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Birkhoff style calculi for hybrid logics

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Abstract. We develop an abstract proof calculus for hybrid logics whose sentences are (*hybrid*) *Horn clauses*, and we prove a *Birkhoff completeness* theorem for hybrid logics in the general setting provided by the *institution theory*. This result is then applied to particular cases of hybrid logics with user-defined sharing, where the first-order variables in quantified sentences are interpreted uniformly across worlds.

Keywords: Institution, Horn clause, Birkhoff calculus, Hybrid logic, Reconfigurable system

1. Introduction

In 1935, Birkhoff [Bir35] first proved a completeness theorem for equational logic, in the unsorted case. Goguen and Meseguer [GM85], giving a sound and complete system of proof rules for many-sorted equational deduction, generalised the completeness theorem of Birkhoff to the completeness of many-sorted equational logic and provided simultaneously a full algebraisation of many-sorted equational deduction. Codescu and Găină [CG08] cast the result in the category-based setting of the institution theory [GB92], separating clearly the details of concrete logics from the logical-independent aspects of the completeness property. Institution theory is a category-based model theory that arose about three decades ago within formal methods as a response to the explosion in the population of logics in use there; its original aim is to develop as much computing science as possible in a general uniform way independently of particular logical systems. In this paper, we define an abstract notion of Horn clause and we prove a Birkhoff completeness result for hybrid logics in the general setting provided by the institution theory.

Hybrid logics [Bla00] are extensions of standard modal logics, involving symbols that name individual states in models. Their history can be traced back to work of Arthur Prior in the fifties [Pri67]. The subject was further developed in contributions such as [PT91, AB01a, Bra11]. Recently, hybrid logics were developed at an abstract institution theoretic level in works such as [MMDB11, Dia16b, DM16, GĬ5c, GĬ5b]. The ability to refer to specific states has several advantages from the point of views of logic and formal specification. For example, it has been argued [Bra11] that hybrid logics allow a more uniform proof theory than non-hybrid modal logics. From a software engineering perspective, hybrid logics offer a generic framework to approach the specification of reconfigurable systems, i.e. systems with reconfigurable features managing the dynamic evolution of their configurations in response to external stimuli or internal performance indicators. See [SC11] for an overview of the software reconfiguration paradigm.

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The hybridization development presented in [MMDB11, Dia16b, DM16] defines a hybrid logical system over an arbitrary institution. The parameters of this construction method are very general yielding to an abstract framework that can be instantiated to many concrete hybrid logics. However, the definition of hybrid institution given in [GĬ5b] provides a more general 'top-down' approach to the development of logical and computing science results in the spirit of 'universal logic' trend (in the sense envisaged by Béziau [Béz06, Béz12]), and it captures examples of hybrid logics that are not instances of the hybridisation process. Hybrid institutions are *stratified institutions* [AD07, Dia16a] with nominal and frame extraction. While the notion of stratified institution describes modal logics, the definition of hybrid institution allows reference to the states of the models and formalises the idea of hybrid logic.

As in [GĬ5b, GĬ5c], we are interested in extracting a significant fragment from a given hybrid logic with good computational properties, in the sense that it can be used not only as a declarative language but also as a executable/programming language. We define a notion of Horn clause for hybrid logics by noting that classically, Horn clauses are constructed from atomic formulae by applying certain sentence building operators in a specific order. The abstract Birkhoff entailment system is developed on top of an entailment system for atomic sentences by adding proof rules for each sentence operator that occurs in Horn clauses. The calculus proposed here will give a natural operational interpretation of the specifications written with (hybrid) Horn clauses. It is worth mentioning that this contribution targets the standard rigid quantification (e.g. [Bra11]) where the possible worlds share a common domain and the variables are interpreted identically across the worlds. This approach is in contrast with the world-line semantics of [Sch11], where the quantified variables may be interpreted differently across distinct worlds.

Concerning practical applications of this work, the general Birkhoff calculus developed in this paper provides the foundations of a methodology for the formal specification and verification of reconfigurable systems. Software systems with reconfigurable capabilities can be seen as transition structures, each node corresponding to a configuration. One may think as such nodes as local specifications of system configurations while the global transition structure describes how the software evolves from one configuration to another. A typical example of reconfigurable system is given by the cloud-based applications that flexibly react to client demands by allocating, for example, new server units to meet higher rates of service requests. The model implemented over the cloud is pay-per-usage, which means that the users will pay only for using the services. Therefore, the cloud service providers have to maintain a certain level of quality of service to keep up the reputation. The operating systems of modern cars offer a second example: in each vehicle dozens of electronic control units are connected together by a network and must operate in different modes, depending on the current situation - such as driving on a highway or in town where different speed regulations are applied. Switching between these modes is an example of dynamic reconfiguration. An error in the operating system of a car could cause loss of human life. Reconfigurable systems are safety and security-critical systems with strong qualitative requirements, and consequently, formal verification is needed. The cloud-based applications and the operating systems of modern cars are mentioned as examples of reconfigurable systems in other works such as [Mad13, MMBH15, NMMB16].

The paper is organised as follows. In Section 2 we recall the definition of institution that formalises the intuitive idea of logical system. In Section 3 we recall the definition of hybrid institution that constitutes the framework of the present work. Section 4 introduces the fundamental concepts that are necessary for our general results. Section 5 is dedicated to the development of the abstract Birkhoff completeness. Section 6 presents applications of the general theorems to concrete hybrid logics. Section 7 concludes the paper and discusses future work.

2. Institutions

The notion of logical system is formalised here as an institution.

2.1. Definition and examples

The concept of institution formalises the intuitive notion of logical system, and has been defined by Goguen and Burstall in the seminal paper [GB92].

Definition 2.1 (*Institution*) An *institution* $\mathtt{I} = (\mathtt{Sig}^{\mathtt{I}}, \mathtt{Sen}^{\mathtt{I}}, \mathtt{Mod}^{\mathtt{I}}, \models^{\mathtt{I}})$ consists of

(1) a category $\mathtt{Sig}^{\mathtt{I}}$, whose objects are called *signatures*,
(2) a functor $\mathtt{Sen}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathbb{S}et$, providing for each signature Σ a set whose elements are called (Σ-)*sentences*,
(3) a functor $\mathtt{Mod}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$, providing for each signature Σ a category whose objects are called (Σ-) *models* and whose arrows are called (Σ-)*homomorphisms*,
(4) a family of relations $\models^{\mathtt{I}} = \{\models^{\mathtt{I}}_{\Sigma}\}_{\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|}$, where $\models^{\mathtt{I}}_{\Sigma} \subseteq |\mathtt{Mod}^{\mathtt{I}}(\Sigma)| \times \mathtt{Sen}^{\mathtt{I}}(\Sigma)$ is called (Σ-)*satisfaction*, such that the following *satisfaction condition* holds:

$$M' \models^{\mathtt{I}}_{\Sigma'} \mathtt{Sen}^{\mathtt{I}}(\varphi)(e) \text{ iff } \mathtt{Mod}^{\mathtt{I}}(\varphi)(M') \models^{\mathtt{I}}_{\Sigma} e$$

for all $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathtt{Sig}^{\mathtt{I}}$, $M' \in |\mathtt{Mod}^{\mathtt{I}}(\Sigma')|$ and $e \in \mathtt{Sen}^{\mathtt{I}}(\Sigma)$.

When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\mathtt{Sig}^{\mathtt{I}}$ may be simply denoted by $\mathtt{Sig}$. We denote the *reduct* functor $\mathtt{Mod}(\varphi)$ by $\_\upharpoonright_{\varphi}$ and the sentence translation $\mathtt{Sen}(\varphi)$ by $\varphi(\_)$. When $M = M'\upharpoonright_{\varphi}$ we say that M is the φ-*reduct* of M' and M' is a φ-*expansion* of M.

Example 2.1 (First-Order Logic (**FOL**) [GB92]) The signatures are triplets of the form (S, F, P), where S is the set of sorts, $F = \{F_{\mathtt{ar}\to s}\}_{(\mathtt{ar},s)\in S^*\times S}$ is the ($S^* \times S$-indexed) set of operation symbols, and $P = \{P_{\mathtt{ar}}\}_{\mathtt{ar}\in S^*}$ is the (S^*-indexed) set of relation symbols.[1] If $\mathtt{ar} = \varepsilon$ then an element of $F_{\mathtt{ar}\to s}$ is called a *constant symbol*, or a *constant*. By a slight notational abuse, we let F and P also denote $\bigcup_{(\mathtt{ar},s)\in S^*\times S} F_{\mathtt{ar}\to s}$ and $\bigcup_{\mathtt{ar}\in S^*} P_{\mathtt{ar}}$, respectively. A signature morphism between (S, F, P) and (S', F', P') is a triplet $\varphi = (\varphi^{st}, \varphi^{op}, \varphi^{rl})$, where $\varphi^{st} : S \to S'$, $\varphi^{op} = \{\varphi^{op}_{\mathtt{ar}\to s} : F_{\mathtt{ar}\to s} \to F'_{\varphi^{st}(\mathtt{ar})\to\varphi^{st}(s)} \mid \mathtt{ar} \in S^*, s \in S\}$, $\varphi^{rl} = \{\varphi^{rl}_{\mathtt{ar}} : P_{\mathtt{ar}} \to P'_{\varphi^{st}(\mathtt{ar})} \mid \mathtt{ar} \in S^*\}$. When there is no danger of confusion, we may let φ denote each of φ^{st}, $\varphi^{op}_{\mathtt{ar}\to s}$, $\varphi^{rl}_{\mathtt{ar}}$. Given a signature $\Sigma = (S, F, P)$, a Σ-model is a triplet

$$M = \left(\{M_s\}_{s\in S}, \{M^{\mathtt{ar},s}_{\sigma}\}_{(\mathtt{ar},s)\in S^*\times S, \sigma\in F_{\mathtt{ar}\to s}}, \{M^{\mathtt{ar}}_{\pi}\}_{\mathtt{ar}\in S^*, \pi\in P_{\mathtt{ar}}}\right)$$

interpreting each sort s as a set M_s, each operation symbol $\sigma \in F_{\mathtt{ar}\to s}$ as a function $M^{\mathtt{ar},s}_{\sigma} : M^{\mathtt{ar}} \to M_s$ (where $M^{\mathtt{ar}}$ stands for $M_{s_1} \times \ldots \times M_{s_n}$ if $\mathtt{ar} = s_1 \cdots s_n$), and each relation symbol $\pi \in P_{\mathtt{ar}}$ as a relation $M^{\mathtt{ar}}_{\pi} \subseteq M^{\mathtt{ar}}$. When there is no danger of confusion we may let M_{σ} and M_{π} denote $M^{\mathtt{ar},s}_{\sigma}$ and $M^{\mathtt{ar}}_{\pi}$, respectively. Morphisms between models are the usual Σ-morphisms, i.e., S-sorted functions that preserve the structure. The Σ-algebra of terms is denoted by T_{Σ}. The Σ-sentences are obtained from (a) equations $t =_s t'$, where $t \in (T_{\Sigma})_s$, $t' \in (T_{\Sigma})_s$, $s \in S$, and (b) relations $\pi(t_1, \ldots, t_n)$, where $\pi \in P_{s_1\ldots s_n}$, $t_i \in (T_{\Sigma})_{s_i}$ and $s_i \in S$, by applying for a finite number of times Boolean operators and quantification over finite sets of variables. When there is no danger of confusion we may omit the subscript s from $t =_s t'$. Satisfaction is the usual first-order satisfaction and it is defined using the natural interpretations of ground terms t as elements M_t in models M. The definitions of functors $\mathtt{Sen}$ and $\mathtt{Mod}$ on morphisms are the natural ones: for any signature morphism $\varphi : \Sigma \to \Sigma'$, $\mathtt{Sen}(\varphi) : \mathtt{Sen}(\Sigma) \to \mathtt{Sen}(\Sigma')$ translates sentences symbol-wise, and $\mathtt{Mod}(\varphi) : \mathtt{Mod}(\Sigma') \to \mathtt{Mod}(\Sigma)$ is the forgetful functor.

Example 2.2 (REL). The institution REL is the sub-institution of single-sorted first-order logic with signatures having only constants and relational symbols.

Example 2.3 (Propositional Logic (**PL**)) The institution **PL** is the fragment of **FOL** determined by the signatures with empty sets of sort symbols.

Example 2.4 (First-Order Logic with user-defined Sharing (**FOLS**)) This institution is used in Example 3.3 to define a hybrid logic and it is not intended for any other application. The signatures $(S^r, F^r, P^r) \subseteq (S, F, P)$ consist of **FOL** signatures (S, F, P) enhanced with a sub-signature (S^r, F^r, P^r) of 'rigid' symbols. Signature morphisms $\varphi : (S^r, F^r, P^r) \subseteq (S, F, P) \to (S'^r, F'^r, P'^r) \subseteq (S', F', P')$ are **FOL** signature morphisms $(S, F, P) \to (S', F', P')$ that map rigid symbols to rigid symbols. The set $\mathtt{Sen}^{\mathbf{FOLS}}((S^r, F^r, P^r) \subseteq (S, F, P))$ consists of those sentences in $\mathtt{Sen}^{\mathbf{FOL}}(S, F, P)$ that contain only quantifiers over rigid variables. The category of models $\mathtt{Mod}^{\mathbf{FOLS}}((S^r, F^r, P^r) \subseteq (S, F, P))$ is $\mathtt{Mod}^{\mathbf{FOL}}(S, F, P)$. The satisfaction relation in **FOLS** is induced from the satisfaction relation in **FOL**, i.e. $\models^{\mathbf{FOLS}}_{(S^r,F^r,P^r)\subseteq(S,F,P)} = \models^{\mathbf{FOL}}_{(S,F,P)}$.

[1] If S is a set then S^* is the set of strings over symbols in S, including the empty string ε.

Example 2.5 (Preorder Algebra (**POA**) [DF02]) The **POA** signatures are just ordinary algebraic signatures, i.e. **FOL** signatures without relation symbols. The **POA** models are *preordered algebras* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. A *preordered algebra morphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra morphism. The sentences have two kinds of atoms: equations and *transitions*. A transition $t \Rightarrow t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leq M_{t'}$. Full sentences are constructed from equations and transitions by applying Boolean operators and first-order quantification.

2.2. Quantification category

Quantification comes with some subtle issues related to the translation of quantified sentences along signature morphisms that will be discussed in this subsection.

Definition 2.2 (Quantification category [Dia16b]) Given an institution $\mathtt{I}$, a *broad subcategory*[2] $\mathtt{Q} \subseteq \mathtt{Sig}$ is a *quantification category* for $\mathtt{I}$ when for each $\Sigma \xrightarrow{\chi} \Sigma' \in \mathtt{Q}$ and $\Sigma \xrightarrow{\varphi} \Sigma_1 \in \mathtt{Sig}$ there is a designated pushout

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\varphi[\chi]} & \Sigma'_1 \\ {\scriptstyle \chi}\uparrow & & \uparrow{\scriptstyle \chi(\varphi)} \\ \Sigma & \xrightarrow[\varphi]{} & \Sigma_1 \end{array}$$

with $\chi(\varphi) \in \mathtt{Q}$ and such that the horizontal composition of such designated pushouts is again a designated pushout: (a) $\chi(1_\Sigma) = \chi$, $1_\Sigma[\chi] = 1_{\Sigma'}$, and (b) for the following pushouts

$$\begin{array}{ccccc} \Sigma' & \xrightarrow{\varphi[\chi]} & \Sigma'_1 & \xrightarrow{\theta[\chi(\varphi)]} & \Sigma'_2 \\ {\scriptstyle \chi}\uparrow & & \uparrow{\scriptstyle \chi(\varphi)} & & \uparrow{\scriptstyle \chi(\varphi)(\theta)} \\ \Sigma & \xrightarrow[\varphi]{} & \Sigma_1 & \xrightarrow[\theta]{} & \Sigma_2 \end{array}$$

we have $\varphi[\chi];\ \theta[\chi(\varphi)] = (\varphi;\ \theta)[\chi]$ and $\chi(\varphi)(\theta) = \chi(\varphi;\ \theta)$.

In concrete examples of institutions the quantification category is fixed.
A first-order variable for a signature $\Sigma = (S, F, P)$ is a triple (x, s, Σ), where x is the name of the variable and $s \in S$ is the sort of the variable. Let $\chi : \Sigma \hookrightarrow \Sigma[X]$ be a signature extension with variables from X, where $X = \{X_s\}_{s \in S}$ is a S-sorted set of variables, $\Sigma[X] = (S, F \cup X, P)$ and $(F \cup X)_{\mathtt{ar} \to s} = \begin{cases} F_{\mathtt{ar} \to s} & \text{if } \mathtt{ar} \in (S^* - \{\varepsilon\}) \\ F_{\mathtt{ar} \to s} \cup X_s & \text{if } \mathtt{ar} = \varepsilon \end{cases}$ for all $(\mathtt{ar}, s) \in S^* \times S$. The quantification category $\mathtt{Q}^{\mathbf{FOL}}$ for **FOL** consists of signature extensions with a finite set of variables. Given a signature morphism $\Sigma \xrightarrow{\varphi} \Sigma_1$, where $\Sigma_1 = (S_1, F_1, P_1)$, then

(a) $\chi(\varphi) : \Sigma_1 \hookrightarrow \Sigma_1[X^\varphi]$, where $X^\varphi = \{(x, \varphi(s), \Sigma_1) \mid (x, s, \Sigma) \in X\}$, and
(b) $\varphi[\chi]$ is the extension of φ that maps each (x, s, Σ) to $(x, \varphi(s), \Sigma_1)$.

Remark 2.1 Let Σ be a **FOL** signature, $(\forall X)\gamma$ a Σ-sentence, and M a Σ-model.

(1) From the institution theory perspective, $(\forall X)\gamma$ is just an abbreviation for $(\forall \chi)\gamma$, where $\chi : \Sigma \hookrightarrow \Sigma[X]$ is the inclusion, and
(2) $M \models_\Sigma (\forall \chi)\gamma$ iff for all χ-expansions M' we have $M' \models_{\Sigma[X]} \gamma$.

When quantified sentences get translated along signature morphisms using this approach, one avoids clashing of variables with the constant symbols from the target signature.

2.3. Substitutions

We recall the notion of substitution in institutions.

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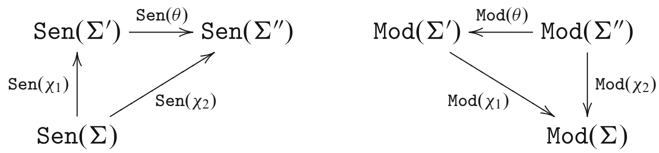
[2] A category $\mathcal{C}$ is a broad subcategory of $\mathcal{C}'$ if $\mathcal{C}$ is a subcategory of $\mathcal{C}'$ and $\mathcal{C}$ contains all objects of $\mathcal{C}'$, i.e. $\mid \mathcal{C} \mid = \mid \mathcal{C}' \mid$.

Definition 2.3 (Substitution [Dia04]) Let $\mathtt{I}$ be an institution. For any signature morphisms $\Sigma \overset{\chi_1}{\to} \Sigma' \in \mathtt{Sig}^{\mathtt{I}}$ and $\Sigma \overset{\chi_2}{\to} \Sigma'' \in \mathtt{Sig}^{\mathtt{I}}$, a Σ-*substitution* $\theta : \chi_1 \to \chi_2$ consists of a pair $(\mathtt{Sen}(\theta), \mathtt{Mod}(\theta))$, where $\mathtt{Sen}(\theta) : \mathtt{Sen}(\Sigma') \to \mathtt{Sen}(\Sigma'')$ is a function and $\mathtt{Mod}(\theta) : \mathtt{Mod}(\Sigma'') \to \mathtt{Mod}(\Sigma')$ is a functor, such that both of them preserve Σ, i.e. the following diagrams commute:



and such that the following *satisfaction condition* holds:

$\mathtt{Mod}(\theta)(M_2) \models \gamma_1$ iff $M_2 \models \mathtt{Sen}(\gamma_1)$

for each Σ''-model M_2 and each Σ'-sentence γ_1.

Note that a substitution $\theta : \chi_1 \to \chi_2$ is uniquely identified by its domain χ_1, codomain χ_2 and the pair $(\mathtt{Sen}(\theta), \mathtt{Mod}(\theta))$. When there is no danger of confusion, we let $\_ \upharpoonright_\theta$ denote the functor $\mathtt{Mod}(\theta)$, and let θ denote the sentence translation $\mathtt{Sen}(\theta)$.

