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Institution Morphisms

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Abstract. Institutions formalise the intuitive notion of logical system, including syntax, semantics, and the relation of satisfaction between them. Our exposition emphasises the natural way that institutions can support deduction on sentences, and inclusions of signatures, theories, etc.; it also introduces terminology to clearly distinguish several levels of generality of the institution concept. A surprising number of different notions of morphism have been suggested for forming categories with institutions as objects, and an amazing variety of names have been proposed for them. One goal of this paper is to suggest a terminology that is uniform and informative to replace the current chaotic nomenclature; another goal is to investigate the properties and interrelations of these notions in a systematic way. Following brief expositions of indexed categories, diagram categories, twisted relations and Kan extensions, we demonstrate and then exploit the duality between institution morphisms in the original sense of Goguen and Burstall, and the 'plain maps' of Meseguer, obtaining simple uniform proofs of completeness and cocompleteness for both resulting categories. Because of this duality, we prefer the name 'comorphism' over 'plain map'; moreover, we argue that morphisms are more natural than comorphisms in many cases. We also consider 'theoroidal' morphisms and comorphisms, which generalise signatures to theories, based on a theoroidal institution construction, finding that the 'maps' of Meseguer are theoroidal comorphisms, while theoroidal morphisms are a new concept. We introduce 'forward' and 'semi-natural' morphisms, and develop some of their properties. Appendices discuss institutions for partial algebra, a variant of order sorted algebra, two versions of hidden algebra, and a generalisation of universal algebra; these illustrate various points in the main text. A final appendix makes explicit a greater generality for the institution concept, clarifies certain details and proves some results that lift institution theory to this level.

Keywords: Abstract model theory; Category theory; Institution; Kan extension; Logic; Specification

1. Introduction

Many different logics are used in computer science, including (many variants of) first order, second order, higher order, Horn clause, type theoretic, equational, temporal, modal and infinitary logics. The goal of institution theory is to do as much computer science (and model theory) as possible, independent of what the underlying logic may be [GoB92]. Institutions formalise the notion of a logical system with varying non-logical

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symbols (collections of such symbols are traditionally called ‘signatures’ in this field). The main ingredient of an institution is a satisfaction relation between its models and its sentences, an abstract form of Tarski’s classic semantic definition of truth [Tar44], and the main requirement is that this relation should be consistent with respect to signature morphisms, which intuitively means that satisfaction is invariant under change of notation. The formalisation only assumes abstract categories (or classes) of signatures, sentences and models, without assuming any particular structure for them; the covariance of sentences and contravariance of models under change of signature is captured by appropriate functors. The literature shows that many interesting results can be developed at this level of generality.

Many papers have been written on institutions, both theoretical and applied, in the more than twenty years since the earliest formulation [BuG78, BuG80]. For example, institutions have been used to study lambda calculus, second order logic and many variants of equational logic, modal logic, higher order logic and first order logic. Mosses showed that his unified algebra forms an institution [Mos89], Goguen showed that hidden algebra forms an institution [Gog91], Roşu gave an institution for order sorted algebra [Ros94], and Mossakowski gave a hierarchy of institutions for total, partial and order sorted algebras [Mos96b]. The main original paper on institutions [GoB92] contains several significant results, including generalisations and equivalent formulations of institutions, cocompleteness for categories of theories, colimit preservation for the functor on theories induced by a signature morphism, a theory of constraints (including freeness and generation constraints, among others), and a ‘borrowing’ result, which allows passing theorem proving tasks from one institution to another; moreover, [GoB92] introduced deductions as sentence morphisms (see the discussion after Definition 3.1), despite the apparently common misconception that institutions do not handle deduction.

One important application of institutions is a uniform approach to modularisation for specifications; in fact, this was a major motivation for institutions [BuG78, GoB84a, GoB92]; among many papers on this topic, we mention [DGS93, GoT00] and [Ros97a], which each add inclusion systems to institutions. The papers [Gog85] and [GoT00] extend this approach from specification to programming. Much other interesting work with institutions has been done by Tarlecki [Tar84, Tar86a, Tar86b, Tar86c, Tar96a], Sannella and Tarlecki [SaT86, SaT87, SaT88a], Cerioli [Cer93], Mossakowski [Mos96b] and Diaconescu [Dia94, Dia96, Dia98], among others; [Tar96a] in particular is an important paper having some goals and results similar to those of this paper. Burstall and Diaconescu [BuD94] generalise ‘hiding’ from algebra to an arbitrary institution, and use this to obtain hidden institutions for both many sorted and order sorted algebra.

Many variations on the institution concept have appeared. For example, Mayoh introduced ‘galleries’ [May85], which Goguen and Burstall further extended to ‘generalized institutions’ [GoB86], allowing non-Boolean values for satisfaction. Poigné’s ‘foundations’ and ‘rich institutions’ [Poi89] further abstracted institutions by requiring that sentences form a fibration, although this gets very complex; Fiadeiro and Sernadas [FiS88] introduced ‘π-institutions’ and Meseguer [Mes89] studied ‘general logics’, each combining institutions with classical entailment relations; [Mes89] is a gem that contains many interesting ideas. Salibra and Scollo introduced ‘pre-institutions’ [SaS92], where the ‘iff’ in the satisfaction condition is split into two implications, which are then studied separately, combined or both dropped; Ehrig et al. introduced ‘specification logics’ [EBC92], which are indexed categories of models, with no sentences; Căzănescu introduced ‘truth systems’ [Caz94], a sort of compromise between institutions and the charters of [GoB86], which allow inference in a designated model; and Pawlowski introduced ‘context institutions’ [Paw96] to deal with variable contexts and substitutions. Diaconescu introduced ‘many sorted institutions’ [Dia94], which assign a sort set to each abstract signature, and Grothendieck (or fibred) institutions [Dia00], which combine multiple institutions in a single structure; the latter was developed for the semantics of the multi-paradigm CafeOBJ language [DiF02].

Although many variants of institutions have interesting properties and are no doubt worth studying, some can be seen as special kinds of institution in the classic sense,[1] and others have close natural relationships to the classic institutions. It seems to us that the original institution concept captures the essence of logical system, which is the intimate dance between syntax and semantics, including deduction. There are tendencies both to focus on syntax at the expense of semantics, and on semantics at the expense of syntax; the first occurs especially in intuitionistic logic and type theory, while the second is more common in some areas of theoretical computer science; we may call these the ‘weak variants’. We consider institutions to be a careful balance of syntax and semantics, and we believe that most concepts and results are easily adapted to those variants and generalisations that preserve this viewpoint (note that the notions of charter and parchment

[1] For example, [GoB86] shows that the ‘π-institutions’ of [FiS88] are a special kind of institution.

from [GoB86] formalise genuinely different notions, which are still closely related to institutions, in that they generate institutions). We feel that most of the weak variants of institution are somehow pathological.[2] Section 3 and especially Appendix E of this paper discuss the 'close variants' of the institution concept, which share its main mathematical properties.

Over the last fifteen years, there have been even more variations on institution morphisms than on institutions, even discounting those that are adaptations of morphism concepts to other institution-like formalisms; moreover, these notions have been given many different names, including morphism, map, mapping, coding, encoding, representation, representation map, embedding, simulation, transformation and more, most of which do little or nothing to suggest their nature. This paper tries to bring some order to this chaos by exploring the properties and interrelationships of these notions, and by proposing names that suggest their meaning. Goguen and Burstall introduced 'morphisms' [GoB92], which are perhaps the most natural, because they include structure forgetting (and hence embedding or representation); but because institution morphisms in this sense do not capture all the important relationships, researchers have introduced many other notions. Perhaps the most important of these is the dual of institution morphisms, introduced by Meseguer [Mes89] under the name 'plain map', and later renamed 'representation' by Tarlecki [Tar96a] and 'plain representation' by Mossakowski [Mos96b], but because of the duality we prefer the name *comorphism*. Cerioli introduced the weaker notion of 'simulations' [Cer93], Tarlecki introduced 'codings' [Tar87], a further weakening, and Meseguer introduced 'simple institution maps' [Mes89], which generalise comorphisms by mapping signatures to theories; some variations, including 'conjunctive maps' which take a sentence to a set of sentences, were studied by Mossakowski [Mos96b], who with Kreowski also introduced 'embeddings of institutions' [KrM95], to formalise equivalence of logical frameworks; Sannella and Tarlecki introduced 'semi-morphisms' [SaT88b, Tar96a], which only have models, for relating specification and implementation languages, and Salibra and Scollo introduced 'transformations' [SaS92], which map models to sets of models. Diaconescu introduced 'extra theory morphisms' [Dia98] for the semantics of multi-paradigm languages like CafeOBJ [DiF98]. It is extremely helpful to look at examples to gain an understanding of this rocky terrain, and we shall often do so.

We had originally hoped to survey and systematise all the distinct notions of morphism that apply to the close variants of institutions; although we found even this limited goal impractical, we do hope to have covered the most important notions. Section 2 gives brief expositions of indexed and diagram categories, twisted relations and Kan extensions, followed in Section 3 by several equivalent definitions for institutions and their 'close variants', especially as functors from signatures to twisted relations; a subsection considers 'inclusive institutions', which support a natural notion of inclusion for signatures and theories. The functor formulation allows easy proofs for completeness and cocompleteness results in Section 4; we also advance and support the hypothesis that morphisms are in general more natural than comorphisms. Section 5 considers 'theoroidal' morphisms and comorphisms, which lift signature morphisms to theory morphisms; what we call theoroidal comorphisms were introduced by Meseguer, while theoroidal morphisms appear to be a new concept. Section 6 introduces another new notion, forward morphisms, while Section 7 considers semi-natural morphisms and comorphisms, which weaken morphisms by removing one naturality condition. Section 8 summarises the paper, and lists some open problems. Appendices A and B discuss partial algebra, a variant of order sorted algebra that supports partiality, their corresponding institutions, and an institution morphism between them; Appendix C gives two institutions for hidden algebra, while Appendix D introduces a new abstract institution for universal algebra. The institutions in these appendices, which draw on the authors' prior work on certain concrete applications, are used in some examples in the body of this paper. Appendix E gives an abstract formulation for institutions which makes the notion of close variant precise, proves some results at this level of generality, and clarifies certain details of the institution concept.

2. Preliminaries

We assume the reader familiar with basic categorical concepts, including limits, colimits, functor categories and adjoints. We use semicolon for morphism composition, written in diagrammatic order, that is, if $f\colon a \to b$ and $g\colon b \to c$ are morphisms, then $f;g\colon a \to c$ is their composition. $\mathbf{C}(a,b)$ denotes the morphisms $a \to b$

[2] For example, the main example used to motivate the 'pre-institutions' of [SaS92] is an unnatural version of hidden algebra where the morphisms fail to preserve all the relevant structure.

in category **C**, and we let $|\mathbf{C}|$ denote the objects of **C**; also we use ';' for vertical composition of natural transformations and '⨾' for their horizontal composition. **Cat** and **Set** denote the categories of all locally small categories (i.e., those with a *set* of morphisms between any two objects) and of sets, respectively, while **cat** denotes the category of all small categories. Similarly, **SET** denotes the category of classes (including both small and 'large' sets) and **CAT** denotes the category of all (large and small, not necessarily locally small) categories. All of these (and thus also their opposites) are both complete and cocomplete [Lan71].

2.1. Indexed Categories

Institutions, with their variation of syntax and semantics over signatures of non-logical symbols, are an instance of a general categorical notion capturing structures that vary over other structures. Let **Ind** be any category, with objects that we will call **indices**.

Definition 2.1. An **indexed category** is a functor $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$; when $i \in |\mathbf{Ind}|$, we may write C_i for $C(i)$. Given an indexed category C, then $\mathbf{Flat}(C)$ is the category having pairs (i, a) as objects, where i is an object in **Ind** and a is an object in C_i, and having pairs $(\alpha, f)\colon (i, a) \to (i', a')$ as morphisms, where $\alpha \in \mathbf{Ind}(i, i')$ and $f \in C_i(a, C_\alpha(a'))$.

The flattening of an indexed category is often called the **Grothendieck construction**, after [Gro63], though the notion of indexed category used there is much more general than the one used here. The following gives sufficient conditions for the flattening of an indexed category to be complete or cocomplete [TBG91]:

Theorem 2.2. If $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ is an indexed category, then:

1. If **Ind** is complete, if C_i is complete for each $i \in |\mathbf{Ind}|$, and if $C_\alpha\colon C_j \to C_i$ is continuous for each $\alpha\colon i \to j$, then $\mathbf{Flat}(C)$ is complete.
2. If **Ind** is cocomplete, if C_i is cocomplete for each $i \in |\mathbf{Ind}|$, and if $C_\alpha\colon C_j \to C_i$ has a left adjoint for each $\alpha\colon i \to j$, then $\mathbf{Flat}(C)$ is cocomplete.

Given an indexed category $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$, define the indexed category $C^{op}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ by C^{op}_i is $(C_i)^{op}$ and $C^{op}_\alpha\colon C^{op}_j \to C^{op}_i$ is $(C_\alpha)^{op}$ for $\alpha \in \mathbf{Ind}(i, j)$. The following is immediate from Theorem 2.2, but is worth stating explicitly because it is so easy to become confused by the dualities that are involved:

Corollary 2.3. If $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ is an indexed category, then:

1. If **Ind** is complete, if C_i is cocomplete for each $i \in |\mathbf{Ind}|$, and if $C_\alpha\colon C_j \to C_i$ is cocontinuous for each $\alpha\colon i \to j$, then $\mathbf{Flat}(C^{op})$ is complete.
2. If **Ind** is cocomplete, if C_i is complete for each $i \in |\mathbf{Ind}|$, and if $C_\alpha\colon C_j \to C_i$ has a right adjoint for each $\alpha\colon i \to j$, then $\mathbf{Flat}(C^{op})$ is cocomplete.

2.2. Functor Categories and Kan Extensions

Given categories **T** and **S**, let $\mathbf{T}^{\mathbf{S}}$ denote the category of functors from **S** to **T**, with natural transformations as morphisms, and for any functor $\Phi\colon \mathbf{S} \to \mathbf{S}'$, let $\mathbf{T}^{\Phi}\colon \mathbf{T}^{\mathbf{S}'} \to \mathbf{T}^{\mathbf{S}}$ denote the functor defined by $\mathbf{T}^{\Phi}(I') = \Phi; I'$ for a functor $I'\colon \mathbf{S}' \to \mathbf{T}$, and by $\mathbf{T}^{\Phi}(\sigma) = 1_\Phi ⨾ \sigma$ for a natural transformation $\sigma\colon I' \Rightarrow J'$, where $I', J'\colon \mathbf{S}' \to \mathbf{T}$ are functors. Also let $\mathbf{T}-\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$ denote the functor that takes a category **S** to $\mathbf{T}^{\mathbf{S}}$ and a functor $\Phi\colon \mathbf{S} \to \mathbf{S}'$ to $\mathbf{T}^{\Phi}$. Note that $\mathbf{T}-\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$ is an indexed category for any category **T**.

Proposition 2.4. If **T** is complete (cocomplete) then $\mathbf{T}^{\mathbf{S}}$ is complete (cocomplete) for any category **S**, and $\mathbf{T}^{\Phi}$ is continuous (cocontinuous) for any functor $\Phi\colon \mathbf{S} \to \mathbf{S}'$.

Proof. Hint: Limits and colimits in $\mathbf{T}^{\mathbf{S}}$ are computed pointwise, e.g., see [Lan71], p. 112. □

After the definition below, we give some general categorical results that are used later in the paper, the first of which is proved in [Lan71]. The next subsection gives some applications to diagram categories.

Definition 2.5. Given functors $\Phi\colon \mathbf{S} \to \mathbf{S}'$ and $I\colon \mathbf{S} \to \mathbf{T}$, a **right Kan extension of** I **along** Φ is a pair containing a functor $I'\colon \mathbf{S}' \to \mathbf{T}$ and a natural transformation $\mu\colon \Phi; I' \Rightarrow I$ which is universal from $\mathbf{T}^{\Phi}$ to I, that is, for every $J'\colon \mathbf{S}' \to T$ and $\mu'\colon \Phi; J' \Rightarrow I$ there is a unique natural transformation $\sigma\colon J' \Rightarrow I'$ such that

$\mu' = (1_\Phi ⨟ \sigma);\mu$. Dually, a **left Kan extension of** I **along** Φ is a functor $I'\colon \mathbf{S}' \to \mathbf{T}$ and a natural transformation $\mu\colon I \Rightarrow \Phi;I'$ which is universal from I to $\mathbf{T}^\Phi$, that is, for every $J'\colon \mathbf{S}' \to \mathbf{T}$ and $\mu'\colon I \Rightarrow \Phi;J'$ there is a unique natural transformation $\sigma\colon I' \Rightarrow J'$ such that $\mu' = \mu;(1_\Phi ⨟ \sigma)$.

Proposition 2.6. Given a small category **S**, then:

1. If **T** is complete then any functor $I\colon \mathbf{S} \to \mathbf{T}$ has a right Kan extension along any $\Phi\colon \mathbf{S} \to \mathbf{S}'$, and also $\mathbf{T}^\Phi$ has a right adjoint.
2. If **T** is cocomplete then any functor $I\colon \mathbf{S} \to \mathbf{T}$ has a left Kan extension along any $\Phi\colon \mathbf{S} \to \mathbf{S}'$, and also $\mathbf{T}^\Phi$ has a left adjoint.

Theorem 2.7. $\mathbf{T}^-$ contravariantly lifts adjoints to functor category adjoints.

Proof. Hint: If $\langle \Phi, \Phi', \eta, \epsilon \rangle\colon \mathbf{S} \to \mathbf{S}'$ is an adjoint pair of functors (with Φ' a left adjoint to Φ), then so is $\langle \mathbf{T}^{\Phi'}, \mathbf{T}^{\Phi}, \mathbf{T}^{\eta}, \mathbf{T}^{\epsilon} \rangle\colon \mathbf{T}^{\mathbf{S}} \to \mathbf{T}^{\mathbf{S}'}$, where $(\mathbf{T}^\eta)_{I'} = \eta ⨟ 1_{I'}$ and $(\mathbf{T}^\epsilon)_I = \epsilon ⨟ 1_I$ for all functors $I\colon \mathbf{S} \to \mathbf{T}$ and $I'\colon \mathbf{S}' \to \mathbf{T}$. □

Then using the same notation as in the above proof, we also have:

Corollary 2.8. $Nat(\Phi;I', I) \simeq Nat(I', \Phi';I)$, naturally in both I and I'. More precisely, a natural transformation $\mu\colon \Phi;I' \Rightarrow I$ goes to $(\eta ⨟ 1_{I'});(1_{\Phi'} ⨟ \mu)$, and conversely, a natural transformation $\mu'\colon I' \Rightarrow \Phi';I$ goes to $(1_\Phi ⨟ \mu');(\epsilon ⨟ 1_I)$.

