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ORIGINAL ARTICLE

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A continuum mechanical theory for turbulence: a generalized Navier–Stokes-α equation with boundary conditions

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Abstract We develop a continuum-mechanical formulation and generalization of the Navier–Stokes-α equation based on a recently developed framework for fluid-dynamical theories involving higher-order gradient dependencies. Our flow equation involves two length scales α and β. The first of these enters the theory through the specific free-energy $\alpha^2|\mathbf{D}|^2$, where $\mathbf{D}$ is the symmetric part of the gradient of the filtered velocity, and contributes a dispersive term to the flow equation. The remaining scale is associated with a dissipative hyperstress which depends linearly on the gradient of the filtered vorticity and which contributes a viscous term, with coefficient proportional to β^2, to the flow equation. In contrast to Lagrangian averaging, our formulation delivers boundary conditions and a complete structure based on thermodynamics applied to an isothermal system. For a fixed surface without slip, the standard no-slip condition is augmented by a wall-eddy condition involving another length scale ℓ characteristic of eddies shed at the boundary and referred to as the wall-eddy length. As an application, we consider the classical problem of turbulent flow in a plane, rectangular channel of gap $2h$ with fixed, impermeable, slip-free walls and make comparisons with results obtained from direct numerical simulations. We find that $\alpha/\beta \sim Re^{0.470}$ and $\ell/h \sim Re^{-0.772}$, where Re is the Reynolds number. The first result, which arises as a consequence of identifying the specific free-energy with the specific turbulent kinetic energy, indicates that the choice $\beta = \alpha$ required to reduce our flow equation to the Navier–Stokes-α equation is likely to be problematic. The second result evinces the classical scaling relation $\eta/L \sim Re^{-3/4}$ for the ratio of the Kolmogorov microscale η to the integral length scale L.

Keywords Turbulence · Hyperstress · Hyperviscosity · Second-grade fluid

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1 Introduction

The Lagrangian averaged Navier–Stokes-α model for (statistically homogeneous and isotropic) turbulent flow yields a governing equation for the filtered velocity $\mathbf{u}$ that can be written in the form[1]

$$\rho\dot{\mathbf{u}} = -\mathrm{grad}\,p + \mu(1-\alpha^2\Delta)\Delta\mathbf{u} + 2\rho\alpha^2\mathrm{div}\,\overset{\circ}{\mathbf{D}}; \tag{1}$$

we refer to (1) as the *Navier–Stokes-α equation*.[2] In this equation $\mathbf{u}$ is subject to the incompressibility constraint

$$\mathrm{div}\,\mathbf{u} = 0, \tag{2}$$

$\dot{\varphi} = \partial\varphi/\partial t + \mathbf{u}\cdot\mathrm{grad}\,\varphi$ is the filtered material time derivative of an arbitrary scalar field φ (so that, in particular, $\dot{\mathbf{u}} = \partial\mathbf{u}/\partial t + (\mathrm{grad}\,\mathbf{u})\mathbf{u}$), p is the filtered pressure, Δ is the Laplace operator, $\mathbf{D} = \frac{1}{2}(\mathrm{grad}\,\mathbf{u} + (\mathrm{grad}\,\mathbf{u})^\top)$ is the filtered stretch-rate,

$$\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{DW} - \mathbf{WD}, \tag{3}$$

with $\mathbf{W} = \frac{1}{2}(\mathrm{grad}\,\mathbf{u} - (\mathrm{grad}\,\mathbf{u})^\top)$ the filtered spin, is the corotational rate of $\mathbf{D}$.

Aside from the density ρ and the shear viscosity μ of the fluid, the flow equation (1) involves an additional parameter $\alpha > 0$ carrying dimensions of length. Within the framework of Lagrangian averaging, α is the statistical correlation length of the excursions taken by a fluid particle away from its phase-averaged trajectory. More intuitively, α can be interpreted as the characteristic linear dimension of the smallest eddies that the model is capable of resolving. Like equations arising from Reynolds averaging, the Navier–Stokes-α equation provides an approximate model that resolves motions only above some critical scale while relying on filtering to approximate effects at smaller scales. A thorough synopsis of properties and advantages of the Navier–Stokes-α equation is provided by Holm et al. [6].

The structure of (1) is formally suggestive of a conservation law expressing the balance of linear momentum, and one might ask whether there is a complete continuum mechanical framework in which the Navier–Stokes-α equation is embedded along with suitable boundary conditions. Based on experience with theories for structured media, the presence of a term involving the fourth-order spatial gradient of the velocity indicates that any such framework should involve a hyperstress in addition to the classical stress.

To see the need for an additional hyperstress, note first that the weak form of the classical momentum balance:

$$\mathrm{div}\,\mathbf{T} + \mathbf{b} = \mathbf{0} \tag{4}$$

has the form

$$\underbrace{\int_{\partial R} \mathbf{t_n}\cdot\boldsymbol{\phi}\,\mathrm{d}a + \int_R \mathbf{b}\cdot\boldsymbol{\phi}\,\mathrm{d}v}_{\mathcal{W}_{\mathrm{ext}}(R,\boldsymbol{\phi})} = \underbrace{\int_R \mathbf{T}\colon\mathrm{grad}\,\boldsymbol{\phi}\,\mathrm{d}v}_{\mathcal{W}_{\mathrm{int}}(R,\boldsymbol{\phi})}, \tag{5}$$

where $\mathbf{t_n} = \mathbf{Tn}$ is the classical surface-traction of Cauchy, and where here and throughout this study we assume that the underlying frame is inertial; we neglect non-inertial body forces; we stipulate that $\mathbf{b}$ account for inertia, so that

$$\mathbf{b} = -\rho\dot{\mathbf{u}}. \tag{6}$$

Granted smoothness, (5) holds for all *virtual velocities* (i.e., test fields) $\boldsymbol{\phi}$ and all control volumes R if and only if the balance (4) and the traction condition $\mathbf{t_n} = \mathbf{Tn}$ (for each unit vector $\mathbf{n}$) are satisfied at all points of the fluid. Moreover, the requirement of frame-indifference—that is, the requirement that the theory be invariant under changes in observer—applied to (5) yields the symmetry of the stress $\mathbf{T}$.

