Ann Reg Sci (2011) 46:397–416
DOI 10.1007/s00168-009-0335-2

ORIGINAL PAPER

Discriminatory prices and the prisoner dilemma problem

Stefano Colombo

Received: 14 February 2009 / Accepted: 6 October 2009 / Published online: 23 October 2009

Abstract We ask whether the tendency of Hotelling duopolists to choose uniform pricing (discriminatory pricing) when the pricing policy is chosen before (after) the location is robust to the case of imperfect or third-degree price discrimination. By using a general framework encompassing both perfect and imperfect price discrimination for any degree of imperfectness, we show that both firms choose uniform pricing when the pricing policy is chosen before the location for any degree of imperfectness of price discrimination. When the pricing policy is chosen after the location and price discrimination is precise enough both firms choose price discrimination; if price discrimination is highly imprecise, an equilibrium exists where both firms commit not to price discriminate.

JEL Classification D43 · L11

1 Introduction

Price discrimination is a widely used business practice. However, in oligopoly it may be possible that the equilibrium discriminatory prices are all lower than the equilibrium uniform price. This phenomenon is called *all-out competition* (Corts 1998).

I am indebted to Luca Lambertini, Massimo Motta, Carlo Scarpa, Michele Grillo, the editor Borje Johansson and two anonymous referees for their comments and suggestions. I thank seminar participants at the Universities of Montpellier, Milano-Bicocca, Cattolica, Torino, Siena and Pavia for helpful comments. All remaining errors are my own.

S. Colombo (✉)
Catholic University of Milan, Largo A. Gemelli 1, 20123 Milano, Italy
e-mail: stefano.colombo@unicatt.it

When *all-out competition* occurs, equilibrium profits under the discriminatory price regime are lower than the equilibrium profits under the uniform price regime.

All-out competition typically emerges in the Hotelling (1929) framework with linear or quadratic transportation costs. Thisse and Vives (1988) study the case of price discrimination within the Hotelling (1929) model when firms are maximally differentiated. First, they show that when firms can price discriminate and simultaneously choose the price schedule, uniform pricing is never an equilibrium. Then, Thisse and Vives (1988) assume a two-stage game, where in the first stage each firm chooses the pricing policy, while in the second stage the price schedules are set. They show that even when every firm may credibly commit to uniform pricing before setting the price schedule, the discriminatory prices still arise in equilibrium, since no-commitment is the dominant strategy for each firm in the first stage of the game conditioned on the equilibrium path in the second stage of the game. This situation gives rise to a typical prisoner dilemma: both firms would be better off setting uniform prices, but the dominant strategy of each firm induces the discriminatory equilibrium. Eber (1997) studies two different versions of a three-stage game. In the first version, firms first simultaneously choose where to locate, then they choose whether to price discriminate or not, then they set the price schedules. In the second version of the game, the first two stages are reversed: firms first choose the pricing policy and then they choose the location. The author shows that the unique equilibrium of the first game is characterized by price discrimination by both firms (Prisoner Dilemma arises), while the unique equilibrium of the second game is characterized by uniform pricing by both firms (no Prisoner Dilemma arises).

The main limitation of the analysis of Thisse and Vives (1988) and Eber (1997) is that only *perfect* price discrimination is considered. As Eber 1997, p 420 points out: "*like Thisse and Vives (1988), we only study the choice between one price and an infinity of prices, so that we rule out imperfect discriminatory pricing from the set of possible choices*". That is, firms are assumed to be able to set a different price for each consumer, which is far from being a realistic assumption. In this paper we fill this gap by using a more general price discrimination framework which allows considering both imperfect or third-degree price discrimination and perfect price discrimination as a limit case. We show that the second-game result of Eber (1997) extends for any degree of imperfectness: when the pricing policy decision occurs before the location decision, uniform pricing emerges as the unique equilibrium, regardless of the degree of imperfectness of price discrimination. The reason for this can be found in what we call the *spatial competition effect*: the possibility to price discriminate induces lower distance between the firms, which in turn increases the competition and makes the no-commitment strategy less profitable for each firm. Moreover, we show that the first-game result of Eber (1997) extends to imperfect price discrimination when the degree of imperfectness is sufficiently low; instead, if price discrimination is highly imprecise, an equilibrium exists where both firms commit not to price discriminate: in this case no Prisoner Dilemma arises. Other results concern equilibrium locations, equilibrium prices and equilibrium welfare. When firms can imperfectly price discriminate, the equilibrium locations are very close to the locations that maximize total welfare and they converge to 1/4 and 3/4 when price discrimination tends to perfectness; equilibrium prices are all lower under price discrimination than under uniform pricing (*all-out competition* occurs); consumer surplus and total welfare are higher under price discrimination than under uniform pricing.

This paper is organized as follows. In Sect. 2 we describe the model and we briefly recall the well-known location-price equilibrium under the hypothesis of uniform price regime. In Sect. 3 we analyse the location-price equilibrium when the firms can price discriminate. In Sect. 4 we analyse the two versions of the three-stage game. Section 5 concludes.

2 Uniform price

Assume a linear market of length 1. Consumers are uniformly distributed along the segment. Assume density one. Define with $x \in [0, 1]$ the location of each consumer. Each consumer consumes exactly one unit of the good.

There are two firms, A and B, competing in the market. Firms sell spatially differentiated, but otherwise homogeneous, products. Both firms have zero marginal costs. Define with a the location chosen by firm A and with b the location chosen by firm B. Without loss of generality, assume: $0 \leq a \leq b \leq 1$. Define with $\bar{p}^A$ the uniform price set by firm A and with $\bar{p}^B$ the uniform price set by firm B. *Free-on-board* prices are assumed.

The utility of a consumer located at x when he buys from firm A is given by $u_x = v - \bar{p}^A - t(x-a)^2$, while the utility of a consumer located at x when he buys from firm B is given by $u_x = v - \bar{p}^B - t(x-b)^2$, with $t > 0$. Suppose that v is large enough to guarantee that each consumer always buys the good. Define with x^* the consumer which is indifferent between buying from firm A or from firm B for a given couple of locations, a and b, and for a given couple of uniform prices, $\bar{p}^A$ and $\bar{p}^B$. Equating the utility in the two cases and solving for x it follows:

$$x^* = \frac{a+b}{2} + \frac{\bar{p}^B - \bar{p}^A}{2t(b-a)}$$

Given the uniform distribution of the consumers, x^* is the demand function of firm A and $1 - x^*$ is the demand function of firm B. It is well known that in a two-stage game in which firms first choose locations and then choose the uniform price, the unique sub-game perfect equilibrium implies maximal distance between the firms, as the following proposition indicates:

Proposition 1 (D'Aspremont et al. 1979) *in a two-stage game in which the firms first simultaneously choose the location and then simultaneously decide the* [uniform] *price, there is a unique sub-game perfect equilibrium, defined by* $a^* = 0$ *and* $b^* = 1$, *and* $\bar{p}^{A*} = \bar{p}^{B*} = t$.

Given the equilibrium locations and the equilibrium prices, the equilibrium profits of each firm are $\Pi^A = \Pi^B = t/2$.