Example 2.6 (**FOL** substitutions [Dia04]) Consider two signature extensions with constants $\chi_1 : \Sigma \hookrightarrow \Sigma[C_1]$ and $\chi_2 : \Sigma \hookrightarrow \Sigma[C_2]$, where $\Sigma = (S, F, P) \in| \mathtt{Sig}^{\mathbf{FOL}} |$, C_i is a set of constant symbols different from the constants in F. A function $\theta : C_1 \to T_\Sigma(C_2)$ represents a substitution between χ_1 and χ_2:

(1) On the syntactic side, $\theta : C_1 \to T_\Sigma(C_2)$ can be canonically extended to a function $\mathtt{Sen}(\theta) : \mathtt{Sen}(\Sigma[C_1]) \to \mathtt{Sen}(\Sigma[C_2])$ which preserves Σ and substitutes $\Sigma(C_2)$-terms for constants in C_1.
(2) On the semantics side, θ determines a model functor $\mathtt{Mod}(\theta) : \mathtt{Mod}(\Sigma[C_2]) \to \mathtt{Mod}(\Sigma[C_1])$ such that for all $\Sigma[C_2]$-models M we have (a) $\mathtt{Mod}(\theta)(M)_x = M_x$, for each sort $x \in S$, or operation symbol $x \in F$, or relation symbol $x \in P$, and (b) $\mathtt{Mod}(\theta)(M)_{c_1} = M_{\theta(c_1)}$ for each $c_1 \in C_1$.

Substitution functors. Let $\mathtt{I}$ be an institution. For any signature $\Sigma \in| \mathtt{Sig}^{\mathtt{I}} |$, Σ-substitutions form a category $\mathtt{Sb}^{\mathtt{I}}(\Sigma)$, where *the objects* are signature morphisms $\Sigma \overset{\chi}{\to} \Sigma' \in \mathtt{Sig}$, and *the arrows* are substitutions $\theta : \chi_1 \to \chi_2$ described in Definition 2.3. Given $\Sigma_0 \overset{\varphi}{\to} \Sigma \in \mathtt{Sig}^{\mathtt{I}}$ there exists a reduct functor $\mathtt{Sb}^{\mathtt{I}}(\varphi) : \mathtt{Sb}^{\mathtt{I}}(\Sigma) \to \mathtt{Sb}^{\mathtt{I}}(\Sigma_0)$ that maps each Σ-substitution $\theta : \chi_1 \to \chi_2$ to the Σ_0-substitution $\mathtt{Sb}(\varphi)(\theta) : \varphi; \chi_1 \to \varphi; \chi_2$ such that $\mathtt{Sen}(\mathtt{Sb}^{\mathtt{I}}(\varphi)(\theta)) = \mathtt{Sen}(\theta)$ and $\mathtt{Mod}(\mathtt{Sb}^{\mathtt{I}}(\varphi)(\theta)) = \mathtt{Mod}(\theta)$.

Fact 2.2 $\mathtt{Sb}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ is a functor.

In applications not all substitutions are of interest, and it is often assumed a substitution sub-functor $\mathtt{St}^{\mathtt{I}} : \mathtt{D}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ of $\mathtt{Sb}^{\mathtt{I}}$ to work with, where $\mathtt{D}^{\mathtt{I}} \subseteq \mathtt{Sig}^{\mathtt{I}}$ is a broad subcategory of signature morphisms. When there is no danger of confusion we may drop the superscript $\mathtt{I}$ from the notations.

Example 2.7 (FOL substitution functor **[GĬ5a])** Let $\mathtt{D}^{\mathbf{FOL}} \subseteq \mathtt{Sig}^{\mathbf{FOL}}$ be the broad subcategory of signature extensions with constants. The first-order substitutions are represented by functions $\theta : C_1 \to T_\Sigma(C_2)$, where $\Sigma \in| \mathtt{Sig}^{\mathbf{FOL}} |$ and C_i are finite sets of new constants for Σ. Let $\mathtt{St}^{\mathbf{FOL}} : \mathtt{D}^{\mathbf{FOL}} \to \mathbb{CAT}^{op}$ denote the substitution functor which maps each signature Σ to the subcategory of Σ-substitutions represented by functions $\theta : C_1 \to T_\Sigma(C_2)$ as in Example 2.6.

Substitution systems. In theorem proving it is often useful to convert sentences into so-called normal forms. An example of normal form in propositional logic is the conjunctive normal form. In order to reason about such conversions, we formalise them as substitutions.

Definition 2.4 A *substitution system* $\Theta^{\mathtt{I}} = \{\Theta^{\mathtt{I}}_\Sigma\}_{\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|}$ for an institution $\mathtt{I}$ is a family of substitutions $\Theta^{\mathtt{I}}_\Sigma : 1_\Sigma \to 1_\Sigma$ indexed by the signatures $\Sigma \in| \mathtt{Sig}^{\mathtt{I}} |$ with the domain and codomain consisting of identity signature morphisms 1_Σ.

When there is no danger of confusion we may drop the superscript $\mathtt{I}$ and/or the subscript Σ from the notation $\Theta^{\mathtt{I}}_\Sigma$. Examples of substitution systems may be found in Section 5.

3. Hybrid institutions

Given a base logic one can construct freely its hybridized version by applying the hybridisation process defined in [MMDB11, DM16]. In this paper, we use the definition of hybrid logic given in [GĬ5b], which provides a higher level of generality and a top-down approach to *Kripke semantics* in the spirit of universal logic.

3.1. Definition and examples

Informally, hybrid institutions are institutions whose signatures are equipped with nominals and modalities and whose models are Kripke structures. Hybrid institutions are refinements of stratified institutions that were introduced in [AD07] to enhance the concept of institution with 'states' for the models.

Definition 3.1 (Hybrid institution [GĬ5b]) A *hybrid institution* $\mathtt{HI} = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ consists of

(1) a category $\mathtt{Sig}^{\mathtt{HI}}$, whose objects are called *signatures*,

(2) a functor $\mathtt{F} : \mathtt{Sig}^{\mathtt{HI}} \to \mathtt{Sig}^{\mathbf{REL}}$, which extracts from each signature Δ its relational part $\mathtt{F}(\Delta) = (\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$, where $\mathtt{Nom}^{\Delta}$ is a set of *nominals* and $\Lambda^{\Delta} = \{\Lambda^{\Delta}_n\}_{n\in\mathbb{N}}$ is a family of sets of *modalities*,

(3) a sentence functor $\mathtt{Sen}^{\mathtt{HI}} : \mathtt{Sig}^{\mathtt{HI}} \to \mathbb{S}et$, providing for each signature Δ a set whose elements are called (Δ-)*sentences*,

(4) a model functor $\mathtt{Mod}^{\mathtt{HI}} : \mathtt{Sig}^{\mathtt{HI}} \to \mathbb{CAT}^{op}$, providing for each signature Δ a category whose objects are called (Δ-)*models* and whose arrows are called (Δ-)*homomorphisms*,

(5) a natural transformation $\mathtt{K} : \mathtt{Mod}^{\mathtt{HI}} \Rightarrow (\mathtt{F};\ \mathtt{Mod}^{\mathbf{REL}})$, providing for each signature Δ a frame functor $\mathtt{K}_{\Delta} : \mathtt{Mod}^{\mathtt{HI}}(\Delta) \to \mathtt{Mod}^{\mathbf{REL}}(\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$, which extracts from each Δ-model M its *frame* $\mathtt{K}_{\Delta}(M)$ consisting of a set of *states/worlds* $|\mathtt{K}_{\Delta}(M)|$ together with their *accessibility relations* $\mathtt{K}_{\Delta}(M)_{\lambda}$, where $\lambda \in \Lambda^{\Delta}_n$ and $n \in \mathbb{N}$,

(6) a satisfaction relation $\models^{\mathtt{HI}} = \{M \models^{-}_{\Delta} -\}_{\Delta\in|\mathtt{Sig}^{\mathtt{HI}}|, M\in|\mathtt{Mod}^{\mathtt{HI}}(\Delta)|}$, where $M \models^{-}_{\Delta} - \subseteq |\mathtt{K}_{\Delta}(M)| \times \mathtt{Sen}^{\mathtt{HI}}(\Delta)$ such that the following *local satisfaction condition* holds:

$$M' \models^{w'}_{\Delta'} \mathtt{Sen}^{\mathtt{HI}}(\varphi)(e) \text{ iff } \mathtt{Mod}^{\mathtt{HI}}(\varphi)(M') \models^{w'}_{\Delta} e$$

for all models $M' \in |\mathtt{Mod}^{\mathtt{HI}}(\Delta')|$, states $w' \in \mathtt{K}_{\Delta'}(M')$ and sentences $e \in \mathtt{Sen}^{\mathtt{HI}}(\Delta)$.

Like for ordinary institutions, when appropriate we shall also use simplified notations without superscripts or subscripts that are clear from the context. The consistency of the local satisfaction condition from Definition 3.1 (6) is given by the following lemma.

Lemma 3.1 For all signature morphisms $\Delta \xrightarrow{\varphi} \Delta'$, every Δ'-model M' has exactly the same set of states as its reduct $M\upharpoonright_{\varphi}$.

Proof Note that for all signature morphisms $\Delta \xrightarrow{\varphi} \Delta'$ and Δ'-models M', by the definition of $\mathtt{Mod}^{\mathbf{REL}}$, we have $|\mathtt{K}_{\Delta'}(M')| = |\mathtt{K}_{\Delta'}(M')\upharpoonright_{F(\varphi)}|$, and since $\mathtt{K}$ is a natural transformation, we obtain $|\mathtt{K}_{\Delta'}(M')\upharpoonright_{F(\varphi)}| = |\mathtt{K}_{\Delta}(M'\upharpoonright_{\varphi})|$. It follows that $|\mathtt{K}_{\Delta'}(M')| = |\mathtt{K}_{\Delta}(M'\upharpoonright_{\varphi})|$. □

Given a hybrid institution $\mathtt{HI}$, we define the following *global satisfaction relation*

$$M \models^{\mathtt{HI}}_{\Delta} e \text{ iff } M \models^{w}_{\Delta} e \text{ for all states } w \in |\mathtt{K}_{\Delta}(M)|$$

where $\Delta \in |\mathtt{Sig}^{\mathtt{HI}}|$, $M \in |\mathtt{Mod}^{\mathtt{HI}}(\Delta)|$, and $e \in \mathtt{Sen}^{\mathtt{HI}}(\Delta)$. We overload the notation and let $\models^{\mathtt{HI}}$ to denote both families of relations $\{M \models^{-}_{\Delta} -\}_{\Delta\in|\mathtt{Sig}^{\mathtt{HI}}|, M\in|\mathtt{Mod}^{\mathtt{HI}}|}$ and $\{- \models^{\mathtt{HI}}_{\Delta} -\}_{\Delta\in|\mathtt{Sig}^{\mathtt{HI}}|}$. Hybrid institutions determine canonically institutions.

Fact 3.2 If $\mathtt{HI}$ is a hybrid institution then $(\mathtt{Sig}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}}, \mathtt{Mod}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ is an institution.

The following definition provides a pattern for describing the semantics of hybrid logics. More concretely, it constructs a *Kripke* model functor from a *base* model functor such that the Kripke structures are pairs consisting of a set of states and a mapping that associates to each state a *base* model. However, our approach is top-down meaning that the results will be developed at a more abstract level provided by Definition 3.1, where the Kripke structures are implicitly assumed (not constructed).

Definition 3.2 (Kripke structures) Let $\mathtt{Mod}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ be a *base* model functor. The *Kripke* model functor $\mathtt{Mod}^{\mathtt{I}}_{\kappa} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ over $\mathtt{Mod}^{\mathtt{I}}$ is defined as follows:

(1) for each signature $(\mathtt{Nom}, \Lambda, \Sigma) \in\mid \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}} \mid$, where $(\mathtt{Nom}, \Lambda) \in\mid \mathtt{Sig}^{\mathbf{REL}} \mid$ and $\Sigma \in\mid \mathtt{Sig}^{\mathtt{I}} \mid$, $\mathtt{Mod}^{\mathtt{I}}_{\kappa}(\mathtt{Nom}, \Lambda, \Sigma)$ is the category that consists of

   (a) Kripke models of the form (W, M), where $W \in\mid \mathtt{Mod}^{\mathbf{REL}}(\mathtt{Nom}, \Lambda) \mid$ and $M :\mid W \mid\to\mid \mathtt{Mod}^{\mathtt{I}}(\Sigma) \mid$ is a mapping from the set of *states/worlds* $\mid W \mid$ to the class of models $\mid \mathtt{Mod}(\Sigma) \mid$, and
   (b) homomorphisms $h : (W, M) \to (W', M')$ of the form $(h^{\mathbf{REL}}, h^{mod})$, where $h^{\mathbf{REL}} : W \to W'$ is a homomorphism in $\mathbf{REL}$, and $h^{mod} : M \Rightarrow M' \circ h^{\mathbf{REL}}$ is a natural transformation.

(2) for each signature morphism $(\mathtt{Nom}, \Lambda, \Sigma) \xrightarrow{\varphi} (\mathtt{Nom}', \Lambda', \Sigma') \in \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}}$, where $\varphi^{\mathbf{REL}} : (\mathtt{Nom}, \Lambda) \to (\mathtt{Nom}', \Lambda') \in \mathtt{Sig}^{\mathbf{REL}}$ and $\varphi^{\mathtt{I}} : \Sigma \to \Sigma' \in \mathtt{Sig}^{\mathtt{I}}$, the reduct functor $\mathtt{Mod}^{\mathtt{I}}_{\kappa}(\varphi) : \mathtt{Mod}^{\mathtt{I}}_{\kappa}(\mathtt{Nom}', \Lambda', \Sigma') \to \mathtt{Mod}^{\mathtt{I}}_{\kappa}(\mathtt{Nom}, \Lambda, \Sigma)$ is defined by

   (a) $\mathtt{Mod}^{\mathtt{I}}_{\kappa}(\varphi)(W', M') = (W, M)$ for all $(W', M') \in\mid \mathtt{Mod}^{\mathtt{I}}_{\kappa}(\mathtt{Nom}', \Lambda', \Sigma') \mid$, where the model $(W, M) \in\mid \mathtt{Mod}^{\mathtt{I}}_{\kappa}(\mathtt{Nom}, \Lambda, \Sigma) \mid$ is defined by
      (i) $W = W' \upharpoonright_{\varphi^{\mathbf{REL}}}$, i.e. $\mid W \mid = \mid W' \mid$ and $W_{\lambda} = W'_{\varphi^{\mathbf{REL}}(\lambda)}$ for all $\lambda \in \Lambda$, and
      (ii) $M_w = M'_w \upharpoonright_{\varphi^{\mathtt{I}}}$, for all states $w \in\mid W \mid$.
   (b) $\mathtt{Mod}^{\mathtt{I}}_{\kappa}(\varphi)(h') = h$ for all homorphisms $h' \in \mathtt{Mod}^{\mathtt{I}}_{\kappa}(\mathtt{Nom}', \Lambda', \Sigma')$, where $h^{\mathbf{REL}} = h'^{\mathbf{REL}} \upharpoonright_{\varphi^{\mathbf{REL}}}$ and $h^{mod} = \{h'_w \upharpoonright_{\varphi^{\mathtt{I}}}\}_{w \in \mid W \mid}$.

In our examples of hybrid institutions, the model functor is a sub-functor of some Kripke functor $\mathtt{Mod}^{\mathtt{I}}_{\kappa} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$, the functor $\mathtt{F}^{\mathtt{HI}} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}} \to \mathtt{Sig}^{\mathbf{REL}}$ is the first projection, and for all signatures Δ, the frame functor $\mathtt{K}_{\Delta}$ is the forgetful functor mapping each Kripke structure (W, M) to W.

Assumption 1 Throughout this paper we assume that $\mathtt{HI}$ range hybrid institutions with a quantification category $\mathtt{Q}^{\mathtt{HI}}$ that satisfy the following *exactness property*: for every signature $\Delta \in\mid \mathtt{Sig}^{\mathtt{HI}} \mid$ and each variable j for $(\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$ there exists a designated signature morphism $\chi[j] : \Delta \to \Delta[j] \in \mathtt{Q}^{\mathtt{HI}}$ such that (a) $\mathtt{F}(\chi[j]) = \chi[j]^{\mathbf{REL}}$, where $\chi[j]^{\mathbf{REL}} : (\mathtt{Nom}^{\Delta}, \Lambda^{\Delta}) \hookrightarrow (\mathtt{Nom}^{\Delta} \cup \{j\}, \Lambda^{\Delta})$, and (b) for any Δ-model M and any $\chi[j]^{\mathbf{REL}}$-expansion W' of $\mathtt{K}_{\Delta}(M)$ there exists a unique $\chi[j]$-expansion M' of M such that $\mathtt{K}_{\Delta[j]}(M') = W'$.

We use Definition 3.2 to justify Assumption 1. In applications, $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$, $\Delta[j] = (\mathtt{Nom} \cup \{j\}, \Lambda, \Sigma)$ and $\chi[j] : (\mathtt{Nom}, \Lambda, \Sigma) \hookrightarrow (\mathtt{Nom} \cup \{j\}, \Lambda, \Sigma)$ is the inclusion. For any Δ-model (W, M) and any $\chi[j]^{\mathbf{REL}}$-expansion W' of W, the model (W', M) is the unique $\chi[j]$-expansion of (W, M) such that $\mathtt{K}_{\Delta[j]}(W', M) = W'$.

Notation 3.3 For every Δ-model M, each state $w \in\mid \mathtt{K}_{\Delta}(M) \mid$ and any variable j for $(\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$, we denote (a) by $\mathtt{K}_{\Delta}(M)^{(j,w)}$ the unique $\chi[j]^{\mathbf{REL}}$-expansion of $\mathtt{K}_{\Delta}(M)$ such that $\mathtt{K}_{\Delta}(M)^{(j,w)}_j = w$, and (b) by $M^{(j,w)}$ the unique $\chi[j]$-expansion of M such that $\mathtt{K}_{\Delta[j]}(M^{(j,w)}) = \mathtt{K}_{\Delta}(M)^{(j,w)}$.

The semantics of each sentence operator is defined at the abstract level provided by Definition 3.1.

Definition 3.4 (Internal logic) Given a hybrid institution $\mathtt{HI}$ with a quantification category $\mathtt{Q}^{\mathtt{HI}}$ then

(1) $M \models^w k_1 = k_2$ iff $\mathtt{K}_{\Delta}(M)_{k_1} = \mathtt{K}_{\Delta}(M)_{k_2}$;
(2) $M \models^w \lambda(k_1, \ldots, k_n)$ iff $(\mathtt{K}_{\Delta}(M)_{k_1}, \ldots, \mathtt{K}_{\Delta}(M)_{k_n}) \in \mathtt{K}_{\Delta}(M)_{\lambda}$;
(3) $M \models^w @_k \rho$ iff $(W, M) \models^{\mathtt{K}_{\Delta}(M)_k} \rho$;
(4) $M \models^w \rho_1 \wedge \rho_2$ iff $M \models^w \rho_1$ and $M \models^w \rho_2$;
(5) $M \models^w \neg\rho$ iff $M \not\models^w \rho$;
(6) $M \models^w [\lambda](\rho_1, \ldots, \rho_n)$ iff for all $(w, w_1, \ldots, w_n) \in \mathtt{K}_{\Delta}(M)_{\lambda}$ we have $M \models^{w_i} \rho$ for some $i \in \{1, \ldots, n\}$;
(7) $M \models^w (\forall \chi)\gamma$ iff $M' \models^w \gamma$ for all χ-expansions M' of M;
(8) $M \models^w_{\Delta} (\downarrow j)\rho$ iff $M^{(j,w)} \models^w_{\Delta[j]} \rho$.

where $\Delta \xrightarrow{\chi} \Delta' \in \mathtt{Q}^{\mathtt{HI}}$, $M \in\mid \mathtt{Mod}^{\mathtt{HI}}(\Delta) \mid$, $w \in\mid \mathtt{K}_{\Delta}(M) \mid$, $k \in \mathtt{Nom}^{\Delta}$, $k_i \in \mathtt{Nom}^{\Delta}$, $n \in \mathbb{N}$, $\lambda \in \Lambda_{n+1}$, $\rho \in \mathtt{Sen}^{\mathtt{HI}}(\Delta)$, $\rho_i \in \mathtt{Sen}^{\mathtt{HI}}(\Delta)$, $\gamma \in \mathtt{Sen}^{\mathtt{HI}}(\Delta')$, and j is a variable for $(\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$.