When $\boldsymbol{\phi}$ represents the velocity $\mathbf{u}$, the weak balance (5) is a physical balance

$$\underbrace{\int_{\partial R} \mathbf{t_n}\cdot\mathbf{u}\,\mathrm{d}a + \int_R \mathbf{b}\cdot\mathbf{u}\,\mathrm{d}v}_{\mathcal{W}_{\mathrm{ext}}(R)} = \underbrace{\int_R \mathbf{T}\colon\mathrm{grad}\,\mathbf{u}\,\mathrm{d}v}_{\mathcal{W}_{\mathrm{int}}(R)} \tag{7}$$

between:

[1] The Lagrangian averaged Euler equation, which is (1) with $\mu = 0$, was first derived by Holm et al. [1,2]. Chen et al. [3–5] added the viscous term to the Lagrangian averaged Euler equation, giving (1).

[2] The Navier–Stokes-α equation is most frequently written as a system for the filtered and unfiltered velocities $\mathbf{u}$ and $\mathbf{v}$, with the latter given in terms of $\mathbf{u}$ by $\mathbf{v} = (1-\alpha^2\Delta)\mathbf{u}$. In this regard see (143), which reduces to the system form of the Navier–Stokes-α equation on setting β equal to α.

- the *external power* $\mathcal{W}_{\text{ext}}(R)$, which represents: (a) power expended on R by tractions acting on ∂R, and (b) power expended by the inertial body force $\mathbf{b}$ directly on the interior points of R;
- the *internal power* $\mathcal{W}_{\text{int}}(R)$, the integrand of which represents the classical stress power $\mathbf{T}\colon\operatorname{grad}\mathbf{u}$ expended within R by the stress field $\mathbf{T}$.

Here and in what follows, we write $\mathcal{W}_{\text{ext}}(R)$ for the external power associated with an actual flow and $\mathcal{W}_{\text{ext}}(R,\boldsymbol{\phi})$ for the (virtual) external power associated with a virtual velocity field $\boldsymbol{\phi}$.[3]

The balance (5) represents a nonstandard form of the classical principle of virtual power developed by Gurtin [7]. This nonstandard form was generalized by Fried and Gurtin [8] to develop a gradient theory for fluid flows at small length scales—a key ingredient in their work is the addition of a term linear in $\operatorname{grad}^2\mathbf{u}$ to the integrand of $\mathcal{W}_{\text{int}}(R)$ in (7). When combined with the simplest possible purely dissipative constitutive relations, this nonstandard virtual-power principle results in a partial differential equation slightly more general than (1) but with the term involving the corotational rate of $\mathbf{D}$ removed.

To capture the internal power associated with the formation of eddies during turbulent flow, we introduce the filtered vorticity

$$\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$$

and generalize the classical theory by including, in the internal power, a term linear in its gradient $\operatorname{grad}\boldsymbol{\omega}$. Specifically, we introduce a second-order tensor-valued *hyperstress* $\mathbf{G}$ via an internal power expenditure, per unit volume, of the form $\mathbf{G}\colon\operatorname{grad}\boldsymbol{\omega}$ and rewrite the power expended within R in the form

$$\mathcal{W}_{\text{int}}(R) = \int_R (\mathbf{T}\colon\operatorname{grad}\mathbf{v} + \mathbf{G}\colon\operatorname{grad}\boldsymbol{\omega})\,\mathrm{d}v. \tag{8}$$

In conjunction with the internal power expenditure (8), we introduce a corresponding external power expenditure

$$\mathcal{W}_{\text{ext}}(R) = \int_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{u} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{u}}{\partial n}\right)\mathrm{d}a + \int_R \mathbf{b}\cdot\mathbf{u}\,\mathrm{d}v, \tag{9}$$

in which $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ represent tractions on the bounding surface $\mathcal{S} = \partial R$ of R, while $\mathbf{b}$ represents the inertial body force (6). Here the term $\mathbf{m}_{\mathcal{S}}\cdot\partial\mathbf{u}/\partial n$, which is not present in classical theories, is needed to balance the effects of the internal power term $\mathbf{G}\colon\operatorname{grad}\boldsymbol{\omega}$, which involves the second gradient of $\mathbf{u}$. In fact, on any subsurface of $\mathcal{S}$ for which $\mathbf{u} = \mathbf{0}$ and $\mathbf{m}_{\mathcal{S}}$ is tangent to $\mathcal{S}$,[4] the integral over $\mathcal{S}$ in (9) takes the form

$$\int_{\mathcal{S}} (\mathbf{n}\times\mathbf{m}_{\mathcal{S}})\cdot\boldsymbol{\omega}\,\mathrm{d}a. \tag{10}$$

Thus, in this special but important case, *the power expended at the boundary by tractions is due solely to vorticity* and results as a consequence of the *hypercouple* $\mathbf{n}\times\mathbf{m}_{\mathcal{S}}$ acting in concert with $\boldsymbol{\omega}$.