3 Discriminatory prices

Following Liu and Serfes (2004), we suppose that there is an information technology which allows firms to partition the consumers into different groups. We assume that the information technology partitions the linear market into n sub-segments indexed by m, with $m = 1, \ldots, n$. Each sub-segment is of equal length, $1/n$. It follows that sub-segment m can be expressed as the interval $\left[\frac{m-1}{n}; \frac{m}{n}\right]$. A firm can price discriminate between consumers belonging to different sub-segments, but not between the consumers belonging to the same sub-segment. The cost of using the information technology is zero. Define with p_m^J the price set by firm $J = A, B$ on consumers belonging to sub-segment m. Clearly, when firm J cannot price discriminate, it must be $p_m^J = p_{m'}^J$, $\forall m, m'$. Finally, assume that $n = 4 + 6k$, with $k = 0, 1, 2, 3, 4 \ldots$ This segmentation allows us to keep the analysis tractable (see footnote 8), while maintaining at the same time the basic idea of Liu and Serfes (2004): n measures the precision of consumer information, since the higher the n, the higher the information precision (the lower is the degree of imperfectness of price discrimination).[1] Note that when $n \to \infty$ we are in the situation of perfect price discrimination analysed by Thisse and Vives (1988) and Eber (1997).

We study now the location-price equilibrium when both firms can price discriminate between consumers. We suppose a two-stage game, in which the firms first decide where to locate and then compete on prices. The utility of the consumer x belonging to sub-segment m when he buys from firm A is $u_x = v - p_m^A - t(x-a)^2$, while his utility when he buys from firm B is given by $u_x = v - p_m^B - t(x-b)^2$. Consider segment m. Define x_m^* as the consumer on segment m which is indifferent between buying from firm A or from firm B for a given couple of locations, a and b, and for a given couple of discriminatory prices, p_m^A and p_m^B. Equating the utility in the two cases and solving for x it follows

$$x_m^* = \frac{a+b}{2} + \frac{p_m^B - p_m^A}{2t(b-a)}$$

Therefore, the demand of firm A and firm B on sub-segment m is, respectively,

$$d_m^A = \frac{a+b}{2} + \frac{p_m^B - p_m^A}{2t(b-a)} - \frac{m-1}{n} \tag{1}$$

$$d_m^B = \frac{m}{n} - \frac{a+b}{2} - \frac{p_m^B - p_m^A}{2t(b-a)} \tag{2}$$

It follows that the profits of firm A and firm B on sub-segment m are, respectively,

$$\Pi_m^A = p_m^A d_m^A = p_m^A \left[\frac{a+b}{2} + \frac{p_m^B - p_m^A}{2t(b-a)} - \frac{m-1}{n}\right]$$

[1] Liu and Serfes (2004) assume $n = 2^k$, with $k = 0, 1, 2, 3, 4 \ldots$.

$$\Pi_m^B = p_m^B d_m^B = p_m^B \left[\frac{m}{n} - \frac{a+b}{2} - \frac{p_m^B - p_m^A}{2t(b-a)} \right]$$

The following proposition defines the equilibrium price schedules:

Proposition 2 *The equilibrium prices and the equilibrium demand of each firm in the second stage of the game are the following:*

- $$\begin{aligned} & m_A < m < m_B \\ & p_m^{A*} = \frac{t(b-a)}{3}\left(\frac{4-2m}{n} + a + b\right), \quad d_m^{A*} = \frac{2-m}{3n} + \frac{a+b}{6} \\ & p_m^{B*} = \frac{t(b-a)}{3}\left(\frac{2+2m}{n} - a - b\right), \quad d_m^{B*} = \frac{m+1}{3n} - \frac{a+b}{6} \end{aligned}$$

- $$\begin{aligned} & m \le m_A \\ & p_m^{A*} = t(b-a)\left(a + b - \frac{2m}{n}\right), \quad d_m^{A*} = \frac{1}{n} \\ & p_m^{B*} = 0, \quad d_m^{B*} = 0 \end{aligned}$$

- $$\begin{aligned} & m \ge m_B \\ & p_m^{A*} = 0, \quad d_m^{A*} = 0 \\ & p_m^{B*} = t(b-a)\left(\frac{2m-2}{n} - a - b\right), \quad d_m^{B*} = \frac{1}{n} \end{aligned}$$

where $m_A = \frac{n(a+b)}{2} - 1$ *and* $m_B = \frac{n(a+b)}{2} + 2$.

Proof We refer directly to the proof provided by Liu and Serfes (2004) for their Proposition 1. In order to keep tractability, m_A (the last sub-segment monopolized by firm A) and m_B (the first sub-segment monopolized by firm B) have to be integers. Therefore, the analysis is limited to symmetric location equilibria, which (together with the assumption of even n) guarantee that m_A and m_B are integer.[2] We check later (see Proposition 3) that in equilibrium firms locate symmetrically. □

Using Proposition 2, the firms' profits can be written directly as functions of a and b (the subscript indicates that both firms are price discriminating). They are

[2] We thank one anonymous referee for this clarification.

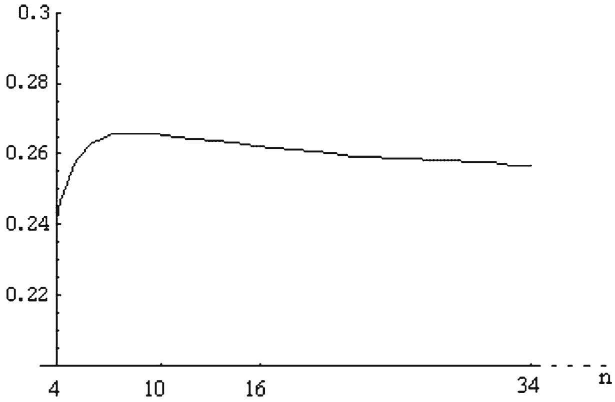

Fig. 1 Equilibrium location of firm A

$$\Pi_{\mathrm{DD}}^{A}* = \sum_{m=1}^{m_A} \frac{t(b-a)}{n}\left(a+b-2\frac{m}{n}\right) + \sum_{m=m_A+1}^{m_B-1} \frac{t(b-a)}{3}\left(\frac{4-2m}{n}+a+b\right)\left(\frac{a+b}{6}+\frac{2-m}{3n}\right) = \frac{t(b-a)[9n^2(a+b)^2-18n(a+b)+40]}{36n^2} \quad (3)$$

$$\Pi_{\mathrm{DD}}^{B}* = \sum_{m=m_A+1}^{m_B-1} \frac{t(b-a)}{3}\left(\frac{2+2m}{n}-a-b\right)\left(\frac{m+1}{3n}-\frac{a+b}{6}\right) + \sum_{m=m_B}^{n} \frac{t(b-a)}{n}\left(\frac{2m-2}{n}-a-b\right) = \frac{t(b-a)[9n^2(2-a-b)^2-18n(2-a-b)+40]}{36n^2} \quad (4)$$

We state the following proposition:

Proposition 3 *The equilibrium locations are* $a^* = \frac{9n^2-40}{36n^2-36n}$ *and* $b^* = 1-a^*$.