We say that that $k_1 = k_2$ is a *nominal equation* and $\lambda(k_1, \ldots, k_n)$ is a *nominal relation*. The operator @ is called 'retrieve' because it changes the point of evaluation for a model. For any modality λ, the operator $[\lambda]$ is called traditionally 'necessity'. The operator $\downarrow$ is called 'store' because it allows us to give a name to the current state that can be referred in sentences. The concepts of Boolean operators and quantifications in ordinary institutions (e.g. from [Dia03, Dia08, Tar86a] etc.) arise as an instance of Definition 3.4 when the underlying set of states $|\,\mathtt{K}_\Delta(M)\,|$ of each model M is a singleton set.

Example 3.1 (Hybrid first-order logic (**HFOL**)) This hybrid institution is a variation of *first-order hybrid logic* of [BM02] where models may have different carrier sets across the states. The functor $\mathtt{Mod}^{\mathbf{HFOL}}$ is $\mathtt{Mod}_\kappa^{\mathbf{FOL}} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathbf{FOL}} \to \mathbb{CAT}^{op}$. Given a signature $\Delta = (\mathtt{Nom}, \Lambda, \Sigma) \in |\,\mathtt{Sig}^{\mathbf{HFOL}}\,|$, the atomic sentences in $\mathtt{Sen}^{\mathbf{HFOL}}(\Delta)$ are the atomic sentences in $\mathtt{Sen}^{\mathbf{FOL}}(\Sigma)$. The sentences are constructed from atomic sentences, nominal equations and nominal relations by applying Boolean operators, retrieve, store, necessity and quantification over nominal variables. The satisfaction of atomic sentences is defined by $(W, M) \models^w \rho$ iff $M_w \models^{\mathbf{FOL}} \rho$ for all signatures $\Delta \in |\,\mathtt{Sig}^{\mathbf{HFOL}}\,|$, atomic sentences $\rho \in \mathtt{Sen}^{\mathbf{HFOL}}(\Delta)$, models $(W, M) \in |\,\mathtt{Mod}^{\mathbf{HFOL}}(\Delta)\,|$ and states $w \in |\,W\,|$.

Example 3.2 (Hybrid Propositional Logic (**HPL**) [AB01a]) This institution is a particular case of **HFOL** with empty set of sorts. The signatures $(\mathtt{Nom}, \Lambda, \mathtt{Prop})$ consist of a set of nominals $\mathtt{Nom}$, a family of sets of modalities $\Lambda = \{\Lambda_n\}_{n\in\mathbb{N}}$, and a set of propositional symbols $\mathtt{Prop}$. Let $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop})$ be a **HPL** signature. The Δ-models are Kripke structures of the form (W, M), where W is a $(\mathtt{Nom}, \Lambda)$-model in **REL** and $M : |\,W\,| \to |\,\mathtt{Mod}^{\mathbf{PL}}(\mathtt{Prop})\,|$ is a mapping. The atomic Δ-sentences consist of propositional symbols $p \in \mathtt{Prop}$. The satisfaction relation for atomic sentences is defined by $(W, M) \models^w p$ iff $p \in M_w$ for all models $(W, M) \in |\,\mathtt{Mod}^{\mathbf{HPL}}(\Delta)\,|$, states $w \in |\,W\,|$ and propositional symbols $p \in \mathtt{Prop}$.

Example 3.3 (Hybrid First-Order Logic with user-defined Sharing (**HFOLS**) [MMDB11]) The model functor $\mathtt{Mod}^{\mathbf{HFOLS}} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathbf{FOLS}} \to \mathbb{CAT}^{op}$ is a sub-functor of $\mathtt{Mod}_\kappa^{\mathbf{FOLS}} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathbf{FOLS}} \to \mathbb{CAT}^{op}$ which restricts the models and the homomorphisms of $\mathtt{Mod}_\kappa^{\mathbf{FOLS}}$ such that the rigid symbols are interpreted uniformly across the states, i.e. for all $\Delta \in |\,\mathtt{Sig}^{\mathbf{HFOLS}}\,|$, we have (1) $(W, M) \in |\,\mathtt{Mod}^{\mathbf{HFOLS}}(\Delta)\,|$ iff for all states $w_1, w_2 \in |\,W\,|$ and rigid symbols x in Δ we have $(M_{w_1})_x = (M_{w_2})_x$, and (2) $h : (W, M) \to (W', M') \in \mathtt{Mod}^{\mathbf{HFOLS}}(\Delta)$ iff for all states $w_1, w_2 \in |\,W\,|$ and rigid sorts $\mathtt{sr}$ in Δ we have $(h_{w_1}^{mod})_{\mathtt{sr}} = (h_{w_2}^{mod})_{\mathtt{sr}}$.

Given a signature $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$, the atomic sentences in $\mathtt{Sen}^{\mathbf{HFOLS}}(\Delta)$ are the atomic sentences in $\mathtt{Sen}^{\mathbf{FOLS}}(\Sigma)$. The sentences are constructed from atomic sentences, nominal equations and nominal relations by applying Boolean operators, retrieve, store, necessity and quantification over nominal variables and rigid variables. The satisfaction of atomic sentences is defined as follows: $(W, M) \models^w \rho$ iff $M_w \models^{\mathbf{FOLS}} \rho$, for all signatures Δ, atomic sentences $\rho \in \mathtt{Sen}^{\mathbf{HFOLS}}(\Delta)$, models $(W, M) \in |\,\mathtt{Mod}^{\mathbf{HFOLS}}(\Delta)\,|$ and states $w \in |\,W\,|$.

Example 3.4 (Hybrid First-Order Logic with user-defined Sharing and Annotation (**HFOLSA**)) This institution was defined in [Gǐ5b] and it has the same model functor as **HFOLS**, i.e. $\mathtt{Mod}^{\mathbf{HFOLSA}} = \mathtt{Mod}^{\mathbf{HFOLS}}$. Let $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$ be a **HFOLSA** signature, where $\Sigma = (S^r, F^r, P^r) \subseteq (S, F, P)$ is a **FOLS** signature. For all nominals $k \in Nom$, we define the S-sorted sets T_k^Δ of hybrid terms:

(1) $\dfrac{\tau_1 \in (T_k^\Delta)_{\mathtt{sr}_1}, \ldots, \tau_n \in (T_k^\Delta)_{\mathtt{sr}_n}}{\varsigma(\tau_1, \ldots, \tau_n) \in (T_k^\Delta)_{\mathtt{sr}}}$ for all rigid symbols $\varsigma \in F^r_{\mathtt{sr}_1\ldots\mathtt{sr}_n\to\mathtt{sr}}$,

(2) $\dfrac{t_1 \in (T_k^\Delta)_{s_1}, \ldots, t_n \in (T_k^\Delta)_{s_n}}{\sigma_k(t_1, \ldots, t_n) \in (T_k^\Delta)_s}$ for all non-rigid symbols $\sigma \in (F_{s_1\ldots s_n\to s} - F^r_{s_1\ldots s_n\to s})$,

(3) $\dfrac{\tau \in (T_{k_1}^\Delta)_{\mathtt{sr}}}{\tau \in (T_k^\Delta)_{\mathtt{sr}}}$, where $\mathtt{sr} \in S^r$.

The set of atomic Δ-sentences consist of

(a) hybrid equations $t =_s^k t'$, where $t \in (T_k^\Delta)_s$, $t' \in (T_k^\Delta)_s$, $k \in \mathtt{Nom}$ and $s \in S$,
(b) rigid relations $\varpi(\tau_1, \ldots, \tau_n)$, where $\varpi \in P^r_{\mathtt{sr}_1\ldots\mathtt{sr}_n}$, $\tau_i \in (T_k^\Delta)_{\mathtt{sr}_i}$, $\mathtt{sr}_i \in S^r$ and $k \in \mathtt{Nom}$, and
(c) non-rigid relations $\pi_k(t_1, \ldots, t_n)$, where $\pi \in P_{s_1\ldots s_n}$, $t_i \in (T_k^\Delta)_{s_i}$, $s_i \in S$ and $k \in \mathtt{Nom}$.

When there is no danger of confusion we omit the subscript s and/or the superscript k from the notation $t =_s^k t'$. Full sentences are constructed from atomic sentences, nominal equations and relations by applying Boolean operators, retrieve, store, necessity and quantification over nominal variables and rigid variables.

Given a model $(W, M) \in| \mathtt{Mod}^{\mathbf{HFOLSA}}(\Delta) |$, the interpretation of a Δ-term into (W, M) is defined inductively on the structure of the Δ-terms:

(1) $(W, M)_{\varsigma(\tau_1,\ldots,\tau_n)} = (M_{W_k})_\varsigma((W, M)_{\tau_1}, \ldots, (W, M)_{\tau_n})$,
(2) $(W, M)_{\sigma_k(t_1,\ldots,t_n)} = (M_{W_k})_\sigma((W, M)_{t_1}, \ldots, (W, M)_{t_n})$, and
(3) assuming that $(W, M)_\tau \in (M_{W_{k_1}})_{\mathtt{sr}}$ is defined, then since $(M_{W_{k_1}})_{\mathtt{sr}} = (M_{W_k})_{\mathtt{sr}}$, the interpretation of $\tau \in (T_k^\Delta)_{\mathtt{sr}}$ into (W, M) is also $(W, M)_\tau \in (M_{W_{k_1}})_{\mathtt{sr}}$.

The satisfaction relation for atomic sentences is defined as follows:

(a) $(W, M) \models t = t'$ iff $(W, M)_t = (W, M)_{t'}$;
(b) $(W, M) \models \varpi(\tau_1, \ldots \tau_n)$ iff $((W, M)_{\tau_1}, \ldots, (W, M)_{\tau_n}) \in (M_k)_\varpi$;
(c) $(W, M) \models \pi_k(t_1, \ldots, t_n)$ iff $((W, M)_{t_1}, \ldots, (W, M)_{t_n}) \in (M_k)_\pi$.

Example 3.5 (SHARE) This institution is obtained from **HFOLSA** by restricting the signatures such that all sorts are rigid.

Example 3.6 (Hybrid Preorder Algebra with user-defined Sharing and Annotation (**HPOASA**)) This institution was defined in [GĬ15b] and it is obtained by replicating the construction of **HFOLSA** in the context of preorder algebra. The Kripke structures of **HPOASA** upgrade the Kripke structures of **HFOLSA** with preorder relations for the carrier sets of each sort. Given a **HPOASA** signature $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$, where $\Sigma = (S^r, F^r) \subseteq (S, F)$, the atomic Δ-sentences consist of (a) equations $t =_s^k t'$ and (b) transitions $t \Rightarrow_s^k t'$, where $t \in (T_k^\Delta)_s$, $t' \in (T_k^\Delta)_s$, $k \in \mathtt{Nom}$ and $s \in S$. The sentences are constructed from atomic sentences, nominal equations and nominal relations by applying Boolean operators, retrieve, store, necessity and quantification over nominal variables and rigid variables. The satisfaction relation for atomic sentences is defined by (a) $(W, M) \models (t = t')$ iff $(W, M)_t = (W, M)_{t'}$ and (b) $(W, M) \models (t \Rightarrow t')$ iff $(W, M)_t \leq (W, M)_{t'}$.

3.2. Hybrid quantification category

Let $\mathtt{HI}$ be a hybrid institution. Given a signature Δ, a nominal variable j for $\mathtt{F}^{\mathtt{HI}}(\Delta)$ is a pair $(x, \mathtt{F}^{\mathtt{HI}}(\Delta))$, where x is the name of the variable.

The quantification category $\mathtt{Q}^{\mathbf{HFOLSA}}$ for **HFOLSA** consists of signature extensions with a finite number of nominal variables and rigid variables of the form $\Delta \hookrightarrow \Delta[J, X]$, where $\Delta = (\mathtt{Nom}, \Lambda, \Sigma) \in| \mathtt{Sig}^{\mathbf{HFOLSA}} |$, $\Delta[J, X] = (\mathtt{Nom} \cup J, \Lambda, \Sigma[X]) \in| \mathtt{Sig}^{\mathbf{HFOLSA}} |$, $\Sigma = (S^r, F^r, P^r) \subseteq (S, F, P) \in| \mathtt{Sig}^{\mathbf{FOLS}} |$, J is a finite set of variables for $(\mathtt{Nom}, \Lambda)$, X is a finite set of rigid variables for Σ and $\Sigma[X] = (S^r, F^r \cup X, P^r) \subseteq (S, F \cup X, P)$.

The quantification category $\mathtt{Q}^{\mathbf{HPL}}$ for **HPL** consists of signature extensions with a finite number of nominal variables $\Delta \hookrightarrow \Delta[J]$, where $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop}) \in| \mathtt{Sig}^{\mathbf{HPL}} |$, $\Delta[J] = (\mathtt{Nom} \cup J, \Lambda, \mathtt{Prop}) \in| \mathtt{Sig}^{\mathbf{HPL}} |$ and J is a finite set of variables for $(\mathtt{Nom}, \Lambda)$.

3.3. Standard approach vs. annotation

The hybrid logics with annotated syntax are recently introduced in [GĬ15b]. They have the advantage of having good logical properties that brings them closer to the first-order logics, i.e. the existence of initial model of Horn clauses, completeness, compactness, etc. In what follows we show that hybrid logics with annotated syntax are more expressive than their classic versions using **HFOLSA** as a benchmark example.

Let $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$ a **HFOLSA** signature and j a variable for $(\mathtt{Nom}, \Lambda)$, where $\Sigma = (S^r, F^r, P^r) \subseteq (S, F, P)$. Let $\mathtt{at}_j : T_{(S,F,P)} \to T_j^{\Delta[j]}$ be the function from the set of first-order (S, F, P)-terms $T_{(S,F,P)}$ to the set of hybrid Δ-terms $T_j^{\Delta[j]}$ defined by $\mathtt{at}_j(\sigma(t_1, \ldots, t_n)) = \begin{cases} \sigma(\mathtt{at}_j(t_1), \ldots, \mathtt{at}_j(t_n)) & \text{if } \sigma \in F^r \\ \sigma_j(\mathtt{at}_j(t_1), \ldots, \mathtt{at}_j(t_n)) & \text{if } \sigma \in F - F^r \end{cases}$.

Let $\mathbf{HFOLS}_a$ be the restriction of **HFOLS** to atomic sentences. The function $\alpha_\Delta : \mathtt{Sen}^{\mathbf{HFOLS}_a}(\Delta) \to \mathtt{Sen}^{\mathbf{HFOLSA}}(\Delta)$ is defined by extending $\mathtt{at}_j$ to atomic sentences:

(a) $\alpha_\Delta(t_1 = t_2) = (\downarrow j)(\mathtt{at}_j(t_1) = \mathtt{at}_j(t_2))$, and
(b) $\alpha_\Delta(\pi(t_1, \ldots, t_n)) = \begin{cases} (\downarrow j)\pi(\mathtt{at}_j(t_1), \ldots, \mathtt{at}_j(t_n)) & \text{if } \pi \in P^r \\ (\downarrow j)\pi_j(\mathtt{at}_j(t_1), \ldots, \mathtt{at}_j(t_n)) & \text{if } \pi \in P - P^r \end{cases}$,

where $t_i \in T_{(S,F,P)}$.

Proposition 3.3 [GĬ5b]. For any Δ-model (W, M), state $w \in| W |$ and sentence $\rho \in \mathtt{Sen}^{\mathbf{HFOLS}_a}(\Delta)$ we have $(W, M) \models^w \rho$ iff $(W, M) \models^w \alpha_\Delta(\rho)$.

This result says that the **HFOLS**-sentence ρ has the same expressive power as the **HFOLSA**-sentence $\alpha_\Delta(\rho)$. Proposition 3.3 can be easily extended to all sentences of **HFOLS**, which implies that **HFOLSA** is more expressive than **HFOLS**. Proposition 3.3 shows that the operator store is somehow integrated in the atomic sentences of **HFOLS** while in **HFOLSA** it is not the case. This is one of the main reasons **HFOLSA** allows a more structured approach to model-theoretic properties. It is worth mentioning that **HFOLSA** is not an instance of the hybridisation process described in [MMDB11, DM16, Dia16b].

3.4. Hybrid substitutions

In this subsection we define an abstract notion of hybrid substitution, which upgrades the notion of substitution defined in Section 2.3 to hybrid institutions.

Definition 3.5 Let HI be a hybrid institution. A *hybrid substitution* between the signature morphisms $\Delta \xrightarrow{\chi_1} \Delta' \in \mathtt{Sig}^{\mathtt{HI}}$ and $\Delta \xrightarrow{\chi_2} \Delta'' \in \mathtt{Sig}^{\mathtt{HI}}$ is a substitution $\chi_1 \xrightarrow{\theta} \chi_2$ such that the following *local satisfaction condition* holds: $M'' \models^w \theta(\rho')$ iff $M'' \upharpoonright_\theta \models^w \rho$, for all $M'' \in| \mathtt{Mod}^{\mathtt{HI}}(\Delta'') |$, $\rho \in \mathtt{Sen}^{\mathtt{HI}}(\Delta')$ and $w \in| \mathtt{K}_{\Delta''}(M'') |$.

Note that $| \mathtt{K}_{\Delta''}(M'') \upharpoonright_\theta | = | (\mathtt{K}_{\Delta''}(M'') \upharpoonright_\theta) \upharpoonright_{\chi_1} | = | \mathtt{K}_{\Delta''}(M'') \upharpoonright_{\chi_2} |$, which makes the above definition consistent.

Example 3.7 (HFOLSA substitutions [GĬ5b]) Consider two signature extensions with nominals and rigid constant symbols $\chi_1 : \Delta \hookrightarrow \Delta[C_1, D_1]$ and $\chi_2 : \Delta \hookrightarrow \Delta[C_2, D_2]$, where $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$, $\Sigma = (S^r, F^r, P^r) \subseteq (S, F, P)$, C_i is a set of nominals different from the elements of Nom, and D_i is a set of rigid constants different from the constants in F. A pair of functions $\theta = (\theta_a : C_1 \to \mathtt{Nom} \cup C_2, \theta_b : D_1 \to T_k^{\Delta[C_2, D_2]})$, where $k \in \mathtt{Nom}$, represents a substitution between χ_1 and χ_2:

(1) On the syntactic side, θ determines a sentence translation $\mathtt{Sen}^{\mathbf{HFOLSA}}(\theta) : \mathtt{Sen}^{\mathbf{HFOLSA}}(\Delta[C_1, D_1]) \to \mathtt{Sen}^{\mathbf{HFOLSA}}(\Delta[C_2, D_2])$, which preserves Δ and substitutes (a) nominals in $\mathtt{Nom} \cup C_2$ for nominals in C_1 corresponding to θ_a, and (b) $\Delta[C_2, D_2]$-terms for constants in D_1 according to θ_b.
(2) On the semantics side, θ determines a model functor $\mathtt{Mod}^{\mathbf{HFOLSA}}(\theta) : \mathtt{Mod}^{\mathbf{HFOLSA}}(\Delta[C_2, D_2]) \to \mathtt{Mod}^{\mathbf{HFOLSA}}(\Delta[C_1, D_1])$ such that for all $\Delta[C_2, D_2]$-models (W'', M'') the model $(W'', M'') \upharpoonright_\theta$ interprets (a) each symbol x in Δ as $(W'', M'')_x$, (b) each nominal $c_1 \in C_1$ as $(W'', M'')_{\theta_a(c_1)}$, and (c) any rigid symbol $d_1 \in D_1$ as $(W'', M'')_{\theta_b(d_1)}$.

Hybrid substitution functors. Given a hybrid institution HI, for any signature $\Delta \in| \mathtt{Sig}^{\mathtt{HI}} |$, the hybrid Δ-substitutions form a subcategory $\mathtt{HSb}^{\mathtt{HI}}(\Delta)$ of $\mathtt{Sb}^{\mathtt{HI}}(\Delta)$.

Fact 3.4 . $\mathtt{HSb}^{\mathtt{HI}} : \mathtt{Sig}^{\mathtt{HI}} \to \mathbb{CAT}^{op}$ is a sub-functor of $\mathtt{Sb}^{\mathtt{HI}}$.

In applications we work with a substitution sub-functor $\mathtt{HSt}^{\mathtt{HI}} : \mathtt{D}^{\mathtt{HI}} \to \mathbb{CAT}^{op}$ of $\mathtt{HSb}^{\mathtt{HI}}$, where $\mathtt{D}^{\mathtt{HI}} \subseteq \mathtt{Sig}^{\mathtt{HI}}$ is a broad subcategory of signature morphisms. When there is no danger of confusion we may drop the superscript HI from notations.