The *principle of virtual power* replaces $\mathbf{u}$ by $\boldsymbol{\phi}$ and (hence) $\boldsymbol{\omega}$ by $\operatorname{curl}\boldsymbol{\phi}$ and is based on the requirement that

$$\mathcal{W}_{\text{ext}}(R,\boldsymbol{\phi}) = \mathcal{W}_{\text{int}}(R,\boldsymbol{\phi}) \tag{11}$$

for all control volumes R and any choice of the virtual velocity field $\boldsymbol{\phi}$. Consequences of the virtual power principle and the requirement that the internal power expenditure be frame-indifferent are that:[5]

[3] Note that, by (6), the negative inertial power is the kinetic energy rate.

[4] Cf. $(63)_1$ and its consequence (66).

[5] Within the restricted framework of finite deformations of an *elastic solid* with couple-stress, the balance (12) was first derived by the brothers Cosserat [9]; cf. Mindlin and Tiersten [10]. (The relation between the present study and that of the Cosserats is discussed in Appendix A.2.) The balance (12) and the traction conditions (13) differ only slightly from (5.11) and (5.12) of Fried and Gurtin [8]. The equations (5.11) and (5.12) of Fried and Gurtin [8] were derived previously by Toupin [11,12] for an *elastic solid* undergoing deformations involving finite strains and rotations. Because the derivation of Toupin is based on a variational argument involving the strain energy, its consequences are not directly applicable to neither the present study nor to that of Fried and Gurtin [8].

- The classical momentum balance $\rho\dot{\mathbf{u}} = \operatorname{div}\mathbf{T}$ must be replaced by the balance

$$\rho\dot{\mathbf{u}} = \operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G}, \tag{12}$$

  with $\mathbf{T}$ symmetric as in the classical theory.
- Cauchy's classical condition $\mathbf{t_n} = \mathbf{Tn}$ for the traction across a surface $\mathcal{S}$ with unit normal $\mathbf{n}$ must be replaced by the conditions

$$\left.\begin{aligned} \mathbf{t}_{\mathcal{S}} &= \mathbf{Tn} + \operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times) + \mathbf{n}\times(\operatorname{div}\mathbf{G} - 2H\mathbf{Gn}), \\ \mathbf{m}_{\mathcal{S}} &= \mathbf{n}\times\mathbf{Gn} \quad \text{or equivalently} \quad \mathbf{n}\times\mathbf{m}_{\mathcal{S}} = -\mathbf{PGn}, \end{aligned}\right\} \tag{13}$$

  in which $\operatorname{div}_{\mathcal{S}}$ is the divergence operator on $\mathcal{S}$, H is the mean curvature of $\mathcal{S}$, and $\mathbf{P} = \mathbf{1} - \mathbf{n}\otimes\mathbf{n}$ is (at any point) the projection onto the plane tangent to $\mathcal{S}$.

Our next step in developing a theory for turbulence is to account for the underlying thermodynamics. As our interest is a mechanical theory, we begin with a free-energy imbalance based on the first two laws for a continuum under isothermal conditions. This imbalance, formulated for an arbitrary region that convects with the fluid, reduces, when localized, to a *local free-energy imbalance*

$$\rho\dot{\psi} - \mathbf{T}:\mathbf{D} - \mathbf{G}:\operatorname{grad}\boldsymbol{\omega} \le 0 \tag{14}$$

in which ψ is the specific free-energy.

Restricting attention to incompressible fluids, we invoke the standard decomposition

$$\mathbf{T} = \mathbf{S} - p\mathbf{1}, \quad \operatorname{tr}\mathbf{S} = 0 \tag{15}$$

of the stress into a traceless *extra stress* $\mathbf{S}$ and a *pressure* p.

In laying down relations for ψ, $\mathbf{S}$, and $\mathbf{G}$ relevant to the modeling of turbulent flow, we are guided by the Navier–Stokes-α model as discussed by Foias et al. [13] and the theory of second-order fluids[6] as developed in Sect. 3 of Dunn and Fosdick [18]—in particular, these references are suggestive of relations giving the specific free-energy ψ and the extra stress $\mathbf{S}$ as functions of $\dot{\mathbf{D}}$ and $\operatorname{grad}\mathbf{u}$, among other variables. On the other hand, the local free-energy imbalance (14) suggests a relation involving the hyperstress $\mathbf{G}$ and $\operatorname{grad}\boldsymbol{\omega}$. Based on these observations and on a desire to avoid ad hoc assumptions, we write

$$\mathbf{L} = \operatorname{grad}\mathbf{u}, \quad \mathbf{J} = \operatorname{grad}\boldsymbol{\omega}, \tag{16}$$

and begin with the general relations

$$\left.\begin{aligned} \psi &= \hat{\psi}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J}), \\ \mathbf{S} &= \hat{\mathbf{S}}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J}), \\ \mathbf{G} &= \hat{\mathbf{G}}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J}), \end{aligned}\right\} \tag{17}$$

leaving it to the underlying physics to dictate simplifications and separations that might occur.[7]

The following bullet, which expresses a central result of this paper, shows that the underlying physics does indeed dictate a strong separation of effects:

- Consider the general relations (17) with $\mathbf{S}$ and $\mathbf{G}$ linear. Then these relations are frame indifferent, isotropic, and consistent with the local free-energy imbalance (14) if and only they have the following simple, specific forms:[8]

$$\psi = \alpha^2|\mathbf{D}|^2, \quad \mathbf{S} = 2\mu\mathbf{D} + 2\rho\alpha^2\overset{\circ}{\mathbf{D}}, \quad \mathbf{G} = \mu\beta^2(\operatorname{grad}\boldsymbol{\omega} + \gamma(\operatorname{grad}\boldsymbol{\omega})^{\top}). \tag{18}$$

  Here $\overset{\circ}{\mathbf{D}}$, the corotational rate of $\mathbf{D}$, is defined by (3), while μ, α, β, and γ are scalar moduli with μ, α, and β nonnegative and $|\gamma| \le 1$.