Proof The equilibrium locations come from the solution of $\begin{cases} \partial\Pi_{\mathrm{DD}}^{A}*/\partial a = 0 \\ \partial\Pi_{\mathrm{DD}}^{B}*/\partial b = 0 \end{cases}$. □

Figure 1 illustrates the equilibrium location of firm A.[3] Firm A locates just below 1/4 when the market is partitioned in four sub-segments; it locates just above 1/4 when the market is partitioned into ten sub-segments and afterwards the equilibrium location decreases monotonically with n and converges to 1/4 at the limit.

[3] Even if n does not take all values, in order to better illustrate the pattern of the equilibrium locations as n increases we draw a continuous line.

The next propositions compare the location-price equilibrium when price discrimination is possible with the location-price equilibrium under the uniform price regime.

Proposition 4 *All equilibrium prices are lower under price discrimination than under uniform price (all-out competition). Profits are lower under price discrimination than under uniform price.*

Proof Consider firm A. The equilibrium prices (Proposition 2) can be written as

$$p_m^{A*} = \begin{cases} \frac{t(1-2a^*)}{3}\left(\frac{4-2m}{n}+1\right) & \text{if } \; m_A < m < m_B \\ t(1-2a^*)\left(1-2\frac{m}{n}\right) & \text{if } \; m \leq m_A \end{cases}$$

First, note that the equilibrium price for $m \leq m_A$ is always higher than the equilibrium price for $m_A < m < m_B$. In fact, since p_m^{A*} is decreasing in m, the lowest equilibrium price for $m \leq m_A$ occurs when $m = m_A$, while the highest equilibrium price for $m_A < m < m_B$ occurs when $m = m_A + 1$. Substituting a^* and b^* into m_A, and then substituting m_A into p_m^{A*}, we have $p^A_{m_A(a^*,b^*)}* = \frac{2t(1-2a^*)}{n} > \frac{4t(1-2a^*)}{3n} = p^A_{m_A(a^*,b^*)+1}*$.

Therefore, the comparison between the uniform equilibrium price, $\bar{p}^{A*} = t$, and the discriminatory prices can be limited to the comparison between $\bar{p}^{A*}$ and the highest discriminatory price for $m \leq m_A$. The highest discriminatory price occurs when $m = 1$. Substituting $m = 1$ into p_m^{A*}, we get $t(1-2a^*)(1-2/n)$, which is always lower than t since both terms in the round brackets are positive and lower than 1. □

Proposition 5 *Consumer surplus and total welfare are higher under price discrimination than under uniform pricing.*

Proof The consumer surplus and the total welfare can be written, respectively, as $\text{CS} = v - \Pi^T - TC$ and $W = \Pi^T + \text{CS} = \Pi^T + v - \Pi^T - \text{TC} = v - \text{TC}$, where TC are the total transportation costs and $\Pi^T \equiv \Pi^A_{\text{DD}} + \Pi^B_{\text{DD}}$ is the sum of the profits of each firm. From Proposition 4, we know that profits are lower under price discrimination than under uniform price. Moreover, transportation costs are lower under price discrimination than under uniform price, since firms locate near to the socially optimal locations, 1/4 and 3/4. It follows that consumer surplus and total welfare increase passing from the uniform price regime to the discriminatory price regime. □

4 The three-stage model

In Sect. 3 we have shown that price discrimination yields lower profits then uniform pricing. Now, we suppose that each firm can choose the pricing policy before setting the price schedule. In a context of *perfect* price discrimination and firms exogenously located at the endpoints of the segment, Thisse and Vives (1988) show that even if there is the possibility to commit to uniform pricing, price discrimination emerges in equilibrium.[4] In the context of *perfect* price discrimination and endogenous locations, Eber (1997) shows that (a) if the location choice occurs before the commitment decision, price discrimination emerges as the unique equilibrium; (b) if the commitment decision occurs before the location choice, uniform pricing emerges as the unique equilibrium. In this section we ask whether these results are still valid when a more general framework of price discrimination is considered. Both versions of the three-stage game *a la* Eber (1997) are considered: in the first version, firms first choose location, then they choose whether to price discriminate or not, and finally they set the prices; in the second version, firms first choose the pricing policy, then choose the location and finally set the price schedules.

4.1 Game 1

Timing: At time 1, both firms simultaneously choose the location; at time 2 both firms simultaneously decide whether to commit to uniform pricing (U) or not (D); at time 3 both firms simultaneously choose the price schedule.

We solve the game by backward induction. Consider the third stage of the game. At the third stage firms compete on prices, given the locations and the commitment decision. We need to calculate the equilibrium prices when one firm has committed and the other has not committed. The following proposition defines the equilibrium prices in such case:

Proposition 6 *If firm A has committed and firm B has not committed, the equilibrium prices in the third stage of the game are the following:*

$$\bar{p}^{A*} = \frac{t(b-a)(a+b)}{2} + \frac{t(b-a)}{2n}$$

$$p_m^{B*} = \begin{cases} \dfrac{t(b-a)}{n} & if \quad m = m^\wedge - 1 \\ \dfrac{t(b-a)}{2}\left(\dfrac{4m-3}{n} - a - b\right) & if \quad m \geq m^\wedge \end{cases}$$

where $m^\wedge = [n(a+b)+7]/4$

If firm A has not committed and firm B has committed, the equilibrium prices in the third stage of the game are the following:

[4] There are many ways in which a firm may commit to uniform pricing. First, one may imagine an explicit contract between the firm and the consumers. An example of such contract is the *most-favoured nation clause*, which engages a firm to offer a consumer the same price as its other consumers: if the clause is not respected, the firm must pay back the consumer the difference between the price he effectively paid and the lowest price fixed by the firm. Other forms of commitment to uniform pricing are no-haggle policies, in which the firm promises the customers that the posted sticker price is the final price (Corts 1998). Moreover, one may consider a more subtle (and perhaps more common) type of commitment to uniform pricing. Since a firm can price discriminate only if it is able to identify consumers (or groups of them), when a firm has not such specific consumers' information, it is prevented from price discrimination. Therefore, when a firm does not acquire consumers' information, it commits to uniform pricing (Liu and Serfes 2004).

$$p_m^{A*} = \begin{cases} \frac{t(b-a)}{n} & if \;\; m = m^\circ + 1 \\ \frac{t(b-a)}{2}(2 + a + b + \frac{1-4m}{n}) & if \;\; m \le m^\circ \end{cases}$$
$$\bar{p}^{B*} = \frac{t(b-a)(2-a-b)}{2} + \frac{t(b-a)}{2n}$$

where $m^\circ = [n(2+a+b)-3]/4$

Proof Suppose that firm A has committed while firm B has not committed. Consider the sub-segment m. The demand of firm B is

$$d_m^B = \frac{m}{n} - \frac{a+b}{2} - \frac{p_m^B - \bar{p}^A}{2t(b-a)} \tag{5}$$