2.3. Diagram Categories

When **Ind** is $\mathbf{cat}^{op}$, the opposite of the category of small categories, and when for some fixed category **T**, each $C(I)$ is the functor category $\mathbf{T}^I$, with $C(\alpha) = \mathbf{T}^\alpha\colon \mathbf{T}^{I'} \to \mathbf{T}^I$ for $\alpha\colon I \to I'$, then **Flat**(C) is the classical **diagram category**, here denoted **dgm**(**T**). We are also interested in the case where **Ind** is $\mathbf{Cat}^{op}$, the opposite of the category of locally small categories, for which we use the notation **Dgm**(**T**); this was used in an abstract formulation of institution in [GoB92], as also discussed here in Appendix E. An important variation takes $C(\_)$ to be $(\mathbf{T}^-)^{op}$, for which we will use the notations **coDgm**(**T**) and **coDgm**(**T**), respectively. The following consequence of results above is therefore of interest:

Corollary 2.9. The category **dgm**(**T**) is complete if **T** is, and is cocomplete if **T** is. Also the category **Dgm**(**T**) is complete if **T** is. Furthermore, **codgm**(**T**) is complete if **T** is cocomplete, and is cocomplete if **T** is complete, while **coDgm**(**T**) is complete if **T** is cocomplete.

Proof. Proposition 2.4 implies that for each $\alpha\colon I \to J$ in **cat** (or in **Cat**), the functor $\mathbf{T}^\alpha$ is continuous and cocontinuous, Proposition 2.6 implies that each $\mathbf{T}^\alpha$ has a right adjoint, and then Theorem 2.2 and Corollary 2.3 complete the proof. □

Completeness of **dgm**(**T**) was also shown in [Gog73] in the context of a general systems theory. One can be more precise, in that **dgm**(**T**) has whatever limits and colimits **T** has, and dually for **codgm**(**T**), while **Dgm**(**T**) has whatever limits **T** has, and **coDgm**(**T**). Delicate set theoretic issues arise for the cocompleteness of **Dgm**(**T**) and **coDgm**(**T**), as discussed in Example 4.10. The following is also needed later:

Proposition 2.10. The forgetful functors on the diagram categories, **dgm**(**T**) and **Dgm**(**T**), which extract the underlying 'shape' category, preserve whatever limits and colimits exist in the diagram categories. The same holds for the codiagram categories. Also, if given a functor $F\colon T \to T'$, we let $\mathbf{dgm}(F)\colon \mathbf{dgm}(T) \to \mathbf{dgm}(T')$ denote the functor that composes diagrams over T with F to get diagrams over T', then **dgm**(F) is both continuous and cocontinuous if F is; moreover, the same holds for **Dgm**(F), **codgm**(F), and **coDgm**(F).

Proof. This again uses the fact that limits and colimits are computed pointwise in functor categories. □

2.4. Twisted Relation Categories

Twisted relations were introduced in [GoB92] and were also explored in [Ros99]:

Definition 2.11. Let **Trel** be the category of **twisted relations**, with triples $\langle A, \mathscr{R}, B\rangle$ as its objects, where A is a category, B is a set and $\mathscr{R} \subseteq |A| \times B$, and with pairs $\langle F, g\rangle\colon \langle A, \mathscr{R}, B\rangle \to \langle A', \mathscr{R}', B'\rangle$ as its morphisms where $F\colon A' \to A$ is a functor and $g\colon B \to B'$ is a function such that the diagram

$$\begin{array}{ccc} |A| & \overset{\mathscr{R}}{\text{———}} & B \\ {\scriptstyle F}\uparrow & & \downarrow{\scriptstyle g} \\ |A'| & \underset{\mathscr{R}'}{\text{———}} & B' \end{array}$$

commutes, in the sense that for any $a' \in |A'|$ and $b \in B$, we have $a'\mathscr{R}'g(b)$ iff $F(a')\mathscr{R}b$.

There are four natural variants of this definition, arising from the four choices of one of sets or categories for the left and right components of the triples; let us call these the **original variants**, since they already appear in [GoB92], and even earlier in [GoB86]. Those variants where the right component is category-valued give rise to institutions that allow deduction, whereas those where the left component is category-valued give rise to institutions that allow morphisms of models (see the discussion after Definition 3.1). It is not hard to see that the following holds for all four original variants, by generalizing the proof given in [Ros99], as already suggested there:

Proposition 2.12. **Trel** is both complete and cocomplete.

3. Institutions

Here finally is (a restricted formulation of) the main basic concept of this paper:

Definition 3.1. An **institution** $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ consists of a category **Sign** whose objects are called **signatures**, a functor $\mathbf{Mod}\colon \mathbf{Sign} \to \mathbf{Cat}^{op}$ giving for each signature Σ a category of **Σ-models**, a functor $\mathbf{Sen}\colon \mathbf{Sign} \to \mathbf{Set}$ giving for each signature a set of **Σ-sentences**, and a $|\mathbf{Sign}|$-indexed relation $\models\; = \{\models_\Sigma \mid \Sigma \in |\mathbf{Sign}|\}$ with $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)$, such that for any signature morphism $\varphi\colon \Sigma \to \Sigma'$, the following diagram commutes,

$$\begin{array}{ccccc} \Sigma & \quad & |\mathbf{Mod}(\Sigma)| & \overset{\models_\Sigma}{\text{————}} & \mathbf{Sen}(\Sigma) \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle \mathbf{Mod}(\varphi)}\uparrow & & \downarrow{\scriptstyle \mathbf{Sen}(\varphi)} \\ \Sigma' & & |\mathbf{Mod}(\Sigma')| & \underset{\models_{\Sigma'}}{\text{————}} & \mathbf{Sen}(\Sigma') \end{array}$$

that is, the following **satisfaction condition**

$$m' \models_{\Sigma'} \mathbf{Sen}(\varphi)(f) \quad \text{iff} \quad \mathbf{Mod}(\varphi)(m') \models_\Sigma f$$

holds for all $m' \in |\mathbf{Mod}(\Sigma')|$ and $f \in \mathbf{Sen}(\Sigma)$.

We call the above the **classic** notion of institution, as opposed to the variants discussed below, since this notion appeared in the earliest versions of [GoB92], including [GoB84a]. Note that in using this definition, we may implicitly replace **Mod** by $\mathbf{Mod}^{op}$ in order to get the expected arrow, following an established convention for contravariant functors in categorical formulae. We often write only φ instead of $\mathbf{Sen}(\varphi)$ and $\_\!\restriction_\varphi$ instead of $\mathbf{Mod}(\varphi)$; the functor $\_\!\restriction_\varphi$ is called the **reduct** functor associated to φ. With this notation, the satisfaction condition becomes

$$m' \models_{\Sigma'} \varphi(f) \quad \text{iff} \quad m'\!\restriction_\varphi \models_\Sigma f$$

We also use the satisfaction notation with a set of sentences F on its right side, letting $m \models_\Sigma F$ mean that m satisfies each sentence in F, and further extend this notation by letting $F \models_\Sigma F'$ mean that $m \models_\Sigma F'$ for any Σ-model m with $m \models_\Sigma F$. We may omit the subscript Σ in $\models_\Sigma$ when it can be inferred from context. The

closure of a set of Σ-sentences F, denoted $F^\bullet$, is the set of all f in **Sen**(Σ) such that $F \models_\Sigma f$. The sentences in $F^\bullet$ are often called the **theorems** of F. Closure is obviously a closure operator.[3]

Notice the asymmetry in Definition 3.1, between treating sentences using sets and treating models using categories; this is inconvenient in cases where classes instead of sets of sentences are needed, or where the additional structure provided by morphisms is needed. Fortunately, natural variants of the institution notion arise from choosing either **Cat** or **Set** for the targets of the functors **Sen** and **Mod**, of course with **Mod** always contravariant; we call these four variants the **original variants**, because they already appear in [GoB92] (page 111). The two variants where **Sen** is **Cat**-valued allow deduction via morphisms among sentences (as advocated for example by Lambek and (Phil) Scott [LaS86]) with (possibly infinite) conjunction appearing as categorical product. From any such category, we can obtain another, which is a partial ordering, by defining $f \vdash_\Sigma f'$ iff there is a morphism from f to f'; this yields an entailment relation. Such categories can also be obtained directly from a system of rules of deduction, and we will refer to all such categories as **entailment variants**. We could instead use multicategories[4] as advocated by Meseguer [Mes89] with their forgetful functor to sets, or any other structure that allows appropriate proof representations. Twisted relations are easily adapted to such variants, as are Proposition 2.12 and the later completeness results that build upon it. We will informally call these, and any other variants that can arise just by substituting other appropriate categories in the twisted category definition (see Theorem 4.6 below), the **close variants** of the institution concept, because technically they proceed in the same way. The main original paper on institutions, [GoB92], gave an even more categorical definition of institution, in which twisted relation categories were reconstructed as certain comma categories; general properties of comma categories then elegantly replaced detailed arguments about twisted relations. This material is reviewed in Appendix E, where it enables us to give a precise definition for the notion of close variant, and to prove some of its properties in a general way.

Example 3.2. We briefly discuss some institutions that are especially relevant to this paper.

1. Classical unsorted universal algebra, the institution of which we denote $\mathbb{USA}$, goes back to Birkhoff [Bir35]; it is the one sorted special case of the many sorted algebra discussed below.
2. Many sorted algebra, the institution of which we denote $\mathbb{MSA}$, was first shown to be an institution in early drafts of [GoB92]. Here signatures and algebras are the usual overloaded many sorted signatures and algebras (but we do allow empty carriers), which go back to Goguen [Gog75]; sentences are explicitly universally quantified pairs of terms, and satisfaction is defined in the obvious way. Proving the satisfaction condition does take a bit of work (see [GoB92]), but as with many other examples, this can be alleviated by using charters [GoB86].
3. Order sorted algebra, the institution for which we denote $\mathbb{OSA}$, has overloaded order sorted signatures and algebras, with explicitly universally quantified pairs of terms as sentences, e.g., see [GoD94] for details. The first proof that this is an institution was probably given by Yan [Yan93] for a case that also included so called sort constraints; see also the proofs in [Ros94] and [Mos96b], noting that there are many variants of order sorted algebra [GoD94].
4. Among the many variants of first order logic, we first mention the one with many sorted function and predicate symbols in its signature, plus of course the usual logical symbols and the models (though we allow empty carriers); let $\mathbb{MSFOL}$ denote this institution, and let $\mathbb{FOL}$ denote its unsorted variant; proofs for their satisfaction conditions are sketched in [GoB92].
5. Many sorted first order logic with equality, denoted $\mathbb{MSFOLE}$, enriches $\mathbb{MSFOL}$ by allowing equations as atoms, rather than just predicates; a proof that this forms an institution is sketched in [GoB92]. The unsorted special case is denoted $\mathbb{FOLE}$.
6. Many sorted Horn clause logic is the same as $\mathbb{MSFOL}$ except that only Horn clauses are allowed as sentences; let us denote this institution $\mathbb{MSHCL}$, its unsorted variant by $\mathbb{HCL}$, its variant with equations as additional atoms $\mathbb{MSHCLE}$, and its unsorted variant with equations as atoms $\mathbb{HCLE}$; proof sketches again may be found in [GoB92].
7. Partial algebra, denoted $\mathbb{PA}$, is discussed in Appendix A.
8. Supersorted order sorted algebra, denoted $\mathbb{OSA}^?$, is discussed in Appendix B.
9. Two versions of hidden algebra, denoted $\mathbb{HA}_1$ and $\mathbb{HA}_2$, are discussed in Appendix C.

[3] That is, it is extensive, monotonic and idempotent, i.e., $F \subseteq F^\bullet$, $F \subseteq F'$ implies $F^\bullet \subseteq F'^\bullet$, and $F^{\bullet\bullet} = F^\bullet$.
[4] See the end of Section 3.1 for some further details.

Of course there are many many other examples, some of which have very different characters.

3.1. Some Basics

We review some basics from [GoB92]:

Proposition 3.3. For any morphism $\varphi\colon \Sigma \to \Sigma'$ and sets F, F' of Σ-sentences:

1. *Closure Lemma:* $\varphi(F^\bullet) \subseteq \varphi(F)^\bullet$;
2. $\varphi(F^\bullet)^\bullet = \varphi(F)^\bullet$;
3. $(F^\bullet \cup F')^\bullet = (F \cup F')^\bullet$.

Definition 3.4. A **specification** or **presentation** is a pair (Σ, F) where Σ is a signature and F is a set of Σ-sentences. A **specification morphism** from (Σ, F) to (Σ', F') is a signature morphism $\varphi\colon \Sigma \to \Sigma'$ such that $\varphi(F) \subseteq F'^\bullet$. Specifications and specification morphisms give a category denoted **Spec**. A **theory** (Σ, F) is a specification with $F = F^\bullet$; the full subcategory of theories in **Spec** is denoted **Th**.

The inclusion functor $\mathscr{U}\colon$ **Th** $\to$ **Spec** is an equivalence of categories, having a left-adjoint-left-inverse $\mathscr{F}\colon$ **Spec** $\to$ **Th**, given by $\mathscr{F}(\Sigma, F) = (\Sigma, F^\bullet)$ on objects and identity on morphisms; note that $\mathscr{F}$ is also a right adjoint of $\mathscr{U}$, so that **Th** is a reflective and coreflective subcategory of **Spec**. It is also known [GoB92] that **Th** is cocomplete whenever **Sign** is cocomplete, and that **Th** has pushouts whenever **Sign** does. The following construction for pushouts in **Th** is a special case of the general colimit creation result proved in [GoB92]:

Proposition 3.5. Given theory morphisms $\varphi_1\colon (\Sigma, F) \to (\Sigma_1, F_1)$ and $\varphi_2\colon (\Sigma, F) \to (\Sigma_2, F_2)$, if

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \varphi_1'} \\ \Sigma_2 & \xrightarrow[\varphi_2']{} & \Sigma' \end{array}$$

is a pushout in **Sign**, then

$$\begin{array}{ccc} (\Sigma, F) & \xrightarrow{\varphi_1} & (\Sigma_1, F_1) \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \varphi_1'} \\ (\Sigma_2, F_2) & \xrightarrow[\varphi_2']{} & (\Sigma', F') \end{array}$$

is a pushout in **Th**, where $F' = (\varphi_1'(F_1) \cup \varphi'_2(F_2))^\bullet$.

Definition 3.6. A theory morphism $\varphi\colon (\Sigma, F) \to (\Sigma', F')$ is **conservative** iff for any (Σ, F)-model m there is some (Σ', F')-model m' such that $m'\restriction_\varphi = m$. A signature morphism $\varphi\colon \Sigma \to \Sigma'$ is **conservative** iff it is conservative as a morphism of void theories, i.e., $\varphi\colon (\Sigma, \emptyset^\bullet) \to (\Sigma', \emptyset'^\bullet)$.

The following is not difficult to prove (see [Ros97a]):

Proposition 3.7. Given $\varphi\colon \Sigma \to \Sigma'$, $f \in$ **Sen**(Σ) and $F \subseteq$ **Sen**(Σ), then:

1. $F \models_\Sigma f$ implies $\varphi(F) \models_{\Sigma'} \varphi(f)$.
2. If φ is conservative, then $F \models_\Sigma f$ iff $\varphi(F) \models_{\Sigma'} \varphi(f)$.

The next result (explicit in [Ros99] for the classic institutions of Definition 3.1, and implicit in [GoB92] for all the close variants) says that an institution over a category of signatures **Sign** can be regarded as a functor with target **Trel**, and vice versa; this also holds for the close variants of the institution and twisted relation concepts (when they are appropriately correlated). Theorem 4.6 extends this result from objects to morphisms and comorphisms (see also [Tar86a]).

Proposition 3.8. There is a bijection (i.e., a one-to-one correspondence between classes) between institutions over **Sign** and functors **Sign** $\to$ **Trel**.

Every institution (**Sign**, **Mod**, **Sen**, $\models$) has an associated functor **Sign** $\rightarrow$ **Trel** taking a signature $\Sigma \in |\mathbf{Sign}|$ to the triple $\langle \mathbf{Mod}(\Sigma), \models_\Sigma, \mathbf{Sen}(\Sigma)\rangle$, and taking a signature morphism $\varphi\colon \Sigma \rightarrow \Sigma'$ to the 'twisted' morphism $\langle \mathbf{Mod}(\varphi), \mathbf{Sen}(\varphi)\rangle$; also, every functor $I\colon$ **Sign** $\rightarrow$ **Trel** has an associated institution (**Sign**, **Mod**, **Sen**, $\models$) such that if $I(\Sigma) = \langle A_\Sigma, \mathscr{R}_\Sigma, B_\Sigma\rangle$, then $\mathbf{Mod}(\Sigma) = A_\Sigma$, $\mathbf{Sen}(\Sigma) = B_\Sigma$ and $\models_\Sigma = \mathscr{R}_\Sigma$, and such that for a signature morphism $\varphi\colon \Sigma \rightarrow \Sigma'$, if $I(\varphi) = \langle F_\varphi, g_\varphi\rangle$, then $\mathbf{Mod}(\varphi) = F_\varphi$ and $\mathbf{Sen}(\varphi) = g_\varphi$. Moreover, these two associations are inverse to each other. Therefore we can use the tuple and functor notations for institutions interchangeably.

An institution where the **Sen** functor is category-valued is said to be **complete** iff for any two Σ-sentences f, f', we have

$$f \vdash_\Sigma f' \quad \text{iff} \quad f \models_\Sigma f'$$

We can define compactness in the same style, provided $\mathbf{Sen}(\Sigma)$ has suitable extra structure, such as that of an infinitary multicategory:[5] an institution is **compact** iff whenever $f \vdash_\Sigma f'$ then $f_0 \vdash_\Sigma f'$ for some finite $f_0 \subseteq f$.

3.2. Inclusive Institutions

In many categories, among the monics are some especially simple and natural maps which may be called *inclusions*. Although many professional category theorists are loath to consider them, because of their desire to identify things that are isomorphic, inclusions are in fact a natural concept, the use of which can greatly simplify some applications, especially where syntax is the object of study. For example, we really do prefer a *subsignature* to be given by an inclusion, so that the exact same symbols are involved; and the same holds for modules in both programming and specification. At the end of [GoB92], axiomatising and then exploiting inclusions for modularisation was listed among the open problems. A first solution was given in [DGS93] with the formal notion of *inclusion system*, which was then used to significantly simplify the semantics of module systems over an institution. The abstract notion of inclusion system was further studied and simplified in a series of papers [Hil96, CaR97, CaR00, Ros97b]. Here we briefly summarise the current state, and sketch some applications.

There is a well-known correspondence between certain small categories and partially ordered sets, or **posets** for short; these categories have exactly one object A for each element a in the set, a morphism from A to B iff $a \leqslant b$, and they satisfy anti-symmetry, in that if there is a morphism from A to B and another from B to A then $A = B$; hereafter, we will identify posets with their corresponding categories. Sums and products correspond to unions and intersections, respectively, and a poset with finite sums and products is a lattice, with all the usual properties thereof. Of course, things generalise from sets to classes, which we will call **poclasses**; we let $\hookrightarrow$ denote the poclass morphisms.