[6] Cf. Rivlin and Ericksen [14] and Sect. 119 of Truesdell and Noll [15]. That there is a connection between second-order fluids and turbulence was first noted by Rivlin [16, 17].

[7] We therefore follow Truesdell's principle of equipresence; cf. Sect. 96 of Truesdell and Noll [15], where it is asserted that: "This principle …reflects on the scale of gross phenomena the fact that all observed effects result from a common structure such as the motions of molecules."

[8] Actually, the Eq. $(18)_1$ holds modulo an arbitrary additive constant.

Assuming that the moduli μ, α, and β are constant, we use $(18)_{2,3}$ in (12) to arrive at the *flow equation*

$$\rho\dot{\mathbf{u}} = -\mathrm{grad}\, p + \mu(1-\beta^2\Delta)\Delta\mathbf{u} + 2\rho\alpha^2\mathrm{div}\,\mathring{\mathbf{D}}, \tag{19}$$

which for the particular choice $\beta = \alpha$ specializes to the Navier–Stokes-α equation (1). However, since the modulus β is dissipative while α is energetic, this particular choice embodies a questionable assumption concerning a relationship between the relative effects of free energy and dissipation. We refer to (19) as the *Navier–Stokes-$\alpha\beta$ equation*.

The parameter γ, which is dimensionless, does not enter the flow equation (19). However, as is clear from $(13)_2$ and $(18)_3$, γ would generally be present in boundary conditions prescribing the hypertraction.[9]

Based on the form of the external power expenditure (9)—in particular on the integral over $\mathcal{S}$ (with $\mathcal{S} = \partial B$)—we consider specific boundary conditions in which a portion $\mathcal{S}_{\mathrm{free}}$ of ∂B is a free surface and the remainder $\mathcal{S}_{\mathrm{nslp}}$ is a fixed, impermeable surface without slip. On $\mathcal{S}_{\mathrm{free}}$ we find that the classical condition $\mathbf{Tn} = \sigma H\mathbf{n}$ must be replaced by the conditions

$$\mathbf{Tn} + \mathrm{div}_{\mathcal{S}}(\mathbf{Gn}\times) + \mathbf{n}\times\mathrm{div}\,\mathbf{G} = 2\sigma H\mathbf{n} \quad \text{and} \quad \mathbf{PGn} = \mathbf{0}, \tag{20}$$

where σ represents the surface tension of the free surface, while on $\mathcal{S}_{\mathrm{nslp}}$ we propose the conditions

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad \mathbf{PGn} = -\mu\ell\boldsymbol{\omega}. \tag{21}$$

We refer to $(21)_2$ as the *wall-eddy condition* and to ℓ (which carries dimensions of length) as the *wall-eddy length*.

To display some key features of the theory, we consider the classical problem of steady, turbulent flow in a plane channel of gap $2h$. We invoke the kinematical assumptions standard for plane Couette flow. Further, we assume that the channel walls are fixed, impermeable, and slip-free. The flow equation (19) and boundary conditions (21) yield a fourth-order boundary-value-problem for the downstream component of the filtered velocity as a function of the coordinate normal to the channel walls. Experiments and DNS simulations of channel flow show that, for suitably normalized laminar and turbulent velocity profiles, the slopes of the turbulent profiles at the channel walls have magnitudes greater than their laminar counterparts.[10] A central result of our work is that the solution of the channel problem is consistent with this "wall-slope requirement" only for positive values of the wall-eddy modulus ℓ:

$$\ell > 0. \tag{22}$$

For a fixed subsurface $\mathcal{S}$ of $\mathcal{S}_{\mathrm{nslp}}$, conventional arguments yield a free-energy imbalance of the form

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{S}} \psi^{\mathrm{x}}\,\mathrm{d}a}_{\text{free-energy rate}} - \underbrace{\int\limits_{\mathcal{S}} (-(\mathbf{n}\times\mathbf{m}^{\mathrm{env}}_{\partial B})\cdot\boldsymbol{\omega})\,\mathrm{d}a}_{\text{power expenditure}} \le 0, \tag{23}$$

where ψ^{x} denotes the excess free-energy, measured per unit area of the fluid at the surface $\mathcal{S}_{\mathrm{nslp}}$. On this basis, $-(\mathbf{n}\times\mathbf{m}^{\mathrm{env}}_{\partial B})\cdot\boldsymbol{\omega}$ would ordinarily be thought of as *dissipative* and, thus positive. However, by $(13)_2$ and (21), $-(\mathbf{n}\times\mathbf{m}^{\mathrm{env}}_{\partial B})\cdot\boldsymbol{\omega} = -\mu\ell|\boldsymbol{\omega}|^2$. Thus, when (22) holds, $-(\mathbf{n}\times\mathbf{m}^{\mathrm{env}}_{\partial B})\cdot\boldsymbol{\omega}$ is *negative*! We attribute this discrepancy to a neglect of small-scale effects. Specifically, because $\mathbf{u}$ and $\boldsymbol{\omega}$ are filtered velocity and vorticity fields, the left side of (23) represents a difference of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\begin{matrix}\text{filtered free}\\ \text{energy of } \mathcal{S}\end{matrix}\right\} - \left\{\begin{matrix}\text{power expended on } \mathcal{S}\\ \text{over the filtered vorticity}\end{matrix}\right\} \tag{24}$$

and hence does not account for power and energy associated with the actual motion of the fluid at the small scales (i.e., those scales which have been filtered and are not included). While it is to be expected that a free-energy imbalance should be satisfied in any flow, laminar or turbulent, it would seem unreasonable to require that filtered variables obey a free-energy imbalance at a wall. This motivates us to reconsider the notion of free-energy imbalance at a wall and, ultimately, based on the recognition that our theory is formulated for

[9] In this regard, consider the condition $(21)_2$.