The profits obtained by firm B from the sub-segment m are therefore

$$\Pi_m^B = p_m^B \left[\frac{m}{n} - \frac{a+b}{2} - \frac{p_m^B - \bar{p}^A}{2t(b-a)} \right] \tag{6}$$

Maximizing Eq. (6) with respect to p_m^B, we obtain the optimal discriminatory price in sub-segment m given the price set by the non-discriminating firm. We get

$$p_m^B = t(b-a)\left(\frac{m}{n} - \frac{a+b}{2} \right) + \frac{\bar{p}^A}{2} \tag{7}$$

Inserting Eq. (7) into Eq. (1), we get firm A's demand in each sub-segment:

$$d_m^A = \frac{a+b}{4} - \frac{m}{2n} - \frac{\bar{p}^A}{4t(b-a)} + \frac{1}{n} \tag{8}$$

The demand of firm A is zero in the most at the right sub-segments. More precisely,

$$d_m^A \le 0 \quad \leftrightarrow \quad m \ge m^\wedge \equiv n \left[\frac{a+b}{2} - \frac{\bar{p}^A}{2t(b-a)} + \frac{2}{n} \right]$$

In order to maintain tractability we assume that $m^\wedge$ is integer.[5] Therefore, the analysis is limited to location equilibria under which $m^\wedge$ is integer (similarly, for the case where firm A has not committed and firm B has committed the analysis is limited to location equilibria under which m° is integer). We check later (see footnote 8) that $m^\wedge$ and m° are indeed integers under the location equilibrium emerging when the firms adopt asymmetric pricing policies. The demand of firm A is $1/n$ in the most at the left

[5] Again, we thank one referee for this clarification.

sub-segments. More precisely

$$d_m^A \geq 1/n \quad \leftrightarrow \quad m \leq m^{\wedge\wedge} \equiv n\left[\frac{a+b}{2} - \frac{\bar{p}^A}{2t(b-a)}\right]$$

Note that $m^{\wedge} - m^{\wedge\wedge} = 2$. Therefore, when $m^{\wedge}$ is integer, $m^{\wedge\wedge}$ is integer as well. Moreover, both firms have a positive demand only in sub-segment $m^{\wedge} - 1$. The profits of firm A are therefore defined by the following equation (the subscript indicates that firm A sets a uniform price while firm B price discriminates):

$$\begin{aligned}\Pi_{\text{UD}}^A &= \bar{p}^A \sum_{m=1}^{m^{\wedge\wedge}} \frac{1}{n} + \bar{p}^A\left[\frac{a+b}{4} - \frac{m^{\wedge}-1}{2n} - \frac{\bar{p}^A}{4t(b-a)} + \frac{1}{n}\right] \\ &= \bar{p}^A\left[\frac{a+b}{2} - \frac{\bar{p}^A}{2t(b-a)} + \frac{1}{2n}\right] \qquad (9)\end{aligned}$$

Maximizing Eq. (9) with respect to $\bar{p}^A$ we obtain the optimal uniform price set by the non-discriminating firm:

$$\bar{p}^{A*} = \frac{t(b-a)(a+b)}{2} + \frac{t(b-a)}{2n} \qquad (10)$$

Inserting Eq. (10) in Eq. (7), and substituting m with $m^{\wedge} - 1$ (in which we insert Eq. (10) again), we obtain the optimal discriminatory price in the only sub-segment in which both firms sell a positive amount. That is

$$p_{m^{\wedge}-1}^B = t(b-a)/n \qquad (11)$$

The demand of firm B in sub-segment $m^{\wedge} - 1$ is obtained inserting Eq. (11) in Eq. (5). It follows

$$d_{m^{\wedge}-1}^B = 1/2n$$

The optimal discriminatory prices in sub-segments $m \geq m^{\wedge}$ are obtained by solving $d_m^B(\bar{p}^{A*}) = 1/n$. It follows

$$p_m^{B*} = \frac{t(b-a)}{2}\left(\frac{4m-3}{n} - a - b\right)$$

The proof of the second part of the proposition proceeds in the same way. □

By Proposition 6, the profits of the two firms when one sets a uniform price while the other discriminates follow immediately.

Table 1 Equilibrium firms' profits in Game 1

Π^A	Π^B U	D
U	$\frac{t(b-a)(2+a+b)^2}{18}$; $\frac{t(b-a)(4-a-b)^2}{18}$	$\frac{t(b-a)\left[n^2(a+b)^2+2n(a+b)+1\right]}{8n^2}$; $\frac{t(b-a)\left[5-2n(4-a-b)+n^2(4-a-b)^2\right]}{16n^2}$
D	$\frac{t(b-a)\left[5-2n(2+a+b)+n^2(2+a+b)^2\right]}{16n^2}$; $\frac{t(b-a)\left[n^2(2-a-b)^2+2n(2-a-b)+1\right]}{8n^2}$	$\frac{t(b-a)\left[9n^2(a+b)^2-18n(a+b)+40\right]}{36n^2}$; $\frac{t(b-a)\left[9n^2(2-a-b)^2-18n(2-a-b)+40\right]}{36n^2}$

Corollary of Proposition 6 *if firm A has committed and firm B has not committed, the equilibrium profits are*

$$\Pi^A_{UD}* = t(b-a)\left[n^2(a+b)^2+2n(a+b)+1\right]/8n^2$$
$$\Pi^B_{UD}* = t(b-a)\left[n^2(4-a-b)^2-2n(4-a-b)+5\right]/16n^2$$

If firm A has not committed and firm B has committed, the equilibrium profits are

$$\Pi^A_{DU}* = t(b-a)\left[n^2(2+a+b)^2-2n(2+a+b)+5\right]/16n^2$$
$$\Pi^B_{DU}* = t(b-a)\left[n^2(2-a-b)^2+2n(2-a-b)+1\right]/8n^2$$

We can write the firms' profits directly as functions of a and b in the four possible cases: UU, UD, DU and DD.[6] We do it in Table 1:

We state the following proposition:

Proposition 7 *If* $n \geq 10$ *the unique equilibrium is characterized by both firms price discriminating; if* $n = 4$ *both firms setting a uniform price is an equilibrium.*

Proof [7]Consider $n \geq 10$. Let call the two firms generically "firm I" and "firm J" and their respective locations i and j, where $I, J \in \{A, B\}$ and $i, j \in \{a, b\}$: if in equilibrium $i < j$, then $I = A$ and $J = B$, while if in equilibrium $i > j$, then $I = B$ and $J = A$. We show that in the first stage of the game no location equilibrium exists that induces an asymmetric pricing policy equilibrium in the second stage. We start with the following observation coming directly from Table 1: if firms locate symmetrically, in the second stage of the game the equilibrium is DD. Consider now the necessary conditions on i and j for DU to arise in the second stage of the

[6] The profit functions in UU can be obtained by standard calculations (see Tirole 1988, p 281).