Definition 3.9. An **inclusive category C** is a category with a broad subcategory[6] **I** which is a poclass, called its **subcategory of inclusions**, having finite intersections and unions, such that for every pair of objects A, B, their union in **I** is a pushout in **C** of their intersection in **I**. **C** is **distributive** iff **I** is distributive. A functor between two inclusive categories is an **inclusive functor** (or **preserves inclusions**) iff it takes inclusions in the source category to inclusions in the target category.

This notion of inclusion is similar to that of (weak) inclusion systems [DGS93, Hil96, CaR97, CaR00, Ros97b], except that no factorisation properties are assumed; however, the weaker notion is adequate for many purposes. Also, sums and products are not needed for many applications. Inclusive categories can play a similar role to factorisation systems [HeS73, Nem82], but tend to have smoother proofs.

The following enriches an institution with inclusions [Ros97a]:

Definition 3.10. An **inclusive institution** is an institution with its category of signatures and its **Sen** functor both inclusive. An inclusive institution is **distributive** iff its category of signatures is distributive, and is **semiexact** iff the functor $\mathbf{Mod}\colon \mathbf{Sign} \rightarrow \mathbf{Cat}^{op}$ preserves pushouts, i.e., it takes pushouts in **Sign** to pullbacks in **Cat**.

The above applies the general institutional notion of semiexactness introduced in [DGS93] to inclusive institutions; semiexactness is a weakening of *exactness*, which says that **Mod** preserves colimits, and seems

[5] While an ordinary multicategory has finite lists as objects, our notion of **infinitary multicategory** is a monoidal category with arbitrary subsets of a given infinite set as its objects, and with union as its multiplication; we hope to develop this notion, which in this form works only as an entailment variant, in detail at some later time.

[6] In the sense that it has the same objects as **C**.

to have first appeared in [SaT88a]; it was used by Tarlecki [Tar86a, Tar86b, Tar86c] for abstract algebraic institutions, and by Meseguer [Mes89] for general logics. Although many sorted logics tend to be exact, their unsorted variants tend to be only semiexact.

The category of theories, **Th**, inherits many properties from **Sign**. One of the most important of these is that **Th** is cocomplete if **Sign** is. Moreover, taking theory inclusions to be theory morphisms which are signature inclusions, we have

Proposition 3.11. For an inclusive institution:

1. **Th** is inclusive; and
2. **Th** has pushouts that preserve inclusions if **Sign** has pushouts that preserve inclusions.

Note that it is often more convenient to speak of a **theory extension** than of a theory inclusion.

Inspired by Goguen and Tracz's 'implementation oriented' (i.e., more concrete) semantics for modularisation [GoT00], Roşu [Ros97a] introduced the notion of *module specification* as a generalisation of a standard specification, having both public (or visible) and private symbols via inclusions of signatures, and then explored their properties and gave semantics for module composition over an arbitrary inclusive institution. More precisely, a module specification in an inclusive institution is a triple (Σ, F, Σ'), where $\Sigma' \hookrightarrow \Sigma$ and F is a set of Σ-sentences. The *visible theorems* (or the visible consequences) of a module (Σ, F, Σ') are the Σ'-sentences satisfied by F over Σ, and a model of (Σ, F, Σ') is a Σ'-model of its visible consequences.

For another application, inclusive institutions are an attractive alternative to Mossakowski's 'institutions with symbols' [Mos00], which assign a set of symbols to each signature, as part of a semantics for the CASL language [CoF00], since inclusions will automatically keep track of shared symbols in subsignatures, while allowing all the usual operations on modules, including renaming, to be (more) easily and naturally expressed. It is our view that inclusive institutions provide the most natural and easy way to formulate the semantics of specification languages like OBJ3 [GWM02], CASL [CoF00], CafeOBJ [DiF98], and BOBJ [GLR00].

4. Institution Morphisms and Comorphisms

Perhaps the two best-known kinds of morphism between institutions are the original 'morphisms' of Goguen and Burstall [GoB92], and the 'plain maps' of Meseguer [Mes89], later given the better name 'representations' by Tarlecki [Tar96a, Tar87]. We show a natural duality between these two, by viewing their categories with institutions as objects as flattened indexed categories; this motivates our preference for the institution comorphism terminology, and also yields easy proofs of completeness and cocompleteness, using the fact that given a functor between signature categories, any institution over the source signature category extends to an institution over the target signature category along that functor in two canonical ways, given by the left and right Kan extensions. Arrais and Fiadeiro [ArF96] showed that given an adjunction between signature categories, an institution morphism gives rise to an institution comorphism and vice versa. We show that this result is a natural consequence of the fact that an adjoint between signature categories lifts contravariantly to functor categories.

The original morphisms for institutions introduced with the institution concept in [GoB92] seem to be the most natural notion. In particular, they include structure forgetting, and hence structure embedding or representation relationships. Our examples will show that morphic formulations are usually simpler and more natural in other contexts as well.

Definition 4.1. Given institutions $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ and $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$, an **institution morphism** from $\mathbb{I}$ to $\mathbb{I}'$ consists of a functor $\Phi\colon \mathbf{Sign} \to \mathbf{Sign}'$, a natural transformation $\beta\colon \mathbf{Mod} \Rightarrow \Phi;\mathbf{Mod}'$, and a natural transformation $\alpha\colon \Phi;\mathbf{Sen}' \Rightarrow \mathbf{Sen}$, such that the following **satisfaction condition** holds for each $\Sigma \in |\mathbf{Sign}|$, $m \in |\mathbf{Mod}(\Sigma)|$ and $f' \in \mathbf{Sen}'(\Phi(\Sigma))$:

$$m \models_\Sigma \alpha_\Sigma(f') \quad \text{iff} \quad \beta_\Sigma(m) \models'_{\Phi(\Sigma)} f'$$

We let $\mathcal{INS}$ denote the category of institutions with institution morphisms.

Note that the functor Φ on signatures and the natural transformation β on models go in the same direction in this definition, while the natural transformation α goes in the opposite direction.

Meseguer [Mes89] introduced a dual of the institution morphisms of Goguen and Burstall under the name ‘plain map’, later renamed ‘representation’ by Tarlecki [Tar96a, Tar96b]; however, we prefer the name ‘comorphism’ in order to emphasise the important duality between these two concepts.

Definition 4.2. Given institutions $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ and $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$, an **institution comorphism** from $\mathbb{I}$ to $\mathbb{I}'$ consists of $\Phi\colon \mathbf{Sign} \to \mathbf{Sign}'$, a natural transformation $\beta\colon \Phi;\mathbf{Mod}' \Rightarrow \mathbf{Mod}$, and a natural transformation $\alpha\colon \mathbf{Sen} \Rightarrow \Phi;\mathbf{Sen}'$, such that the following **(co-)satisfaction condition** holds for each $\Sigma \in |\mathbf{Sign}|$, $m' \in |\mathbf{Mod}'(\Phi(\Sigma))|$, and $f \in \mathbf{Sen}(\Sigma')$:

$$\beta_\Sigma(m') \models_\Sigma f \quad \text{iff} \quad m' \models'_{\Phi(\Sigma)} \alpha_\Sigma(f)$$

We let $co\mathcal{INS}$ denote the category of institutions and institution comorphisms.

Cerioli introduced the related notion of *simulation* [Cer93], which requires that β be a surjective partial natural transformation; we conjecture that many, or even most, examples of this notion will be at least as well captured by forward institution morphisms or some variant thereof (see Section 6 below). The basis for this conjecture is the intuition that it will often be possible to reformulate natural examples to use something like an inverse of β.

It is characteristic of our subject that the same example can often be presented in more than one way. For example, consider the relationship between the institutions of equational logic and first order logic with equality, for simplicity restricted to the unsorted versions. Since signatures for first order logic with equality are pairs (Π, Σ) where Π gives the predicate symbols and Σ gives the function symbols, we can capture the relationship between the two kinds of signature with a forgetful functor sending (Π, Σ) to Σ, or with an embedding functor sending Σ to $(\emptyset, \Sigma)$. An apparently insufficiently well-known small insight from category theory is that it is very often better to work with forgetful functors than with functors going the other way, even if they are adjoint. For example, the forgetful functor from groups to sets better expresses the relationship between these two than the free group functor; and we can see a similar phenomenon in our little example where the forgetful functor avoids the (admittedly rather small) arbitrariness of introducing the empty set. Although intuitively we have an embedding of equational signatures into first order with equality signatures, it is more natural to use the forgetful functor than the embedding functor. The examples below extend this insight from signatures to institutions.

Example 4.3. We give some examples of morphisms and comorphisms for embeddings.

1. First some more details of the embedding of equational logic into first order logic with equality. Let Φ denote the forgetful functor which on objects sends (Π, Σ) to Σ, let β_Σ be the forgetful functor sending a (Π, Σ)-model to the corresponding Σ-algebra, and let $\alpha_{(\Pi,\Sigma)}$ send a Σ-equation to the same equation viewed as a (Π, Σ)-sentence (which may require adding quantifiers). It is now easy to check the naturality and satisfaction conditions.
2. A contrasting case is the embedding of unsorted equational logic into many sorted equational logic, because here there is no natural forgetful functor for the signatures; therefore this is better seen as a comorphism, with Φ mapping an unsorted signature to the corresponding one sorted signature, and with the obvious α and β.
3. On the other hand, if we modify the many sorted equational logic institution to provide distinguished elements in its sort sets,[7] then there is a natural forgetful functor from many sorted signatures to unsorted signatures, and we get an institution morphism. We encourage the reader to work out the details of this as an exercise.
4. An example similar to the first above (but simpler) is the embedding of Horn clause logic into first order logic. Here the signature categories are the same in the two institutions, consisting of just indexed sets of predicate symbols, and Φ is the identity functor. The two model categories are also the same, and β_Π consists of all identity functors (where Π is a signature of predicate symbols). Finally, each α_Π is the inclusion of the Π-Horn clauses into the first order Π-sentences. Since so many of the structures in this example are the same, there is no significant difference between using this morphism and using the corresponding comorphism to represent the relationship of the two institutions; moreover, these two are dual in the sense of Section 4.2.

[7] This is by no means an unnatural concept. For example, in the OBJ3 system, every module has a ‘principal sort’, which is needed for computing default views [GWM02]. We can therefore argue that these ‘pointed sort sets’ are actually *more* natural, at least for many computer science applications.

5. There is also a comorphism from equational logic to first order logic with equality. Let Φ' send an equational signature Σ to the first order signature $(\emptyset, \Sigma)$, let α' send a Σ-equation to the corresponding $(\emptyset, \Sigma)$-sentence, and let β' send a $(\emptyset, \Sigma)$-model to the corresponding Σ-algebra. We will see in Section 4.2 that this comorphism is dual to the morphism of item 1 above in a very natural way.

There are many more examples of a similar character. In general, it appears that the forgetful morphism versions are somewhat simpler and more natural than the comorphism versions.

An important application for institution morphisms, first studied in [GoB92], is to support the re-use of a theorem prover for an institution $\mathbb{I}$ on a problem '$f' \in F'^{\bullet}$' in another institution $\mathbb{I}'$ by translating it from $\mathbb{I}'$ to $\mathbb{I}$. The following is Proposition 35 (page 126) of [GoB92]:

Theorem 4.4. Given an institution morphism $(\Phi, \beta, \alpha)\colon \mathbb{I} \to \mathbb{I}'$ such that β_Σ is surjective on objects for all Σ, and given an $\mathbb{I}'$-specification (Σ', F) and a Σ'-sentence f', then

$$f' \in F'^{\bullet} \quad \text{iff} \quad \alpha_\Sigma(f') \in \alpha_\Sigma(F')^{\bullet}$$

where Σ is such that $\Sigma' = \Phi(\Sigma)$.

Results like this have come to be called 'borrowing' theorems, after the work of Cerioli and Meseguer [CeM97], who gave a borrowing theorem for comorphisms. Many interesting comorphisms that support borrowing for the CASL system appear in [Mos02]. It seems that almost any notion of institution morphism will support some kind of borrowing theorem; see [Tar96a, Mos02], and also [Mes98] for 'generalised maps of institutions' which embody some very general (but rather complex) conditions that support borrowing. Although borrowing is not a major concern of the present paper, and we hope to treat it carefully in a later work, for now we note that Theorem 4.4 can be very useful in practice, e.g., when (Φ, β, α) is a forgetful morphism.

Example 4.5. There is an institution comorphism from $\mathbb{OSA}^?$ to $\mathbb{PA}$ (these institutions of partial algebra, and of supersorted order sorted algebra, are defined in Appendices A and B, which also review the notation from [Gog97] that we use here). Given a supersorted signature (S, Σ) and a partial (S', Σ')-algebra A', it is natural to extend A' to an order sorted (S, Σ)-algebra by adding a special symbol $\star$ called the **error element**, to the carrier of each supersort $s?$, and extending all partial operations to total operations having the value $\star$ where they were undefined, and propagate error elements. A disadvantage of this construction is that it does not provide information about the origin of errors.

For any supersorted signature (S, Σ) and partial (S', Σ')-algebra A', let $\beta'_\Sigma(A')$ be the S-sorted family given by

1. $(\beta'_\Sigma(A'))_{s'} = A'_{s'}$ for all $s' \in S'$, and
2. $(\beta'_\Sigma(A'))_{s'?} = A'_{s'} \cup \{\star\}$ for all $s' \in S'$.

Then $\beta'_\Sigma(A')$ can be given an (S, Σ)-algebra structure as follows, where σ is an operation in Σ:

1. $(\beta'_\Sigma(A'))_\sigma(a_1, \ldots, a_n) = A'_\sigma(a_1, \ldots, a_n)$ if $a_1, \ldots, a_n$ are different from the error element $\star$ and $A'_\sigma(a_1, \ldots, a_n)$ is defined; and
2. $(\beta'_\Sigma(A'))_\sigma(a_1, \ldots, a_n) = \star$ if any of $a_1, \ldots, a_n$ is equal to $\star$ or if $A'_\sigma(a_1, \ldots, a_n)$ is not defined.

We call $\beta'_\Sigma(A')$ the **single error superextension** of A', and it is easily seen that $\beta'_\Sigma(A')$ is a strict Σ-algebra. As shown in [Gog97], β'_Σ can be organised as a functor β'_Σ: $\mathbf{PAlg}(\Sigma^\flat) \to \mathbf{OSAlg}(\Sigma)$ which is left inverse to $\mathscr{U}_\Sigma$, and which is right adjoint to $\mathscr{U}_\Sigma$ restricted to strict algebras;[8] moreover, β' is a natural transformation.

Now we can check that (Φ', β', α), with α as defined in Appendix B, and with Φ' the forgetful functor $\_^\flat$ of Appendix B, is a comorphism $\mathbb{OSA}^? \to \mathbb{PA}$. When the signature is clear from context, we prefer to write $\_^\star$ for β'_Σ and to omit α_Σ. Then the satisfaction condition for this comorphism is as follows, for $A' \in \mathbf{PAlg}(\Sigma^\flat)$ and $(\gamma, e) \in \mathbf{Sen}^?(\Sigma)$:

$$A'^{\star} \models_\Sigma (\gamma, e) \quad \text{iff} \quad A' \models\!\!\!\!\!\Leftarrow_{\Sigma^\flat} (\gamma, e)$$

This not entirely trivial result is proved in [Gog97].

[8] A subtle point arising here is that initial order sorted algebras may not be strict.

However, a simpler relationship between these institutions is given by an institution morphism $\mathbb{PA} \to \mathbb{OSA}^?$ that we will now define. Given a many sorted signature (D, Δ) and a partial Δ-algebra A, it is natural to extend Δ to a supersorted order sorted signature $\Delta^? = (D \cup D^?, \Delta)$ by adding an error supersort $d^?$ for each sort $d \in D$, extending A to an order sorted $(D \cup D^?, \Delta)$-algebra by adding the error element $\star$ to the carrier of each supersort $d^?$, and extending all partial operations to total operations taking the value $\star$ where they were undefined. As above, errors are propagated by these operations, and information about the origin of errors is lost.

Given a partial Δ-algebra A, let $\beta_\Delta(A)$ be the $(D \cup D^?)$-sorted family given by

1. $(\beta_\Delta(A))_d = A_d$ for all $d \in D$, and
2. $(\beta_\Delta(A))_{d^?} = A_d \cup \{\star\}$ for all $d \in D$.

Then $\beta_\Delta(A)$ can be made a $\Delta^?$-algebra by defining $(\beta_\Delta(A))_\sigma(a_1, \ldots, a_n)$ as $A_\sigma(a_1, \ldots, a_n)$ when $A_\sigma(a_1, \ldots, a_n)$ is defined, and $\star$ when $A_\sigma(a_1, \ldots, a_n)$ is not defined, for $\sigma \in \Delta$. We call $\beta_\Delta(A)$ the **single error superextension** of A, and it is easy to check that it is a strict $\Delta^?$-algebra, and that β_Δ can be organised as a functor β_Δ: $\mathbf{PAlg}(\Delta) \to \mathbf{OSAlg}(\Delta^?)$ which is left inverse to $\mathscr{U}_{\Delta^?}$, and right adjoint to $\mathscr{U}_{\Delta^?}$ restricted to strict algebras; moreover, β is a natural transformation.

Now we can check that (Φ, β, α), with α as in Appendix B, and with Φ the functor defined above, is a morphism $\mathbb{PA} \to \mathbb{OSA}^?$. As above, when the signature is clear, we may write $\_^\star$ for β_Δ and omit α_Δ, so the satisfaction condition for this institution morphism, for $A \in \mathbf{PAlg}(\Delta)$ and $(\gamma, e) \in \mathbf{Sen}^?(\Delta^?)$, is

$$A \mathrel{\models\!\!\!\!\circ}_\Delta (\gamma, e) \quad \text{iff} \quad A^\star \models_{\Delta^?} (\gamma, e)$$

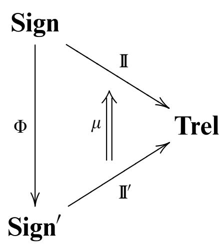
which is not difficult to check.

Let us now compare the morphism and the comorphism. It is clear from the constructions that there are many similarities. But it is also clear that β is significantly simpler to construct than β', and that Φ is simpler than Φ'. It also turns out that the morphism satisfaction condition is significantly easier to check than the comorphism condition. All this seems consistent with our hypothesis about the greater naturality of morphisms over comorphisms.[9]

The following extends Proposition 3.8 to morphisms and to comorphisms, and of course, it holds for all close variants; proofs for the classic case of Definition 3.1 can be found in [Ros99].

Theorem 4.6. $\mathscr{INS}$ is isomorphic to the category $\mathbf{Dgm}(\mathbf{Trel}) = \mathbf{Flat}((\mathbf{Trel}-)^{op})$, and $co\mathscr{INS}$ is isomorphic to $\mathbf{coDgm}(\mathbf{Trel}) = \mathbf{Flat}(\mathbf{Trel}-)$.