[10] Cf. Pope [19].

filtered fields and therefore neglects the energy supplied at a wall at the filtered scales, to *consider the theory as complete without* (23).

The flow equation (19) and boundary conditions (21) involve three problem-dependent length scales α, β, and ℓ. Assuming that (22) holds, we use the method of least-squares to fit our solution of the channel flow problem to the mean downstream velocity for turbulent channel flow predicted by the direct numerical simulations (DNS) of Kim et al. [20] and Moser et al. [21] for three values of the friction Reynolds number Re_τ. The fits show that the ratios of ℓ and β to the half-gap h are on the order of 10^{-2}. These ratios therefore correspond to dimensionless lengths in the lower half of the buffer layer. Moreover, the fits combine to yield data relating ℓ/h to Re_τ and a power-law fit of this data yields $\ell/h \sim Re_\tau^{-0.882}$. Invoking Blasius' [22] empirical resistance law $Re_\tau \sim Re^{7/8}$, where Re denotes the Reynolds number, we find that

$$\frac{\ell}{h} \sim Re^{-0.772}. \tag{25}$$

If we identify the half-gap h with the integral length L and ℓ with the Kolmogorov microscale η, the result (189) is then strikingly reminiscent of Kolmogorov's [23–25] classical scaling relation

$$\frac{\eta}{L} \sim Re^{-3/4} \tag{26}$$

for the ratio of the smallest to largest length scales present in a turbulent flow. Another interesting feature of the fitted data suggests that ℓ increases monotonically with Re_τ and should most likely obey the limit $\lim_{Re_\tau \to \infty} \ell = \beta$. Granted this limit, the wall-eddy length ℓ would be less than or equal to the dissipative length scale β:

$$\ell \leq \beta. \tag{27}$$

This is consistent with the view that the distribution of eddy scales represented near the boundary of a flow domain should be dominated by the smallest scales present in the flow, as seen for example in extensive statistical studies of DNS data recently reported by Das et al. [26].

While the velocity profile shows good agreement with the DNS data, we find that the Reynolds shear stress in the downstream plane of the channel agrees with the DNS data only outside the viscous wall region. As Chen et al. [4] observe in similar work concerning the Navier–Stokes-α equation, this discrepancy might be attributed to the presence of statistically inhomogeneous and/or anisotropic fluctuations within the viscous wall region.

Due to the idealized kinematics of plane channel flow, the velocity field is independent of the energetic length-scale α. Information concerning that scale can nonetheless be obtained by identifying the specific free-energy $\psi = \alpha^2|\mathbf{D}|^2$ with the specific turbulent kinetic-energy. Using a least-squares fit based on this identification, we arrive at data relating α/β and Re_τ. A power-law fit then yields $\alpha/\beta \sim Re_\tau^{0.538}$ and, if we again invoke Blasius' [22] resistance law, we find that

$$\frac{\alpha}{\beta} \sim Re^{0.471}. \tag{28}$$

For turbulent flow ($Re \gg 1$), this result suggests that dissipative length scale β should be less than the energetic length scale α, viz.,

$$\beta < \alpha. \tag{29}$$

Agreement with the DNS data thus requires that the energetic length scale α be substantially larger than the dissipative length scale β. The importance of allowing the energetic and dissipative length scales to differ is underscored by the foregoing results. For the Navier–Stokes-α model, $\alpha = \beta$ is determined by fitting the velocity profile. Since the fitted values of β/h are less than those of α/h by two orders-of-magnitude, the corresponding peak values of the dimensionless specific free-energy for the Navier–Stokes-α model must be lower by four orders-of-magnitude than those obtained for the Navier–Stokes-$\alpha\beta$ model. In this sense, it would be unphysical to identify the specific-free energy with the specific turbulent kinetic-energy in the Navier–Stokes-α model.

The extent to which the hierarchy

$$\ell \leq \beta < \alpha \tag{30}$$

of scales suggested by the study of the channel-flow problem should apply under more generic flow conditions remains a matter for further investigation. We are currently using numerical methods to explore this important issue.

2 Preliminaries

To simplify our calculations, we use direct notation. However, for clarity, we also present key definitions and results in component form.

2.1 Notation

The following notation is useful: given any vector $\mathbf{a}$ we write $(\mathbf{a}\times)$ for the tensor defined by

$$(\mathbf{a}\times)\mathbf{q} = \mathbf{a} \times \mathbf{q}, \tag{31}$$

for every vector $\mathbf{q}$, so that $(\mathbf{a}\times)_{ij} = \varepsilon_{irj}a_r$; then given any *skew* tensor $\mathbf{A}$ there is a unique vector $\mathbf{a}$ such that

$$\mathbf{A} = (\mathbf{a}\times). \tag{32}$$

We find it most convenient to work spatially; i.e., to use what is commonly called an Eulerian description. We write $\rho(\mathbf{x}, t)$ for the *mass density*, $\mathbf{u}(\mathbf{x}, t)$ for the velocity,[11]

$$\mathbf{D} = \tfrac{1}{2}(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^{\top}) \quad \text{and} \quad \mathbf{W} = \tfrac{1}{2}(\operatorname{grad}\mathbf{u} - (\operatorname{grad}\mathbf{u})^{\top}) \tag{33}$$

for the *stretching* and *spin*, and

$$\boldsymbol{\omega} = \operatorname{curl}\mathbf{u} \tag{34}$$

for the *vorticity*. It then follows that

$$\boldsymbol{\omega}\times = 2\mathbf{W}. \tag{35}$$

We use a superposed dot for the *material time-derivative*; e.g., for $\varphi(\mathbf{x}, t)$ a scalar field

$$\dot{\varphi} = \frac{\partial\varphi}{\partial t} + \mathbf{u}\cdot\operatorname{grad}\varphi.$$