[7] Here we provide the intuition of the proof. The complete proof can be found in the Appendix.

game. The necessary conditions can be obtained by solving $\Pi_{DU}^{B} > \Pi_{DD}^{B}$. Without loss of generality, assume that the non discriminating firm is J and define the R^2-set inducing DU with S, that is, $S \equiv \{(i, j) \in R^2 : \Pi_{DU}^{B} > \Pi_{DD}^{B}, j > i\}$. Now, consider firm J: given i, is the best reply of firm J to set j such that $(i, j) \in S$? If at least one $\tilde{j}$ such that $(i, \tilde{j}) \notin S$ exists that guarantees higher profits to firm J when the rival plays i, then j such that $(i, j) \in S$ cannot be the best reply of firm J, and $(i, j) \in S$ cannot be an equilibrium. We consider $\tilde{j} = 1 - i$: since symmetric locations induce DD equilibrium in the second stage, we are sure that $(i, \tilde{j}) \notin S$. Then, we compare $\Pi_{DU}^{J}((i, j) \in S)$ with $\Pi_{DD}^{J}((i, \tilde{j}))$. It can be shown that $\Pi_{DU}^{J}((i, j) \in S)$ is always lower than $\Pi_{DD}^{J}((i, \tilde{j}))$: therefore, choosing j such that $(i, j) \in S$ cannot be the best reply of firm J when firm I plays i. No location equilibrium inducing DU can exist in the first stage of the game. The proof for the non-existence of location equilibria inducing UD in the second stage equilibrium is analogous. Moreover, it can be easily verified by looking at Table 1 that for any possible couple of locations, UU is never an equilibrium in the second stage of the game when $n \geq 10$. Therefore, only a location equilibrium inducing DD can exist in the first stage of the game. Consider now the location equilibrium defined in Proposition 3. When firm I plays $i*$, the equilibrium emerging in the second stage of the game is DD for any possible j (this can be easily verified by substituting i^* into Table 1 and noting that neither $\Pi_{DU}^{B} > \Pi_{DD}^{B}$ nor $\Pi_{UD}^{A} > \Pi_{DD}^{A}$ are possible). Therefore, firm J plays $j*$ as shown in Proposition 3, and (i^*, j^*) is the unique equilibrium. It follows that in the second stage of the game both firms choose not to commit. When $n = 4$ direct computations show instead that UU is an equilibrium, while neither UD nor DU are an equilibrium. Unfortunately, we have not been able to prove whether DD can arise as an equilibrium or not. □

Proposition 7 shows that the Prisoner Dilemma argument of Thisse and Vives (1988) and Eber (1997) cannot be extended for any degree of imperfectness of price discrimination: when price discrimination is highly imperfect, an equilibrium exists where the Prisoner Dilemma does not occur.

4.2 Game 2

Timing: At time 1 both firms simultaneously decide whether to commit or not; at time 2 both firms simultaneously choose the location; at time 3 both firms simultaneously choose the price schedule.

As usual, in order to solve the game we start from the last stage. We already have the equilibrium prices and locations when both firms set a uniform price (Proposition 1) and when both price discriminate (Propositions 2 and 3). Moreover, we already know the equilibrium prices when one firm has committed and the other has not (Proposition 6). It remains to calculate the equilibrium locations in the sub-games arising when only one firm has committed in the first stage. Equilibrium locations in these sub-games are defined by the following proposition:

Proposition 8 *If firm A has chosen U and firm B has chosen D, the equilibrium locations at the second stage of the game are* $a^* = \frac{1}{3} - \frac{1}{3n}$ *and* $b^* = 1$.

Table 2 Equilibrium firms' profits in Game 2

Π^A	Π^B U		Π^B D	
U	$\frac{t}{2};\ \frac{t}{2}$		$\frac{t(1+2n)^3}{54n^3};$	$\frac{t(1+2n)\left(5-4n+8n^2\right)}{54n^3}$
D	$\frac{t(1+2n)\left(5-4n+8n^2\right)}{54n^3};$	$\frac{t(1+2n)^3}{54n^3}$	$\frac{t\left(9n^2-18n+40\right)^2}{648n^3(n-1)};$	$\frac{t\left(9n^2-18n+40\right)^2}{648n^3(n-1)}$

If firm A has chosen D and firm B has chosen U, the equilibrium locations at the second stage of the game are $a^*=0$ *and* $b^*=\frac{2}{3}+\frac{1}{3n}$.[8]

Proof Suppose that firm A has chosen U and firm B has chosen D in the first stage of the game. Then, taking the derivative of $\Pi^B_{\text{UD}}*$ with respect to b, the result is $\partial\Pi^B_{\text{UD}}*/\partial b = t[5+4n(b-2)+n^2(16-16b-a^2+2ab+3b^2)]/16n^2$, which is always positive. Therefore, the discriminating firm, B, locates at the right endpoint of the market. Substituting $b^*=1$ into $\Pi^A_{\text{UD}}*$ and maximizing it with respect to a, we get $a^*=1/3-1/3n$. The second part of the proposition can be proved in the same way. □

Since we have the equilibrium prices (third stage) and the equilibrium locations (second stage) in all possible cases, we can write the equilibrium profits of each firm directly as functions of the pricing policy at the first stage of the game, by substituting the equilibrium prices and the equilibrium locations in the appropriate profit functions. The equilibrium profits are summarised in Table 2.

We state the following proposition:

Proposition 9 *The* (*unique*) *sub-game perfect Nash equilibrium entails uniform pricing by both firms.*

Proof Suppose that firm A chooses U. The pattern of the profits of firm B as function of n when it chooses U and when it chooses D is depicted in Fig. 2.[9] Suppose now that firm A chooses D. The pattern of the profits of firm B as function of n when it chooses U and when it chooses D is depicted in Fig. 3. From Figs. 2 and 3 it follows immediately that firm B always prefers to commit, regardless of firm A's pricing policy.

□

Therefore, when the pricing policy is decided before the location, discriminatory prices do not arise in equilibrium, independently of the numbers of partitions of the consumers.

[8] In order to verify that $m^\wedge$ and m° are integer under the equilibrium locations, substitute the appropriate equilibrium locations from Proposition 8 into $m^\wedge$ and m°. We get $m^\wedge=(n+5)/3$ and $m^\circ=(2n-2)/3$, which are always integer when $n=4+6k$, with $k=0,1,2,3,4\ldots$.

[9] Both in Fig. 2 and in Fig. 3, the graphs are drawn for continuous n though we consider only a subset of n.