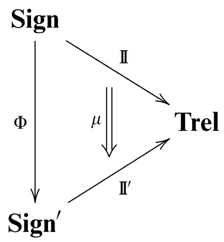
Therefore we can use morphisms in $\mathbf{Flat}((\mathbf{Trel}-)^{op})$ instead of institution morphisms whenever this simplifies the exposition. The intuition behind this isomorphism is that any institution morphism (Φ, β, α) as in Definition 4.1 corresponds to a morphism (Φ, μ) in $\mathbf{Flat}((\mathbf{Trel}-)^{op})$,



where μ: $\Phi; \mathbb{I}' \Rightarrow \mathbb{I}$ is the natural transformation defined as $\mu_\Sigma = \langle \beta_\Sigma, \alpha_\Sigma \rangle$ for each Σ in **Sign**.

Similarly, we can use morphisms in $\mathbf{Flat}(\mathbf{Trel}-)$ instead of institution comorphisms whenever this simplifies the exposition. The intuition is that any institution comorphism (Φ, β, α) as in Definition 4.2 corresponds to a morphism $\langle \Phi, \mu \rangle$ in $\mathbf{Flat}(\mathbf{Trel}-)$,

9 On the other hand, it is interesting to note that it is the comorphism that involves the forgetful functor here, and that the authors uncovered the morphism relatively late in writing this paper. Perhaps such phenomena help to explain why much of the literature seems to prefer comorphisms over morphisms. We thank Till Mossakowski for pointing out that [Mos02] contains a different, simpler comorphism for this example, going the opposite way; although the logics in [Mos02] are first order, they can be lowered to equational logics to facilitate comparison.



where $\mu\colon \mathbb{I} \Rightarrow \Phi;\mathbb{I}'$ is the natural transformation defined by $\mu_\Sigma = \langle \beta_\Sigma, \alpha_\Sigma \rangle$.

4.1. Completeness and Cocompleteness

The following is an important consequence of Theorem 4.6, using Corollary 2.9 plus Propositions 2.12 and 2.4; of course it holds for all the close variants of institutions.

Corollary 4.7. $\mathcal{INS}$ and $co\mathcal{INS}$ are both complete.

Completeness of $\mathcal{INS}$ was first shown by Tarlecki in [Tar86a] for the classic notion of Definition 3.1, and completeness of $co\mathcal{INS}$ by Tarlecki in [Tar96b], again for the classic case; proofs for the classic case can also be found in [Ros99].

Recall that **cat** denotes the category of small categories, $\mathcal{SINS}$ the category of institutions over small signature categories and institution morphisms, and $co\mathcal{SINS}$ the category of institutions over small signature categories and institution comorphisms. Since **cat** is both complete and cocomplete, and since the proof of Theorem 4.6 works just as well for categories of small signatures, we obtain the following, which also holds for all close variants of the institution concepts:

Proposition 4.8. $\mathcal{SINS}$ and $co\mathcal{SINS}$ are both complete.

Cocompleteness is more difficult, due to set theoretical issues, as discussed in Example 4.10. The result below, which first appeared in [Ros99] for the classic case of Definition 3.1, now follows similarly, and also holds for all close variants:

Theorem 4.9. $\mathcal{SINS}$ and $co\mathcal{SINS}$ are both cocomplete.

For practical applications, there is little or no reason to be interested in anything beyond finite limits and colimits. Nevertheless, results about larger diagrams are of some theoretical interest, and might have some applications in the future.

The following example[10] shows that $\mathcal{INS}$ and $co\mathcal{INS}$ are not cocomplete for the classic institutions of Definition 3.1, so that the above result is maximal for this case; however, the example does not apply to the variants that allow categories for sentences, because classes (i.e., 'large' sets) are allowed here. The example also fails if we require signatures to be small categories.

Example 4.10. Let $\mathbf{S}$ be a category with only identity morphisms, whose objects form a proper class, let $\pi_1, \pi_2\colon \mathbf{S} \times \mathbf{S} \to \mathbf{S}$ be the two projections, and consider the following four institutions:

- $\mathbb{I}_1$ has $\mathbf{S}$ for signatures, with one sentence and no models for each signature;
- $\mathbb{I}_2$ has $\mathbf{S}$ for signatures, with two sentences and no models for each signature;
- $\mathbb{I}_0^\times$ has $\mathbf{S} \times \mathbf{S}$ for signatures, with no sentences and no models;
- $\mathbb{I}_1^\times$ has $\mathbf{S} \times \mathbf{S}$ for signatures, with one sentence and no models for each signature.

There are two institution morphisms from $\mathbb{I}_1^\times$ to $\mathbb{I}_2$, one for each of π_1, π_2, taking the two sentences of $\Sigma_i = \pi_i(\langle \Sigma_1, \Sigma_2 \rangle)$ to the unique sentence of $\langle \Sigma_1, \Sigma_2 \rangle$. These two morphisms cannot have a coequaliser because: (1) a coequaliser institution would have to have exactly one object in its category of signatures; and (2) the sentences for that unique signature would need to be in one-one correspondence with the $\mathbf{S}$-indexed product of all the cardinality two sentence sets of $\mathbb{I}_2$, which is not a set.

[10] In a personal communication from Andrzej Tarlecki, for which we are very grateful.

Similarly, there are two comorphisms of institutions from $\mathbb{I}_0^\times$ to $\mathbb{I}_1$ associated to the two projections. These two comorphisms cannot have a coequaliser for similar reasons: (1) a coequaliser institution would have to have exactly one object in its category of signatures; and (2) the sentences of this signature would need to be in one-one correspondence to the disjoint sum of all the sentences of $\mathbb{I}_1$, which is not a set.

The two assertions (1) above follow from Proposition E.7 of Appendix E, while the two assertions (2) follow from Proposition E.8 there, with **C** the category of classes. Note that the above constructions are not a counter example for notions of institution in the style of Appendix E, or that restrict their signatures to small categories. This example therefore highlights two seemingly innocent assumptions built into the classic notion of institution, and bolsters our confidence in the alternative approach of Appendix E.

4.2. Adjointness and Institution Morphisms and Comorphisms

Arrais and Fiadeiro [ArF96] noted that an adjunction on signature categories induces a bijection of the morphisms and comorphisms between their institutions. This nice result has a simple abstract proof, using the fact that the functor **Trel**− contravariantly lifts adjoint pairs to functor categories (Theorem 2.7), via Corollary 2.8; details are in [Ros99] for the classic case of Definition 3.1, and of course it holds for all close variants.

Theorem 4.11. If Φ: **Sign** → **Sign**′ has a left adjoint Φ': **Sign**′ → **Sign** then for any institutions $\mathbb{I}$: **Sign** → **Trel** and $\mathbb{I}'$: **Sign**′ → **Trel** there is a bijection between institution morphisms $\langle \Phi, \mu \rangle\colon \mathbb{I} \to \mathbb{I}'$ and institution comorphisms $\langle \Phi', \mu' \rangle\colon \mathbb{I}' \to \mathbb{I}$. Moreover, this bijection is natural in $\mathbb{I}$ and $\mathbb{I}'$.

We can describe this bijection as follows: Corollary 2.8 takes a natural transformation $\mu\colon \Phi; \mathbb{I}' \Rightarrow \mathbb{I}$ to $(\eta ⨾ 1_{\mathbb{I}'}); (1_{\Phi'} ⨾ \mu)$, and it has an inverse that takes $\mu'\colon \mathbb{I}' \Rightarrow \Phi'; \mathbb{I}$ to $(1_\Phi ⨾ \mu'); (\epsilon ⨾ 1_{\mathbb{I}})$, where η and ϵ are the unit and the counit of the adjunction, respectively. If we translate this to institutions using the isomorphisms of Theorem 4.6, we get exactly the construction of [ArF96]:

1. A morphism $(\Phi, \beta, \alpha)\colon \mathbb{I} \Rightarrow \mathbb{I}'$ yields a comorphism $(\Phi', \beta', \alpha')\colon \mathbb{I}' \to \mathbb{I}$, where $\beta'_{\Sigma'} = \beta_{\Phi'(\Sigma')}; \mathbf{Mod}'(\eta_{\Sigma'})$ and $\alpha'_{\Sigma'} = \mathbf{Sen}'(\eta_{\Sigma'}); \alpha_{\Phi'(\Sigma')}$ for all $\Sigma' \in |\mathbf{Sign}'|$.
2. A comorphism $(\Phi', \beta', \alpha')\colon \mathbb{I}' \Rightarrow \mathbb{I}$ yields a morphism $(\Phi, \alpha, \beta)\colon \mathbb{I} \to \mathbb{I}'$, where for all $\Sigma \in |\mathbf{Sign}|$, $\beta_\Sigma = \mathbf{Mod}(\epsilon_\Sigma); \beta'_{\Phi(\Sigma)}$, $\alpha_\Sigma = \alpha'_{\Phi(\Sigma)}; \mathbf{Sen}(\epsilon_\Sigma)$.

Example 4.12. The morphisms and comorphisms of Example 4.3 provide some good examples of the above bijection:

1. The functor Φ' in item 5 of Example 4.3, from equational to first order signatures, is left adjoint to the functor Φ in item 1 of Example 4.3, and the morphism (item 1) and comorphism (item 5) between these institutions correspond exactly as above.
2. The same holds for the morphism of item 3 of Example 4.3, from many sorted algebra to unsorted algebra, and the corresponding modification of the comorphism of item 2 of Example 4.3, from unsorted algebra to many sorted algebra.
3. The same also holds for the morphism and comorphism of item 4 of Example 4.3, between Horn clause logic and first order logic.

There are of course many other examples of a similar kind. On the other hand, the morphism and comorphism of Example 4.5 do not correspond in this way, despite the fact that their functors Φ and Φ' are adjoint.

4.3. Kan Extensions of Institutions

Given a morphism from its signature category, an institution can be translated in two distinct canonical ways, given by the two Kan extensions associated to the signature category morphism. The result below follows from Proposition 2.6, plus Proposition 2.12, that **Trel** is both complete and cocomplete; as usual, everything in this subsection holds for all close variants.

Proposition 4.13. Given a small category **Sign** and a functor Φ: **Sign** → **Sign**′, any institution $\mathbb{I}$: **Sign** → **Trel** has both a right and a left Kan extension along Φ, and the functor $\mathbf{Trel}^\Phi$ has both a right and a left adjoint.

The limitation to small categories is insignificant for practical purposes, even though it is inconsistent with the usual examples of signature categories; for example, in forming the category of equational signatures, we can restrict symbols to those that could be expressed in ASCII, or in an idealised LaTeX, both of which form countable sets. In fact, since the intuition for signatures is that they provide the atomic symbols for sentences, it seems highly reasonable to assume that they are small categories, and highly unreasonable to assume that they are not.

5. Theoroidal Morphisms

This section generalises to morphisms that map theories instead of just signatures. As mentioned before, the 'maps' of Meseguer [Mes89] are comorphisms generalised in this way, which we prefer to call 'theoroidal'. In particular, we consider completeness and cocompleteness of categories with theoroidal morphisms and comorphisms. We first define the theoroidal institution of an institution, and then theoroidal morphisms and theoroidal comorphisms. The next subsection treats simple theoroidal morphisms and comorphisms using the Kleisli composition. Similar constructions appear in [Tar86c] and [Mos96a] for comorphisms, but the treatment of morphisms and the observation that everything generalises to all close variants seem to be new.

Definition 5.1. The **theoroidal institution** $\mathbb{I}^{th}$ of an institution $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ is the institution $(\mathbf{Th}, \mathbf{Mod}^{th}, \mathbf{Sen}^{th}, \models^{th})$, where $\mathbf{Th}$ is the category of theories of $\mathbb{I}$, $\mathbf{Mod}^{th}$ is the extension of $\mathbf{Mod}$ to theories, $\mathbf{Sen}^{th}$ is $sign;\mathbf{Sen}$, and $\models^{th}$ is $|sign|;\models$, where $sign\colon \mathbf{Th} \to \mathbf{Sign}$ is the functor which forgets the sentences of a theory and $|sign|\colon |\mathbf{Th}| \to |\mathbf{Sign}|$ is its restriction to objects. We may omit superscripts *th*, so that $\mathbb{I}^{th}$ appears as $(\mathbf{Th}, \mathbf{Mod}, \mathbf{Sen}, \models)$.

It follows that theories of $\mathbb{I}^{th}$ are pairs $((\Sigma, F_1), F_2)$ where F_1, F_2 are sets of Σ-sentences, and that the models of $((\Sigma, F_1), F_2)$ in $\mathbb{I}^{th}$ are $(\Sigma, (F_1 \cup F_2))$-models in $\mathbb{I}$. The following natural notions are important for this section:

Definition 5.2. Given institutions $\mathbb{I}$ and $\mathbb{I}'$, a functor $\Phi\colon \mathbf{Th} \to \mathbf{Th}'$ is **signature preserving** iff there is a functor $\Phi^\diamond\colon \mathbf{Sign} \to \mathbf{Sign}'$ such that $\Phi; sign' = sign; \Phi^\diamond$.

The reader may check that $\Phi^\diamond$ is unique if it exists. Now we introduce the main concepts:

Definition 5.3. A **theoroidal morphism** (**comorphism**) from $\mathbb{I}$ to $\mathbb{I}'$ is a morphism (comorphism) (Φ, β, α) from $\mathbb{I}^{th}$ to $\mathbb{I}'^{th}$ such that Φ is signature preserving. We let $th\mathcal{INS}$ and $thco\mathcal{INS}$ denote the categories of institutions with theoroidal morphisms and comorphisms, respectively, and we let $\_^{th}\colon th\mathcal{INS} \to \mathcal{INS}$ and $\_^{th}\colon thco\mathcal{INS} \to co\mathcal{INS}$ denote the associated functors to $\mathcal{INS}$ and to $co\mathcal{INS}$, respectively.

To be explicit, the theoroidal morphism satisfaction condition says that for any $\mathbb{I}$-theory (Σ, F), any model $m \in \mathbf{Mod}(\Sigma, F)$, and any formula $f' \in \mathbf{Sen}'(\Sigma')$ where $\Sigma' = \Phi^\diamond(\Sigma)$,

$$m \models_\Sigma \alpha_\Sigma(f') \quad \text{iff} \quad \beta_{(\Sigma,F)}(m) \models'_{\Sigma'} f'$$

while the theoroidal comorphism satisfaction condition states that for any $\mathbb{I}$-theory (Σ, F), any model $m' \in \mathbf{Mod}(\Sigma', F')$ and any formula $f \in \mathbf{Sen}(\Sigma)$, where $\Sigma' = \Phi^\diamond(\Sigma)$,

$$\beta_{(\Sigma,F)}(m') \models_\Sigma f \quad \text{iff} \quad m' \models'_{\Sigma'} \alpha_\Sigma(f)$$

It is immediate that institutions with theoroidal morphisms (or comorphisms) form a category. But despite the simplicity of Definition 5.3, it can be difficult to check the satisfaction condition directly; however, it fortunately reduces to checking the condition for just the empty theories, as shown in the next two results:

Proposition 5.4. Given institutions $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ and $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$, a signature preserving functor $\Phi\colon \mathbf{Th} \to \mathbf{Th}'$, a natural transformation $\beta\colon \mathbf{Mod} \Rightarrow \Phi; \mathbf{Mod}'$ and a natural transformation $\alpha\colon \Phi; \mathbf{Sen}' \Rightarrow \mathbf{Sen}$, then (Φ, β, α) is a theoroidal morphism if and only if

$$m \models_\Sigma \alpha_\Sigma(f') \quad \text{iff} \quad \beta_{(\Sigma,\emptyset^\bullet)}(m) \models'_{\Sigma'} f'$$

for any empty theory $(\Sigma, \emptyset^\bullet) \in \mathbf{Th}$, any model $m \in \mathbf{Mod}(\Sigma, \emptyset^\bullet)$ and any formula $f' \in \mathbf{Sen}'(\Sigma')$, where $\Sigma' = \Phi^\diamond(\Sigma)$.

Proof. The 'only if' part follows from the definition of theoroidal morphism. Conversely, let (Σ, F) be any theory in **Th**, let $m \in \mathbf{Mod}(\Sigma, F)$ and let $f' \in \mathbf{Sen}'(\Sigma')$. Then

$$\begin{array}{lll} m \models_\Sigma \alpha_\Sigma(f') & \text{iff} & \beta_{(\Sigma,\emptyset^\bullet)}(m) \models'_{\Sigma'} f' \qquad \text{(by hypothesis)} \\ & \text{iff} & \mathbf{Mod}'(\Phi(\iota))(\beta_{(\Sigma,F)}(m)) \models'_{\Sigma'} f' \qquad \text{(by the naturality of } \beta) \\ & \text{iff} & \beta_{(\Sigma,F)}(m) \models'_{\Sigma'} f' \qquad \text{(by the satisfaction condition in } \mathbb{I}') \end{array}$$

where ι is the theory inclusion $(\Sigma, \emptyset^\bullet) \hookrightarrow (\Sigma, F)$.

Proposition 5.5. Given institutions $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ and $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$, a signature preserving functor Φ: $\mathbf{Th} \to \mathbf{Th}'$, a natural transformation β: $\Phi;\mathbf{Mod}' \Rightarrow \mathbf{Mod}$ and a natural transformation α: $\mathbf{Sen} \Rightarrow \Phi;\mathbf{Sen}'$, then (Φ, β, α) is a theoroidal comorphism if and only if

$$\beta_{(\Sigma,\emptyset^\bullet)}(m') \models_\Sigma f \quad \text{iff} \quad m' \models'_{\Sigma'} \alpha_\Sigma(f)$$

for any empty theory $(\Sigma, \emptyset^\bullet) \in \mathbf{Th}$, any model $m' \in \mathbf{Mod}(\Phi(\Sigma, \emptyset^\bullet))$ and any formula $f \in \mathbf{Sen}(\Sigma)$, where $\Sigma' = \Phi^\diamond(\Sigma)$.

Proof. The 'only if' part follows from the definition of theoroidal comorphism. Conversely, let $(\Sigma, F) \in \mathbf{Th}$, let $m' \in \mathbf{Mod}(\Sigma', F')$, and let $f \in \mathbf{Sen}(\Sigma)$, where $\Phi(\Sigma, F) = (\Sigma', F')$. Then

$$\begin{array}{lll} \beta_{(\Sigma,F)}(m') \models_\Sigma f & \text{iff} & \beta_{(\Sigma,F)}(m') \models_\Sigma \mathbf{Sen}(\iota)(f) \qquad (\mathbf{Sen}(\iota) \text{ is an identity}) \\ & \text{iff} & \mathbf{Mod}(\iota)(\beta_{(\Sigma,F)}(m')) \models_\Sigma f \qquad \text{(by the satisfaction condition in } \mathbb{I}^{th}) \\ & \text{iff} & \beta_{(\Sigma,\emptyset^\bullet)}(\mathbf{Mod}'(\Phi(\iota))(m')) \models_\Sigma f \qquad \text{(by the naturality of } \beta) \\ & \text{iff} & \mathbf{Mod}'(\Phi(\iota))(m') \models'_{\Sigma'} \alpha_\Sigma(f) \qquad \text{(by hypothesis)} \\ & \text{iff} & m' \models'_{\Sigma'} \mathbf{Sen}'(\Phi(\iota))(\alpha_\Sigma(f)) \qquad \text{(by the satisfaction condition in } \mathbb{I}'^{th}) \\ & \text{iff} & m' \models'_{\Sigma'} \alpha_\Sigma(f) \qquad (\mathbf{Sen}(\Phi(\iota)) \text{ is an identity}) \end{array}$$

where ι is the theory inclusion $(\Sigma, \emptyset^\bullet) \hookrightarrow (\Sigma, F)$. □

Meseguer [Mes89] defined[11] his maps as in Proposition 5.5, but with the additional requirement that Φ be α-*sensible*,[12] which seems not only natural, but also technically desirable for proving properties beyond the above, as in the following:

Conjecture 5.6. With appropriate restrictions on morphisms, such as sensibility, $th\mathcal{SINS}$ and $thco\mathcal{SINS}$ are complete and cocomplete.