Example 3.8 (HFOLSA substitution functor) Given a signature $\Delta = (\mathtt{Nom}, \Lambda, (S^r, F^r, P^r) \subseteq (S, F, P))$, only substitutions represented by pairs of functions $(\theta_a : C_1 \to \mathtt{Nom} \cup C_2, \theta_b : D_1 \to T_k^{\Delta[C_2, D_2]})$ as described in Example 3.7 are relevant for the present study. Let $\mathtt{D}^{\mathbf{HFOLSA}} \subseteq \mathtt{Sig}^{\mathbf{HFOLSA}}$ be the broad subcategory of signature extensions with nominals and rigid constants. Let $\mathtt{HSt}^{\mathbf{HFOLSA}} : \mathtt{D}^{\mathbf{HFOLSA}} \to \mathbb{CAT}^{op}$ denote the substitution functor which maps each signature Δ to the category of hybrid substitutions represented by pairs of functions $(\theta_a : C_1 \to \mathtt{Nom} \cup C_2, \theta_b : D_1 \to T_k^{\Delta[C_2, D_2]})$ as described in Example 3.7.

Example 3.9 (HPL substitution functor) Let $\mathtt{D}^{\mathbf{HPL}}$ be the broad subcategory of signature extensions with nominals. Let $\mathtt{HSt}^{\mathbf{HPL}} : \mathtt{D}^{\mathbf{HPL}} \to \mathbb{CAT}^{op}$ denote the substitution functor which maps each signature $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop})$ to the category of hybrid substitutions represented by functions $\theta : C_1 \to \mathtt{Nom} \cup C_2$, where C_i is a set of nominals different from the elements of Nom.

Hybrid substitution systems. We formalise sentence conversions in hybrid institutions as hybrid substitutions.

Definition 3.6 Given a hybrid institution HI, a *hybrid substitution system* is a substitution system that consists only of hybrid substitutions.

Examples of hybrid substitution systems may be found in Section 5.

4. Institution-independent concepts

We investigate some of the institution-independent notions and concepts that are necessary to prove our abstract results.

4.1. Reachable models

In general, the signatures of institutions provide the vocabulary for defining sentences: one can construct terms from the syntactic elements that compose the signatures, and then build formulas using terms. Each element of a signature has a precise semantics, which implies that every term has a unique interpretation into a model. Note that models consist of elements which may, or may not be, interpretations of terms. The models with elements which are interpretations of terms are called *reachable* [Pet07, GP10, GFO12, GŤ5a, cF15, GŤ5c, GŤ5b]. The following definition originates from [Pet07] and provides an institution-independent characterisation of reachable models.

Definition 4.1 Let I be an institution, $\mathtt{D} \subseteq \mathtt{Sig}$ a broad subcategory of signature morphisms, and $\mathtt{St} : \mathtt{D} \to \mathbb{CAT}^{op}$ a substitution functor for I. Given a signature $\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|$, a Σ-model M is St-*reachable* if for every signature morphism $\chi : \Sigma \to \Sigma' \in \mathtt{D}$ and each χ-expansion M' of M there exists a substitution $\theta : \chi \to 1_\Sigma \in \mathtt{St}(\Sigma)$ such that $M\restriction_\theta = M'$.

If the substitution functor St is fixed, St-reachable models may be called simply reachable. In the following the notion of reachability is applied to concrete logical systems.

Proposition 4.1 [Pet07, GFO12] In **FOL**, a model is $\mathtt{St}^{\mathbf{FOL}}$-reachable iff its elements consist of interpretations of terms.

Proposition 4.1 says that a first order (S, F, P)-model M is reachable iff the unique homomorphism $T_{(S,F,P)} \to M$ is surjective. In **HFOLSA** reachable models consist of interpretations of nominals and hybrid terms.

Proposition 4.2 [GŤ5b] In **HFOLSA**, a model is $\mathtt{St}^{\mathbf{HFOLSA}}$-reachable iff (a) its set of states consists of interpretations of nominals and (b) its carrier sets for the rigid sorts consist of interpretations of terms.

The expansions of models that consist of interpretations of syntactic elements along signature extensions with rigid variables do not generate substitutions in **HFOLS**, in general. The abstract theorems of the following sections are applicable to hybridized institutions with user-defined sharing in their standard versions only if the quantification is restricted to the quantification over nominal variables.

Proposition 4.3 [GŤ5b] In **HPL**, a model is $\mathtt{St}^{\mathbf{HPL}}$-reachable iff its states consists of interpretations of nominals.

4.2. Basic sentences

In concrete examples of institutions, basic sentences are the simplest sentences, which are intimately linked to the internal structure of models, in the sense that their satisfaction is preserved by homomorphisms. Basic sentences [Dia03] tend to be the starting building blocks from which the complex sentences are constructed by using Boolean operators, quantification, or other sentence operators. Basic sentences were initially introduced in [Tar86b] under the name of *positive elementary sentences*.

Definition 4.2 Let $\mathtt{I}$ be an institution. A set of sentences $B \subseteq \mathtt{Sen}(\Sigma)$ is *basic* if there exists a Σ-model M^B such that

$$M \models B \text{ iff there exists a homomorphism } M^B \to M$$

for all Σ-models M. We say that M^B is a *basic model* of B. If in addition the morphism $M^B \to M$ is unique then the set B is called *epi-basic*.

We show that the sets of atomic sentences of the institutions presented above are epi-basic. In addition, the basic model M^B is reachable, since M^B is obtained by factorising the term model by some congruence relation generated by B.

Proposition 4.4 [Dia03] Any set of atomic sentences in $\mathbf{FOL}$ is epi-basic, and the basic models are $\mathtt{St}^{\mathbf{FOL}}$-reachable.

Proof. Let B be a set of atomic Σ-sentences in $\mathbf{FOL}$. The basic model M_B is the initial model of B and it is constructed as follows: on the quotient $T_\Sigma/_{\equiv_B}$ of the term model T_Σ by the congruence generated by the equational atoms of B, we interpret each relation symbol π of Σ by $(M_B)_\pi = \{(t_1/_{\equiv_B}, \ldots, t_n/_{\equiv_B}) \mid \pi(t_1, \ldots, t_n) \in B\}$. $\square$

Proposition 4.5 [GĬ5b] Let $\mathbf{HFOLSA}_b$ be the restriction of $\mathbf{HFOLSA}$ to sentences obtained by applying the operator retrieve to atomic sentences, nominal equations and nominal relations. Any set of sentences in $\mathbf{HFOLSA}_b$ is epi-basic and the basic models are $\mathtt{HSt}^{\mathbf{HFOLSA}}$-reachable.

Proposition 4.6 [GĬ5b]. Let $\mathbf{HPL}_b$ be the restriction of $\mathbf{HPL}$ to sentences obtained by applying the operator retrieve to nominal equations, nominal relations and sentences of the form $@_k p$, where k is a nominal and p is a propositional symbol. Any set of sentences in $\mathbf{HPL}_b$ is epi-basic and the basic models are $\mathtt{HSt}^{\mathbf{HPL}}$-reachable.

The following result shows that the semantic consequences of a basic set of sentences can be reduced to the satisfaction by a base model.

Lemma 4.7 [GĬ5b] Let $\mathtt{I}$ be an institution. Consider a signature $\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|$, a basic set B of Σ-sentences and a basic model M^B for B. Then for all basic sentences $\gamma \in \mathtt{Sen}(\Sigma)$ we have $B \models_\Sigma \gamma$ iff $M^B \models_\Sigma \gamma$.

4.3. Entailment systems

Institutions with proof-theoretic structure provide a complete formal notion for the intuitive notion of logic, including both the model and the proof-theoretic sides. To account for proofs and their construction, Meseguer [Mes89] introduced the notion of *entailment system*. The *proof systems* defined by Diaconescu [Dia06] provide a more refined framework by discriminating between different proofs and addressing to their internal structure. Our approach is slightly more general than [Mes89] since we consider entailment relations $E_1 \vdash E_2$ such that not only E_1 but also E_2 is a set of sentences rather than a single sentence.

Definition 4.3 An *entailment system* $\mathcal{E} = (\mathtt{Sig}, \mathtt{Sen}, \vdash)$ consists of a category of signatures $\mathtt{Sig}$, a sentence functor $\mathtt{Sen} : \mathtt{Sig} \to \mathbb{S}et$, and a family of entailment relations $\vdash = \{\vdash_\Sigma\}_{\Sigma \in |\mathtt{Sig}|}$ between sets of sentences with the following properties:

$$(\mathit{Monotonicity})\frac{E \subseteq E_1}{E_1 \vdash_\Sigma E} \qquad (\mathit{Transitivity})\frac{E_1 \vdash_\Sigma E_2,\ E_2 \vdash_\Sigma E_3}{E_1 \vdash_\Sigma E_3}$$

$$(\mathit{Unions})\frac{E_1 \vdash_\Sigma E_2,\ E_1 \vdash_\Sigma E_3}{E_1 \vdash_\Sigma E_2 \cup E_3} \qquad (\mathit{Translation})\frac{E_1 \vdash_\Sigma E_2}{\varphi(E_1) \vdash_{\Sigma'} \varphi(E_2)}$$

where $E_i \subseteq \mathtt{Sen}(\Sigma)$, $E \subseteq \mathtt{Sen}(\Sigma)$ and $\varphi : \Sigma \to \Sigma' \in \mathtt{Sig}$.

When there is no danger of confusion we may omit the subscript Σ from $\vdash_\Sigma$. We call the entailment system $\mathcal{E} = (\mathtt{Sig}, \mathtt{Sen}, \vdash)$ *compact* whenever for every $\Gamma \subseteq \mathtt{Sen}(\Sigma)$ and each finite $E_f \subseteq \mathtt{Sen}(\Sigma)$ if $\Gamma \vdash_\Sigma E_f$ then there exists $\Gamma_f \subset \Gamma$ finite such that $\Gamma_f \vdash_\Sigma E_f$. For each entailment system $\mathcal{E} = (\mathtt{Sig}, \mathtt{Sen}, \vdash)$ one can easily construct the *compact entailment subsystem* $\mathcal{E}^c = (\mathtt{Sig}, \mathtt{Sen}, \vdash^c)$ by defining the entailment relation $\vdash^c$ as follows: $\Gamma \vdash^c_\Sigma E$ iff for each finite set $E_f \subseteq E$ there exists a finite set $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash_\Sigma E_f$.

Lemma 4.8 (Compact entailment subsystem [Dia06]) $\mathcal{E}^c = (\mathtt{Sig}, \mathtt{Sen}, \vdash^c)$ is an entailment system.

The semantic entailment system of an institution $\mathtt{I}$ consists of $(\mathtt{Sig}^{\mathtt{I}}, \mathtt{Sen}^{\mathtt{I}}, \models^{\mathtt{I}})$. For every signature morphism $\varphi \in \mathtt{Sig}^{\mathtt{I}}$, we sometimes let φ denote the sentence translation $\mathtt{Sen}^{\mathtt{I}}(\varphi)$. An entailment system $\mathcal{E}^{\mathtt{I}} = (\mathtt{Sig}^{\mathtt{I}}, \mathtt{Sen}^{\mathtt{I}}, \vdash^{\mathtt{I}})$ is *sound* (resp. *complete*) for an institution $\mathtt{I}$ if $\Gamma \vdash^{\mathtt{I}}_{\Sigma} \gamma$ implies $\Gamma \models^{\mathtt{I}}_{\Sigma} \gamma$ (resp. $\Gamma \models^{\mathtt{I}}_{\Sigma} \gamma$ implies $\Gamma \vdash^{\mathtt{I}}_{\Sigma} \gamma$) for every signature Σ, each set of Σ-sentences Γ and any Σ-sentence γ.

Proposition 4.9 Consider (a) an institution $\mathtt{I}$, (b) a sub-functor $\mathtt{Sen}^{\mathtt{I}_0} : \mathtt{Sig}^{\mathtt{I}} \to \mathbb{S}et$ of $\mathtt{Sen}^{\mathtt{I}}$, (c) an entailment system $\mathcal{E}^{\mathtt{I}_0} = (\mathtt{Sig}^{\mathtt{I}}, \mathtt{Sen}^{\mathtt{I}_0}, \vdash^{\mathtt{I}_0})$ for the institution $\mathtt{I}_0 = (\mathtt{Sig}^{\mathtt{I}}, \mathtt{Sen}^{\mathtt{I}_0}, \mathtt{Mod}^{\mathtt{I}}, \models^{\mathtt{I}})$, and (d) a substitution system $\Theta^{\mathtt{I}}$ for $\mathtt{I}$. We define $\mathcal{E}^{\mathtt{I}} = (\mathtt{Sig}^{\mathtt{I}}, \mathtt{Sen}^{\mathtt{I}}, \vdash^{\mathtt{I}})$ as the least entailment system over $\mathcal{E}^{\mathtt{I}_0}$ closed to

$$(\mathtt{Subst}_{\Theta^{\mathtt{I}}})\frac{}{\gamma \vdash \Theta^{\mathtt{I}}_{\Sigma}(\gamma)} \quad \text{and} \quad \frac{}{\Theta^{\mathtt{I}}_{\Sigma}(\gamma) \vdash \gamma}$$

where $\gamma \in \mathtt{Sen}^{\mathtt{I}}(\Sigma)$. Then we have:

(1) $\mathcal{E}^{\mathtt{I}}$ is sound if $\mathcal{E}^{\mathtt{I}_0}$ is sound,

(2) $\mathcal{E}^{\mathtt{I}}$ is compact if $\mathcal{E}^{\mathtt{I}_0}$ is compact, and

(3) $\mathcal{E}^{\mathtt{I}}$ is complete if (i) $\mathcal{E}^{\mathtt{I}_0}$ is complete and compact, and (ii) $\Theta^{\mathtt{I}}_{\Sigma}(\gamma) \in \mathtt{Sen}^{\mathtt{I}_0}(\Sigma)$ for all signatures $\Sigma \in\mid \mathtt{Sig}^{\mathtt{I}} \mid$ and sentences $\gamma \in \mathtt{Sen}^{\mathtt{I}}(\Sigma)$.

Proof. For the sake of simplicity, we will drop the superscript $\mathtt{I}$ from the notations.

For soundness, it suffices to show that the proof rules $(\mathtt{Subst}_{\Theta})$ are sound, which follows directly from the satisfaction condition for substitutions.

For compactness, let $\mathcal{E}^c = (\mathtt{Sig}, \mathtt{Sen}, \vdash^c)$ be the compact entailment subsystem of $\mathcal{E} = (\mathtt{Sig}, \mathtt{Sen}, \vdash)$. It is easy to see that $\mathcal{E}^c$ is closed to proof rules $(\mathtt{Subst}_{\Theta})$. Since $\mathcal{E}^{\mathtt{I}_0}$ is compact, $\mathcal{E}^c$ includes all deductions from $\mathcal{E}^{\mathtt{I}_0}$. Hence, $\mathcal{E} = \mathcal{E}^c$.

For completeness, suppose that $\Gamma \models_{\Sigma} \gamma$, where $\Sigma \in\mid \mathtt{Sig} \mid$, $\Gamma \subseteq \mathtt{Sen}(\Sigma)$ and $\gamma \in \mathtt{Sen}(\Sigma)$. By the satisfaction condition for substitutions, $\Theta(\Gamma) \models \Theta(\gamma)$. Since $\mathcal{E}^{\mathtt{I}_0}$ is complete, $\Theta(\Gamma) \vdash^{\mathtt{I}_0} \Theta(\gamma)$. By the compactness of $\mathcal{E}^{\mathtt{I}_0}$ there exists $\Gamma_f \subseteq \Gamma$ finite such that $\Theta(\Gamma_f) \vdash^{\mathtt{I}_0} \Theta(\gamma)$. Since Γ_f is finite and $\Gamma \vdash \Theta(e)$ for all $e \in \Gamma_f$, $\Gamma \vdash \Theta(\Gamma_f)$. It follows that $\Gamma \vdash \Theta(\gamma)$ and since $\Theta(\gamma) \vdash \gamma$, we get $\Gamma \vdash \gamma$. □

This result can be used to extend completeness to a logical system obtained from a given logic by adding a new sentence operator that does not enrich the expressivity of the initial syntax, i.e. the new sentence operator can be defined using the sentence operators of the given logic. Applications of Proposition 4.9 can be found in Section 5.

5. Completeness

The notion of *Horn clause* is generalised to the level of hybrid institutions, and an abstract *Birkhoff calculus* is developed to reason formally about Horn clauses. The entailment system developed here consists of two layers:

(1) the base layer corresponding to atomic sentences, nominal equations and nominal relations which is assumed in the abstract setting but it is developed in concrete examples, and

(2) the abstract layer corresponding to Horn clauses, which is developed on top of the base layer at the general level provided by the institution theory by adding proof rules for each sentence operator.

Definition 5.1 Let $\mathtt{HI}$ be a hybrid institution and $\mathtt{Sen}^{\mathtt{HI}_b} : \mathtt{Sig}^{\mathtt{HI}} \to \mathbb{S}et$ a sub-functor of $\mathtt{Sen}^{\mathtt{HI}}$. $\mathtt{HI}$ is a *Horn hybrid institution* over $\mathtt{HI}_b = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_b}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ if all sentences of $\mathtt{HI}$ consist of *Horn clauses* over $\mathtt{HI}_b$, i.e. sentences constructed from the sentences of $\mathtt{HI}_b$ by applying at most one time the following sentence building operators: implication, store, universal quantification, necessity over binary modalities and retrieve, respectively.

Given a Horn hybrid institution $\mathtt{HI}$ over $\mathtt{HI}_b$ and a signature $\Delta \in\mid \mathtt{Sig}^{\mathtt{HI}} \mid$, the most complex Δ-sentence is of the form $@_k[\lambda](\forall \chi)(\downarrow j) \bigwedge H \Rightarrow C$, where $k \in \mathtt{Nom}^{\Delta}$, $\lambda \in \Lambda^{\Delta}_2$, $\Delta \xrightarrow{\chi} \Delta' \in \mathtt{Q}^{\mathtt{HI}}$, j is a variable for $F(\Delta')$, $H \subseteq \mathtt{Sen}^{\mathtt{HI}_b}(\Delta'[j])$ and $C \in \mathtt{Sen}^{\mathtt{HI}_b}(\Delta'[j])$.

An example of Horn hybrid institution is $\mathbf{HFOLSA}_h$, the restriction of **HFOLSA** to Horn clauses over $\mathbf{HFOLSA}_b$. Another example of Horn hybrid institution is $\mathbf{HPL}_h$, the restriction of **HPL** to Horn clauses over $\mathbf{HPL}_b$.

Table 1. Hybrid substitution system for Birkhoff institutions

(1) $\Theta^{\mathtt{HI}}_{\Delta}(\gamma)$	=	γ for all $\gamma \in \mathtt{Sen}^{\mathtt{HI}_p}(\Delta)$,
(2) $\Theta^{\mathtt{HI}}_{\Delta}([\lambda](\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C))$	=	$(\forall\chi)(\forall\chi[j])(\downarrow j_1)(\lambda(j_1, j) \wedge \bigwedge_{h\in H} \chi[j_1](@_j h) \Rightarrow \chi[j_1](@_j C))$, where j_1 is a new variable,
(3) $\Theta^{\mathtt{HI}}_{\Delta}([\lambda](\forall\chi)(\bigwedge H' \Rightarrow C'))$	=	$(\forall\chi)(\forall\chi[j])(\downarrow j_1)(\lambda(j_1, j) \wedge \bigwedge_{h'\in H'} (\chi[j];\ \chi[j_1])(@_j h') \Rightarrow (\chi[j];\ \chi[j_1])(@_j C'))$, where j_1 is a new variable,
(4) $\Theta^{\mathtt{HI}}_{\Delta}(@_k((\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)))$	=	$(\forall\chi)(\bigwedge_{h\in H} @_{k'}\psi(h) \Rightarrow @_{k'}\psi(C))$, where $k' = \mathtt{F}(\chi)(k)$, $\psi^{\mathbf{REL}} : \mathtt{F}(\Delta')[j] \to \mathtt{F}(\Delta')$ preserves $\mathtt{F}(\Delta')$, and $\psi^{\mathbf{REL}}(j) = k'$,
(5) $\Theta^{\mathtt{HI}}_{\Delta}(@_k((\forall\chi)(\bigwedge H' \Rightarrow C')))$	=	$(\forall\chi)(\bigwedge_{h'\in H'} @_{k'}h' \Rightarrow @_{k'}C')$, where $k' = \mathtt{F}(\chi)(k)$,
(6) $\Theta^{\mathtt{HI}}_{\Delta}(@_k[\lambda]\gamma)$	=	$\Theta^{\mathtt{HI}}_{\Delta}(@_k\Theta^{\mathtt{HI}}_{\Delta}([\lambda]\gamma))$.