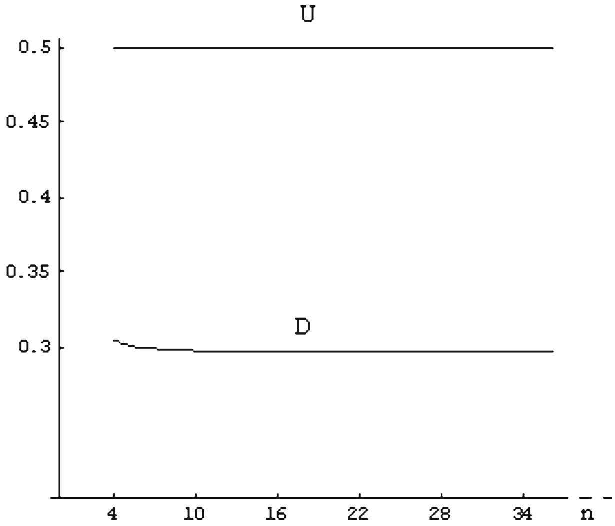

Fig. 2 Firm *B*'s profits when firm *A* chooses U

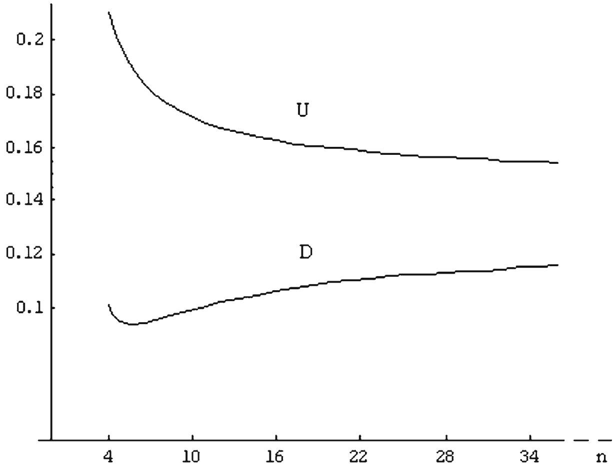

Fig. 3 Firm *B*'s profits when firm *A* chooses D

In what follows, we provide the intuition of the main result of this paper. When locations are fixed at the pricing policy stage (Game 1), the possibility to price discriminate creates two effects: on the one hand, each firm is able to extract more consumer surplus when it can customize the prices, but on the other hand flexibility induces firms to compete more fiercely.[10] When price discrimination is sufficiently precise the first effect dominates and each firm decides to price discriminate.

When locations are chosen after the choice of the pricing policy (Game 2), a third effect arises: we call it the *spatial competition effect*. When firms do not commit,

[10] See Ulph and Vulkan (2000) for more details about these effects.

they are nearer each other in equilibrium, and then competition is fiercer: this induces each firm to commit to uniform pricing at the first stage of the game. Therefore, the occurrence of the *spatial competition effect* seems responsible for the emerging of the uniform pricing equilibrium in Game 2, because it makes price discrimination less attractive for firms. However, one may ask why the *spatial competition effect* arises, or, in other words, why distance is lower when firms price discriminate. In what follows we give the intuition by analysing the incentives that drive the location choice of firm A (for firm B the reasoning is analogous) in the limit case of perfect price discrimination ($n \rightarrow \infty$) by both firms.[11] We observe three different effects. First, firm A wants to move toward the centre because in this way some consumers, which otherwise would be nearer to firm B, become nearer to firm A. This effect is analogous to the *demand effect* emerging under uniform pricing (Tirole 1988, p 281). Consider now the effect of a movement of firm A toward the centre on the equilibrium prices applied on those consumers which are nearer to firm A even before the movement. Note that when $n \rightarrow \infty$ each sub-segment is infinitely small and reduces to the individual consumer. Therefore, there cannot be consumers "shared" between the two firms: each consumer patronizes one firm. Moreover, when $n \rightarrow \infty$ we get $m = nx$. This allows us to rewrite the equilibrium price schedule defined in Proposition 2 as $p_x^{A*} = t(b-a)(a+b-2x)$. This makes clear that the price on consumer x depends on two factors. On the one hand, it depends on the distance between the two firms (the first term): the more the firms are distant, the higher is the price applied on consumer x, because the difference between the two firms perceived by the consumer is higher (and therefore the advantage of firm A over firm B is higher). Clearly, this effect (we call it the *increasing distance effect*) stimulates firm A to move toward the endpoint of the segment. On the other hand, the price applied on consumer x depends on the distance between consumer x and firm A (second term): for a given couple of locations, the more the consumer is near to the firm the higher is the price. This effect (we call it the *minimizing transportation costs effect*) stimulates firm A to minimize its distance from the consumers buying from it. Note that this last incentive is absent under uniform pricing, because in that case the (uniform) price depends only on the distance between the firms, but not on the distance between the firm and its consumers. To sum up, when both firms price discriminate the *demand effect* pushes the firms towards the centre of the segment, the *increasing distance effect* pushes the firms towards the endpoints of the segment, and the *minimizing transportation costs effect* pushes each firm towards the middle of its respective markets. In the perfect price discrimination case, the equilibrium between these three forces arises when firms choose respectively locations at 1/4 and 3/4.

5 Conclusion

We extend the perfect price discrimination analysis of Thisse and Vives (1988) and Eber (1997) to a general framework of imperfect price discrimination encompassing perfect price discrimination as a limit case. We obtain the following results. If firms

[11] The intuition is similar for the cases of asymmetric pricing policies.

cannot commit to uniform pricing before competing on price, price discrimination emerges as the unique sub-game perfect equilibrium (Proposition 2). Firms locate, respectively, in the proximity of 1/4 and 3/4 (Proposition 3). Equilibrium discriminatory prices are all lower than the equilibrium uniform price of a two-stage location-price game where price discrimination is impossible, for any degree of imperfectness of price discrimination (Proposition 4). Therefore, *all-out competition* (Corts 1998) is shown to occur regardless of the degree of imperfectness of price discrimination. If firms can commit to uniform pricing before competing on price but after choosing the location, the unique equilibrium is characterized by price discrimination provided that price discrimination is precise enough: if price discrimination is highly imprecise, an equilibrium exists where both firms commit not to price discriminate (Proposition 7). On the contrary, if firms can commit to uniform pricing before competing on price and before choosing the location, the unique equilibrium is characterized by uniform pricing for any degree of imperfectness of price discrimination (Proposition 9).

Appendix

Proof of Proposition 7 Consider $n \geq 10$. The necessary (though not sufficient) conditions on a and b for DU to arise in the second stage of the game can be obtained by solving $\Pi^B_{\mathrm{DU}} > \Pi^B_{\mathrm{DD}}$. For this inequality to hold, a, b and n have to belong to one of the following sets:

- $$1 - \frac{\sqrt{10}+9}{3n} < a < 1 - \frac{\sqrt{5/2}}{3n} - \frac{3}{2n} \quad \text{and} \quad -\frac{\sqrt{10}+9}{3n} + 2 - a < b \leq 1$$

- $$1 - \frac{\sqrt{5/2}}{3n} - \frac{3}{2n} \leq a \leq 1 + \frac{\sqrt{10}-9}{3n} \quad \text{and} \quad a < b \leq 1$$

- $$1 + \frac{\sqrt{10}-9}{3n} < a < 1 + \frac{\sqrt{5/2}}{3n} - \frac{3}{2n} \quad \text{and} \quad a < b < \frac{\sqrt{10}-9}{3n} + 2 - a$$

