamentally from Eqs. 6 and 7 for the urban system where the slopes are positive, though each corresponding pair of equations adheres to a power function form. Furthermore, in the regional system for $0 < n < 1$, the slope for average density is less steep than the slope for population, which contrasts with the urban system for $h > 0$, where the slope for average density is steeper than the slope for population.

5.2 Relations between the functions

We may briefly explore the slope values in Eqs. 8 and 9 for different values of n. A similar procedure was not undertaken for values of $h > 0$ in Eqs. 6 and 7, which related to the urban system, because the general form of the slopes did not vary with h. Table 1 indicates the range of values for n, as well as the slope of Eq. 8 for average density, and the slope of Eq. 9 for population. In the case where $n = 0$, the slopes of Eqs. 8 and 9 are both horizontal and at the same level of area. Area thus remains constant for all regions, as average density and population both increase. When $0 < n < 1$ (the most commonly-occurring case), the slopes of Eqs. 8 and 9 are both negative, the slope of

[1] Across 67 regions of the US (termed "metropolitan communities") Bogue (1950, p.88) noted a broadly negative relation between area and population, although the form of the relation was not specified.

Eq. 8 being less steep than that of Eq. 9. When $n = 1$, the slope of Eq. 8 is negative but the slope of Eq. 9 is vertical, indicating that all regions have identical populations. And when $1 < n < \infty$, the slope of Eq. 8 is still negative but the slope of Eq. 9 is positive, so that area increases as population increases. Finally, in the limiting case where n approaches infinity, the slope of Eq. 8 is vertical, indicating that all regions have the same average density, whereas Eq. 9 has a positive slope equal to unity, so that area increases as population increases.

Attention is drawn at this point to a complication. This relates to the fact that much of the discussion has dealt with mathematical relationships, whereas empirical analysis usually involves statistical relationships, where goodness of fit and significance levels become critical factors. A single example serves to illustrate the problem. It is assumed that a given best-fitting function for Eq. 8 has a slope within the interval $(-1, 0)$, so that area decreases as average density increases. If the fit is perfect (or at least sufficiently good), the slope of the best-fitting function for population in Eq. 9 will also be negative, indicating that area decreases as population increases. If, however, the fit for Eq. 8 is poor, it is possible for the best-fitting function for Eq. 9 to have a positive slope, so that area increases as population increases.[2] Should it transpire that this outcome is commonplace within other systems of regions (such as those involving different hierarchical levels), the link between the area–density relationship and the area–population relationship would be only a tenuous one, in which case the general argument about contrasting spatial structures in urban and regional systems would have to rely on the former relationship. Interestingly, the link between these two relationships appears to be much stronger within urban systems.

5.3 Random regionalization

Finally, and as a point of reference, it is possible to examine the outcome for a randomly generated system of regions (Cliff et al. 1975). Such a system may be derived by considering a population that is randomly distributed over a bounded space, which is then randomly divided (Vining et al. 1979). In this system of regions, there would tend to be a positive relationship between area A_j and population B_j, simply because the more extensive the region, the greater is its expected population. From the preceding analysis, this would suggest a relationship of the following form:

$$\log A_j = \log S + s\left(\log B_j\right) \qquad (s > 0) \tag{10}$$

where S and s are constants. The corresponding function for average density G_j would therefore become

$$\log A_j = \log T + t\left(\log G_j\right) \qquad (t > 0) \tag{11}$$

[2] Vining et al. (1979, p. 219) reworked the Stephan (1972) data, confining their attention to the 65 nations with 10 or more official regions. Within this set, 61 nations had slopes for Eq. 8 within the interval $(-1, 0)$, but as few as 20 of these had negative slopes for Eq. 9, and in only 12 cases was the relationship significant at the 10 percent level.

where T and t are constants, the value of t being $s/(1-s)$. To assist comparison with Eq. 9, Eq. 10 can be expressed in terms of t as

$$\log A_j = \log S + \frac{t}{1+t}\left(\log B_j\right) \tag{12}$$

As discussed previously, a poor goodness of fit in a regression analysis may obscure the interrelationships between Eqs. 11 and 12. It will be noted that these equations mimic Eqs. 6 and 7 for cities in the urban system, although no particular inference can be drawn from this. Equations 11 and 12 differ from their respective counterparts, Eqs. 8 and 9, in the regional system setting discussed earlier. Such a contrary result should not be surprising since under actual conditions, it is reasonable to assume the presence of a strong element of spatial organization, whether economic or political, as discussed in the following section.