5.1. Simple Theoroidal Morphisms

There is an important special case of theoroidal comorphism that often occurs in practice, called 'simple'[13] by Meseguer [Mes89], that maps signatures to theories instead of theories to theories:

Definition 5.7. A **simple theoroidal morphism** (**comorphism**) from $\mathbb{I}$ to $\mathbb{I}'$ is a morphism (comorphism) (Φ, β, α) from $\mathbb{I}$ to $\mathbb{I}'^{th}$.

Notice that simple theoroidal (co)morphisms reduce to ordinary (co)morphisms where signatures map to theories with no axioms. Also notice that the simple theoroidal morphism satisfaction condition says that for any signature $\Sigma \in \mathbf{Sign}$, any model $m \in \mathbf{Mod}(\Sigma)$ and any formula $f' \in \mathbf{Sen}'(\Sigma')$, where $\Sigma' = \Phi^\diamond(\Sigma)$,

$$m \models_\Sigma \alpha_\Sigma(f') \quad \text{iff} \quad \beta_\Sigma(m) \models'_{\Sigma'} f'$$

while the satisfaction condition for a simple theoroidal comorphism states that for any $\Sigma \in \mathbf{Sign}$, any $m' \in \mathbf{Mod}(\Sigma')$ and any formula $f \in \mathbf{Sen}(\Sigma)$, where $\Sigma' = \Phi^\diamond(\Sigma)$,

$$\beta_\Sigma(m') \models_\Sigma f \quad \text{iff} \quad m' \models'_{\Sigma'} \alpha_\Sigma(f)$$

If (Φ, β, α): $\mathbb{I} \to \mathbb{I}'$ is a simple theoroidal morphism of institutions, then let $(\Phi, \beta, \alpha)^{th}$ be the theoroidal morphism $(\Phi^{th}, \beta^{th}, \alpha^{th})$ from $\mathbb{I}$ to $\mathbb{I}'$ defined as $\Phi^{th}(\Sigma, F) = (\Sigma', (\alpha_\Sigma^{-1}(F) \cup F_\emptyset^\Sigma)^\bullet)$ for each theory $(\Sigma, F) \in \mathbf{Th}$,

[11] However, Meseguer used presentations instead of theories.
[12] This essentially means that Φ is completely determined by its restriction to empty theories and α.
[13] We have not thought of a better name, but we do feel that one is needed, since simple morphisms are not very simple.

where $\Phi(\Sigma) = (\Sigma', F_\emptyset^\Sigma)$, i.e., $F_\emptyset^\Sigma$ is the set of $\mathbb{I}'$-sentences associated by Φ to the $\mathbb{I}$-signature Σ, and where also $\beta^{th}_{(\Sigma,F)}(m) = \beta_\Sigma(m)$ for each (Σ, F)-model m, and α^{th} is exactly α. We let the reader check that indeed Φ^{th} is a signature preserving functor and that β^{th} and α^{th} are natural transformations. The satisfaction condition follows by Proposition 5.4 using that $\beta^{th}_{(\Sigma,\emptyset^\bullet)}$ is exactly β_Σ.

The most natural way to compose simple morphisms is as in Kleisli categories, that is, to compose the first simple theoroidal morphism with the extension of the second to a theoroidal morphism. More precisely, given two simple morphisms of institutions $(\Phi_1, \beta_1, \alpha_1)$ from $\mathbb{I}_1$ to $\mathbb{I}_2$ and $(\Phi_2, \beta_2, \alpha_2)$ from $\mathbb{I}_2$ to $\mathbb{I}_3$, their composition $(\Phi_1, \beta_1, \alpha_1);(\Phi_2, \beta_2, \alpha_2)$ is defined as the institution morphism $(\Phi_1, \beta_1, \alpha_1);(\Phi_2, \beta_2, \alpha_2)^{th}$ from $\mathbb{I}_1$ to $\mathbb{I}_3^{th}$. Unfortunately, in order to prove the associativity of morphism composition, one has to show that $((\Phi_1, \beta_1, \alpha_1);(\Phi_2, \beta_2, \alpha_2)^{th})^{th}$ equals $(\Phi_1, \beta_1, \alpha_1)^{th};(\Phi_2, \beta_2, \alpha_2)^{th}$, which does not seem to follow without further assumptions; at this time, we do not know what the weakest requirements should be.

The situation is better for simple theoroidal comorphisms, because here Φ and α go in the same direction; indeed, simple theoroidal comorphisms form a category without any additional assumptions. If $(\Phi, \beta, \alpha)\colon \mathbb{I} \to \mathbb{I}'$ is a simple theoroidal comorphism of institutions, then let $(\Phi, \beta, \alpha)^{th}$ be the theoroidal comorphism $(\Phi^{th}, \beta^{th}, \alpha^{th})$ from $\mathbb{I}$ to $\mathbb{I}'$ defined as $\Phi^{th}(\Sigma, F) = (\Sigma', (\alpha_\Sigma(F) \cup F_\emptyset^\Sigma)^\bullet)$ for each theory $(\Sigma, F) \in \mathbf{Th}$, where $\Phi(\Sigma) = (\Sigma', F_\emptyset^\Sigma)$, $\beta^{th}_{(\Sigma,F)}(m) = \beta_\Sigma(m)$ for each (Σ, F)-model m, and α^{th} is exactly α. We let the reader check that indeed Φ^{th} is a signature preserving functor, that β^{th} is well defined (the satisfaction condition of (Φ, β, α) is needed), is a natural transformation, and that and α^{th} is also a natural transformation. The satisfaction condition follows by Proposition 5.5 using that $\beta^{th}_{(\Sigma,\emptyset^\bullet)}$ is exactly β_Σ.

Simple comorphisms can be composed as expected from Kleisli, composing the first with the extension of the second, i.e., given simple comorphisms $(\Phi_1, \beta_1, \alpha_1)$ from $\mathbb{I}_1$ to $\mathbb{I}_2$ and $(\Phi_2, \beta_2, \alpha_2)$ from $\mathbb{I}_2$ to $\mathbb{I}_3$, their composition $(\Phi_1, \beta_1, \alpha_1);(\Phi_2, \beta_2, \alpha_2)$ is defined to be $(\Phi_1, \beta_1, \alpha_1);(\Phi_2, \beta_2, \alpha_2)^{th}$ from $\mathbb{I}_1$ to $\mathbb{I}_3^{th}$. To show associativity, one must show that $((\Phi_1, \beta_1, \alpha_1);(\Phi_2, \beta_2, \alpha_2)^{th})^{th}$ equals $(\Phi_1, \beta_1, \alpha_1)^{th};(\Phi_2, \beta_2, \alpha_2)^{th}$, which after some calculation reduces to showing that

$$(\alpha_{2,\Sigma_2}(\alpha_{1,\Sigma_1}(F_1)) \cup (\alpha_{2,\Sigma_2}(F_\emptyset^{\Sigma_1}) \cup F_\emptyset^{\Sigma_2})^\bullet)^\bullet = (\alpha_{2,\Sigma_2}((\alpha_{1,\Sigma_1}(F_1) \cup F_\emptyset^{\Sigma_1})^\bullet) \cup F_\emptyset^{\Sigma_2})^\bullet$$

where $\Phi_1(\Sigma_1) = (\Sigma_2, F_\emptyset^{\Sigma_1})$ and $\Phi_2(\Sigma_2) = (\Sigma_3, F_\emptyset^{\Sigma_2})$, and where the last assertion follows from paragraph 3 of Proposition 3.3 and the following:

Proposition 5.8. If (Φ, β, α) is a comorphism (or a simple theoroidal comorphism or a theoroidal comorphism) of institutions from $\mathbb{I}$ to $\mathbb{I}'$ and F is a set of Σ-sentences of $\mathbb{I}$, then $\alpha_\Sigma(F^\bullet) \subseteq \alpha_\Sigma(F)^\bullet$ and $\alpha_\Sigma(F^\bullet)^\bullet = \alpha_\Sigma(F)^\bullet$.

Proof. Let $m' \models_{\Sigma'} \alpha_\Sigma(F)$, where $\Sigma' = \Phi(\Sigma)$ in the case of comorphisms and $\Sigma' = \Phi^\diamond(\Sigma)$ in the case of (simple) theoroidal comorphisms. Then $m' \models_{\Sigma'} \alpha_\Sigma(F)$ iff (by the satisfaction condition) $\beta_\Sigma(m') \models_\Sigma F$ iff $\beta_\Sigma(m') \models_\Sigma F^\bullet$ iff (by the satisfaction condition) $m' \models_{\Sigma'} \alpha_\Sigma(F^\bullet)$. Therefore $\alpha_\Sigma(F) \models_{\Sigma'} \alpha_\Sigma(F^\bullet)$, which proves the inclusion. Then the equality is immediate. □

Example 5.9. We consider the relationship between $\mathbb{FOLE}$ and $\mathbb{FOL}$, unsorted first order algebra with and without equality, respectively. First observe that there is a very simple and natural morphism $\mathbb{FOLE} \to \mathbb{FOL}$, where the functor Φ forms the disjoint union of an $\mathbb{FOLE}$ signature Σ with the symbol '='; for notational convenience, we may denote this signature by $\Sigma^=$ and we assume that '=' does not occur in any $\mathbb{FOLE}$ signature, but is reserved for equality in $\mathbb{FOLE}$-sentences. Given an $\mathbb{FOLE}$ signature Σ and a $\mathbb{FOLE}$ Σ-model M, we define $\beta_\Sigma(M)$ to be the $\Phi(\Sigma)$-model $M^=$ with the equality symbol interpreted as actual identity in M; it is easy to see that β is natural. Given any $\mathbb{FOL}$ $\Sigma^=$-sentence f', let $\alpha_\Sigma(f')$ be just f', but with '=' now viewed as the symbol used to form equational atoms. The satisfaction condition follows easily.

Although it is simple, this morphism fails to capture the familiar trick of axiomatising equality when moving from $\mathbb{FOLE}$ to $\mathbb{FOL}$, as is needed to borrow a first order theorem prover for the translations of $\mathbb{FOLE}$ sentences. But it is easy to extend it to a simple theoroidal morphism, the theories of which contain axioms for equality, such as reflexivity, symmetry, and congruence: let the signature map send Σ to $\Psi(\Sigma) = (\Phi(\Sigma), T(\Sigma))$ where Φ is as above and where $T(\Sigma)$ is a $\Phi(\Sigma)$-theory of equality. But there is something strange about this, because the satisfaction condition holds no matter what axioms we give, including none at all – unless some of them are wrong!

On the other hand, we can view this situation as a simple theoroidal comorphism $\mathbb{FOLE} \to \mathbb{FOL}$, with equality axioms to ensure the satisfaction condition. Use Ψ as above for the signature to theory map, and given a $\mathbb{FOL}$ $\Psi(\Sigma)$-model M', define $\beta_\Sigma(M')$ to be the quotient of M' by the congruence generated by the equality axioms; it is easy to see that β is natural. Also, given an $\mathbb{FOLE}$ Σ-sentence f, let $\alpha_\Sigma(f)$ be f with '=' viewed as the new predicate symbol in $\Sigma^=$. The axioms in $T(\Sigma)$ must include all instances of the congruence axiom, but it is impossible to guarantee that '=' will be interpreted as identity.[14]

We can get the same effect as the above with a 'forgetful' theoroidal morphism $(\Phi, \beta, \alpha)\colon \mathbb{FOL}^= \to \mathbb{FOLE}$, where $\mathbb{FOL}^=$ is $\mathbb{FOL}$ with signatures that all include the special equality symbol =, where Φ forgets the special equality, where α_Σ adds all the congruence equations, and where β_Σ forgets the interpretation of the special equality. Theorem 4.4 now allows us to borrow a $\mathbb{FOL}$ theorem prover.

This example seems to confirm our hypothesis that morphisms are often simpler and more natural than comorphisms, but it also demonstrates that morphisms may not encapsulate all the information we need. The first theoroidal morphism is simple, and it can include all the information we want, but it is not possible to make proper use of this information. The comorphism is more complex because it needs equality axioms and a quotienting operation; moreover, it cannot force the equality to be identity. The second theoroidal morphism is very simple and it does what we want, by restricting the class of signatures to those where the borrowing makes sense. One moral of this story is that we must be very careful in choosing morphisms, and of course also comorphisms. We should also recall that simple theoroidal morphisms are not so simple, and perhaps are even problematic.

Theorem 4.11 gives a rather general correspondence, so that under its hypotheses every morphism gives rise to a comorphism, and vice versa. Of course, one of the two might still be 'aesthetically' better in some sense, but this could be considered subjective. An example of this might be the assertion that for borrowing it is nice to have the signature and sentence translations go in the same direction. Also, in writing this paper, we found that interesting further structure was sometimes revealed when trying to construct a good morphism, as with the pointed sort sets of Example 4.3; this observation might be considered a kind of evidence for the priority of morphisms.

6. Forward and Backward Morphisms

For both institution morphisms and comorphisms, the syntactic and semantic components go in opposite directions; but there are examples where it seems natural for these to go the same direction, and we will speak of 'forward morphisms' when both go in the forward direction. The following is the theoroidal version of this concept, though there is of course also a version at the ordinary level (as in [Tar96a, Tar00]); as usual, everything works for all close variants, at both levels:

Definition 6.1. Given institutions $\mathbb{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ and $\mathbb{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$, then a **theoroidal forward institution morphism**, **from $\mathbb{I}$ to $\mathbb{I}'$**, consists of

- $\Phi\colon \mathbf{Sign} \to \mathbf{Th}(\mathbb{I}')$ is signature preserving,
- $\beta\colon \mathbf{Mod} \Rightarrow \Phi; \mathbf{Mod}'$ is a natural transformation, and
- $\alpha\colon \mathbf{Sen} \Rightarrow \Phi; \mathbf{Sen}'$ is a natural transformation,

such that for any signature $\Sigma \in \mathbf{Sign}$, any sentence $f \in \mathbf{Sen}(\Sigma)$ and any model $m \in \mathbf{Mod}(\Sigma)$, the **satisfaction condition** holds:

$$m \models_\Sigma f \quad \text{iff} \quad \beta_\Sigma(m) \models'_{\Phi(\Sigma)} \alpha_\Sigma(f)$$

Example 6.2. There is a nice relation between the two institutions, $\mathbb{HA}_1$ and $\mathbb{HA}_2$, for hidden algebra described in Appendix C:

- since congruent operations are declared as sentences, any signature in the first institution translates to a specification in the second;
- any model A of (Σ, Γ) in the first institution gives a model of the second, namely $(A, \equiv^\Gamma_\Sigma)$;
- any (Σ, Γ)-sentence is a Σ-sentence;

and we can see that for any (Σ, Γ)-sentence f and any hidden Σ-algebra A, we get $A \models^\Gamma_\Sigma f$ iff $(A, \equiv^\Gamma_\Sigma) \models_\Sigma f$. All these say that there is a theoroidal forward morphism from $\mathbb{HA}_1$ to $\mathbb{HA}_2$.

[14] We thank Till Mossakowski for this last remark, for finding a bug in an earlier version, and for pointing out that this comorphism also occurs in [Mos02].

Of course, we can also define forward theoroidal comorphisms in much the same way, as well as simple theoroidal versions, and these will work for all close variants. Moreover, we can 'untwist' the definitions and results about twisted relations, institutions, morphisms and comorphisms to obtain forward versions of all the main results, including completeness and cocompleteness of the categories with institutions as objects, and with morphisms or comorphisms.

It is easy to give corresponding definitions for backward notions, but this is unnecessary, because a backward morphism is just a forward comorphism, and a backward comorphism is just a forward morphism; because of these relationships, the backward terminology is not necessary, although we may use it if convenient.

7. Semi-Natural Institution Morphisms and Comorphisms

The following weakens comorphisms by eliminating one of the naturality conditions; as usual, everything in this section holds for all close variants.

Definition 7.1. Given institutions $\mathbb{I} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \models \rangle$ and $\mathbb{I}' = \langle \mathbf{Sign}', \mathbf{Sen}', \mathbf{Mod}', \models' \rangle$, a **semi-natural institution comorphism** $(\Phi, \alpha, \beta)\colon \mathbb{I} \to \mathbb{I}'$ consists of

- a functor $\Phi\colon \mathbf{Sign} \to \mathbf{Sign}'$,
- a family of functors $\beta = \{\beta_\Sigma\colon \mathbf{Mod}'(\Phi(\Sigma)) \to \mathbf{Mod}(\Sigma)\}_{\Sigma \in |\mathbf{Sign}|}$, and
- a natural transformation $\alpha\colon \mathbf{Sen} \Rightarrow \Phi; \mathbf{Sen}'$,

such that for every $\Sigma \in |\mathbf{Sign}|$, $m' \in \mathbf{Mod}'(\Phi(\Sigma))$ and $f \in \mathbf{Sen}(\Sigma)$, the following **(co-)satisfaction condition** holds:

$$\beta_\Sigma(m') \models_\Sigma f \quad \text{iff} \quad m' \models'_{\Phi(\Sigma)} \alpha_\Sigma(f)$$

The point to note here is that β need not be natural; this condition is not satisfied in some examples, and is not needed to ensure some significant properties. The following shows that the free superextension of a partial algebra to an order sorted algebra [Gog97] gives rise to a semi-natural institution comorphism.

Example 7.2. Another natural expansion of a partial algebra to a supersorted algebra is the free extension, which freely adds supersorted terms for operations when they are undefined. We formalise this construction in the following.

For any supersorted signature (S, Σ) and partial (S', Σ')-algebra A', let $\beta_\Sigma(A')$ be the smallest S-sorted family such that:

1. $(\beta_\Sigma(A'))_{s'} = A'_{s'}$ for all $s' \in S'$ – let us call the elements of $(\beta_\Sigma(A'))_{s'}$ the **pure** elements;
2. $\beta_\Sigma(A')_{s'} \subseteq \beta_\Sigma(A')_s$ whenever $s' \leqslant s$; and
3. $\sigma\underline{(}a_1, \ldots, a_n\underline{)}$ is in $\beta_\Sigma(A')_s$ and is called **impure** whenever any of $a_1, \ldots, a_n$ are impure or $A'_\sigma(a_1, \ldots, a_n)$ is not defined, where $\sigma\colon w \to s$ is an operation with $|w| = n$ and where A'_σ is the partial map which interprets $\sigma\colon w' \to s'$ in A'.