Balance of mass is then the requirement that

$$\dot{\rho} + \rho\operatorname{div}\mathbf{u} = 0. \tag{36}$$

2.2 Control volume R and differential geometry on ∂R

We denote by R an arbitrary region, fixed in time, that is contained in the region of space occupied by the body over some time interval. We refer to R as a *control volume* and write

$$\mathcal{S} = \partial R$$

for the boundary of R and $\mathbf{n}$ for the outward unit normal on $\mathcal{S}$, which we assume to be smooth. We let $\mathbf{P}$ denote the projection onto the plane tangent to $\mathcal{S}$:

$$\mathbf{P} = \mathbf{1} - \mathbf{n}\otimes\mathbf{n} \quad (P_{ij} = \delta_{ij} - n_i n_j). \tag{37}$$

Then, by (31), we are led to the identity

$$\mathbf{P} = -(\mathbf{n}\times)(\mathbf{n}\times). \tag{38}$$

The operator $\operatorname{grad}_{\mathcal{S}}$ defined on any vector field $\mathbf{g}$ by

$$\operatorname{grad}_{\mathcal{S}}\mathbf{g} = (\operatorname{grad}\mathbf{g})\mathbf{P} \quad \big((\operatorname{grad}_{\mathcal{S}}\mathbf{g})_{ij} = g_{i,j} - g_{i,k}n_k n_j\big)$$

is the *surface gradient*;

$$\operatorname{div}_{\mathcal{S}}\mathbf{g} = \operatorname{tr}(\operatorname{grad}_{\mathcal{S}}\mathbf{g}) = \mathbf{P}:\operatorname{grad}\mathbf{g} = \operatorname{div}\mathbf{g} - \mathbf{n}\cdot(\operatorname{grad}\mathbf{g})\mathbf{n} = g_{i,i} - g_{i,k}n_i n_k$$

[11] We defer identifying $\mathbf{u}$ as a filtered-velocity.

defines the *surface divergence*; $\partial/\partial n$ defined by

$$\frac{\partial \mathbf{g}}{\partial n} = (\operatorname{grad}\mathbf{g})\mathbf{n}$$

is the *normal derivative*. Then

$$\operatorname{grad}\mathbf{g} = \operatorname{grad}_{\mathcal{S}}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial n} \otimes \mathbf{n} \quad \text{and} \quad \operatorname{div}\mathbf{g} = \operatorname{div}_{\mathcal{S}}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial n} \cdot \mathbf{n}. \tag{39}$$

The *surface divergence* of a tensor field $\mathbf{A}$ is the vector field defined by

$$\operatorname{div}_{\mathcal{S}}\mathbf{A} = (\operatorname{grad}\mathbf{A})\mathbf{P} \quad \big((\operatorname{div}_{\mathcal{S}}\mathbf{A})_i = A_{ij,k}P_{kj}\big)\,. \tag{40}$$

The domains of $\operatorname{grad}_{\mathcal{S}}\mathbf{g}$, $\operatorname{div}_{\mathcal{S}}\mathbf{g}$, $\partial\mathbf{g}/\partial n$, and $\operatorname{div}_{\mathcal{S}}\mathbf{A}$ are restricted to the surface $\mathcal{S}$.

Granted a smooth extension[12] of the unit normal $\mathbf{n}$, $\operatorname{grad}\mathbf{n}$ is defined in a neighborhood of $\mathcal{S}$ and the field

$$\mathbf{K} = -\operatorname{grad}_{\mathcal{S}}\mathbf{n} = -(\operatorname{grad}\mathbf{n})\mathbf{P}$$

is the *curvature tensor* of $\mathcal{S}$; as is well known, $\mathbf{K}$ obeys

$$\mathbf{K} = \mathbf{K}^{\top} \quad \text{and} \quad \mathbf{K}\mathbf{n} = \mathbf{0}. \tag{41}$$

The scalar field

$$H = \tfrac{1}{2}\operatorname{tr}\mathbf{K} = -\tfrac{1}{2}\operatorname{div}_{\mathcal{S}}\mathbf{n}$$

is the *mean curvature* of $\mathcal{S}$.

Let $\mathbf{A}$ be a (second-order) tensor field and let $\mathbf{g}$ be a vector field. We make considerable use of the identities

$$\left.\begin{aligned} \operatorname{div}_{\mathcal{S}}(\mathbf{A}\mathbf{P}) &= \operatorname{div}_{\mathcal{S}}\mathbf{A} + 2H\mathbf{A}\mathbf{n},\\ \operatorname{div}_{\mathcal{S}}(\mathbf{A}^{\top}\mathbf{g}) &= \mathbf{g}\cdot\operatorname{div}_{\mathcal{S}}\mathbf{A} + \mathbf{A} : \operatorname{grad}_{\mathcal{S}}\mathbf{g}, \end{aligned}\right\} \tag{42}$$

and, in particular, their specializations

$$\operatorname{div}_{\mathcal{S}}\mathbf{P} = 2H\mathbf{n}, \quad \operatorname{div}_{\mathcal{S}}(\mathbf{A}^{\top}\mathbf{n}) = \mathbf{n}\cdot\operatorname{div}_{\mathcal{S}}\mathbf{A} - \mathbf{A} : \mathbf{K}, \tag{43}$$

which arise, respectively, on choosing $\mathbf{A} = \mathbf{1}$ in $(42)_1$ and $\mathbf{g} = \mathbf{n}$ in $(42)_2$.