Assumption 2 Throughout this paper, we assume that $\mathtt{HI}$ range over hybrid institutions satisfying another mild property: given a signature Δ and a variable j for $(\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$, a signature morphism $\psi^{\mathbf{REL}} : (\mathtt{Nom}^{\Delta} \cup \{j\}, \Lambda^{\Delta}) \to (\mathtt{Nom}^{\Delta}, \Lambda^{\Delta}) \in \mathtt{Sig}^{\mathbf{REL}}$ which preserves $(\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$ defines a signature morphism $\psi : \Delta[j] \to \Delta \in \mathtt{Sig}^{\mathtt{HI}}$ such that $\mathtt{F}(\psi) = \psi^{\mathbf{REL}}$ and $\chi[j];\ \psi = 1_{\Delta}$.

Assumptions 1 and 2 lead to the following lemma which is useful to prove results about sentences constructed with store and retrieve.

Lemma 5.1 Let M be a Δ-model, where Δ is a signature of some hybrid institution $\mathtt{HI}$. Consider a nominal $k \in \mathtt{Nom}^{\Delta}$, a variable j for $(\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$, and a signature morphism $\psi^{\mathbf{REL}} : (\mathtt{Nom}^{\Delta} \cup \{j\}, \Lambda^{\Delta}) \to (\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$ such that $(\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$ is preserved and $\psi^{\mathbf{REL}}(j) = k$. Then the unique expansion of M along $\chi[j]$ which satisfies $j = k$ is the reduct of M along ψ (i.e. $M^{(j,w)} = M\upharpoonright_{\psi}$, where $\mathtt{K}_{\Delta}(M)_k = w$).

Proof. Note that $\mathtt{K}_{\Delta[j]}(M\upharpoonright_{\psi}) = \mathtt{K}_{\Delta}(M)\upharpoonright_{\psi^{\mathbf{REL}}} = \mathtt{K}_{\Delta}(M)^{(j,w)}$. Since $\chi[j];\ \psi = 1_{\Delta}$, the model $(M\upharpoonright_{\psi})$ is a $\chi[j]$-expansion of M. Since $M^{(j,w)}$ is the unique $\chi[j]$-expansion of M such that $\mathtt{K}_{\Delta[j]}(M^{(j,w)}) = \mathtt{K}_{\Delta}(M)^{(j,w)}$, we have $M\upharpoonright_{\psi} = M^{(j,w)}$. □

5.1. Plain Horn clauses

Let $\mathtt{HI}$ be a Horn hybrid institution over $\mathtt{HI}_b$ such that $\mathtt{HI}_b$ is closed to retrieve (i.e. $@_k\rho \in \mathtt{Sen}^{\mathtt{HI}_b}(\Delta)$ for all $\Delta \in|\ \mathtt{Sig}^{\mathtt{HI}}\ |$, $k \in \mathtt{Nom}^{\Delta}$ and $\rho \in \mathtt{Sen}^{\mathtt{HI}_b}(\Delta)$). We say that a Horn clause is *plain* if it is constructed from the sentences of $\mathtt{HI}_b$ without necessity and retrieve. Let $\mathtt{HI}_p$ be the restriction of $\mathtt{HI}$ to plain Horn clauses. We will prove that $\mathtt{HI}$ and $\mathtt{HI}_p$ have the same expressivity power by defining a hybrid substitution system for $\mathtt{HI}$ that maps every Horn clause to a plain Horn clause. Consider the hybrid substitution system $\Theta^{\mathtt{HI}}$ defined in Table 1. For the sake of simplicity, the cases where the sentences do not contain implication or quantification are omitted as these cases can be seen as instances of more complex cases which have been considered. This is due to the fact that sentences of the form $\emptyset \Rightarrow C$ and $(\forall 1_{\Delta})\rho$ are semantically equivalent to C and ρ, respectively.

Proposition 5.2 $\Theta^{\mathtt{HI}}$ is a hybrid substitution system.

Proof. Given a Horn clause ρ, a model M, and a state w of M, one needs to show that $M \models^w \rho$ iff $M \models^w \Theta^{\mathtt{HI}}(\rho)$ by taking into account all six cases considered in Table 1. See Section A for full proof. □

Table 2. Plain Birkhoff calculus

(*Generalisation*)

$$\frac{\chi[j](\Gamma) \vdash_{\Delta'[j]} \bigwedge_{h\in H} (@_j h) \Rightarrow (@_j C)}{\Gamma \vdash_{\Delta} (\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)}$$

$$\frac{\Gamma \vdash_{\Delta} (\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)}{\chi[j](\Gamma) \vdash_{\Delta'[j]} \bigwedge_{h\in H} (@_j h) \Rightarrow (@_j C)}$$

(*Quantification*)

$$\frac{\chi(\Gamma) \vdash_{\Delta'} \bigwedge H' \Rightarrow C'}{\Gamma \vdash_{\Delta} (\forall\chi)(\bigwedge H' \Rightarrow C')}$$

$$\frac{\Gamma \vdash_{\Delta} (\forall\chi)(\bigwedge H' \Rightarrow C')}{\chi(\Gamma) \vdash_{\Delta'} \bigwedge H' \Rightarrow C'}$$

(*Substitutivity*)

$$\frac{}{(\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C) \vdash_{\Delta} \bigwedge_{h\in H} \theta(@_j h) \Rightarrow \theta(@_j C)}$$

where $\theta : (\chi;\ \chi[j]) \to 1_{\Delta}$

$$\frac{}{(\forall\chi)(\bigwedge H' \Rightarrow C') \vdash_{\Delta} \bigwedge \theta'(H') \Rightarrow \theta'(C')} \text{ where } \theta' : \chi \to 1_{\Delta}$$

(*Implication*)

$$\frac{\Gamma \vdash \bigwedge_{e\in E} (@_k e) \Rightarrow (@_k \rho)}{\Gamma \cup (\bigcup_{e\in E} @_k e) \vdash (@_k \rho)}$$

$$\frac{\Gamma \cup (\bigcup_{e\in E} @_k e) \vdash (@_k \rho)}{\Gamma \vdash \bigwedge_{e\in E} (@_k e) \Rightarrow (@_k \rho)}$$

(*Retrieve*)

$$\frac{}{\bigwedge E \Rightarrow \rho \vdash \bigwedge_{e\in E} (@_k e) \Rightarrow (@_k \rho)}$$

Fact 5.3 Notice that $\Theta^{\mathtt{HI}}_{\Delta}(\rho)$ is a plain Horn clause for all Horn clauses ρ.

Fact 5.3 and Proposition 5.2 show that $\mathtt{HI}$ and $\mathtt{HI}_p$ have the same expressivity power. If we define an entailment system for $\mathtt{HI}_p$ then we obtain an entailment system for $\mathtt{HI}$ by adding $(\mathtt{Subst}_{\Theta^{\mathtt{HI}}})$ proof rules. Moreover, soundness and completeness can be extended from $\mathtt{HI}_p$ to $\mathtt{HI}$ using Proposition 4.9.

5.2. Birkhoff entailment systems

Consider a Horn hybrid institution $\mathtt{HI}$ over $\mathtt{HI}_b$ such that $\mathtt{HI}_b$ is closed to retrieve, and an entailment system $\mathcal{E}^{\mathtt{HI}_b} = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_b}, \vdash^{\mathtt{HI}_b})$ for $\mathtt{HI}_b$.

Definition 5.2 $\mathcal{E}^{\mathtt{HI}_p} = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_p}, \vdash^{\mathtt{HI}_p})$ is the least entailment system over $\mathcal{E}^{\mathtt{HI}_b}$ closed to the proof rules defined in Table 2.

Soundness and compactness can be prove directly without any extra assumptions.

Theorem 5.4 (1) $\mathcal{E}^{\mathtt{HI}_p}$ is sound if $\mathcal{E}^{\mathtt{HI}_b}$ is sound, and
(2) $\mathcal{E}^{\mathtt{HI}_p}$ is compact if $\mathcal{E}^{\mathtt{HI}_b}$ is compact.

Proof. For soundness, it suffices to prove that the proof rules in Table 2 are sound. For compactness, let $\mathcal{E}^c = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_p}, \vdash^c)$ be the compact entailment subsystem of $\mathcal{E}^{\mathtt{HI}_p}$. Since $\mathcal{E}^{\mathtt{HI}_b}$ is compact, it suffices to prove that $\mathcal{E}^c$ is closed to the proof rules defined in Tabel 2. See Section A for full proof. □

For completeness, supplementary assumptions based on institution-independent concepts defined in Section 4 are needed.

Theorem 5.5 Assume a hybrid substitution functor $\mathtt{HSt} : \mathtt{D} \to \mathbb{CAT}^{op}$ for $\mathtt{HI}_b$ such that $\mathtt{Q}^{\mathtt{HI}} \subseteq \mathtt{D}$. The entailment system $\mathcal{E}^{\mathtt{HI}_p}$ is complete if

(1) each set of $\mathtt{HI}_b$ sentences is basic and has a basic model that is reachable, and
(2) $\mathcal{E}^{\mathtt{HI}_b}$ is complete and compact.

Proof. We will focus on the most complex case as the remaining cases can be proved similarly. We assume that $\Gamma \models (\forall\,\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$, where $\Delta \xrightarrow{\chi} \Delta' \in \mathtt{Q}^{\mathtt{HI}}$.

Suppose towards a contradiction that $\Gamma \not\vdash (\forall\,\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. In what follows, we construct a model M such that $M \models^{\mathtt{HI}} \Gamma$ and $M \not\models^{\mathtt{HI}} (\forall\,\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. This is a contradiction with the initial assumption, which implies $\Gamma \vdash (\forall\,\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$.

Let Γ_b be the set of all $\mathtt{HI}_b$ sentences entailed by $(\chi;\ \chi[j])(\Gamma)$ and $\bigcup_{h\in H} @_j h$, i.e. $\Gamma_b = \{\rho \in \mathtt{Sen}^{\mathtt{HI}_b}(\Delta'[j]) \mid (\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h\in H} @_j h) \vdash \rho\}$. Let M^b be a basic model of Γ_b that is reachable. It is straightforward to show that

(1) $\Gamma_b \not\vdash @_j C$, and
(2) $M^b \models^{\mathtt{HI}} (\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h\in H} @_j h)$.

By (1) and completeness of $\mathtt{HI}_b$, we have $\Gamma_b \not\models @_j C$, and by Lemma 4.7, $M^b \not\models^{\mathtt{HI}} @_j C$. By (2), $M^b \models^{\mathtt{HI}} \bigcup_{h\in H} @_j h$, which implies $M^b \upharpoonright_{(\chi;\,\chi[j])} \not\models^{\mathtt{HI}} (\forall\,\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. By (2), we have $M^b \models^{\mathtt{HI}} (\chi;\ \chi[j])(\Gamma)$, which implies $M^b \upharpoonright_{(\chi;\,\chi[j])} \models^{\mathtt{HI}} \Gamma$. See Section A for full proof. □ The entailment system of the institution $\mathtt{HI}$ is obtained by adding the proof rules $(\mathtt{Subst}_{\Theta^{\mathtt{HI}}})$ to the calculus for $\mathtt{HI}_p$.

Definition 5.3 $\mathcal{E}^{\mathtt{HI}} = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}}, \vdash^{\mathtt{HI}})$ is the least entailment system over $\mathcal{E}^{\mathtt{HI}_b}$ closed to the proof rules defined in Table 2 and $(\mathtt{Subst}_{\Theta^{\mathtt{HI}}})$, where $\Theta^{\mathtt{HI}}$ is defined in Table 1.

Soundness and compactness are direct consequences of Proposition 4.9 and Theorem 5.4.

Theorem 5.6 (Birkhoff soundness)

(1) $\mathcal{E}^{\mathtt{HI}}$ is sound if $\mathcal{E}^{\mathtt{HI}_b}$ is sound, and
(2) $\mathcal{E}^{\mathtt{HI}}$ is compact if $\mathcal{E}^{\mathtt{HI}_b}$ is compact.

Completeness is a corollary of Proposition 4.9, Theorem 5.4 and Theorem 5.5.

Theorem 5.7 (Birkhoff Completeness) Consider a subcategory of signature morphisms $\mathtt{D}$ such that $\mathtt{Q}^{\mathtt{HI}} \subseteq \mathtt{D}$, and let $\mathtt{HSt} : \mathtt{D} \to \mathbb{CAT}^{op}$ be a hybrid substitution functor for $\mathtt{HI}_b$ such that (1) each set of sentences of $\mathtt{HI}_b$ is basic and has a basic model that is $\mathtt{HSt}$-reachable, and (2) $\mathcal{E}^{\mathtt{HI}_b}$ is complete and compact. Then $\mathcal{E}^{\mathtt{HI}}$ is complete.

Proof. By Theorem 5.4, $\mathcal{E}^{\mathtt{HI}_p}$ is compact, and By Theorem 5.5, $\mathcal{E}^{\mathtt{HI}_p}$ is complete. For all $\Delta \in\mid \mathtt{Sig}^{\mathtt{HI}} \mid$ and $\rho \in \mathtt{Sen}^{\mathtt{HI}}(\Delta)$ we have $\Theta^{\mathtt{HI}}_{\Delta}(\rho) \in \mathtt{Sen}^{\mathtt{HI}_p}(\Delta)$. By Proposition 4.9, $\mathcal{E}^{\mathtt{HI}}$ is complete. □

6. Instances of Birkhoff entailment systems

In order to develop concrete sound and complete Birkhoff entailment systems we need to set the parameters of the completeness theorem for each example.

Table 3. Hybrid system for $\mathbf{HPL}_b$

$\Theta^{\mathbf{HPL}_b}_{\Delta}(\rho)$	$= \rho$ for all $\rho \in \mathtt{Sen}^{\mathbf{HPL}_0}(\Delta)$,
$\Theta^{\mathbf{HPL}_b}_{\Delta}(@_k(k_1 = k_2))$	$= (k_1 = k_2)$,
$\Theta^{\mathbf{HPL}_b}_{\Delta}(@_k\, \lambda(k_1, \ldots, k_n))$	$= \lambda(k_1, \ldots, k_n)$,
$\Theta^{\mathbf{HPL}_b}_{\Delta}(@_{k_1} @_{k_2} \gamma)$	$= @_{k_2}\gamma$ for all $\gamma \in \mathtt{Sen}^{\mathbf{HPL}_b}(\Delta)$.

Table 4. Proof rules for $\mathbf{HPL}_0$

(*Reflexivity*)	$\dfrac{}{\Gamma \vdash k_1 = k_1}$
(*Transitivity*)	$\dfrac{\Gamma \vdash k_1 = k_2 \text{ and } \Gamma \vdash k_2 = k_3}{\Gamma \vdash k_1 = k_3}$
(*Symmetry*)	$\dfrac{\Gamma \vdash k_1 = k_2}{\Gamma \vdash k_2 = k_1}$
(*Congruence*)	$\dfrac{\Gamma \vdash \lambda(k_1, \ldots, k_n) \text{ and } \Gamma \vdash k_i = k_i' \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \lambda(k_1', \ldots, k_n')}$
(*Congruence$_p$*)	$\dfrac{\Gamma \vdash k_1 = k_2 \text{ and } \Gamma \vdash @_{k_1} p}{\Gamma \vdash @_{k_2} p}$

6.1. Birkhoff entailment system for HPL

We set the parameters of the completeness theorem for **HPL** as follows:

- the institution $\mathtt{HI}$ is $\mathbf{HPL}_h$,
- the institution $\mathtt{HI}_b$ is $\mathbf{HPL}_b$, and
- the functor $\mathtt{HSt} : \mathtt{D} \to \mathbb{CAT}^{op}$ is $\mathtt{HSt}^{\mathbf{HPL}_b} : \mathtt{D}^{\mathbf{HPL}} \to \mathbb{CAT}^{op}$, the restriction of $\mathtt{HSt}^{\mathbf{HPL}} : \mathtt{D}^{\mathbf{HPL}} \to \mathbb{CAT}^{op}$ (defined in Example 3.9) to $\mathbf{HPL}_b$.

Let $\mathbf{HPL}_0$ be the restriction of **HPL** to nominal equations, nominal relations and sentences of the form $@_k p$, where k is a nominal and p is a propositional symbol. We show that $\mathbf{HPL}_0$ has the same expressivity power as $\mathbf{HPL}_b$ by defining a hybrid substitution system for $\mathbf{HPL}_b$ that maps each sentence of $\mathbf{HPL}_b$ to a sentence of $\mathbf{HPL}_0$. The hybrid substitution system $\Theta^{\mathbf{HPL}_b}$ for $\mathbf{HPL}_b$ is defined in Table 3.

Proposition 6.1. $\Theta^{\mathbf{HPL}_b}$ is a hybrid substitution system.

Proof. Straightforward. □

We define an entailment system for $\mathbf{HPL}_0$, and we prove its soundness, compactness and completeness.

Definition 6.1 $\mathcal{E}^{\mathbf{HPL}_0} = (\mathtt{Sig}^{\mathbf{HPL}}, \mathtt{Sen}^{\mathbf{HPL}_0}, \vdash^{\mathbf{HPL}_0})$ is the least entailment system of $\mathbf{HPL}_0$ closed to the proof rules defined in Table 4.

Theorem 6.2 $\mathcal{E}^{\mathbf{HPL}_0}$ is sound, compact and complete.

Proof. The proof of soundness is straightforward as it is easy to show that the proof rules defined in Table 4 are sound.

For compactness, let $\mathcal{E}^c = (\mathtt{Sig}^{\mathbf{HPL}}, \mathtt{Sen}^{\mathbf{HPL}_0}, \vdash^c)$ be the compact entailment subsystem of $\mathcal{E}^{\mathbf{HPL}_0}$. We prove that $\mathcal{E}^c$ is closed to the proof rules defined in Table 4, which implies $\mathcal{E}^{\mathbf{HPL}_0} = \mathcal{E}^c$. For (*Reflexivity*), $\Gamma \vdash^c (k_1 = k_1)$ as for any finite $\Gamma_f \subseteq \Gamma$ we have $\Gamma_f \vdash^c (k_1 = k_1)$. For (*Symmetry*), assume that $\Gamma \vdash^c (k_1 = k_2)$; there exists

$\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash (k_1 = k_2)$, which implies $\Gamma_f \vdash (k_2 = k_1)$, and we get $\Gamma \vdash^c (k_2 = k_1)$. The remaining cases can be proved similarly.

For completeness, we assume that $\Gamma \models_\Delta \rho$, where $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop})$. We construct a Δ-model (W^Γ, M^Γ) such that (a) $(W^\Gamma, M^\Gamma) \models^{\mathbf{HPL}} \Gamma$ and (b) $(W^\Gamma, M^\Gamma) \models^{\mathbf{HPL}} \rho$ iff $\Gamma \vdash \rho$. The Δ-model (W^Γ, M^Γ) is defined as follows:

(1) $|W^\Gamma| = \mathtt{Nom}/_{\equiv_\Gamma}$, where $\equiv_\Gamma = \{(k_1, k_2) \mid \Gamma \vdash k_1 = k_2\}$,

(2) $W^\Gamma_\lambda = \{(\widehat{k_1}, \ldots, \widehat{k_n}) \mid \Gamma \vdash \lambda(k_1, \ldots, k_n)\}$ for all $n \in \mathbb{N}at$ and $\lambda \in \Lambda_n$, where $\widehat{k_i}$ is the equivalence class of k_i,

(3) the $\mathtt{Prop}$-model $M^\Gamma_{\widehat{k}}$ interprets $p \in \mathtt{Prop}$ as true iff $\Gamma \vdash @_k p$, where $k \in \mathtt{Nom}$.

Since $\Gamma \models \rho$ and $(W^\Gamma, M^\Gamma) \models^{\mathbf{HPL}} \Gamma$, we have $(W^\Gamma, M^\Gamma) \models^{\mathbf{HPL}} \rho$, which implies $\Gamma \vdash \rho$. See Section A for full proof of completeness. □

By adding the proof rules ($\mathtt{Subst}_{\Theta^{\mathbf{HPL}_b}}$) to the entailment system of $\mathbf{HPL}_0$, we obtain an entailment system for $\mathbf{HPL}_b$.