Then $\beta_\Sigma(A')$ can be given an (S, Σ)-algebra structure as follows:

1. $(\beta_\Sigma(A'))_\sigma(a_1, \ldots, a_n) = A'_\sigma(a_1, \ldots, a_n)$ if $a_1, \ldots, a_n$ are all pure and $A'_\sigma(a_1, \ldots, a_n)$ is defined, and
2. $(\beta_\Sigma(A'))_\sigma(a_1, \ldots, a_n) = \sigma\underline{(}a_1\underline{,} \ldots\underline{,} a_n\underline{)}$ if any of $a_1, \ldots, a_n$ are impure or if $A'_\sigma(a_1, \ldots, a_n)$ is not defined,

where σ is as above. We call the (S, Σ)-algebra $\beta_\Sigma(A')$ the **free superextension** of A'. As shown in [Gog97], β_Σ can be organised as a functor $\beta_\Sigma\colon \mathbf{PAlg}(\Sigma^\flat) \to \mathbf{OSAlg}(\Sigma)$ which is left inverse left adjoint to $\mathscr{U}_\Sigma$. When the signature is clear from the context, we prefer to use the notation $\_^\natural$ instead of β_Σ.

Although all these constructions are very natural, β is still not a natural transformation. To see this, let $\varphi = (f, g)\colon (S_1, \Sigma_1) \to (S_2, \Sigma_2)$ be a morphism of supersorted signatures, and let A' be a partial Σ'_2-algebra. Then the free superextension of the φ-reduct of A' involves operation symbols in Σ_1 but the φ-reduct of the free superextension of A' involves operation symbols in Σ_2, so that these two Σ_1-algebras cannot be equal.

Now the satisfaction condition for the semi-natural institution comorphism from $\mathbb{OSA}^?$ to $\mathbb{PA}$ can be formulated as follows: for every $A' \in \mathbf{PAlg}(\Sigma^\flat)$ and $(\gamma, e) \in \mathbf{Sen}^?(\Sigma)$,

$$A'^\natural \models_\Sigma (\gamma, e) \quad \text{iff} \quad A' \models\!\!\!\!\!\!\approx_{\Sigma^\flat} (\gamma, e)$$

This result is proved in [Gog97].

Although the relationship between institutions is not quite so neat for the free superextension construction as for the single error superextension of the previous subsection, the former is more useful for many purposes, because it preserves information about why functions are undefined that is very useful for doing proofs, as well as for other purposes.

The notion that we call semi-naturality was introduced in the context of membership algebra by Meseguer with his 'general maps of institutions' [Mes98], where α is not required to be natural, but only a signature-indexed family of functions, just as with β in our Definition 7.1. At present, it is unclear how important semi-natural morphisms or comorphisms may be, or what are the general properties of their institutions. This, plus the fact that we do not know any good examples of semi-natural morphisms, seem to be further points in favour of the morphism concept.

8. Summary and Further Research

Mathematicians, and even logicians, have not shown much interest in the theory of institutions, perhaps because their tendency towards Platonism inclines them to believe that there is just one true logic and model theory; it also doesn't much help that institutions use category theory extensively.

On the other hand, computer scientists, having been forcibly impressed with the need to work with a number of different logics, often for very practical reasons, have written hundreds of papers that apply or further develop the theory of institutions. Institution morphisms become especially relevant when multiple logical systems need to be used for the same application and somehow coordinated, as often occurs in complex systems, where different logics are used for different aspects, including functional requirements, safety and liveness properties, concurrency control, real time response, data type design and architectural structure.

We wish to emphasise certain points made in this paper which, though not really new, do not seem to have been sufficiently appreciated in the current literature:

1. The notion of institution easily accommodates inference for logical systems; this was already noted in the basic early paper on institutions [GoB92], and it is further emphasised here through our notion of close variant. This fact makes it unnecessary to combine institutions with other more traditional machinery for inference.
2. It is easy to add a notion of inclusion to a category, and hence to an institution, and this can greatly simplify many typical applications of institutions, such as giving semantics to a specification language. In every single practical example we know, the category of signatures has a natural and obvious notion of inclusion, so it is quite harmless to assume an inclusive institution when doing specification semantics over an arbitrary institution.
3. In many cases, institution morphisms in the original sense [GoB92] provide more natural formulations of important relationships between institutions than other more recent notions; this appears to be especially true when the institution morphism is forgetful.
4. Results about institutions can often be pulled out of a general categorical hat, after a little translation, generalisation and/or massaging. Indeed, we now feel unsatisfied unless we have managed to do this for the major results. The use of indexed categories in Section 2.1 is one good example, the duality of morphisms and comorphisms is another, the construction of theoroidal morphisms and comorphisms using the theoroidal institution is a third, and the general results about close variants in Appendix E are a fourth.
5. Small categories of signatures are much more reasonable than large ones, at least for any remotely traditional applications to logic and computer science.
6. Finite limits and colimits of institutions are of more interest for practical applications than those of larger diagrams.

In this paper we have tried to bring some additional order to the menagerie of morphisms between institutions, starting with but not limited to, an improved taxonomy for the various genres and species, bringing out some unexpected relationships, and some new properties. Our new nomenclature includes the forms co-, semi-, theoroidal and forward, among which all combinations are meaningful, and some special cases, such as simple. All of these could be adapted to various institution-like formalisms, but we argue that there is no good reason to do so.

As is often the case, it seems that our research has opened far more questions than it has closed, including the following:

- One broad class of questions concerns properties of the various categories of institutions, the most immediate of which is how complete and cocomplete they are. Another such question is, which ones can be seen as flattened indexed categories? It is also of interest to determine their monos and epis, and to investigate their factorisation systems.
- One can ask for each category of institutions, which of its morphisms admit Kan extensions. However, one should also seek significant applications of translating a whole logical system along a mapping of its syntax in this way.
- To what extent do the various morphisms support the borrowing of logics and theorem provers in the style suggested in [GoB92] and later in [CeM97]?
- Can Theorem 4.11 be further generalised, for example, to theoroidal morphisms?
- To what extent do the various morphisms support the 'extra theory morphisms' and 'Grothendieck construction' in [Dia98] and [Dia00], respectively?
- Is there any value to coinstitutions, e.g., by dualising the material in Appendix D?
- Could the machinery of this paper be applied to the rapidly evolving field of coalgebra?
- Finally, it would be interesting to apply the theory of this paper to combining modules from different specification and programming languages.

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Dedication

This paper is dedicated, most warmly and respectfully, to Prof. Rod Burstall on the occasion of his retirement from the University of Edinburgh. Rod was the cofounder of the institution of institutions and has always been an enthusiastic supporter of its further development. He is also a very close and very dear friend, and one of the most insightful, kind and intelligent people we have ever known. We salute his very distinguished past, and we wish him every success and happiness in his future.

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Appendix A. Partial Algebra

This appendix presents partial algebra, while Appendix Appendix B presents a variant of order sorted algebra; in each case, we give the corresponding institution, following [Gog97]. Section B.4 describes the elements of a nice institution comorphism from the first to the second. This material is used in Example 4.5.

A.1. Partial Algebra

Given a many sorted signature[15] Δ, a **partial Δ-algebra** A is just the same as an ordinary Δ-algebra, except that the interpretations of the symbols from Δ in A may be *partial* functions instead of total functions; note that even constants can be partial – which just means they are undefined. Given a many sorted signature Δ, let $\mathbf{PAlg}_\Delta$ denote the category of all partial Δ-algebras with total Δ-homomorphisms. Unfortunately, there are other choices for homomorphisms, with no clear way to decide among them; for example, they might be indexed sets of partial functions. However, we choose to require that they are total and that they both reflect and preserve definedness.

Two classic references on partial algebra, by Horst Reichel [Rei84] and by Peter Burmeister [Bur86], are excellent sources. More recently, Cerioli, Mossakowski and Reichel have written a useful survey [CMR99], in which they argue for partial satisfaction and against order sorted algebra and, in particular, against the use of retracts; we of course disagree with their view.

A.2. Partial Satisfaction

One of the frustrations of partial algebra is the confusing plethora of definitions of satisfaction. We consider only satisfaction of unconditional equations by partial algebras, over a many sorted signature Δ. We next remind the reader some of the well-known notions of satisfaction for partial algebras, mostly with the purpose to ease the read of the next appendix. Perhaps the most common notion, called **existential satisfaction**,[16] says that a partial Δ-algebra A satisfies a Δ-equation $(\forall X)\ t = t'$ iff for every assignment $a\colon X \to A$, both $a(t)$ and $a(t')$ are defined, and they are equal. This notion has the disadvantage that equations like this inverse law

$$(\forall N : Nat)\ N * (1/N) = 1$$

are not satisfied by the rational numbers, because the left side is undefined for some values where the right side is not (namely $N = 0$). Existentially satisfied equations act as if they were totally satisfied, since they require everything that they talk about to be defined. Therefore existential satisfaction is not in general reflexive. These considerations suggest that existential satisfaction is too strong.

[15] In this section, we use Δ instead of the more usual Σ to avoid conflict with notation in Examples 4.5 and 7.2.
[16] This name is a bit ironic, because many existentialist philosophers had serious doubts about even the possibility of genuine satisfaction.

Another notion, called **strong satisfaction**, says that A satisfies $(\forall X)\ t = t'$ iff for every assignment $a\colon X \to A$, if *either* $a(t)$ or $a(t')$ is defined, then so is the other, and they are equal. For example, the equation

$$(\forall\ N, M : Nat)\ 1/(N * M) = (1/N) * (1/M)$$

is not existentially satisfied by the rationals, but it is strongly satisfied, because the two sides are defined for exactly the same assignments (namely when $N \neq 0$ and $M \neq 0$) and they are equal for all these assignments. However, the inverse law above fails to be strongly satisfied by the rationals, because the two sides are defined for different values. Similarly, the equation

$$(\forall\ N, M : Nat) N * M = 1/((1/N) * (1/M))$$

is neither strongly nor existentially satisfied by the rationals, because the left side is defined for some assignments where the right is not (namely whenever $N = 0$ or $M = 0$). These examples suggest that strong satisfaction is also too strong.

A third notion, called **weak satisfaction**, is that A satisfies $(\forall X)\ t = t'$ iff for every assignment $a\colon X \to A$, if *both* $a(t)$ and $a(t')$ are defined, then they are equal. The difference between weak and strong satisfaction is illustrated by the equation

$$(\forall M, N : Nat)\ M - N = N - M$$

which is weakly satisfied on the natural numbers, because both sides are defined iff $N = M$; however, it is neither strongly nor existentially satisfied by the naturals. Our intuition is that equations like the above should *not* be true, which implies that weak satisfaction is too weak. It is well known and easy to check that given a partial Δ-algebra A and a Δ-equation e, if A existentially satisfies e then A strongly satisfies e, and if A strongly satisfies e then A weakly satisfies e.

A.3. A Partial Algebra Institution

Let **Sign** be the category of many sorted signatures, and let **Sen**: **Sign** $\to$ **Set** be the functor that gives for each signature Δ the set of all pairs (γ, e) where γ is a type of satisfaction, i.e., an element in the set $\{$*weak*, *strong*, *existenatial*$\}$), and e is a Δ-equation. Let **PAlg**: **Sign** $\to$ **Cat**^{op} be the functor that gives for any signature Δ the category of partial Δ-algebras. If A is a partial Δ-algebra and e is a Δ-equation, let us write $A \models_\Delta (\gamma, e)$ whenever A partially γ-satisfies e. Then

Proposition A.1. $\mathbb{PA} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{PAlg}, \{\models_\Delta\}_{\Delta \in |\mathbf{Sign}|}\rangle$ is an institution.

Appendix B. Supersorted Order Sorted Algebra

Goguen [Gog97] showed that order sorted algebra with retracts can effectively handle both calculations and proofs for partial functions. There are two order sorted approaches to partiality, one using subsorts of definition and the other using error supersorts [Gog97]. Here we concentrate on the second, and show that partial algebra concepts fit very naturally into (total) order sorted algebra. This gives rise to an institution called supersorted order sorted algebra and denoted $\mathbb{OSA}^?$ (see [Gog97]).

B.1. Supersorted Signatures

Given an order sorted signature Σ, let $\mathbf{OAlg}_\Sigma$ denote the category of all Σ-algebras with Σ-homomorphisms. Call an order sorted signature Σ with sort set S **supersorted** iff S is the disjoint union of subsets S' and $S^?$ such that S' and $S^?$ are isomorphic (as ordered sets), with $<$ the least ordering on S including S' and $S^?$ (as ordered sets) such that $s' < s^?$ whenever $s' \in S'$ and $s^? \in S^?$ are corresponding sort symbols. Call the sorts in S' **pure**. By abuse of notation, if s is any sort in S then let s' be its pure correspondent, that is, s' is exactly s when s is pure and s' is s'_0 if $s = s^?_0$; also, if $w = s_1 s_2 \ldots s_n$ then let $w' = s'_1 s'_2 \ldots s'_n$. Similarly, any supersorted signature Σ can be 'purified' to an S'-signature which we denote Σ'. Given a Σ-algebra A, call its elements having sorts in S' its **pure** elements. Also, let us call a Σ-algebra A **strict** iff each of its operations returns an impure value whenever (one or more) of its arguments is impure.

Let a **morphism of supersorted signatures**, from (S_1, Σ_1) to (S_2, Σ_2), be a pair (f, g) where $f\colon S_1 \to S_2$ is such that $f(s_1) \in S'_2$ and $f(s_1^?) = (f(s_1))^?$ for each $s_1 \in S'_1$, and where $g = \{g_{w,s}\colon (\Sigma_1)_{w,s} \to (\Sigma_2)_{f(w),f(s)}\}$ is such that $g_{w,s}(\sigma) = g_{u,t}(\sigma)$ whenever $w' = u'$, $s' = t'$ and $\sigma \in (\Sigma_1)_{w,s} \cap (\Sigma_1)_{u,t}$. Notice that f can be restricted to source S'_1 and target S'_2, and let $f'\colon S'_1 \to S'_2$ denote such a restriction of f; note that $f(w)' = f'(w')$ for every $w \in S_1^*$. If $\mathbf{Sign}^?$ denotes the supersorted signatures and their morphisms, then

Fact B.1. $\mathbf{Sign}^?$ is a category.

B.2. Supersatisfaction

We present order sorted versions for the various kinds of partial satisfaction presented in Section A.1. Given a Σ-equation $e = (\forall X)\ t = t'$, we can make the following definitions: A **existentially supersatisfies** e iff for every pure assignment $a\colon X \to A$, both $a(t)$ and $a(t')$ are pure and they are equal. Similarly, A **strongly supersatisfies** e iff for every pure assignment $a\colon X \to A$, if either $a(t)$ or $a(t')$ are pure, then both are pure and they are equal. And finally, A **weakly supersatisfies** e iff for every pure assignment $a\colon X \to A$, if $a(t)$ and $a(t')$ are both pure, then they are equal.

B.3. The Supersorted Order Sorted Algebra Institution

Let $\mathbf{Sen}^?\colon \mathbf{Sign}^? \to \mathbf{Set}$ denote the functor that maps a supersorted signature to the set of all pairs (γ, e) where γ is a type of supersatisfaction (i.e., an element in the set $\{\mathit{weak},\ \mathit{strong},\ \mathit{existenatial}\}$) and e is a standard equation over that signature quantified with variables of non-error sorts.[17] Let $\mathbf{OSAlg}\colon \mathbf{Sign}^? \to \mathbf{Cat}^{op}$ be the usual functor that gives for any supersorted signature Σ the category of order sorted Σ-algebras. If A is an order sorted Σ-algebra and e is a Σ-equation, let us write $A \models_\Sigma (\gamma, e)$ when A γ-satisfies e. Then we have

Fact B.2. $\mathbb{OSA}^? = \langle \mathbf{Sign}^?, \mathbf{Sen}^?, \mathbf{OSAlg}, \{\models_\Sigma\}_{\Sigma \in |\mathbf{Sign}^?|} \rangle$ is an institution.

B.4. Forgetting the Errors

Given supersorted signature (S, Σ), let $(S, \Sigma)^\flat = (S', \Sigma')$ and note that $(S, \Sigma)^\flat$ is a signature whenever (S, Σ) is a supersorted signature, because the operations in Σ' only involve sorts in S'. If $(f, g)\colon (S_1, \Sigma_1) \to (S_2, \Sigma_2)$ is a morphism of supersorted signatures, define $(f, g)^\flat$ to be the pair (f', g') where g' is the family $\{g'_{w',s'}\colon (\Sigma'_1)_{w',s'} \to (\Sigma'_2)_{f(w'),f(s')}\}$ with $g'_{w',s'}(\sigma) = g_{w,s}(\sigma)$. Then

Fact B.3. $\_^\flat\colon \mathbf{Sign}^? \to \mathbf{Sign}$ is a functor.

We now define a natural transformation $\alpha\colon \mathbf{Sen}^? \Rightarrow \_^\flat; \mathbf{Sen}$ as follows: for any supersorted signature (S, Σ) and any (S, Σ)-equation (γ, e), let $\alpha_\Sigma(\gamma, e)$ be the (S', Σ')-equation obtained from (γ, e) by replacing each operation $\sigma\colon w \to s$ by $\sigma\colon w' \to s'$. Then

Fact B.4. $\alpha\colon \mathbf{Sen}^? \Rightarrow \_^\flat; \mathbf{Sen}$ is a natural transformation.

There is a natural a map $\mathscr{U}_\Sigma\colon \mathbf{OSAlg}(\Sigma) \to \mathbf{PAlg}(\Sigma^\flat)$ that forgets the impure elements of order sorted algebras, letting operations be undefined on some elements. Then [Gog97] shows:

Fact B.5. Given a supersorted signature (S, Σ), then $\mathscr{U}_\Sigma\colon \mathbf{OSAlg}(\Sigma) \to \mathbf{PAlg}(\Sigma^\flat)$ is a functor, and moreover, $\mathscr{U}\colon \mathbf{OSAlg} \Rightarrow \_^\flat; \mathbf{PAlg}$ is a natural transformation.

Example 4.5 shows how to put these elements together to form a nice institution comorphism from $\mathbb{OSA}^?$ to $\mathbb{PA}$.

[17] For $\mathbf{Sen}^?$ to be a functor, we need the rather technical fact that the equations quantified by non-error variables are mapped to equations quantified by non-error sorts. However, this is a consequence of the fact that non-error sorts are mapped to non-error sorts.

Appendix C. Two Hidden Algebra Institutions

A thorough exposition of hidden algebra may be found [Ros00]. Here we give only a few notions needed to present the two institutions. Intuitively, a hidden algebra is a 'blackbox' that can only be observed through experiments.

Definition C.1. Given disjoint sets V, H called **visible** and **hidden sorts**, a **loose data hidden** (V, H)**-signature** is a many sorted $(V \cup H)$-signature. A **fixed data hidden** (V, H)**-signature** is a pair (Σ, D) where Σ is a loose data hidden (V, H)-signature and D, called the **data algebra**, is a many sorted $\Sigma\restriction_V$-algebra. A **loose data hidden subsignature of** Σ is a loose data hidden (V, H)-signature Γ with $\Gamma \subseteq \Sigma$ and $\Gamma\restriction_V = \Sigma\restriction_V$. A **fixed data hidden subsignature of** (Σ, D) is a fixed data hidden (V, H)-signature (Γ, D) over the same data with $\Gamma \subseteq \Sigma$ and $\Gamma\restriction_V = \Sigma\restriction_V$.