We now show that if $\mathbf{u} = \mathbf{0}$ on a subsurface $\mathcal{S}_0$ of $\mathcal{S}$, then

$$\boldsymbol{\omega}\cdot\mathbf{n} = 0 \quad \text{and} \quad \boldsymbol{\omega}\times\mathbf{n} = \mathbf{P}\frac{\partial \mathbf{u}}{\partial n} \quad \text{on } \mathcal{S}_0. \tag{44}$$

To verify this assertion assume that $\mathbf{u} = \mathbf{0}$ on $\mathcal{S}_0$. Then on that subsurface $\mathbf{u}\cdot\mathbf{n} = 0$, so that $\operatorname{grad}_{\mathcal{S}}(\mathbf{u}\cdot\mathbf{n}) = \mathbf{0}$ and

$$(\operatorname{grad}_{\mathcal{S}}\mathbf{u})^{\top}\mathbf{n} = \mathbf{P}(\operatorname{grad}\mathbf{u})^{\top}\mathbf{n} = \mathbf{0};$$

hence $\boldsymbol{\omega}\times\mathbf{n} = \mathbf{P}(\boldsymbol{\omega}\times\mathbf{n}) = 2\mathbf{P}\mathbf{W}\mathbf{n} = \mathbf{P}(\operatorname{grad}\mathbf{u})\mathbf{n} = \mathbf{P}(\partial\mathbf{u}/\partial n)$ which is $(44)_2$. Next, $\mathbf{u} = \mathbf{0}$ on $\mathcal{S}_0$ implies that $\operatorname{grad}_{\mathcal{S}}\mathbf{u} = \mathbf{0}$ and hence that, by (39), $\operatorname{grad}\mathbf{u} = (\partial\mathbf{u}/\partial n)\otimes\mathbf{n}$; thus $\operatorname{curl}\mathbf{u} = \mathbf{n}\times(\partial\mathbf{u}/\partial n)$, which implies $(44)_1$.

[12] Any smooth vector field on $\mathcal{S}$ can be extended smoothly to a neighborhood of $\mathcal{S}$; cf. Sect. 3.4 of Cermelli et al. [27].

3 Principle of virtual power

Throughout this section, R—with boundary $\mathcal{S}$ and outward unit normal $\mathbf{n}$—is an arbitrary control volume.

Consistent with the discussion leading to (8) and (9), we introduce internal and external power expenditures

$$\left.\begin{aligned} \mathcal{W}_{\rm int}(R) &= \int_R (\mathbf{T} : \operatorname{grad}\mathbf{u} + \mathbf{G} : \operatorname{grad}\boldsymbol{\omega})\,\mathrm{d}v, \\ \mathcal{W}_{\rm ext}(R) &= \int_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}} \cdot \mathbf{u} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{u}}{\partial n}\right) \mathrm{d}a + \int_R \mathbf{b} \cdot \mathbf{u}\,\mathrm{d}v, \end{aligned}\right\} \tag{45}$$

with stress $\mathbf{T}$ and hyperstress $\mathbf{G}$ defined over the deformed body for all time. Since $\operatorname{tr}(\operatorname{grad}\boldsymbol{\omega}) = \operatorname{div}\boldsymbol{\omega} = 0$, we may, without loss in generality, require that $\mathbf{G}$ be *traceless*:

$$\operatorname{tr}\mathbf{G} = 0. \tag{46}$$

To state the principle of virtual power, assume that, at some arbitrarily chosen but *fixed time*, the region occupied by the body is known, as are the tractions $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$, the body force $\mathbf{b}$, and the stresses $\mathbf{T}$ and $\mathbf{G}$, and consider the velocity $\mathbf{u}$ as a virtual field $\boldsymbol{\phi}$ that may be specified *independently of the actual evolution of the body*. Then the *principle of virtual power* is the requirement that *the external and internal powers be balanced*: given any control volume R,

$$\underbrace{\int_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}} \cdot \boldsymbol{\phi} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \boldsymbol{\phi}}{\partial n}\right) \mathrm{d}a + \int_R \mathbf{b} \cdot \boldsymbol{\phi}\,\mathrm{d}v}_{\mathcal{W}_{\rm ext}(R,\boldsymbol{\phi})} = \underbrace{\int_R (\mathbf{T} : \operatorname{grad}\boldsymbol{\phi} + \mathbf{G} : \operatorname{grad}\operatorname{curl}\boldsymbol{\phi})\,\mathrm{d}v}_{\mathcal{W}_{\rm int}(R,\boldsymbol{\phi})} \tag{47}$$

for any choice of the virtual velocity $\boldsymbol{\phi}$.

3.1 Alternative form of the virtual power balance

Our first step in the derivation of a force balance and traction conditions is to rewrite the internal power in a more useful form. Using the divergence theorem, we obtain

$$\int_R \mathbf{T} : \operatorname{grad}\boldsymbol{\phi}\,\mathrm{d}v = -\int_R \operatorname{div}\mathbf{T} \cdot \boldsymbol{\phi}\,\mathrm{d}v + \int_{\mathcal{S}} \mathbf{T}\mathbf{n} \cdot \boldsymbol{\phi}\,\mathrm{d}a. \tag{48}$$