Definition 6.2 $\mathcal{E}^{\mathbf{HPL}_b} = (\mathtt{Sig}^{\mathbf{HPL}}, \mathtt{Sen}^{\mathbf{HPL}_b}, \vdash^{\mathbf{HPL}_b})$ is the least entailment system over $\mathcal{E}^{\mathbf{HPL}_0}$ closed to the proof rules ($\mathtt{Subst}_{\Theta^{\mathbf{HPL}_b}}$), where $\Theta^{\mathbf{HPL}_b}$ is defined in Table 3.

The following result is a corollary of Proposition 4.9 and Theorem 6.2.

Theorem 6.3 $\mathcal{E}^{\mathbf{HPL}_b}$ is sound, complete and compact.

Proof. By noting that $\Theta_\Delta(\gamma) \in \mathtt{Sen}^{\mathbf{HPL}_0}(\Delta)$ for all $\Delta \in |\mathtt{Sig}^{\mathbf{HPL}}|$ and $\gamma \in \mathtt{Sen}^{\mathbf{HPL}_b}(\Delta)$. □ We can define now the Birkhoff entailment system for $\mathbf{HPL}_h$.

Definition 6.3 $\mathcal{E}^{\mathbf{HPL}_h} = (\mathtt{Sig}^{\mathbf{HPL}}, \mathtt{Sen}^{\mathbf{HPL}_h}, \vdash^{\mathbf{HPL}_h})$ is the least entailment system closed to the proof rules defined in Table 2, ($\mathtt{Subst}_{\Theta^{\mathbf{HPL}_b}}$), where $\Theta^{\mathbf{HPL}_b}$ is defined in Table 3, and ($\mathtt{Subst}_{\Theta^{\mathbf{HPL}_h}}$), where $\Theta^{\mathbf{HPL}_h}$ is defined in Table 1.

Theorem 6.4 $\mathcal{E}^{\mathbf{HPL}_h}$ is sound, compact and complete.

Proof. By Theorem 6.3, $\mathcal{E}^{\mathbf{HPL}_b}$ is sound and compact. By Theorem 5.6, $\mathcal{E}^{\mathbf{HPL}_h}$ is sound and compact.

By Theorem 6.3, $\mathcal{E}^{\mathbf{HPL}_b}$ is complete and compact. By Proposition 4.6, any set of sentences in $\mathbf{HPL}_b$ is epi-basic, and the basic models are $\mathtt{HSt}^{\mathbf{HPL}}$-reachable. In particular, the basic models are $\mathtt{HSt}^{\mathbf{HPL}_b}$-reachable. By Theorem 5.7, the entailment system $\mathcal{E}^{\mathbf{HPL}_h}$ is complete. □

6.2. Birkhoff entailment system for SHARE

Our abstract completeness result is not applicable to **HFOLSA** without any restriction, as our framework does not allow equality of elements that originate from different states. For example, given a non-rigid constant σ of non-rigid sort and two nominals k_1 and k_2, if $k_1 = k_2$ is deducible then there is no way to infer that σ_{k_1} is equal to σ_{k_2} as $\sigma_{k_1} = \sigma_{k_2}$ is not an equation in our framework. We can either restrict the signatures such that all sorts are rigid, or eliminate nominal equations. In this subsection we consider the first case.

We set the parameters of the completeness theorem for **SHARE** as follows:

- the institution $\mathtt{HI}_b$ is $\mathbf{SHARE}_b$, where $\mathbf{SHARE}_b$ is the restriction of **SHARE** to sentences obtained by applying the operator retrieve to atomic sentences, nominal equations and nominal relations,
- the institution $\mathtt{HI}$ is $\mathbf{SHARE}_h$, the restriction of **SHARE** to Horn clauses over $\mathbf{SHARE}_b$, and
- the functor $\mathtt{HSt} : \mathtt{D} \to \mathbb{CAT}^{op}$ is $\mathtt{HSt}^{\mathbf{SHARE}_b} : \mathtt{D}^{\mathbf{SHARE}_b} \to \mathbb{CAT}^{op}$, the restriction of $\mathtt{HSt}^{\mathbf{HFOLSA}} : \mathtt{D}^{\mathbf{HFOLSA}} \to \mathbb{CAT}^{op}$ (defined in Example 3.8) to $\mathbf{SHARE}_b$.

Let $\mathbf{SHARE}_0$ be the restriction of **SHARE** to atomic sentences, nominal equations and nominal relations. We show that $\mathbf{SHARE}_0$ has the same expressivity power as $\mathbf{SHARE}_b$ by defining a hybrid substitution system for $\mathbf{SHARE}_b$ that maps each sentence of $\mathbf{SHARE}_b$ to a sentence of $\mathbf{SHARE}_0$. The hybrid substitution system $\Theta^{\mathbf{SHARE}_b}$ for $\mathbf{SHARE}_b$ is defined in Table 5.

Table 5. Hybrid substitution system for $\mathbf{SHARE}_b$

$\Theta^{\mathbf{SHARE}_b}_{\Delta}(\rho)$	$= \rho$ for all $\rho \in \mathtt{Sen}^{\mathbf{SHARE}_0}(\Delta)$,
$\Theta^{\mathbf{SHARE}_b}_{\Delta}(@_k \gamma)$	$= \gamma$ for all $\gamma \in \mathtt{Sen}^{\mathbf{SHARE}_b}(\Delta)$,

Table 6. Calculus for $\mathbf{SHARE}_b$

$(Reflexivity_n)$	$\dfrac{}{\Gamma \vdash k_1 = k_1}$
$(Transitivity_n)$	$\dfrac{\Gamma \vdash k_1 = k_2 \text{ and } \Gamma \vdash k_2 = k_3}{\Gamma \vdash k_1 = k_3}$
$(Symmetry_n)$	$\dfrac{\Gamma \vdash k_1 = k_2}{\Gamma \vdash k_2 = k_1}$
$(Congruence_n)$	$\dfrac{\Gamma \vdash \lambda(k_1, \ldots, k_n) \text{ and } \Gamma \vdash k_i = k'_i \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \lambda(k'_1, \ldots, k'_n)}$
$(Reflexivity)$	$\dfrac{}{\Gamma \vdash t = t'}$
$(Symmetry)$	$\dfrac{\Gamma \vdash t = t'}{\Gamma \vdash t' = t}$
$(Transitivity)$	$\dfrac{\Gamma \vdash t = t' \text{ and } \Gamma \vdash t' = t''}{\Gamma \vdash t = t''}$
$(Share)$	$\dfrac{\Gamma \vdash \tau =^{k} \tau'}{\Gamma \vdash \tau =^{k'} \tau'}$
$(Congruence)$	$\dfrac{\Gamma \vdash k = k' \text{ and } \Gamma \vdash t_i = t'_i \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \sigma_k(t_1, \ldots, t_n) = \sigma_{k'}(t'_1, \ldots, t'_n)}$, where $\sigma \in (F - F^r)$
	$\dfrac{\Gamma \vdash \tau_i = \tau'_i \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \varsigma(\tau_1, \ldots, \tau_n) = \varsigma(\tau'_1, \ldots, \tau'_n)}$, where $\varsigma \in F^r$
$(Congruence_p)$	$\dfrac{\Gamma \vdash \pi_k(t_1, \ldots, t_n),\ \Gamma \vdash k = k' \text{ and } \Gamma \vdash t_i = t'_i \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \pi_{k'}(t'_1, \ldots, t'_n)}$, where $\pi \in (P - P^r)$
	$\dfrac{\Gamma \vdash \varpi(\tau_1, \ldots, \tau_n) \text{ and } \Gamma \vdash \tau_i = \tau'_i \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \varpi(\tau'_1, \ldots, \tau_n)}$, where $\varpi \in P^r$

λ is a modality, k, k' and k_i are nominals, t, t_i, t', t'_i and t'' are any terms, and τ, τ_i, τ', τ'_i are rigid terms.

Proposition 6.5 $\Theta^{\mathbf{SHARE}_b}$ is a hybrid substitution system.

Proof. Straightforward. □

We define an entailment system for $\mathbf{SHARE}_0$, and we prove its soundness, compactness and completeness.

Definition 6.4 $\mathcal{E}^{\mathbf{SHARE}_0} = (\mathtt{Sig}^{\mathbf{SHARE}}, \mathtt{Sen}^{\mathbf{SHARE}_0}, \vdash^{\mathbf{SHARE}_0})$ is is the least entailment system of $\mathbf{SHARE}_0$ closed to the proof rules defined in Table 6.

Theorem 6.6 $\mathcal{E}^{\mathbf{SHARE}_0}$ is sound, complete and compact.

Proof. The proof of soundness is straightforward.

For compactness, let $\mathcal{E}^c = (\mathtt{Sig}^{\mathbf{SHARE}}, \mathtt{Sen}^{\mathbf{SHARE}_0}, \vdash^c)$ be the compact entailment subsystem of $\mathcal{E}^{\mathbf{SHARE}_0}$. We prove that $\mathcal{E}^c$ is closed to the proof rules defined in Table 6, which implies $\mathcal{E}^{\mathbf{SHARE}_0} = \mathcal{E}^c$. For $(Reflexivity)$,

$\Gamma \vdash^c t = t$ as for any finite $\Gamma_f \subseteq \Gamma$ we have $\Gamma_f \vdash^c t = t$. For (*Symmetry*), assume that $\Gamma \vdash^c t = t'$; there exists $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash t = t'$, which implies $\Gamma_f \vdash t' = t$, and we get $\Gamma \vdash^c t' = t$. The remaining cases can be proved similarly.

For completeness, we assume that $\Gamma \models_\Delta \rho$, where $\Delta = (\mathtt{Nom}, \Lambda, \Sigma) \in |\, \mathtt{Sig}^{\mathbf{SHARE}} \,|$ and $\Sigma = (S, F^r, P^r) \subseteq (S, F, P) \in |\, \mathtt{Sig}^{\mathbf{FOLS}} \,|$. We construct a Δ-model (W^Γ, M^Γ) such that (a) $(W^\Gamma, M^\Gamma) \models^{\mathbf{SHARE}} \Gamma$ and (b) $(W^\Gamma, M^\Gamma) \models^{\mathbf{SHARE}} \rho$ iff $\Gamma \vdash \rho$. Let $\equiv^{\mathtt{Nom}} = \{(k_1, k_2) \mid \Gamma \vdash k_1 = k_2 \text{ and } k_i \in \mathtt{Nom}\}$ be an equivalence on Nom. Let $\equiv^k = \{(t_1, t_2) \mid \Gamma \vdash t_1 = t_2 \text{ and } t_i \in T_k^\Delta\}$ be a congruence on the (S, F, P)-model T_k^Δ, where $k \in \mathtt{Nom}$ is a nominal. The Δ-model (W^Γ, M^Γ) is defined as follows:

(1) $|\, W^\Gamma \,| = \mathtt{Nom}/_{\equiv^{\mathtt{Nom}}}$ and $W^\Gamma_\lambda = \{(\widehat{k_1}, \dots, \widehat{k_n}) \mid \Gamma \vdash \lambda(k_1, \dots, k_n)\}$ for all $n \in \mathbb{N}at$ and $\lambda \in \Lambda_n$, where $\widehat{k_i}$ is the equivalence class of k_i, and
(2) for each $k \in \mathtt{Nom}$, $M^\Gamma_{\widehat{k}}$ consists of the (S, F)-algebra $T_k^\Delta/_{\equiv^k}$, and interprets
   (a) each $\varpi \in P^r$ as $(M^\Gamma_{\widehat{k}})_\varpi = \{(\widehat{\tau_1}, \dots, \widehat{\tau_n}) \mid \Gamma \vdash \varpi(\tau_1, \dots, \tau_n)\}$, and
   (b) each $\pi \in (P - P^r)$ as $(M^\Gamma_{\widehat{k}})_\pi = \{(\widehat{t_1}, \dots, \widehat{t_n}) \mid \Gamma \vdash \pi_k(t_1, \dots, t_n)\}$,

   where $\widehat{\tau_i}$ and $\widehat{t_i}$ are the equivalence classes of τ_i and t_i, respectively.

Since $\Gamma \models \rho$ and $(W^\Gamma, M^\Gamma) \models^{\mathbf{SHARE}} \Gamma$, we have $(W^\Gamma, M^\Gamma)^{\mathbf{SHARE}} \models \rho$, which implies $\Gamma \vdash \rho$. See Section A for full proof of completeness. □

The following result is a corollary of Proposition 4.9 and Theorem 6.6.

Theorem 6.7 $\mathcal{E}^{\mathbf{SHARE}_b}$ is sound, compact and complete.

Proof. By noting that for all $\Delta \in |\, \mathtt{Sig}^{\mathbf{SHARE}} \,|$ and $\gamma \in \mathtt{Sen}^{\mathbf{SHARE}_b}(\Delta)$ we have $\Theta_\Delta^{\mathbf{SHARE}_b}(\gamma) \in \mathtt{Sen}^{\mathbf{SHARE}_0}(\Delta)$. □

We can define now the Birkhoff entailment system for $\mathbf{SHARE}_h$.

Definition 6.5. $\mathcal{E}^{\mathbf{SHARE}_h} = (\mathtt{Sig}^{\mathbf{SHARE}}, \mathtt{Sen}^{\mathbf{SHARE}_h}, \vdash^{\mathbf{SHARE}_h})$ is the least entailment system of $\mathbf{SHARE}_h$ closed to the proof rules defined in Table 2 and Table 6, $(\mathtt{Subst}_{\Theta^{\mathbf{SHARE}_b}})$, where $\Theta^{\mathbf{SHARE}_b}$ is defined in Table 5, and $(\mathtt{Subst}_{\Theta^{\mathbf{SHARE}_h}})$, where $\Theta^{\mathbf{SHARE}_h}$ is defined in Table 1.

Theorem 6.8 $\mathcal{E}^{\mathbf{SHARE}_h}$ is sound, compact and complete.

Proof. By Theorem 6.7, $\mathcal{E}^{\mathbf{SHARE}_b}$ is sound and compact. By Theorem 5.6, $\mathcal{E}^{\mathbf{SHARE}_h}$ is sound and compact.

By Theorem 6.7, $\mathcal{E}^{\mathbf{SHARE}_b}$ is complete and compact. By Proposition 4.5, any set of sentences in $\mathbf{HFOLSA}_b$ is epi-basic and the basic models are $\mathtt{HSt}^{\mathbf{HFOLSA}}$-reachable. In particular, any set of sentences in $\mathbf{SHARE}_b$ is epi-basic and the basic models are $\mathtt{HSt}^{\mathbf{SHARE}_b}$-reachable. By Theorem 5.7, $\mathcal{E}^{\mathbf{SHARE}_h}$ is complete. □

6.3. Birkhoff entailment system for HFOLSA

We set the parameters of the completeness theorem for **HFOLSA** as follows:

- the institution $\mathtt{HI}_b$ is $\mathbf{HFOLSA}'_b$, where $\mathbf{HFOLSA}'_b$ is the restriction of **HFOLSA** to sentences obtained by applying the operator retrieve to atomic sentences and nominal relations,
- the institution HI is $\mathbf{HFOLSA}'_h$, the restriction of **HFOLSA** to Horn clauses over $\mathbf{HFOLSA}'_b$, and
- the functor $\mathtt{HSt} : \mathtt{D} \to \mathbb{CAT}^{op}$ is $\mathtt{HSt}^{\mathbf{HFOLSA}'_b} : \mathtt{D}^{\mathbf{HFOLSA}} \to \mathbb{CAT}^{op}$, the restriction of $\mathtt{HSt}^{\mathbf{HFOLSA}} : \mathtt{D}^{\mathbf{HFOLSA}} \to \mathbb{CAT}^{op}$ (defined in Example 3.8) to $\mathbf{HFOLSA}'_b$.

Let $\mathbf{HFOLSA}'_0$ be the restriction of **HFOLSA** to atomic sentences and nominal relations. We show that $\mathbf{HFOLSA}'_0$ has the same expressivity power as $\mathbf{HFOLSA}'_b$ by defining a hybrid substitution system for $\mathbf{HFOLSA}'_b$ that maps each sentence of $\mathbf{HFOLSA}'_b$ to a sentence of $\mathbf{HFOLSA}'_0$. The hybrid substitution system $\Theta^{\mathbf{HFOLSA}'_b}$ for $\mathbf{HFOLSA}'_b$ is defined in Table 7.

Proposition 6.9 $\Theta^{\mathbf{HFOLSA}'_b}$ is a hybrid substitution system.

Table 7. Hybrid substitution system for $\mathbf{HFOLSA}'_b$

$\Theta_\Delta^{\mathbf{HFOLSA}'_b}(\rho)$	$= \rho$ for all $\rho \in \mathtt{Sen}^{\mathbf{HFOLSA}'_0}(\Delta)$,
$\Theta_\Delta^{\mathbf{HFOLSA}'_b}(@_k\gamma)$	$= \gamma$ for all $\gamma \in \mathtt{Sen}^{\mathbf{HFOLSA}'_b}(\Delta)$,

Table 8. Calculus for $\mathbf{HFOLSA}'_b$

(*Reflexivity*)	$\dfrac{}{\Gamma \vdash t = t'}$
(*Symmetry*)	$\dfrac{\Gamma \vdash t =^k t'}{\Gamma \vdash t' = t}$
(*Transitivity*)	$\dfrac{\Gamma \vdash t = t' \text{ and } \Gamma \vdash t' = t''}{\Gamma \vdash t = t''}$
(*Share*)	$\dfrac{\Gamma \vdash \tau =^k_{\mathtt{sr}} \tau'}{\Gamma \vdash \tau =^{k'}_{\mathtt{sr}} \tau'}$, where $\mathtt{sr}$ is a rigid sort
(*Congruence*)	$\dfrac{\Gamma \vdash t_i = t'_i \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \sigma_k(t_1, \ldots, t_n) = \sigma_k(t'_1, \ldots, t'_n)}$, where $\sigma \in (F - F^r)$
	$\dfrac{\Gamma \vdash \tau_i = \tau'_i \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \varsigma(\tau_1, \ldots, \tau_n) = \varsigma(\tau'_1, \ldots, \tau'_n)}$, where $\varsigma \in F^r$
(*Congruence$_p$*)	$\dfrac{\Gamma \vdash \pi_k(t_1, \ldots, t_n) \text{ and } \Gamma \vdash t_i = t'_i \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \pi_k(t'_1, \ldots, t'_n)}$, where $\pi \in (P - P^r)$
	$\dfrac{\Gamma \vdash \varpi(\tau_1, \ldots, \tau_n) \text{ and } \Gamma \vdash \tau_i = \tau'_i \text{ for all } i \in \{1, \ldots, n\}}{\Gamma \vdash \varpi(\tau'_1, \ldots, \tau_n)}$, where $\varpi \in P^r$

k and k' are nominals, t, t_i, t', t'_i and t'' are terms, and τ, τ_i, τ' and τ'_i are rigid terms.

Proof. Straightforward. □

We define an entailment system for $\mathbf{HFOLSA}'_0$, and we prove its soundness, compactness and completeness.

Definition 6.6 $\mathcal{E}^{\mathbf{HFOLSA}'_0} = (\mathtt{Sig}^{\mathbf{HFOLSA}}, \mathtt{Sen}^{\mathbf{HFOLSA}'_0}, \vdash^{\mathbf{HFOLSA}'_0})$ is is the least entailment system of $\mathbf{HFOLSA}'_0$ closed to the proof rules in Table 8.

Theorem 6.10 $\mathcal{E}^{\mathbf{HFOLSA}'_0}$ is sound, complete and compact.

Proof. The proof is similar to the proof of Theorem 6.6. □

Definition 6.7 $\mathcal{E}^{\mathbf{HFOLSA}'_b} = (\mathtt{Sig}^{\mathbf{HFOLSA}}, \mathtt{Sen}^{\mathbf{HFOLSA}'_b}, \vdash^{\mathbf{HFOLSA}'_b})$ is the least entailment system over $\mathcal{E}^{\mathbf{HFOLSA}'_0}$ closed to $(\mathtt{Subst}_{\Theta^{\mathbf{HFOLSA}'_b}})$, where $\Theta^{\mathbf{HFOLSA}'_b}$ is defined in Table 7.

The following result is a corollary of Proposition 4.9 and Theorem 6.10.

Theorem 6.11 $\mathcal{E}^{\mathbf{HFOLSA}'_b}$ is sound, complete and compact.

Proof. By noting that $\Theta_\Delta^{\mathbf{HFOLSA}'_b}(\gamma) \in \mathtt{Sen}^{\mathbf{HFOLSA}'_0}(\Delta)$ for all $\Delta \in |\mathtt{Sig}^{\mathbf{HFOLSA}}|$ and $\gamma \in \mathtt{Sen}^{\mathbf{HFOLSA}'_b}(\Delta)$. □

We can define now the Birkhoff entailment system for $\mathbf{HFOLSA}'_h$.