Hereafter we may write 'hidden signature' instead of 'loose data hidden (V, H)-signature' or 'fixed data hidden (V, H)-signature', since we don't need to distinguish[18] them, and we may write Σ for (Σ, D).

Definition C.2. A **loose data hidden Σ-algebra** A is a Σ-algebra, and a **fixed data hidden** (Σ, D)**-algebra** A is a Σ-algebra A such that $A\restriction_{\Sigma_V} = D$.

Again we may say just 'hidden algebra'. We next use the mathematical concept of context to formalise the notion of 'experiment', which informally is an observation of an attribute of a system after it has been perturbed by certain methods; here the symbol $\bullet$ is a placeholder for the state being experimented upon.

Definition C.3. Given a hidden subsignature Γ of Σ, an (appropriate) **Γ-context for sort** s is a term in $T_\Gamma(\{\bullet : s\} \cup Z)$ having exactly one occurrence of a special variable[19] $\bullet$ of sort s, where Z is an infinite set of special variables. Let $\mathscr{C}_\Gamma[\bullet : s]$ denote the set of all Γ-contexts for sort s, and $var(c)$ the finite set of variables in a context c except $\bullet$. A Γ-context with visible result sort is called a **Γ-experiment**; let $\mathscr{E}_\Gamma[\bullet : s]$ denote the set of all Γ-experiments for sort s, let $\mathscr{C}_{\Gamma,s'}[\bullet : s]$ denote the Γ-contexts of sort s' for sort s, and let $\mathscr{E}_{\Gamma,v}[\bullet : s]$ denote all the Γ-experiments of sort v for sort s. If $c \in \mathscr{C}_{\Gamma,s'}[\bullet : s]$ and $t \in T_{\Sigma,s}(X)$, then $c[t]$ denotes the term in $T_{\Sigma,s'}(var(c) \cup X)$ obtained from c by substituting t for $\bullet$; formally, $c[t] = (\bullet \to t)^*(c)$, where $(\bullet \to t)^*\colon T_\Sigma(var(c) \cup \{\bullet : s\}) \to T_\Sigma(var(c) \cup X)$ is the unique extension of the map $(\bullet \to t)\colon var(c) \cup \{\bullet : s\} \to T_\Sigma(var(c) \cup X)$ which is identity on $var(c)$ and takes $\bullet : s$ to t. Further, c generates a map $A_c\colon A_s \to [A^{var(c)} \to A_{s'}]$ on each Σ-algebra A, defined by $A_c(a)(\theta) = a_\theta^*(c)$, where a_θ^* is the unique extension of the map (denoted a_θ) that takes $\bullet$ to a and each $z \in var(c)$ to $\theta(z)$.

The interesting experiments are those of hidden sort, i.e., with $s \in H$; experiments of visible sort are allowed just to smooth the presentation. We now define a distinctive feature of hidden logic, behavioural equivalence. Intuitively, two states are behaviourally equivalent iff they cannot be distinguished by any experiment.

Definition C.4. Given a hidden Σ-algebra A and a hidden subsignature Γ of Σ, the equivalence given by $a \equiv_\Sigma^\Gamma a'$ iff $A_\gamma(a)(\theta) = A_\gamma(a')(\theta)$ for all Γ-experiments γ and all maps $\theta\colon var(\gamma) \to A$ is called **Γ-behavioural equivalence on** A. We may write $\equiv$ instead of $\equiv_\Sigma^\Gamma$ when Σ and Γ can be inferred from context, and we write $\equiv_\Sigma$ when $\Sigma = \Gamma$. Given any equivalence $\sim$ on A, an operation σ in $\Sigma_{s_1 \ldots s_n, s}$ is **congruent for** $\sim$ iff $A_\sigma(a_1, \ldots, a_n) \sim A_\sigma(a'_1, \ldots, a'_n)$ whenever $a_i \sim a'_i$ for $i = 1 \ldots n$. An operation σ is **Γ-behaviourally congruent for** A iff it is congruent for $\equiv_\Sigma^\Gamma$. We often write just 'congruent' instead of 'behaviourally congruent'. A **hidden Γ-congruence on** A is an equivalence on A which is the identity on visible sorts and for which each operation in Γ is congruent.

The following supports several results below and works also for operations with more than one hidden argument or that are not behavioural; see [Ros00] for a proof. Since final algebras need not exist in this setting, existence of a largest hidden Γ-congruence does not depend on them, as it does in coalgebra.

Theorem C.5. Given a hidden subsignature Γ of Σ and a hidden Σ-algebra A, then Γ-behavioural equivalence is the largest hidden Γ-congruence on A.

[18] Actually, there are applications where we want some data sorts fixed and others loose, so that a more general notion that includes both cases is needed; it is clear how to do this by combining the two cases in one definition.

[19] These are assumed different from any other variables in a given situation.

Definition C.6. Given a hidden Σ-algebra A and a Σ-equation $(\forall X)\ t = t'$, say e, then A **Γ-behaviourally satisfies** e iff $\theta(t) \equiv^{\Gamma}_{\Sigma} \theta(t')$ for each $\theta\colon X \to A$; in this case we write $A \models^{\Gamma}_{\Sigma} e$. If E is a set of Σ-equations, we write $A \models^{\Gamma}_{\Sigma} E$ if A Γ-behaviourally satisfies each Σ-equation in E.

When Σ and Γ are clear, we may write $\equiv$ and $\models$ instead of $\equiv^{\Gamma}_{\Sigma}$ and $\models^{\Gamma}_{\Sigma}$, respectively. We only consider unconditional equations here, but the theory also allows conditional equations [Ros00].

Definition C.7. A **behavioural** (or **hidden**) **Σ-specification** (or **-theory**) is a triple (Σ, Γ, E) where Σ is a hidden signature, Γ is a hidden subsignature of Σ, and E is a set of Σ-equations. The operations in $\Gamma - \Sigma\restriction_V$ are called **behavioural**. We usually let $\mathscr{B}$, $\mathscr{B}'$, $\mathscr{B}_1$, etc., denote behavioural specifications. A hidden Σ-algebra A **behaviourally satisfies** (or **is a model of**) a behavioural specification $\mathscr{B} = (\Sigma, \Gamma, E)$ iff $A \models^{\Gamma}_{\Sigma} E$, and in this case we write $A \models \mathscr{B}$; we write $\mathscr{B} \models e$ if $A \models \mathscr{B}$ implies $A \models^{\Gamma}_{\Sigma} e$. An operation $\sigma \in \Sigma$ is **behaviourally congruent for** $\mathscr{B}$ iff σ is behaviourally congruent for every $A \models \mathscr{B}$.

The following gives the existence of many congruent operations:

Proposition C.8. If $\mathscr{B} = (\Sigma, \Gamma, E)$ is a behavioural specification, then all operations in Γ, and all hidden constants, are behaviourally congruent for $\mathscr{B}$.

Of course, depending on E, other operations may also be congruent; in fact, our experience is that all operations are congruent in many practical situations.

There are two rather different ways to present hidden algebra as an institution in two interesting ways, depending on whether the declaration of an operation to be behavioural is considered part of the signature, or as a separate sentence; we first approached this issue in [GoR99].

The first institution, denoted $\mathbb{HA}_1$, follows the institution of hidden algebra initially presented in [Gog91], the institution of observational logic in [HeB99], and the coherent hidden algebra approach in [DiF98, DiF00], while the second, which we simply call $\mathbb{HA}_2$, seems more promising for future research. Our approach also avoids the infinitary logic used in observational logic. Only the fixed-data case is investigated here, but we hope to extend it to the loose-data case soon (see [Ros00] for more on the terminology of hidden algebra). We fix a data Ψ-algebra D.

C.1. The First Institution

The institution $\mathbb{HA}_1$ is built as follows:

Signatures: The category **Sign** has hidden signatures over a fixed data algebra D as objects. A morphism of hidden signatures $\phi\colon (\Gamma_1, \Sigma_1) \to (\Gamma_2, \Sigma_2)$ is the identity on the visible signature Ψ, takes hidden sorts to hidden sorts, and if a behavioural operation δ_2 in Γ_2 has an argument sort in $\phi(H_1)$ then there is some behavioural operation δ_1 in Γ_1 such that $\delta_2 = \phi(\delta_1)$. **Sign** is in fact a category, and the composition of two hidden signature morphisms is another. Indeed, let $\psi\colon (\Gamma_2, \Sigma_2) \to (\Gamma_3, \Sigma_3)$ and let δ_3 be an operation in Γ_3 having an argument sort in $(\phi;\psi)(H_1)$. Then δ_3 has an argument sort in $\psi(H_2)$, so there is an operation δ_2 in Γ_2 with $\delta_3 = \psi(\delta_2)$. Also δ_2 has an argument sort in $\phi(H_1)$, so there is some δ_1 in Γ_1 with $\delta_2 = \phi(\delta_1)$. Therefore $\delta_3 = (\phi;\psi)(\delta_1)$, i.e., $\phi;\psi$ is also a morphism of hidden signatures.

Sentences: Given a hidden signature (Γ, Σ), let **Sen**(Γ, Σ) be the set of all Σ-equations. If $\phi\colon (\Gamma_1, \Sigma_1) \to (\Gamma_2, \Sigma_2)$ is a hidden signature morphism, then **Sen**(ϕ) is the function taking a Σ_1-equation $e = (\forall X)\ t = t'$ **if** $t_1 = t'_1, \ldots, t_n = t'_n$ to the Σ_2-equation

$$\phi(e) = (\forall X')\ \phi(t) = \phi(t')\ \textbf{if}\ \phi(t_1) = \phi(t'_1), \ldots, \phi(t_n) = \phi(t'_n)$$

where X' is $\{x : \phi(s) \mid x : s \in X\}$. Then **Sen**: **Sign** → **Set** is indeed a functor.

Models: Given a hidden signature (Γ, Σ), let **Mod**(Γ, Σ) be the category of hidden Σ-algebras and their morphisms. If $\phi\colon (\Gamma_1, \Sigma_1) \to (\Gamma_2, \Sigma_2)$ is a hidden signature morphism, then **Mod**(ϕ) is the usual reduct functor, $\_\restriction_\phi$. Unlike [BBR98, HeB99], etc., this allows models where not all operations are congruent.

Satisfaction Relation: behavioural satisfaction, i.e., $\models_{(\Gamma,\Sigma)} = \models^{\Gamma}_{\Sigma}$.

Theorem C.9. Satisfaction Condition: Given $\phi\colon (\Gamma_1, \Sigma_1) \to (\Gamma_2, \Sigma_2)$ a hidden signature morphism, $d = (\forall X)\ t = t'$ **if** $t_1 = t'_1, \ldots, t_n = t'_n$ a Σ_1-equation, and A a hidden Σ_2-algebra, then $A \models^{\Gamma_2}_{\Sigma_2} \phi(e)$ iff $A\restriction_\phi \models^{\Gamma_1}_{\Sigma_1} e$.

Proof. See [GoR99, Ros00]. □

C.2. The Second Institution

Our second institution views the declaration of a behavioural operation as a new kind of sentence, rather than part of a hidden signature. The notion of model also changes, adding an equivalence relation as in [BBK92]. This is natural for modern software engineering, since languages like Java provide classes with an operation denoted `equals` which serves this purpose. Sentences in [BBK92] are pairs $\langle e, \Delta\rangle$, where Δ is a set of terms (pretty much like a cobasis over the derived signature), which are satisfied by $(A, \sim)$ iff $(A, \sim)$ satisfies e as in our case below (actually e is a first-order formula in their framework) and $\sim \subseteq \equiv_\Delta$. Fix a data algebra D, and proceed as follows:
Signatures: The category **Sign** has hidden signatures over D as objects, with its morphisms $\phi\colon \Sigma_1 \to \Sigma_2$ the identity on the visible signature Ψ, and taking hidden sorts to hidden sorts.
Sentences: Given a hidden signature Σ, let $\mathbf{Sen}(\Sigma)$ be the set of all Σ-equations unioned with Σ. If $\phi\colon \Sigma_1 \to \Sigma_2$ is a hidden signature morphism, then $\mathbf{Sen}(\phi)$ is the function taking a Σ_1-equation $e = (\forall X)\ t = t'$ **if** $t_1 = t'_1, \ldots, t_n = t'_n$ to the Σ_2-equation $\phi(e) = (\forall X')\ \phi(t) = \phi(t')$ **if** $\phi(t_1) = \phi(t'_1), \ldots, \phi(t_n) = \phi(t'_n)$, where X' is the set $\{x\colon \phi(s) \mid x\colon s \in X\}$, and taking $\sigma\colon s_1 \ldots s_n \to s$ to $\phi(\sigma)\colon \phi(s_1)\ldots\phi(s_n) \to \phi(s)$. Then $\mathbf{Sen}\colon \mathbf{Sign} \to \mathbf{Set}$ is indeed a functor.
Models: Given a hidden signature Σ, let $\mathbf{Mod}(\Sigma)$ be the category of pairs $(A, \sim)$ where A is a hidden Σ-algebra and $\sim$ is an equivalence relation on A which is identity on visible sorts, with morphisms $f\colon (A, \sim) \to (A', \sim')$ with $f\colon A \to A'$ a Σ-homomorphism such that $f(\sim) \subseteq \sim'$. If $\phi\colon \Sigma_1 \to \Sigma_2$ is a hidden signature morphism, then $\mathbf{Mod}(\phi)$, often denoted $\_|_\phi$, is defined as $(A, \sim)\restriction_\phi = (A\restriction_\phi, \sim\restriction_\phi)$ on objects, where $A\restriction_\phi$ is the ordinary many sorted algebra reduct and $(\sim\restriction_\phi)_s = \sim_{\phi(s)}$ for all sorts s of Σ_1, and as $f\restriction_\phi : (A, \sim)\restriction_\phi \to (A', \sim')\restriction_\phi$ on morphisms. Notice that $f\restriction_\phi (\sim\restriction_\phi) \subseteq \sim'\restriction_\phi$, so that **Mod** is well defined.
Satisfaction Relation: A Σ-model $(A, \sim)$ satisfies a conditional Σ-equation $(\forall X)\ t = t'$ **if** $t_1 = t'_1, \ldots, t_n = t'_n$ iff for each $\theta\colon X \to A$, if $\theta(t_1) \sim \theta(t'_1), \ldots, \theta(t_n) \sim \theta(t'_n)$ then $\theta(t) \sim \theta(t')$. Also $(A, \sim)$ satisfies a Σ-sentence $\gamma \in \Sigma$ iff γ is congruent for $\sim$.

Theorem C.10. Satisfaction Condition: Let $\phi\colon \Sigma_1 \to \Sigma_2$ be a morphism of hidden signatures, let e be a Σ_1-sentence and let $(A, \sim)$ be a model of Σ_2. Then $(A, \sim) \models_{\Sigma_2} \phi(e)$ iff $(A, \sim)\restriction_\phi \models_{\Sigma_1} e$.

Proof. See [GoR99, Ros00]. □

This institution justifies our belief that asserting an operation behavioural is a kind of sentence, not a kind of syntactic declaration as in the 'extended hidden signatures' of [DiF00].[20] Coinduction now appears in the following elegant guise:

Proposition C.11. Given a hidden subsignature Γ of Σ, a set of Σ-equations E and a hidden Σ-algebra A, then

- $(A, \sim) \models_\Sigma E, \Gamma$ implies $(A, \equiv^\Gamma_\Sigma) \models_\Sigma E, \Gamma$.
- $(A, \equiv^\Gamma_\Sigma) \models_\Sigma \Gamma$.
- $A \models^\Gamma_\Sigma E$ iff $(A, \equiv^\Gamma_\Sigma) \models_\Sigma E$ iff $(A, \equiv^\Gamma_\Sigma) \models_\Sigma E, \Gamma$.

Appendix D. A More Categorical Universal Algebra Institution

This section develops universal algebra more abstractly than usual, using a categorical approach that interprets satisfaction as injectivity. We show that this gives an institution in which the satisfaction condition is 'almost equivalent' to the notion of adjoint functor, thus strengthening our belief in the naturality of institutions [GoB92]. We assume familiarity with the basics of factorisation systems [HeS73, Nem82].

Definition D.1. If $\mathbf{A}$ is a category and $\mathscr{C}$ is a class of morphisms in $\mathbf{A}$, then an object D is **$\mathscr{C}$-injective** iff for any morphism $c\colon A \to B$ in $\mathscr{C}$ and any morphism $f\colon A \to D$ there is some morphism $g\colon B \to D$ such that $f = c; g$.

Definition D.2. If $\langle \mathscr{E}_\mathbf{A}, \mathscr{M}_\mathbf{A}\rangle$ and $\langle \mathscr{E}_\mathbf{B}, \mathscr{M}_\mathbf{B}\rangle$ are factorisation systems for respective categories $\mathbf{A}$ and $\mathbf{B}$, then a functor $F\colon \mathbf{A} \to \mathbf{B}$ is called **$\mathscr{E}$-preserving** iff $F(\mathscr{E}_\mathbf{A}) \subseteq \mathscr{E}_\mathbf{B}$.

[20] However, the most recent version of [DiF02] treats coherence assertions as sentences.

We first note that satisfaction of equations in universal algebra is equivalent to injectivity. Let **A** be the category of universal (many sorted) Σ-algebras over a (many sorted) signature Σ. Then each equation $(\forall X)\ t = t'$ generates a congruence relation on $T_\Sigma(X)$ (the term algebra over variables in X), which gives rise to a surjective morphism $e\colon T_\Sigma(X) \to \bullet$. It can be seen that an algebra D satisfies $(\forall X)\ t = t'$ iff it is $\{e\}$-injective. Conversely, each surjective morphism e with a free algebra source generates an infinite set E of equations over variables in that free algebra, namely all pairs in its kernel. It follows that an algebra is $\{e\}$-injective iff it satisfies all equations in E. Therefore, satisfaction of equations and $\mathscr{C}$-injectivity, where $\mathscr{C}$ contains only surjective morphisms with free sources, are equivalent concepts.

It can be shown [Ros97b] that, given a set of surjective morphisms with not necessarily free sources, $\mathscr{C}$-injectivity is equivalent to the satisfaction of conditional equations, where the conditions are given by the pairs in the kernel of the co-unit morphisms with target in $\mathscr{C}$. The institution given below defines satisfaction as injectivity and therefore covers conditional equations.

Signatures: Sign has small categories with factorisation systems as objects, and $\mathscr{E}$-preserving left adjoint functors as its morphisms.

Sentences: Sen: **Sign** $\to$ **Set** is defined by **Sen**(**A**) $= \mathscr{E}_{\mathbf{A}}$. The reader may check that **Sen** is well defined.