Note that for DU to arise, a has to be particularly high, since the lowest admissible value for a is $a = 7/10 - 1/3\sqrt{10} \cong 0, 59$. Now, consider firm B. Suppose that firm A locates at a belonging to one of the sets defined above, and that firm B *wants to induce* the DU equilibrium. Which is firm B's optimal location? By maximizing Π^B_{DU} with respect to b we get $b = \tilde{b} \equiv [1 + n(2 + a)]/3n$. Provided that firm B wants to induce the equilibrium DU, if a and $\tilde{b}$ belong to one of the sets defined above, $\tilde{b}$ is the optimal choice of firm B. However, it may be that, given a, $\tilde{b}$ does not induce the equilibrium DU because it is outside all the sets defined above. In this case, if $\tilde{b}$ is higher than the highest b which induces the equilibrium DU given a, recalling that the profits of B

are increasing for any b lower than $\tilde{b}$, firm B chooses the most to the right location among those locations which induce DU given a. On the contrary if $\tilde{b}$ is lower than the lowest b which induces the equilibrium DU given a, firm B chooses the most to the left location among those locations which induce DU given a, because the profits of B are decreasing for any b higher than $\tilde{b}$. Therefore, the possible equilibrium firm B's locations are the following:[12]

(1) $b = \tilde{b}$

(2) $b = 1$

(3) $b = 2 - a - \dfrac{\sqrt{10} + 9}{3n}$

(4) $b = 2 - a + \dfrac{\sqrt{10} - 9}{3n}$

However, given a, firm B may decide to leapfrog the rival from the right and locate at the left of firm A. Suppose in particular that firm B chooses a symmetric location with respect to firm A. In this case direct calculations in Table 1 show that when $n \geq 10$, DD emerges as the unique equilibrium in the second stage of the game.[13] By leapfrogging, firm B obtains the following profits:[14]

$$\hat{\Pi}^B_{\mathrm{DD}} = \frac{t(2a-1)(40 - 18n + 9n^2)}{36n^2}$$

If $\hat{\Pi}^B_{\mathrm{DD}}$ is higher than the profits induced in the case of DU, it means that at least one location exists for firm B that guarantees higher profits by inducing DD rather than inducing DU. In this case DU cannot be an equilibrium in the second stage since firm B never chooses an equilibrium location which induces DU, but prefers to leapfrog firm A from the right and induce DD in the second stage. Now we proceed to compare case by case.

(1) $b = \tilde{b}$. Equilibrium profits of firm B are: $\Pi^{B,1}_{\mathrm{DU}} = t[1 + 2n(1-a)]^3/54n^3$.

Define: $\Gamma^1 \equiv \Pi^{B,1}_{\mathrm{DU}} - \hat{\Pi}^B_{\mathrm{DD}} = [1 + 2n(1-a)]^3/54n^3 - t(2a-1)(40 - 18n + 9n^2)/36n^2$. We have to prove that Γ^1 is always negative. First, by taking the derivative of Γ^1 with respect to n we get $\frac{\partial \Gamma^1}{\partial n} = -\frac{t[1+n(44-84a)-n^2(5-10a-4a^2)]}{18n^4}$, which

[12] Note that also $b = a$ is a priori a potential candidate for being an optimal firm B's location. However, it is noteworthy that $\tilde{b} \equiv [1 + (2 + a)n]/3n > a$. Therefore, firm B never chooses to locate in the same point of firm A.

[13] Suppose symmetric locations. The difference between Π^B_{UU} and Π^B_{UD} is $[t(1-2a)(6n - 5 - n^2)]/16n^2$, which is decreasing in n: since it is negative when $n = 10$, it is negative for all $n \geq 10$. The difference between Π^B_{DU} and Π^B_{DD} is given by $[t(1-2a)(54n - 71 - 9n^2)]/72n^2$, which is decreasing in n: since it is negative when $n = 10$, it is negative for all $n \geq 10$.

[14] Note that, given the leapfrog, the identity of the two firms should be inverted. In order to avoid confusion, we maintain the same notation used until now when indicating the two firms. However, one should recall that firm B is now located at the left of firm A. Moreover, note that profits are positive, since $a > 1/2$.

is strictly negative. Therefore, we can limit the analysis to the lowest n, that is $n = 10$. By taking the derivative of $\Gamma^1(n = 10)$ with respect to a, we get $\partial\Gamma^1(n = 10)/\partial a = -t(821 - 840a + 400a^2)/900$, which is always negative. Therefore, we can consider the lowest value of a. The lowest value of a inducing the DU equilibrium is $a = 7/10 - 1/3\sqrt{10}$. By substituting the lowest value of a into Γ^1, we get $\Gamma^1 = t(-111{,}339 + 23{,}246\sqrt{10})/1{,}458{,}000 < 0$.

(2) $b = 1$. Equilibrium profits of firm B are $\Pi_{\text{DU}}^{B,2} = t(1-a)[1+(1-a)n]^2/8n^2$.

Define $\Gamma^2 \equiv \Pi_{\text{DU}}^{B,2} - \hat{\Pi}_{\text{DD}}^{B} = t(1-a)[1+(1-a)n]^2/8n^2 - t(2a-1)(40-18n+9n^2)/36n^2$. We have to prove that Γ^2 is always negative. First, by taking the derivative of Γ^2 with respect to n we get $\frac{\partial\Gamma^2}{\partial n} = -\frac{t[89-9n+9na^2+a(18n-169)]}{36n^3}$, which is strictly negative. Therefore, we can limit the analysis to the lowest n, that is, $n = 10$. By taking the derivative of $\Gamma^2(n = 10)$ with respect to a, we get $\partial\Gamma^2(n = 10)/\partial a = -t(6{,}109 - 5{,}760a + 2{,}700a^2)/7{,}200$, which is always negative. Therefore, we can consider the lowest value of a. The lowest value of a inducing the DU equilibrium is $a = 7/10 - 1/3\sqrt{10}$. By substituting this value into Γ^2, we get $\Gamma^2 = t(-8{,}307 + 1{,}705\sqrt{10})/108{,}000 < 0$.

(3) $b = 2 - a - (\sqrt{10} + 9)/3n$.

Equilibrium profits of firm B are: $\Pi_{\text{DU}}^{B,3} = -t[9 + n(6a - 6 + \sqrt{10}/n)](12\sqrt{10} + 77)/108n^3$. Define $\Gamma^3 \equiv \Pi_{\text{DU}}^{B,3} - \hat{\Pi}_{\text{DD}}^{B} = -\frac{t[9+n(6a-6+\sqrt{10}/n)](12\sqrt{10}+77)}{108n^3} - \frac{t(2a-1)(40-18n+9n^2)}{36n^2}$. We have to prove that Γ^3 is always negative. Consider first $n \geq 16$. Taking the derivative of Γ^3 with respect to n we get $\frac{\partial\Gamma^3}{\partial n} = \frac{t[813+n(468a-388+185\sqrt{10}/n)+6n^2(3+a(8\sqrt{10}/n-6)-8\sqrt{10}/n)]}{36n^4}$, which is strictly negative. Therefore, we can limit the analysis to the lowest n, which is $n = 16$. By taking the derivative of $\Gamma^3(n = 16)$ with respect to a, we get $\partial\Gamma^3(n = 16)/\partial a = -t(711 + 4\sqrt{10})/1{,}536$, which is always negative. Therefore, we can consider the lowest value of a. The lowest value of a among those inducing the DU equilibrium is $a = 1 - (\sqrt{10} + 9)/3n$ which is equal to $a = 13/16 - \sqrt{5/2}/24$ when $n = 16$. By substituting it into $\Gamma^3(n = 16)$, we get $\Gamma^3 = t(-60{,}867 + 4{,}297\sqrt{10})/442{,}368 < 0$. Consider now $n = 10$. By taking the derivative of $\Gamma^3(n = 10)$ with respect to a we get $\partial\Gamma^3/\partial a = -t(279 + 4\sqrt{10})/600$, which is always negative. Then, we can consider the lowest value of a. The lowest value of a inducing the DU equilibrium is $a = 7/10 - 1/3\sqrt{10}$. By substituting it into $\Gamma^3(n = 10)$, we get $\Gamma^3 = t(-8{,}307 + 1{,}705\sqrt{10})/108{,}000 < 0$.