Definition 6.8 $\mathcal{E}^{\mathbf{HFOLSA}'_h} = (\mathtt{Sig}^{\mathbf{HFOLSA}}, \mathtt{Sen}^{\mathbf{HFOLSA}'_h}, \vdash^{\mathbf{HFOLSA}'_h})$ is the least entailment system of $\mathbf{HFOLSA}'_h$ closed to the proof rules defined in Table 2 and Table 8, $(\mathtt{Subst}_{\Theta^{\mathbf{HFOLSA}'_b}})$, where $\Theta^{\mathbf{HFOLSA}'_b}$ is defined in Table 7, and $(\mathtt{Subst}_{\Theta^{\mathbf{HFOLSA}'_h}})$, where $\Theta^{\mathbf{HFOLSA}'_h}$ is defined in Table 1.

Theorem 6.12 $\mathcal{E}^{\mathbf{HFOLSA}'_h}$ is sound, compact and complete.

Proof. By Theorem 6.11, $\mathcal{E}^{\mathbf{HFOLSA}'_b}$ is sound and compact. By Theorem 5.6, $\mathcal{E}^{\mathbf{HFOLSA}'_h}$ is sound and compact.

By Theorem 6.11, $\mathcal{E}^{\mathbf{HFOLSA}'_b}$ is complete and compact. By Proposition 4.5, any set of sentences in $\mathbf{HFOLSA}'_b$ is epi-basic and the basic models are $\mathtt{HSt}^{\mathbf{HFOLSA}}$-reachable. In particular, the basic models are $\mathtt{HSt}^{\mathbf{HFOLSA}'_b}$-reachable. By Theorem 5.7, $\mathcal{E}^{\mathbf{HFOLSA}'_h}$ is complete. □

7. Conclusions

We have proved a Birkhoff completeness theorem for hybrid logics in the general setting of the institution theory. We have instantiated the result to hybrid propositional logic (**HPL**) and hybrid first-order logic with user-defined sharing and annotation (**HFOLSA**). The former result comes in two variants. For the first case, the signatures are restricted such that all sorts are rigid, which implies that models share the same carrier sets across the states. The second case excludes nominal equations from the syntax. The abstract result is applicable also to hybrid first-order logic (**HFOL**), similarly to the case of **HPL**, and hybrid preorder algebra with user-defined sharing and annotation (**HPOASA**), similarly to the case of **HFOLSA**. Many other instances are expected to the hybridization of unconventional logics used in computer science such as order-sorted algebra [GM92], partial algebra [ABK+02] or higher-order logic [Hen50] with intensional Henkin semantics.

The definition of hybrid institution provides a more general framework than the hybridization process [MMDB11, Dia16b, DM16], since the hybrid framework is assumed and not constructed. As a result, hybrid institutions with annotation syntax, such as **HFOLSA**, are not instances of the hybridization but they fall into our framework. hybridization is a method of hierarchical logic combination but it is different from fibring [CCG+08] (which is the major general theory of logic combination in the mathematical logic literature); a discussion comparing them is outside the scope of our paper.

The temporalization method of Finger and Gabbay [FG92] is a simplified version of hybridization performed with different sentence operators, where signature morphisms and homomorphisms are not considered. The work reported in [FG92] proves that soundness and completeness are preserved from the base logic to the temporalized version. A similar result is obtained for the abstract framework provided by the hybridization process [NMMB16]. Both papers, [FG92, NMMB16], do not consider quantified formulas for the logic construction method and are not concerned with extracting a fragment with good computational properties.

The operator retrieve is essential for proving our completeness property, which shows that our results may not be applicable to pure modal logics. By defining an abstract notion of Horn clause, the present work shows how to extract a fragment with good computational properties from a hybrid logic. The Birkhoff entailment system opens the possibility to perform theorem proving based on term rewriting [Gog94] in hybrid logics. In the future we are planning to develop o methodology for proving properties of reconfigurable software systems that are described with hybrid logics. An interesting research topic is borrowing interpolation from first-order logics to hybrid logics in the style of [Dia12] or [GĬ3].

A. Exiled proofs

Proof of Proposition 5.2 We have six cases to consider. The first case $\gamma \in \mathtt{Sen}^{\mathtt{HI}_p}(\Delta)$ is trivial. The third case and the fifth case are instances of the second case and the forth case, respectively. We will focus on the non-trivial cases:

(2) Let $M \in\mid \mathtt{Mod}(\Delta) \mid$, $w_1 \in\mid \mathtt{K}_\Delta(M) \mid$, and $W = \mathtt{K}_\Delta(M)$. Assuming $M \models^{w_1} [\lambda](\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$ we prove $M \models^{w_1} (\forall\chi)(\forall\chi[j])(\downarrow j_1)(\lambda(j_1, j) \wedge (\bigwedge_{h\in H} \chi[j_1](@_j h)) \Rightarrow \chi[j_1](@_j C))$.

Let M^1 be a $(\chi;\ \chi[j];\ \chi[j_1])$-expansion of M such that $M^1 \models^{w_1} \lambda(j_1, j)$, $M^1 \models^{w_1} \bigcup_{h\in H} \chi[j_1](@_j h)$ and $W^1_{j_1} = w_1$, where $W^1 = \mathtt{K}_{\Delta'[j][j_1]}(M^1)$. Let $M' = M^1 \upharpoonright_{\chi[j_1]}$, $W' = W^1 \upharpoonright_{\mathtt{F}(\chi[j_1])} = \mathtt{K}_{\Delta'[j]}(M')$ and $w = W^1_j = W'_j$. Since $M^1 \models^{w_1} \lambda(j_1, j)$, we have $(W^1_{j_1}, W^1_j) \in W^1_\lambda$, i.e. $(w_1, w) \in W^1_\lambda$. Since $W^1_\lambda = W_\lambda$, we have $(w_1, w) \in W_\lambda$. $M \models^{w_1} [\lambda](\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$ and $(w_1, w) \in W_\lambda$ implies $M \models^{w} (\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. By the local satisfaction condition, $M^1 \models^{w_1} \bigcup_{h\in H} \chi[j_1](@_j h)$ is equivalent to $M' \models^{w_1} \bigcup_{h\in H} @_j h$; it follows that $M' \models^{W'_j} H$, i.e. $M' \models^{w} H$. Since M' is a $(\chi;\ \chi[j])$-expansion of M such that $W'_j = w$ and $M' \models^{w} H$, we have $M' \models^{w} C$, i.e. $M' \models^{W'_j} C$; it follows that $M' \models^{w_1} @_j C$, and by the local satisfaction condition, we get $M^1 \models^{w_1} \chi[j_1](@_j C)$.
Assuming $M \models^{w_1} (\forall\chi)(\forall\chi[j])(\downarrow j_1)(\lambda(j_1, j) \wedge (\bigwedge_{h\in H} \chi[j_1](@_j h) \Rightarrow \chi[j_1](@_j C))$ we prove $M \models^{w_1} [\lambda](\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. Let $w \in\mid W \mid$ such that $(w_1, w) \in W_\lambda$ and consider a $(\chi;\ \chi[j])$-expansion M' of M such that $M' \models^{w} H$ and $W'_j = w$, where $W' = \mathtt{K}_{\Delta'[j]}(M')$. It follows that $M' \models^{w_1} \bigcup_{h\in H} @_j h$, and by the local satisfaction condition, we have $M'^{(j_1,w_1)} \models^{w_1} \bigcup_{h\in H} \chi[j_1](@_j h)$. Since $M \models^{w_1} (\forall\chi)(\forall\chi[j])(\downarrow j_1)(\lambda(j_1, j) \wedge \bigwedge_{h\in H} \chi[j_1](@_j h) \Rightarrow \chi[j_1](@_j C))$ and $M'^{(j_1,w_1)}$ is a $(\chi;\ \chi[j];\ \chi[j_1])$-expansion of M such that $M'^{(j_1,w_1)} \models^{w_1} \lambda(j_1, j)$, $M'^{(j_1,w_1)} \models^{w_1} \bigcup_{h\in H} \chi[j_1](@_j h)$ and $W'^{(j_1,w_1)}_{j_1} = w_1$, we have $M'^{(j_1,w_1)} \models^{w_1} \chi[j_1](@_j C)$. By the local satisfaction condition, we obtain $M' \models^{w_1} @_j C$, which implies $M' \models^{W'_j} C$, i.e. $M' \models^{w} C$.

(4) For all Δ-models M and states $w \in\mid W \mid$, where $W = \mathtt{K}_\Delta(M)$, we have:
$M \models^{w} @_k(\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$ iff $M \models^{W_k} (\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$ iff
for all χ-expansions M' of M we have $M' \models^{W_k} (\downarrow j)\bigwedge H \Rightarrow C$ iff
for all χ-expansions M' of M, $M' \models^{W'_{k'}} (\downarrow j)\bigwedge H \Rightarrow C$, where $W' = \mathtt{K}_{\Delta'}(M')$ and $k' = \mathtt{F}(\chi)(k)$ iff
for all χ-expansions M' of M, $M'^{(j,W'_{k'})} \models^{W'_{k'}} \bigwedge H \Rightarrow C$, where $W' = \mathtt{K}_{\Delta'}(M')$ and $k' = \mathtt{F}(\chi)(k)$.
By Lemma 5.1, the last statement is equivalent to the followings:
for all χ-expansions M' of M, $M' \upharpoonright_\psi \models^{W'_{k'}} \bigwedge H \Rightarrow C$, where $W' = \mathtt{K}_{\Delta'}(M')$, $k' = \mathtt{F}(\chi)(k)$ and $\psi^{\mathbf{REL}} : \mathtt{F}(\Delta')[j] \to \mathtt{F}(\Delta')$ preserves $\mathtt{F}(\Delta')$ and $\psi^{\mathbf{REL}}(j) = k'$ iff
for all χ-expansions M' of M, $M' \models^{W'_{k'}} \bigwedge\psi(H) \Rightarrow \psi(C)$, where $W' = \mathtt{K}_{\Delta'}(M')$, $k' = \mathtt{F}(\chi)(k)$ and $\psi^{\mathbf{REL}} : \mathtt{F}(\Delta')[j] \to \mathtt{F}(\Delta')$ preserves $\mathtt{F}(\Delta')$ and $\psi^{\mathbf{REL}}(j) = k'$ iff
for all χ-expansions M' of M we have $M' \models^{w} \bigwedge_{h\in H} @_{k'}\psi(h) \Rightarrow @_{k'}\psi(C)$, where $k' = \mathtt{F}(\chi)(k)$ and $\psi^{\mathbf{REL}} : \mathtt{F}(\Delta')[j] \to \mathtt{F}(\Delta')$ preserves $\mathtt{F}(\Delta')$ and $\psi^{\mathbf{REL}}(j) = k'$ iff
$M \models^{w} (\forall\chi)\bigwedge_{h\in H} @_{k'}\psi(h) \Rightarrow @_{k'}\psi(C)$, where $k' = \mathtt{F}(\chi)(k)$ and $\psi^{\mathbf{REL}} : \mathtt{F}(\Delta')[j] \to \mathtt{F}(\Delta')$ preserves $\mathtt{F}(\Delta')$ and $\psi^{\mathbf{REL}}(j) = k'$.

(6) For all Δ-models M and states $w \in\mid \mathtt{K}_\Delta(M) \mid$ we have:
$M \models^{w} @_k[\lambda]\rho$ iff $M \models^{w} @_k\Theta([\lambda]\rho)$ iff $M \models^{w} \Theta(@_k\ \Theta([\lambda]\rho))$. □

Proof of Theorem 5.4

(1) We show that the proof rules in Table 2 are sound.
$(Generalisation)$: Assuming $\chi[j](\Gamma) \models \bigwedge_{h\in H}(@_j h) \Rightarrow (@_j C)$ we prove $\Gamma \models (\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. Let M be a Δ-model such that $M \models^{\mathtt{HI}} \Gamma$, and let $w \in\mid W \mid$ be an arbitrary state, where $W = \mathtt{K}_\Delta(M)$. Consider a $(\chi;\ \chi[j])$-expansion M' of M such that $W'_j = w$, where $W' = \mathtt{K}_{\Delta'}(M')$. By the satisfaction condition, $M' \models^{\mathtt{HI}} (\chi;\ \chi[j])(\Gamma)$. It follows that $M' \models^{\mathtt{HI}} \bigwedge_{h\in H}(@_j h) \Rightarrow (@_j C)$, which is equivalent to $M' \models^{W'_j} \bigwedge H \Rightarrow C$. Hence, $M' \models^{w} \bigwedge H \Rightarrow C$.
Assuming $\Gamma \models (\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$ we prove $\chi[j](\Gamma) \models \bigwedge_{h\in H}(@_j h) \Rightarrow (@_j C)$. Let M' be a $\Delta'[j]$-model such that $M' \models^{\mathtt{HI}} \chi[j](\Gamma)$. By the satisfaction condition, $M' \upharpoonright_{(\chi;\ \chi[j])} \models^{\mathtt{HI}} \Gamma$, which implies $M' \upharpoonright_{(\chi;\ \chi[j])} \models^{\mathtt{HI}} (\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. We have $M' \upharpoonright_{(\chi;\ \chi[j])} \models^{W'_j} (\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$, where $W' = \mathtt{K}_{\Delta'}(M')$. It follows that $M' \models^{W'_j} \bigwedge H \Rightarrow C$, which is equivalent to $M' \models^{\mathtt{HI}} \bigwedge_{h\in H}(@_j h) \Rightarrow (@_j C)$.
$(Quantification)$: This case is similar to the one above.
$(Substitutivity)$: We prove $(\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C) \models \bigwedge_{h\in H}\theta(@_j h) \Rightarrow \theta(@_j C)$. Let M be a Δ-model such that $M \models^{\mathtt{HI}} (\forall\chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. Let $w \in\mid W \mid$ be a state such that $M \models^{w} \theta(@_j h)$ for all $h \in H$, where $W = \mathtt{K}_\Delta(M)$. We show that $M \models^{w} \theta(@_j C)$. We denote $M \upharpoonright_\theta$ and $\mathtt{K}_{\Delta'[j]}(M' \upharpoonright_\theta)$ by M' and W', respectively. Since M' is a $(\chi;\ \chi[j])$-expansion of M, we have $M' \models^{W'_j} \bigwedge H \Rightarrow C$. By the satisfaction condition for hybrid

substitutions, $M' \models^w @_j h$ for all $h \in H$, which is equivalent to $M' \models^{W'_j} H$. It follows that $M' \models^{W'_j} C$, which is equivalent to $M' \models^w @_j C$. By the satisfaction condition, $M \models^w \theta(@_j C)$.
The other subcase can be proved similarly.
(*Implication*) : Assuming $\Gamma \models \bigwedge_{e\in E}(@_k e) \Rightarrow (@_k \rho)$ we prove $\Gamma \cup (\bigcup_{e\in E} @_k e) \models @_k \rho$. Let M be a Δ-model such that $M \models^{\mathtt{HI}} \Gamma \cup (\bigcup_{e\in E} @_k e)$. Since $M \models^{\mathtt{HI}} \Gamma$, we have $M \models^{\mathtt{HI}} \bigwedge_{e\in E}(@_k e) \Rightarrow (@_k \rho)$, which is equivalent to $M \models^{W_k} \bigwedge E \Rightarrow \rho$, where $W = \mathtt{K}_\Delta(M)$. Since $M \models^{\mathtt{HI}} \bigcup_{e\in E} @_k e$, we get $M \models^{W_k} E$. It follows that $M \models^{W_k} \rho$. Hence, $M \models^{\mathtt{HI}} @_k \rho$.
Assuming $\Gamma \cup (\bigcup_{e\in E} @_k e) \models @_k \rho$ we prove $\Gamma \models \bigwedge_{e\in E}(@_k e) \Rightarrow (@_k \rho)$. Let M be a Δ-model such that $M \models^{\mathtt{HI}} \Gamma$. Let $w \in |W|$ such that $M \models^w \bigcup_{e\in E} @_k e$, where $W = \mathtt{K}_\Delta(M)$. It follows that $M \models^{\mathtt{HI}} \bigcup_{e\in E} @_k e$, and we have $M \models^{\mathtt{HI}} \Gamma \cup (\bigcup_{e\in E} @_k e)$. We obtain $(W, M) \models^{\mathtt{HI}} @_k \rho$. Hence, $M \models^w @_k \rho$.
(*Retrieve*) : Straightforward.

(2) Let $\mathcal{E}^c = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_p}, \vdash^c)$ be the compact entailment subsystem of $\mathcal{E}^{\mathtt{HI}_p}$. We prove that $\mathcal{E}^c$ is closed to the proof rules defined in Tabel 2.
(*Generalisation*) : Assume that $\chi[j](\Gamma) \vdash^c \bigwedge_{h\in H}(@_j h) \Rightarrow (@_j C)$; by the definition of $\mathcal{E}^c$, there exists $\Gamma_f \subseteq \Gamma$ finite such that $\chi[j](\Gamma_f) \vdash \bigwedge_{h\in H}(@_j h) \Rightarrow (@_j C)$; by (*Generalisation*), $\Gamma_f \vdash (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$, which implies $\Gamma \vdash^c (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. Now, if $\Gamma \vdash^c (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$ then by the definition of $\mathcal{E}^c$, there exists $\Gamma_f \subseteq \Gamma$ finite such that $\Gamma_f \vdash (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$; by (*Generalisation*), $\chi[j](\Gamma_f) \vdash \bigwedge_{h\in H}(@_j h) \Rightarrow (@_j C)$, which implies $\chi[j](\Gamma) \vdash^c \bigwedge_{h\in H}(@_j h) \Rightarrow (@_j C)$.
(*Quantification*): Can be proved similarly as above.
(*Substitutivity*) : By the definition of $\mathcal{E}^c$, we have $(\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C) \vdash^c \bigwedge_{h\in H} \theta(@_j h) \Rightarrow \theta(@_j C)$.
(*Implication*) : Assume that $\Gamma \vdash^c \bigwedge_{e\in E}(@_k e) \Rightarrow (@_k \rho)$; by the definition of $\mathcal{E}^c$, there exists $\Gamma_f \subseteq \Gamma$ finite such that $\Gamma_f \vdash \bigwedge_{e\in E}(@_k e) \Rightarrow (@_k \rho)$; by (*Implication*), $\Gamma_f \cup (\bigcup_{e\in E} @_k e) \vdash @_k \rho$, which implies $\Gamma \cup (\bigcup_{e\in E} @_k e) \vdash^c @_k \rho$. Now if, $\Gamma \cup (\bigcup_{e\in E} @_k e) \vdash^c @_k \rho$ then by the definition of $\mathcal{E}^c$, there exists $\Gamma_f \subseteq \Gamma$ finite such that $\Gamma_f \cup (\bigcup_{e\in E} @_k e) \vdash @_k \rho$; by (*Implication*), $\Gamma_f \vdash \bigwedge_{e\in E}(@_k e) \Rightarrow (@_k \rho)$, which implies $\Gamma \vdash^c \bigwedge_{e\in E}(@_k e) \Rightarrow (@_k \rho)$.
(*Retrive*) : Straightforward. □

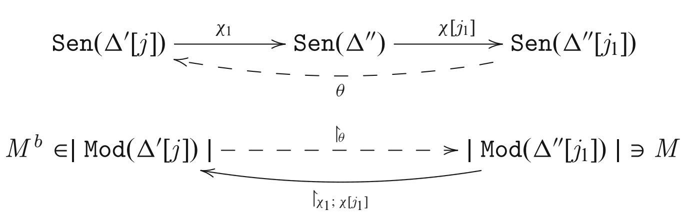
Proof of Theorem 5.5 We assume that $\Gamma \models (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$, where $\Delta \xrightarrow{\chi} \Delta' \in \mathtt{Q}^{\mathtt{HI}}$, j is a nominal variable, $H \subseteq \mathtt{Sen}^{\mathtt{HI}_b}(\Delta'[j])$, $C \in \mathtt{Sen}^{\mathtt{HI}_b}(\Delta'[j])$ and $\Gamma \subseteq \mathtt{Sen}^{\mathtt{HI}_p}(\Delta)$. Suppose towards a contradiction that $\Gamma \not\vdash (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$.