Models: Define **Mod**: **Sign** $\to$ **Cat**^{op} by **Mod**(**A**) = **A** and **Mod**(F) is a right adjoint of F, where the right adjoints are chosen so that **Mod** is functorial.

Satisfaction: Given $\mathbf{A} \in |\mathbf{Sign}|$, then define $\models_{\mathbf{A}} \subseteq |\mathbf{A} \times \mathscr{E}_{\mathbf{A}}$ by $A \models_{\mathbf{A}} e$ iff A is $\{e\}$-injective.

Theorem D.3. Satisfaction Condition: Given an $\mathscr{E}$-preserving functor $F\colon \mathbf{B} \to \mathbf{A}$ which is left adjoint to $U\colon \mathbf{A} \to \mathbf{B}$, an object $A \in |\mathbf{A}|$ and a morphism $e \in \mathscr{E}_B$, then $A \models_{\mathbf{A}} F(e)$ iff $U(A) \models_{\mathbf{B}} e$.

Proof. The proof follows from properties of adjoint functors, and we leave it as an exercise. The following diagrams may help:

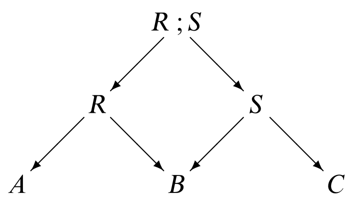
□

Appendix E. A More Categorical Institution Formulation

The major purpose of this appendix is to make the notion of close variant institution precise, in such a way that results about such institutions can be proved in a uniform, abstract manner. The exposition mainly follows ideas from [GoB92], and is divided into three subsections, which discuss relations, comma categories and close variant institutions, respectively.

E.1. Relations and Multi-relations

Since the main feature of an institution is its satisfaction relation, our more categorical formulation of institutions will need a categorical notion of relation. A common way to do this is to define the **category of relations** in an arbitrary category **C** having pullbacks, denoted **Rel**(**C**), as follows: its objects are those of **C**; its morphisms from A to B are pairs $\langle p_1\colon R \to A,\ p_2\colon R \to B\rangle$ of morphisms of **C** having a common source, which is called the **apex**; its identities have both p_1 and p_2 identity morphisms; and its **composition** is obtained by pullback, as indicated in the diagram below. (Strictly speaking, we need to identify relations that have isomorphic apexes, or else associativity will only hold up to isomorphism; however, we tend to ignore this point.)



Proposition E.1. If **C** has pullbacks, then **Rel**(**C**) is a category.

Although most of the literature (e.g., [Bor94]) calls these morphisms 'relations', when one specialises to the usual case $\mathbf{C} = \mathbf{Set}$, one actually gets 'multi-relations', for which a given pair of items may be related in an arbitrary set of ways. Given $\langle p_1 \colon R \to A,\ p_2 \colon R \to B \rangle$, then $\{r \in R \mid p_1(r) = a,\ p_2(r) = b\}$ is the set of all the ways that a, b are related by R (of course, it may be empty). The situation is the same when $\mathbf{C} = \mathbf{SET}$, except that there may be a proper class of ways of relating two elements.

Although multi-relations are not quite the same as the relations used in Definition 3.1 for satisfaction, and at this time we don't know any applications for the additional structure that multi-relations provide, we do not view this as a problem, because ordinary 'singulary' relations are a special case of multi-relations,[21] and moreover, there is a nice functor that 'flattens' multi-relations to singulary relations, as we now show.

We use the following notation for the case $\mathbf{C} = \mathbf{Set}$: $\mathbf{R}$ is the category of ordinary relations, $\mathbf{M}$ is the category **Rel**(**Set**), and $F \colon \mathbf{M} \to \mathbf{R}$ is the flattening functor, defined on objects as the identity, and on morphisms by $F(\langle p_1, p_2 \rangle) = \{\langle a, b \rangle \mid (\exists r \in R)\ p_1(r) = a,\ p_2(r) = b\}$ (the reader may check that this really is a functor). Although there aren't any nice functors $\mathbf{R} \to \mathbf{M}$ [BrG01], if we restrict the sizes of the sets involved we can construct one. Let Ω be an infinite cardinal, let Z be a set of cardinality Ω, let $\mathbf{Set}_\Omega$ be the full subcategory of **Set** whose objects have cardinality less than or equal to Ω, let $\mathbf{R}_\Omega$ denote the full subcategory of $\mathbf{R}$ with objects in $\mathbf{Set}_\Omega$, and let $\mathbf{M}_\Omega$ denote $\mathbf{Rel}(\mathbf{Set}_\Omega)$. Then F restricts to a functor $F_\Omega \colon \mathbf{M}_\Omega \to \mathbf{R}_\Omega$, and we define a functor $G_\Omega \colon \mathbf{R}_\Omega \to \mathbf{M}_\Omega$ as follows: it is the identity on objects; and it maps a relation $R \colon A \to B$ to the multi-relation with apex $M = \coprod_{\langle a, b \rangle \in R} Z_{a,b}$, with $p_1(z) = a$ iff $z \in Z_{a,b}$ and $p_2(z) = b$ iff $z \in Z_{a,b}$. It is easy to see that G_Ω is functorial, that $F_\Omega(G_\Omega(R)) = R$, and that $G_\Omega(F_\Omega(M)) \leqslant M$. In fact, these functors are adjoint in a certain 2-categorical sense made explicit in Appendix B of [Gog99].

Notice that **Rel**(**C**) is self-dual, i.e., there is an equivalence (actually an isomorphism) between **Rel**(**C**) and its dual. It follows that limits and colimits are the same in this category. When $\mathbf{C} = \mathbf{Set}$, the product (and coproduct) of sets A, B is $A \times B + A + B$, where $\times$ and $+$ denote the usual product and coproduct for sets. The same holds for $\mathbf{C} = \mathbf{SET}$. However, the following shows that **Rel**(**Set**) and **Rel**(**SET**) do not have (co)equalisers, and hence are not (co)complete:

Example E.2. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$ be two sets, and define relations from A to B by $R_1 = \{(a_1, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$ and $R_2 = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_2)\}$. Suppose that there is a relation R from a set E to A that is an equaliser of R_1 and R_2. We will show that E must contain at least four elements $\{e_1, e_2, e_3, e_4\}$ and R must contain at least the pairs

$$\{(e_1, a_1), (e_1, a_2), (e_2, a_3), (e_3, a_1), (e_3, a_3), (e_4, a_2), (e_4, a_3)\}$$

by considering the cones over the singleton $C = \{c\}$ with relations $Q_1 = \{(c, a_1), (c, a_2)\}$, $Q_2 = \{(c, a_3)\}$, $Q_3 = \{(c, a_1), (c, a_3)\}$, and $Q_4 = \{(c, a_2), (c, a_3)\}$ from C to A. Notice that $Q_i; R_1 = Q_i; R_2$ for $i = 1, 2, 3, 4$, and that the only way these relations could factor through R is if there exist e_i as above. Next, notice that the relation $Q = \{(c, a_1), (c, a_2), (c, a_3)\}$ equalises R_1 and R_2, but does not factor uniquely through R, since defining $S_1 = \{(c, e_1), (c, e_2)\}$ and $S_2 = \{(c, e_3), (c, e_4)\}$, we have both $S_1; R = Q$ and $S_2; R = Q$. (This example was emailed to us by Roberto Bruni, to whom we give our thanks.)

Similar examples can show the non-(co)completeness of other relation categories.

Finally, we describe some ways to construct functors into relation categories. Given $F \colon \mathbf{A} \to \mathbf{C}$, define $F\!\uparrow \colon \mathbf{A} \to \mathbf{Rel}(\mathbf{C})$ as follows: an object A in $\mathbf{A}$ goes to the object $F(A)$ in **Rel**(**C**); and a morphism $a \colon A \to A'$ in $\mathbf{A}$ goes to a morphism $\langle 1_{F(A)}, F(a) \rangle$ from $F(A)$ to $F(A')$ in **Rel**(**C**). Similarly, given $F \colon \mathbf{A} \to \mathbf{C}$, define $F\!\downarrow \colon \mathbf{A}^{op} \to \mathbf{Rel}(\mathbf{C})$ by: an object A in $\mathbf{A}$ goes to the object $F(A)$ in **Rel**(**C**); and a morphism $a \colon A \to A'$ in $\mathbf{A}^{op}$ goes to a morphism $\langle 1_{F(A')}, F(a) \rangle$ from $F(A')$ to $F(A)$ in **Rel**(**C**). It is easy to see that both constructions are functors. The second will let us 'twist' relations over an arbitrary category.

[21] However, the singulary relations are not a *subcategory* of the multi-relations, because the composition of two singulary relations as multi-relations need not be singulary. But at least when **C** is **Set** or **SET**, instead of just identifying isomorphic apexes, we could identify relations having apexes that define the same singulary relation.

E.2. Comma Categories

To construct twisted relation categories, we will use comma categories, which first appeared in Lawvere's thesis [Law63]. Given functors $F\colon \mathbf{A} \to \mathbf{C}$ and $G\colon \mathbf{B} \to \mathbf{C}$, the **comma category** that Lawvere denoted (F, G), but which we prefer to write (F/G), has as its objects triples (A, c, B) where A is an object of $\mathbf{A}$, B is an object of $\mathbf{B}$, and $c\colon F(A) \to G(B)$ is a morphism of $\mathbf{C}$; also (F/G) has as its morphisms $(A, c, B) \to (A', c', B')$ pairs (a, b) such that $a\colon A \to A'$, $b\colon B \to B'$, and the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\;c\;} & G(B) \\ {\scriptstyle F(a)}\downarrow & & \downarrow{\scriptstyle G(b)} \\ F(A') & \xrightarrow[\;c'\;]{} & G(B') \end{array}$$

The identities of (F/G) are pairs of identities, and its composition is also defined pairwise. An important special case has $\mathbf{A} = \mathbf{C}$, F is the identity functor on $\mathbf{A}$, $\mathbf{B} = \mathbf{1}$, the one morphism category, and G picks out an object A of $\mathbf{A}$; here (F/G) is often written $(\mathbf{A}/A)$, and is called a **slice category**. The following appears in [GoB84b] along with some other general results:

Proposition E.3. (F/G) has whatever limits and colimits exist in $\mathbf{A}$ and $\mathbf{B}$ and are preserved by both F and G. In particular, if $\mathbf{A}$ and $\mathbf{B}$ are both complete and cocomplete and if F and G preserve all limits and colimits, then (F/G) and $(F/G)^{op}$ are both complete and cocomplete.

However, this result is not adequate for our purposes, because F and G do not preserve products even for the classic institutions, since $F = 1_{\mathbf{SET}}\!\downarrow$ does not preserve products, which are different in **Set** than they are in **Rel**. This motivates the following, in which we let $\pi_1\colon \mathbf{A} \times \mathbf{B} \to \mathbf{A}$ and $\pi_2\colon \mathbf{A} \times \mathbf{B} \to \mathbf{B}$ be the projection functors, we let $\rho_1\colon (F/G) \to \mathbf{A}$ and $\rho_2\colon (F/G) \to \mathbf{B}$ be the functors taking objects (A, c, B) to A and to B, respectively, and taking morphisms (a, b) to a and to b, respectively, and we let $\rho\colon (F/G) \to \mathbf{A} \times \mathbf{B}$ be the unique functor such that $\rho;\pi_1 = \rho_1$ and $\rho;\pi_2 = \rho_2$. Notice that by definition, (F/G) has whatever limits and colimits ρ creates. We will use the following:

Assumption [A]: The categories $\mathbf{A}$ and $\mathbf{B}$ are both complete and cocomplete, and ρ creates both limits and colimits.

Although this is a rather strong assumption, it does not require the (co)completeness of $\mathbf{C}$, and it has the following immediate consequence:

Proposition E.4. Under Assumption [A], both (F/G) and $(F/G)^{op}$ are complete and cocomplete.

E.3. Close Variant Institutions

We can combine the relation and comma category constructions with the functors $\uparrow$ and $\downarrow$ to get categories of twisted relations; then functors into these will give us the close variant institutions. The classic institutions of Definition 3.1 arise from the category $(1_{\mathbf{SET}}\!\downarrow / U\!\uparrow)$ where $U\colon \mathbf{Cat} \to \mathbf{SET}$ is the forgetful functor taking each category to its underlying class, and each functor to its underlying function, with **SET** the category of classes. Similarly, the institution variant with both the model and sentence functors class-valued arises from $(1_{\mathbf{SET}}\!\downarrow / 1_{\mathbf{SET}}\!\uparrow)$, where $1_{\mathbf{A}}$ denotes the identity functor on $\mathbf{A}$. The category $(U\!\downarrow / U\!\uparrow)$ allows morphisms between sentences as well as between models. Recall that an institution is a functor into the category of diagrams over the opposite of a twisted relation category, and that the institution morphisms of Definition 4.1 are also exactly those that arise via the diagram category of Section 2.1. In particular, the category of classic institutions of Definition 3.1 is $\mathbf{Dgm}((1_{\mathbf{SET}}\!\downarrow / U\!\uparrow)^{op})$.

Most of the general results of this paper used (co)completeness of the category of twisted relations (Proposition 2.12), but did not rely on the particular way that it was constructed. The main goal of this appendix is to provide a general framework in which one can obtain the results of this paper for any close variant institution. One approach is to use Proposition E.4 to adapt Proposition 2.12 to a given close variant relation category. For the original variants, $\mathbf{A}$ and $\mathbf{B}$ are one of **Set**, **SET**, **Cat**, or **CAT**, each of which is both complete and cocomplete, and so the task becomes that of showing that ρ creates limits and colimits. We now apply this method to the twisted relation category for classic institutions (as in Proposition 2.12):

Proposition E.5. The comma category $(1_{\mathbf{SET}}\!\downarrow / U\!\uparrow)$ is complete and cocomplete.

Proof. This category has $\mathbf{A} = \mathbf{SET}^{op}$, $\mathbf{B} = \mathbf{Cat}$, $\mathbf{C} = \mathbf{Rel}(\mathbf{SET})$, $F = 1_{\mathbf{SET}}\downarrow$, and $G = U\uparrow$. Let $D\colon \mathbf{J} \to (F/G)$ be any diagram, let $J = |\mathbf{J}|$, and let $R_j\colon A_j \to U(B_j)$ denote the relation $D(j)$ for $j \in J$. First, let $(A, \{\alpha_j\}_{j\in J})$ and $(B, \{\beta_j\}_{j\in J})$ be limits of $D;\rho_1$ and $D;\rho_2$, respectively, and let $R\colon A \to U(B)$ be the relation defined by: $(x, y) \in R$ iff for all $j \in J$ and all $x_j \in A_j$, we have $(x_j, \beta_j(y)) \in R_j$ whenever $\alpha_j(x_j) = x$. Notice that for every $x \in A$ there do exist some $j \in J$ and $x_j \in A_j$ such that $\alpha_j(x_j) = x$, because of the way colimits are built in **SET**. The reader may now check that $(R\colon A \to U(B), \{\langle \alpha_j, \beta_j\rangle\}_{j\in J})$ is a limit of D. Therefore ρ creates limits.

Second, let $(\{\alpha_j\}_{j\in J}, A)$ and $(\{\beta_j\}_{j\in J}, B)$ be colimits of $D;\rho_1$ and $D;\rho_2$, respectively, and let $R\colon A \to U(B)$ be the relation defined by: $(x, y) \in R$ iff for all $j \in J$ and all $y_j \in U(B_j)$, we have $(\alpha_j(x), y_j) \in R_j$ whenever $\beta_j(y_j) = y$. Notice that for every $y \in U(B)$ there do exist some $j \in J$ and $y_j \in U(B_j)$ such that $\beta_j(y_j) = y$, because of the way colimits are built in **Cat**. The reader may now check that $(\{\langle \alpha_j, \beta_j\rangle\}_{j\in J}, R\colon A \to U(B))$ is a colimit of D. Therefore ρ creates colimits.

It now follows from Proposition E.5 that (F/G) is both complete and cocomplete. □

Although the constructions in the above proof depend on how limits and colimits are built for sets and categories, it still seems possible that the results also follow from some general (co)completeness results for comma categories, and this should be investigated in the future. In any case, similar constructions certainly work for each of the other original variants. It is therefore natural to make the following:

Definition E.6. A **close variant institution** category has the form $\mathbf{Dgm}((F\downarrow/G\uparrow)^{op})$ where F, G have the same relation category as their target. Similarly, close variant institution categories with comorphisms have the form $\mathbf{coDgm}((F\downarrow/G\uparrow)^{op})$. Moreover, a **small close variant institution** category has the form $\mathbf{dgm}((F\downarrow/G\uparrow)^{op})$ where F, G have the same relation category as their target, while the small close variant institutions with comorphisms constitute a category having the form $\mathbf{codgm}((F\downarrow/G\uparrow)^{op})$.

Close variant institutions can differ significantly from the original variants, for example, if **C** is the category of groups, or of topological spaces. We also recall the arguments at the end of Section 4.3, that the small variants are much more reasonable than their large cousins. It is clear that many results about close variant institutions follow easily from general results about diagram, relation and comma categories. For example, both Theorem 4.9 and Corollary 4.7 follow from Propositions 2.10 and E.4 under Assumption [A], and Proposition 2.10 also has the following consequences:

Proposition E.7. The functor that extracts signature categories from close variant institutions, with either morphisms or comorphisms, preserves all limits and colimits.

Proposition E.8. First notice that the first component projection functor ρ_1 is contravariant on $(F\downarrow/G\uparrow)^{op}$, i.e., it is a functor $((F\downarrow/G\uparrow)^{op})^{op} \to \mathbf{A}$. Then under Assumption [A], the functor $\mathbf{Dgm}(\rho_1)$, which goes from $\mathbf{Dgm}((F\downarrow/G\uparrow)^{op})$ to $\mathbf{Dgm}(\mathbf{A})$, is op-continuous and op-cocontinuous (i.e., it maps limits to colimits and vice versa). The same holds for $\mathbf{coDgm}(\rho_1)$, as well as for $\mathbf{dgm}(\rho_1)$ and $\mathbf{codgm}(\rho_1)$.

Proof. Proposition 2.10 gives the desired result, because ρ_1 is continuous and cocontinuous, because ρ preserves the limits and colimits that it creates via Assumption [A], π_1 preserves limits and colimits, and $\rho_1 = \rho;\pi_1$. □

These two results are used to justify assertions in Example 4.10. Note that the anomalous situation of that example cannot occur for institutions that are consistent with Definition E.6, because this formulation will not allow sets for sentences and classes for models; one can choose **C** to be either **Set** or else **SET**, but there is no way to get part of each. We consider this a good argument for the style of formulating institutions developed in this appendix, as opposed to the more concrete style of Definition 3.1, which nevertheless can be very helpful in understanding the concepts involved. Example 4.10 also highlights the desirability of restricting to institutions that have small signature categories in some cases.

It seems clear that many other institutional results can be proved in a similar way for all close variants, and probably they can all be proved for all the small close variants. But what this appendix provides seems enough to illustrate the principles involved. Note that an even more abstract formulation of institutions based on 'wedges' was discussed in [GoB86], and this might have some further advantages.

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