6 Contrasting spatial structures: some underlying influences

The concern above has been with a class of spatial-structure differences between urban and regional systems. This has been examined from a common perspective, involving the way in which area is related to average density and to population. The results for the urban system and those for the regional system differ markedly. When area is related to average density, each relationship is positive for the urban system, but negative for the regional system (with regard to population in the regional system, the exceptions to this negative relationship have been noted). It is not immediately obvious why such differences between the two types of national system should exist. The following comments, which are concerned primarily with area–density relationships, suggest that the contrast may be related to differences in the significance of space between the two systems.

In the case of the urban system, the area–density relationship is positive. Within this system, the individual city is locationally separate from neighboring cities, so that cities do not compete for territory (in an ecological sense). In terms of the consumption of space, therefore, cities are largely independent of one another and there is no constraint based on the development of a neighboring city. On the other hand, cities are ultimately limited in extent by (among other things) the existence of transportation costs on the movement of people and freight. As a consequence, there is a competition for space on the part of the economic actors involved (households, firms, public agencies, etc.), as revealed in their respective bid-rent functions. The concern here is not with the growth of any individual city but with the variation in spatial structure among cities at a given time. In this connection, it is reasonable to assume that the greater the population of the city, the more intense will be the competition for space. As argued in Sect. 3, evidence suggests that increasing population (reflected in increasingly intense competition) is accommodated by a higher marginal density (and therefore a higher average density) at all relevant locations; see Fig. 1. This manifests itself in an increasing value of C in Eq. 1 and, in an era of relatively low-cost surface transportation, a decreasing value of b (Clark 1951, p. 495). Across cities of increasing population, therefore, the overall effect of heightening competition for space is twofold: it causes expansion in city area and also causes the average density to be higher. These two tendencies thus provide a rationale for the area of a city being positively related to its average density.

For the regional system, however, the relationship between area and average density is negative. By their nature, regions are space consuming and, as defined here, also space filling. Regions can therefore be said to compete for territory, inasmuch as a hypothetical enlargement of a given region would necessarily result in the diminution of one or more neighboring regions. Central to the negative relationship between area and average density is the presence of two maximizing forces. The first results from free-entry competition among regional-scale producers, by which profits are driven to zero. This maximizes the number of producers by limiting the extent of their individual market areas and consequently results in relatively small (large) market areas or nodal regions where average density is high (low). This was an important facet of the Lösch (1944/1954, pp.105–108) analysis of equilibrium market-area size, in which an expression for population density was made explicit. Much empirical support for this negative relationship exists, including the Skinner (1964) study of market areas in China, and such a regularity has been demonstrated graphically by Rushton (1972).

The second force, in a sense the public-sector analog of the first, is concerned with maximizing accessibility to goods and services provided by subnational governments. Stephan (1977) was able to derive Eq. 8 given previously by means of an optimizing model, in which the sum of travel time (incurred by the user population) and the total man-hours (required for the supply of these goods) would be at a minimum. Such a "time-minimizing" outcome could evolve spontaneously from below, although it is frequently organized or imposed by some higher political or administrative authority. Each of the two maximizing forces is subject to a minimum-scale constraint. As already mentioned, there may be a correspondence between the pattern of nodal economic regions and the pattern of political or administrative regions. In such cases, the two patterns tend to be mutually reinforcing, so that the structure of regions becomes well defined. A rather different approach to the emergence of regions was suggested by Alesina and Spolaore (1997, p. 1046), this being a by-product of their analysis of the formation of nations.

7 Concluding comment

In regional science and related disciplines, it is usual to assume that as individual entities, cities and regions can be treated as broadly similar, differing only in terms of scale and mass. Both the city and the region are subnational in character, both represent open economic systems with respect to trade, factor movement, and government transfers, and both lend themselves to scrutiny in terms of a common set of conceptual frameworks and analytical techniques. Here, however, the city and the region have each been considered in a system-wide context, and from this perspective, certain significant spatial–structure differences were shown to be present. The causes of these are not readily obvious, and the explanations offered immediately above are of a general and preliminary nature. It is clear, therefore, that such unexpected contrasts merit further investigation.

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