We define $\Gamma_b = \{\rho \in \mathtt{Sen}^{\mathtt{HI}_b}(\Delta'[j]) \mid (\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h\in H} @_j h) \vdash \rho\}$. We show that $\Gamma_b \not\vdash^{\mathtt{HI}_b} @_j C$. If $\Gamma_b \vdash^{\mathtt{HI}_b} @_j C$ then since $\mathcal{E}^{\mathtt{HI}_b}$ is compact, there exists $\Gamma_f \subseteq \Gamma_b$ finite such that $\Gamma_f \vdash^{\mathtt{HI}_b} @_j C$; we have $(\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h\in H} @_j h) \vdash \rho$ for all $\rho \in \Gamma_f$, and since Γ_f is finite, $(\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h\in H} @_j h) \vdash \Gamma_f$; it follows that $(\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h\in H} @_j h) \vdash @_j C$; by (*Implication*), $(\chi;\ \chi[j])(\Gamma) \vdash \bigwedge_{h\in H}(@_j h) \Rightarrow (@_j C)$; by (*Generalisation*), $\Gamma \vdash (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$, which is a contradiction with our assumption.

Let M^b be a basic model of Γ_b that is reachable. We denote $\mathtt{K}_{\Delta'[j]}(M^b)$ by W^b. Since $\Gamma_b \not\vdash^{\mathtt{HI}_b} @_j C$, by completeness of $\mathcal{E}^{\mathtt{HI}_b}$, we have $\Gamma_b \not\models @_j C$, and by Lemma 4.7, we obtain $M^b \not\models^{\mathtt{HI}} @_j C$. We prove that $M^b \models^{\mathtt{HI}} (\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h\in H} @_j h)$:

(a) For each $h \in H$, we have $@_j h \in \Gamma_b$, which implies $\Gamma_b \models @_j h$, and by Lemma 4.7, we get $M^b \models^{\mathtt{HI}} @_j h$.

(b) Let $(\forall \chi_1)(\downarrow j_1)(\bigwedge H_1 \Rightarrow C_1) \in (\chi;\ \chi[j])(\Gamma)$, where $\Delta'[j] \xrightarrow{\chi_1} \Delta'' \in \mathtt{Q}^{\mathtt{HI}}$, j_1 is a variable for $\mathtt{F}(\Delta'')$, $H_1 \subseteq \mathtt{Sen}^{\mathtt{HI}_b}(\Delta''[j_1])$ and $C_1 \in \mathtt{Sen}^{\mathtt{HI}_b}(\Delta''[j_1])$. Let M be a $(\chi_1;\ \chi[j_1])$-expansion of M^b such that $M \models^{W_{j_1}} H_1$, where $W = \mathtt{K}_{\Delta''[j_1]}(M)$. We show that $M \models^{W_{j_1}} C_1$. Since M^b is reachable, there exists a substitution $(\chi_1;\ \chi[j_1]) \xrightarrow{\theta} 1_{\Delta'[j]} \in \mathtt{HSt}(\Delta'[j])$ such that $M^b\upharpoonright_\theta = M$.



Since $M \models^{W_{j_1}} H_1$, we have $M \models^{\mathtt{HI}} @_{j_1} h_1$ for all $h_1 \in H_1$. By the satisfaction condition for substitutions, $M^b \models^{\mathtt{HI}} \theta(@_{j_1} h_1)$ for all $h_1 \in H_1$, and by Lemma 4.7, $\Gamma_b \models \theta(@_{j_1} h_1)$ for all $h_1 \in H_1$. It follows that $\Gamma_b \models @_j \theta(@_{j_1} h_1)$ for all $h_1 \in H_1$. By completeness of $\mathcal{E}^{\mathtt{HI}_b}$, $\Gamma_b \vdash^{\mathtt{HI}_b} @_j \theta(@_{j_1} h_1)$ for all $h_1 \in H_1$. Since H_1 is finite, $\Gamma_b \vdash^{\mathtt{HI}_b} \bigcup_{h_1 \in H_1} @_j \theta(@_{j_1} h_1)$. By compactness of $\mathcal{E}^{\mathtt{HI}_b}$, there exists a finite set $\Gamma_f \subseteq \Gamma_b$ such that $\Gamma_f \vdash^{\mathtt{HI}_b} \bigcup_{h_1 \in H_1} @_j \theta(@_{j_1} h_1)$. We have $(\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h \in H} @_j h) \vdash \rho$ for all $\rho \in \Gamma_f$, which implies $(\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h \in H} @_j h) \vdash \Gamma_f$. It follows that $(\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h \in H} @_j h) \vdash \bigcup_{h_1 \in H_1} @_j \theta(@_{j_1} h_1)$, and we get

$$(\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h \in H} @_j h) \vdash (\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h \in H} @_j h) \cup (\bigcup_{h_1 \in H_1} @_j \theta(@_{j_1} h_1)) \tag{1}$$

By (*Substitutivity*), $(\chi;\ \chi[j])(\Gamma) \cup \left(\bigcup_{h \in H} @_j h\right) \vdash \bigwedge_{h_1 \in H_1} \theta(@_{j_1} h_1) \Rightarrow \theta(@_{j_1} C_1)$.

By (*Retrieve*), $\bigwedge_{h_1 \in H_1} \theta(@_{j_1} h_1) \Rightarrow \theta(@_{j_1} C_1) \vdash \bigwedge_{h_1 \in H_1} @_j \theta(@_{j_1} h_1) \Rightarrow @_j \theta(@_{j_1} C_1)$,

which implies $(\chi;\ \chi[j])(\Gamma) \cup \left(\bigcup_{h \in H} @_j h\right) \vdash \bigwedge_{h_1 \in H_1} @_j \theta(@_{j_1} h_1) \Rightarrow @_j \theta(@_{j_1} C_1)$.

By (*Implication*), we get

$$(\chi;\ \chi[j])(\Gamma) \cup \left(\bigcup_{h \in H} @_j h\right) \cup \left(\bigcup_{h_1 \in H_1} @_j \theta(@_{j_1} h_1)\right) \vdash @_j \theta(@_{j_1} C_1) \tag{2}$$

By statements (1) and (2), $(\chi;\ \chi[j])(\Gamma) \cup (\bigcup_{h \in H} @_j h) \vdash @_j \theta(@_{j_1} C_1)$, meaning that $@_j \theta(@_{j_1} C_1) \in \Gamma_b$. We have $\Gamma_b \models @_j \theta(@_{j_1} C_1)$, and by Lemma 4.7, $M^b \models^{\mathtt{HI}} @_j \theta(@_{j_1} C_1)$. It follows that $M^b \models^{W_j^b} \theta(@_{j_1} C_1)$, and by the local satisfaction condition, $M \models^{W_j^b} @_{j_1} C_1$. Hence, $M \models^{W_{j_1}} C_1$.

We have $M^b \restriction_{(\chi;\ \chi[j])} \not\models^{\mathtt{HI}} (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$ as $M^b \models^{\mathtt{HI}} @_j h$ for all $h \in H$ and $M^b \not\models^{\mathtt{HI}} @_j C$. Since $M^b \restriction_{(\chi;\ \chi[j])} \models^{\mathtt{HI}} \Gamma$, there is a contradiction with the statement $\Gamma \models (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. It follows that the assumption $\Gamma \not\vdash (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$ is not correct. Hence, $\Gamma \vdash (\forall \chi)(\downarrow j)(\bigwedge H \Rightarrow C)$. □

Proof of Theorem 6.2 We assume that $\Gamma \models_\Delta \rho$ and we prove $\Gamma \vdash_\Delta \rho$, where $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop})$. Let (W^Γ, M^Γ) be the Δ-model such that

(1) $|W^\Gamma| = \mathtt{Nom}/_{\equiv_\Gamma}$, where $\equiv_\Gamma = \{(k_1, k_2) \mid \Gamma \vdash k_1 = k_2\}$,

(2) $W^\Gamma_\lambda = \{(\widehat{k_1}, \ldots, \widehat{k_n}) \mid \Gamma \vdash \lambda(k_1, \ldots, k_n)\}$ for all $n \in \mathbb{N}at$ and $\lambda \in \Lambda_n$, where $\widehat{k_i}$ is the equivalence class of k_i,

(3) the $\mathtt{Prop}$-model $M^\Gamma_{\widehat{k}}$ interprets $p \in \mathtt{Prop}$ as true iff $\Gamma \vdash @_k p$, where $k \in \mathtt{Nom}$.

By (*Reflexivity*), (*Symmetry*) and (*Transitivity*), the relation $\equiv_\Gamma$ is an equivalence on $\mathtt{Nom}$. By (*Congruence*), the definition of W^Γ_λ is consistent for all modalities $\lambda \in \Lambda$. It follows that W^Γ is a **REL** model. By (*Congruence$_p$*), the definition of $M^\Gamma_{\widehat{k}}$ is consistent for each nominal $k \in \mathtt{Nom}$. It follows that $M^\Gamma_{\widehat{k}}$ is a **PL** model, and (W^Γ, M^Γ) is a **HPL** model. We show that $(W^\Gamma, M^\Gamma) \models \Gamma$:

(a) for any $(k_1 = k_2) \in \Gamma$, we have $\Gamma \vdash k_1 = k_2$, which implies $\widehat{k_1} = \widehat{k_2}$; by the definition of (W^Γ, M^Γ), we have $W^\Gamma_{k_1} = \widehat{k_1} = \widehat{k_2} = W^\Gamma_{k_2}$, which implies $(W^\Gamma, M^\Gamma) \models k_1 = k_2$.

(b) for any $\lambda(k_1, \ldots, k_n) \in \Gamma$ we have $\Gamma \vdash \lambda(k_1, \ldots, k_n)$, and by the definition of (W^Γ, M^Γ), we get $(\widehat{k_1}, \ldots, \widehat{k_n}) \in W^\Gamma_\lambda$, which implies $(W^\Gamma, M^\Gamma) \models \lambda(k_1, \ldots, k_n)$;

(c) for any $@_k p \in \Gamma$ we have $\Gamma \vdash @_k p$ and by the definition of (W^Γ, M^Γ), p is true in $M^\Gamma_{\widehat{k}}$, which implies $(W^\Gamma, M^\Gamma) \models @_k p$.

Since $\Gamma \models \rho$ and $(W^\Gamma, M^\Gamma) \models \Gamma$, we have $(W^\Gamma, M^\Gamma) \models \rho$. In order to prove completeness, three cases need to be considered:

(a) if ρ is a nominal equation of the form $k_1 = k_2$ then $W^{\Gamma}_{k_1} = W^{\Gamma}_{k_2}$, which means $\widehat{k_1} = \widehat{k_2}$, and we get $\Gamma \vdash k_1 = k_2$;
(b) if ρ is a nominal relation of the form $\lambda(k_1, \ldots, k_n)$ then $(W^{\Gamma}_{k_1}, \ldots, W^{\Gamma}_{k_1}) \in W^{\Gamma}_{\lambda}$, which means that we have $(\widehat{k_1}, \ldots, \widehat{k_n}) \in W^{\Gamma}_{\lambda}$, and by the definition of (W^{Γ}, M^{Γ}), we obtain $\Gamma \vdash \lambda(k_1, \ldots, k_n)$;
(c) if ρ is of the form $@_k p$, where k is a nominal and p a propositional symbol, then $M^{\Gamma}_{\widehat{k}}$ interprets p as true, and by the definition of (W^{Γ}, M^{Γ}), we obtain $\Gamma \vdash @_k p$. □

Proof of Theorem 6.6 We assume that $\Gamma \models_{\Delta} \rho$ and prove that $\Gamma \vdash_{\Delta} \rho$, where $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$. We define the equivalence $\equiv^{\mathtt{Nom}} = \{(k_1, k_2) \mid \Gamma \vdash k_1 = k_2 \text{ and } k_i \in \mathtt{Nom}\}$ on Nom. By (*Reflexivity$_n$*), (*Symmetry$_n$*) and (*Transitivity$_n$*), the relation $\equiv^{\mathtt{Nom}}$ is indeed an equivalence. For each $k \in \mathtt{Nom}$ we define the congruence $\equiv^k = \{(t_1, t_2) \mid \Gamma \vdash t_1 = t_2 \text{ and } t_i \in T^{\Delta}_k\}$ on the (S, F, P)-model T^{Δ}_k. By (*Reflexivity*), (*Symmetry*), (*Transitivity*) and (*Congruence*), the relation $\equiv^k$ is indeed a congruence on T^{Δ}_k. We define the Δ-model (W^{Γ}, M^{Γ}) as follows:

(1) $|W^{\Gamma}| = \mathtt{Nom}/_{\equiv^{\mathtt{Nom}}}$ and $W^{\Gamma}_{\lambda} = \{(\widehat{k_1}, \ldots, \widehat{k_n}) \mid \Gamma \vdash \lambda(k_1, \ldots, k_n)\}$ for all $n \in \mathbb{N}at$ and $\lambda \in \Lambda_n$, where $\widehat{k_i}$ is the equivalence class of k_i, and
(2) for each $k \in \mathtt{Nom}$, $M^{\Gamma}_{\widehat{k}}$ consists of the (S, F)-algebra $T^{\Delta}_k/_{\equiv^k}$ and interprets
   (a) each $\varpi \in P^r$ as $(M^{\Gamma}_{\widehat{k}})_{\varpi} = \{(\widehat{\tau_1}, \ldots, \widehat{\tau_n}) \mid \Gamma \vdash \varpi(\tau_1, \ldots, \tau_n)\}$, and
   (b) each $\pi \in (P - P^r)$ as $(M^{\Gamma}_{\widehat{k}})_{\pi} = \{(\widehat{t_1}, \ldots, \widehat{t_n}) \mid \Gamma \vdash \pi_k(t_1, \ldots, t_n)\}$,

   where $\widehat{\tau_i}$ and $\widehat{t_i}$ are the equivalence classes of τ_i and t_i, respectively.

By (*Congruence$_n$*), W^{Γ} is a $(\mathtt{Nom}, \Lambda)$-model. By (*Congruence$_p$*), $(M^{\Gamma}_{\widehat{k}})$ is a (S, F, P)-model. By (*Share*), for all $k_1, k_2 \in \mathtt{Nom}$ we have $\Gamma \vdash t_1 =^{k_1} t_2$ iff $\Gamma \vdash t_1 =^{k_2} t_2$, which implies $\equiv^{k_1} = \equiv^{k_2}$, meaning that $T^{\Delta}_{k_1}/_{\equiv^{k_1}}$ and $T^{\Delta}_{k_2}/_{\equiv^{k_2}}$ have the same carrier sets. Hence, the definition of (W^{Γ}, M^{Γ}) is consistent. We show that $(W^{\Gamma}, M^{\Gamma}) \models \Gamma$:

(a) For any nominal equation $(k_1 = k_2) \in \Gamma$ we have $\Gamma \vdash k_1 = k_2$, and by the definition of (W^{Γ}, M^{Γ}), we have $W^{\Gamma}_{k_1} = \widehat{k_1} = \widehat{k_2} = W^{\Gamma}_{k_2}$, which means that $(W^{\Gamma}, M^{\Gamma}) \models k_1 = k_2$.
(b) For any nominal relation $\lambda(k_1, \ldots, k_n) \in \Gamma$, we have $\Gamma \vdash \lambda(k_1, \ldots, k_n)$, and by the definition of (W^{Γ}, M^{Γ}), we have $(\widehat{k_1}, \ldots, \widehat{k_n}) \in W^{\Gamma}_{\lambda}$, which is equivalent to $(W^{\Gamma}_{k_1}, \ldots, W^{\Gamma}_{k_n}) \in W^{\Gamma}_{\lambda}$, and we get $(W^{\Gamma}, M^{\Gamma}) \models \lambda(k_1, \ldots, k_n)$.
(c) For any hybrid equation $(t = t') \in \Gamma$ we have $\Gamma \vdash t = t'$, and by the definition of (W^{Γ}, M^{Γ}), we have $(W^{\Gamma}, M^{\Gamma})_t = \widehat{t} = \widehat{t'} = (W^{\Gamma}, M^{\Gamma})_{t'}$, which means that $(W^{\Gamma}, M^{\Gamma}) \models t = t'$.
(d) For any rigid relation $\varpi(\tau_1, \ldots, \tau_n) \in \Gamma$, where $\varpi \in P^r$, we have $\Gamma \vdash \varpi(\tau_1, \ldots, \tau_n)$. By the definition of (W^{Γ}, M^{Γ}), $(\widehat{\tau_1}, \ldots, \widehat{\tau_n}) \in (M^{\Gamma}_{\widehat{k}})_{\varpi}$ for all $k \in \mathtt{Nom}$. This means $((W^{\Gamma}, M^{\Gamma})_{\tau_1}, \ldots, (W^{\Gamma}, M^{\Gamma})_{\tau_n}) \in (M^{\Gamma}_{\widehat{k}})_{\varpi}$ for all $k \in \mathtt{Nom}$. Hence, $(W^{\Gamma}, M^{\Gamma}) \models \varpi(\tau_1, \ldots, \tau_n)$.
   For any non-rigid relation $\pi_k(t_1, \ldots, t_n) \in \Gamma$, where $\pi \in (P - P^r)$, we have $\Gamma \vdash \pi_k(t_1, \ldots, t_n)$. By the definition of (W^{Γ}, M^{Γ}), we have $(\widehat{t_1}, \ldots, \widehat{t_n}) \in (M^{\Gamma}_{\widehat{k}})_{\pi}$. This means $((W^{\Gamma}, M^{\Gamma})_{t_1}, \ldots, (W^{\Gamma}, M^{\Gamma})_{t_n}) \in (M^{\Gamma}_{\widehat{k}})_{\pi}$. Hence, $(W^{\Gamma}, M^{\Gamma}) \models \pi_k(t_1, \ldots, t_n)$.

Since $\Gamma \models \rho$ and $(W^{\Gamma}, M^{\Gamma}) \models \Gamma$, we get $(W^{\Gamma}, M^{\Gamma}) \models \rho$. In order to prove completeness, four cases are considered:

(a) If ρ is a nominal equation $k_1 = k_2$ then $W^{\Gamma}_{k_1} = W^{\Gamma}_{k_2}$, and by the definition of (W^{Γ}, M^{Γ}), we have $\widehat{k_1} = \widehat{k_2}$, which is equivalent to $\Gamma \vdash k_1 = k_2$.
(b) If ρ is a nominal relation $\lambda(k_1, \ldots, k_n)$ then $(W^{\Gamma}_{k_1}, \ldots W^{\Gamma}_{k_n}) \in W^{\Gamma}_{\lambda}$, and by the definition of (W^{Γ}, M^{Γ}), we have $(\widehat{k_1}, \ldots, \widehat{k_n}) \in W^{\Gamma}_{\lambda}$, which is equivalent to $\Gamma \vdash \lambda(k_1, \ldots, k_n)$.
(c) If ρ is a hybrid equation $(t = t')$ then $(W^{\Gamma}, M^{\Gamma})_t = (W^{\Gamma}, M^{\Gamma})_{t'}$, and by the definition of (W^{Γ}, M^{Γ}), we have $\widehat{t} = (W^{\Gamma}, M^{\Gamma})_t = (W^{\Gamma}, M^{\Gamma})_{t'} = \widehat{t'}$, which is equivalent to $\Gamma \vdash t = t'$.
(d) If ρ is a rigid relation of the form $\varpi(\tau_1, \ldots, \tau_n)$, where $\varpi \in P^r$, then $((W^{\Gamma}, M^{\Gamma})_{\tau_1}, \ldots, (W^{\Gamma}, M^{\Gamma})_{\tau_n}) \in (M^{\Gamma}_{k})_{\varpi}$ for all $k \in \mathtt{Nom}$. By the definition of (W^{Γ}, M^{Γ}), we have $(\widehat{\tau_1}, \ldots, \widehat{\tau_n}) \in (M^{\Gamma}_{\widehat{k}})_{\varpi}$ for all $k \in \mathtt{Nom}$. Hence, $\Gamma \vdash \varpi(\tau_1, \ldots, \tau_n)$.
   Similarly, if ρ is a non-rigid relation of the form $\pi_k(t_1, \ldots, t_n)$, where $\pi \in (P - P^r)$, then we have $((W^{\Gamma}, M^{\Gamma})_{t_1}, \ldots, (W^{\Gamma}, M^{\Gamma})_{t_n}) \in (M^{\Gamma}_{\widehat{k}})_{\pi}$. By the definition of (W^{Γ}, M^{Γ}), we have $(\widehat{t_1}, \ldots, \widehat{t_n}) \in (M^{\Gamma}_{\widehat{k}})_{\pi}$. Hence, $\Gamma \vdash \pi_k(t_1, \ldots, t_n)$. □

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