(4) $b = 2 - a + (\sqrt{10} + 9)/3n$.

Equilibrium profits of firm B are $\Pi_{\text{DU}}^{B,4} = t[n(6 - 6a + \sqrt{10}/n) - 9](77 - 12\sqrt{10})/108n^3$. Define $\Gamma^4 \equiv \Pi_{\text{DU}}^{B,4} - \hat{\Pi}_{\text{DD}}^{B} = \frac{t[n(6-6a+\sqrt{10}/n)-9](77-12\sqrt{10})}{108n^3} - \frac{t(2a-1)(40-18n+9n^2)}{36n^2}$. We have to prove that Γ^4 is always negative. Consider first $n \geq 16$. By taking the derivative of Γ^4 with respect to n we get: $\frac{\partial\Gamma^4}{\partial n} = \frac{t[813+n(468a-388-185\sqrt{10}/n)+6n^2(3-a(6+8\sqrt{10}/n)+8\sqrt{10}/n)]}{36n^4}$, which is strictly negative.

Therefore, we can limit the analysis to the lowest n, that is, $n = 16$. By taking the derivative of $\Gamma^4(n = 16)$ with respect to a, we get $\partial\Gamma^4(n = 16)/\partial a = t(-711 + 4\sqrt{10})/1{,}536$, which is always negative. Therefore, we can consider the lowest value of a. The lowest value of a inducing the DU equilibrium is $a = 1 + (\sqrt{10} + 9)/3n$, which is equal to $a = 13/16 + \sqrt{5/2}/24$ when $n = 16$. By substituting this value into $\Gamma^4(n = 16)$, we get $\Gamma^4 = -t(60{,}867 + 4{,}297\sqrt{10})/442{,}368 < 0$. Consider now $n = 10$. By taking the derivative of $\Gamma^4(n = 10)$ with respect to a we get $\partial\Gamma^4/\partial a = -t(279 - 4\sqrt{10})/600$, which is always negative. Therefore, we can consider the lowest value of a. The lowest value of a inducing the DU equilibrium is $a = 7/10 + 1/3\sqrt{10}$. By substituting this value into $\Gamma^4(n = 10)$, we get $\Gamma^4 = -t(8{,}307 + 1{,}705\sqrt{10})/108{,}000 < 0$.

Therefore, firm B can always obtain higher profits by leapfrogging firm A from the right and inducing the equilibrium DD rather than inducing the equilibrium DU. In the same way, it can be shown that A can always obtain higher profits by leapfrogging firm B from the left and inducing the equilibrium DD rather than inducing the equilibrium UD.

Now we show that UU never arises in the second stage when $n \geq 10$. Consider firm B profits in the case of UU and in the case of UD.

Define: $Z \equiv \Pi^B_{\mathrm{UU}} - \Pi^B_{\mathrm{UD}} = t(b-a)(4-a-b)^2/18 - t(b-a)[5-2n(4-a-b) + n^2(4-a-b)^2]/16n^2$. By taking the derivative of Z with respect to n we get $\partial Z/\partial n = t(b-a)[5 - n(4-a-b)]/8n^3$, which is always negative for $n \geq 10$. Therefore, if Z is negative for $n = 10$, it must be negative for any other n. By substituting $n = 10$ into Z we get $Z = t[20a^3 - 4a^2(31 - 5b) - b(185 - 124b + 20b^2) + 5a(37 - 4b^2)]/2{,}880$ which is always negative given that $1 \geq b \geq a \geq 0$. Therefore, firm B always chooses D when firm A chooses U. Similarly, firm A always chooses D when firm B chooses U. It follows that UU cannot be an equilibrium for any possible couple of locations.

We consider now the case where $n = 4$. Suppose that firm A plays U. Firm B plays D when $\Pi^B_{\mathrm{UD}} > \Pi^B_{\mathrm{UU}}$. By solving the inequality, we get the following conditions on the variables for firm B playing D when firm A plays U: $0 \leq a < 1/8 \cap a < b < 1/4 - a$. Under symmetric locations instead, firm B plays U when firm A plays U (this can be observed by substituting $b = 1 - a$ into the relevant payoff functions and noting that $\Pi^B_{\mathrm{UD}} < \Pi^B_{\mathrm{UU}}$). Under symmetric locations, firm B's profits are, therefore, $\Pi^B_{\mathrm{UU}} = t(1 - 2a)/2$. Define: $G \equiv \Pi^B_{\mathrm{UU}}(a, 1-a) - \Pi^B_{\mathrm{UD}}(a, b; n = 4)$. Taking the derivative of G with respect to a we get $\partial G/\partial a = t[48a^2 - 27 - 16b^2 + 16a(2b - 15)]/256$, which is negative for all the admissible values of a and b. Therefore, if G is positive when $a \to 1/8$ (the highest value of a under which the equilibrium UD may arise in the second stage of the game), it has to be positive for all the other values of a. Note that when $a \to 1/8$, the only value of b which induces the equilibrium UD is $b \to 1/8$. By substituting $a = 1/8$ and $b = 1/8$ into G we get $3t/8$. Therefore, G is positive for all the values of a and b belonging to the set inducing the equilibrium UD in the second stage of the game. It follows that firm B never chooses D when firm A plays U, since it prefers to mimic the strategy of firm A. As a consequence, UU emerges as an equilibrium in the second stage of the game. Clearly, neither DU nor UD can be equilibria. □

References

Corts K (1998) Third-degree price discrimination in oligopoly: all-out competition and strategic commitment. RAND J Econ 29:306–323
D'Aspremont C, Gabszewicz JJ, Thisse JJ (1979) On Hotelling stability in competition. Econometrica 47:1145–1150
Eber N (1997) A note on the strategic choice of spatial price discrimination. Econ Lett 55:419–423
Hotelling H (1929) Stability in competition. Econ J 39:41–57
Liu Q, Serfes K (2004) Quality of information and oligopolistic price discrimination. J Econ Manag Strateg 13:671–702
Thisse JF, Vives X (1988) On the strategic choice of spatial price policy. Am Econ Rev 78:122–137
Tirole J (1988) The theory of industrial organization. MIT Press, Cambridge
Ulph D, Vulkan N (2000) Electronic commerce and competitive first-degree price discrimination, *wp*. Economic and Social Research Council (ESRC) Centre, London