Similarly, the divergence theorem applied twice yields

$$\begin{aligned} \int_R \mathbf{G} : \operatorname{grad}\operatorname{curl}\boldsymbol{\phi}\,\mathrm{d}v &= -\int_R (\operatorname{div}\mathbf{G}) \cdot (\operatorname{curl}\boldsymbol{\phi})\,\mathrm{d}v + \int_{\mathcal{S}} \mathbf{G}\mathbf{n} \cdot \operatorname{curl}\boldsymbol{\phi}\,\mathrm{d}a \\ &= -\int_R (\operatorname{curl}\operatorname{div}\mathbf{G}) \cdot \boldsymbol{\phi}\,\mathrm{d}v + \int_{\mathcal{S}} (\mathbf{G}\mathbf{n} \cdot \operatorname{curl}\boldsymbol{\phi} + (\mathbf{n} \times \operatorname{div}\mathbf{G}) \cdot \boldsymbol{\phi})\,\mathrm{d}a. \end{aligned}$$

Thus

$$\mathcal{W}_{\rm int}(R, \boldsymbol{\phi}) = -\int_R (\operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G}) \cdot \boldsymbol{\phi}\,\mathrm{d}v + \int_{\mathcal{S}} (\mathbf{G}\mathbf{n} \cdot \operatorname{curl}\boldsymbol{\phi} + (\mathbf{T}\mathbf{n} + \mathbf{n} \times \operatorname{div}\mathbf{G}) \cdot \boldsymbol{\phi})\,\mathrm{d}a. \tag{49}$$

Further, by $(39)_1$,

$$\operatorname{grad}\boldsymbol{\phi} = \operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi} + \frac{\partial \boldsymbol{\phi}}{\partial n} \otimes \mathbf{n};$$

thus, letting

$$\mathbf{g} = \mathbf{G}\mathbf{n} \tag{50}$$

and, using (31), we obtain

$$\begin{aligned}\mathbf{g}\cdot\operatorname{curl}\boldsymbol{\phi} &= -(\mathbf{g}\times)\colon\operatorname{grad}\boldsymbol{\phi} = -(\mathbf{g}\times)\colon\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi} - (\mathbf{g}\times)\colon\left(\frac{\partial\boldsymbol{\phi}}{\partial n}\otimes\mathbf{n}\right)\\ &= -(\mathbf{g}\times)\colon\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi} + (\mathbf{n}\times\mathbf{g})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n},\end{aligned}$$

whereby (49) becomes

$$\begin{aligned}\mathcal{W}_{\rm int}(R,\boldsymbol{\phi}) = &-\int_R(\operatorname{div}\mathbf{T}+\operatorname{curl}\operatorname{div}\mathbf{G})\cdot\boldsymbol{\phi}\,\mathrm{d}v - \int_{\mathcal{S}}(\mathbf{g}\times)\colon\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}\,\mathrm{d}a\\ &+\int_{\mathcal{S}}\left((\mathbf{Tn}+\mathbf{n}\times\operatorname{div}\mathbf{G})\cdot\boldsymbol{\phi}+(\mathbf{n}\times\mathbf{g})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\right)\mathrm{d}a. \end{aligned}\tag{51}$$

Our next step is to establish an important identity for the integral $\int_{\mathcal{S}}(\mathbf{g}\times)\colon\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}\,\mathrm{d}a$; specifically, letting

$$\mathbf{A} = -(\mathbf{g}\times),\tag{52}$$

we now show that

$$\int_{\mathcal{S}}\mathbf{A}\colon\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}\,\mathrm{d}a = -\int_{\mathcal{S}}(\operatorname{div}_{\mathcal{S}}\mathbf{A}+2H\mathbf{An})\cdot\boldsymbol{\phi}\,\mathrm{d}a.\tag{53}$$

The verification of (53) is based on the *surface divergence theorem*: let $\boldsymbol{\tau}$ be a *tangential* vector field on $\mathcal{S}$ and let $\mathcal{T}$ be a subsurface of $\mathcal{S}$ with $\boldsymbol{\nu}$ the outward unit normal to the boundary curve $\partial\mathcal{T}$ (so that $\boldsymbol{\nu}$ is tangent to $\mathcal{S}$, normal to $\partial\mathcal{T}$, and directed outward from $\mathcal{T}$); then

$$\int_{\partial\mathcal{T}}\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds = \int_{\mathcal{T}}\operatorname{div}_{\mathcal{S}}\boldsymbol{\tau}\,\mathrm{d}a.\tag{54}$$

To establish (53) note that

$$\boldsymbol{\tau} \overset{\text{def}}{=} \mathbf{PA}^{\top}\boldsymbol{\phi}$$

represents a tangential vector field, so that, by (54),

$$\int_{\partial\mathcal{T}}\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds = \int_{\mathcal{T}}\operatorname{div}_{\mathcal{S}}(\mathbf{PA}^{\top}\boldsymbol{\phi})\,\mathrm{d}a.\tag{55}$$

Further, by $(42)_2$—with $\mathbf{A}$ replaced by $\mathbf{AP}$—and $(42)_1$,

$$\operatorname{div}_{\mathcal{S}}(\mathbf{PA}^{\top}\boldsymbol{\phi}) = (\mathbf{AP})\colon\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}+\boldsymbol{\phi}\cdot\operatorname{div}_{\mathcal{S}}(\mathbf{AP}) = \mathbf{A}\colon\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}+(\operatorname{div}_{\mathcal{S}}\mathbf{A}+2H\mathbf{An})\cdot\boldsymbol{\phi};$$

hence (55) takes the form

$$\int_{\partial\mathcal{T}}\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,\mathrm{d}s = \int_{\mathcal{T}}(\mathbf{A}\colon\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}+(\operatorname{div}_{\mathcal{S}}\mathbf{A}+2H\mathbf{An})\cdot\boldsymbol{\phi})\,\mathrm{d}a.\tag{56}$$

Finally, if we take $\mathcal{T}=\mathcal{S}$, then $\partial\mathcal{T}$ is empty and the left side of (56) vanishes; thus the desired result (53) is satisfied.

Next, recalling, from (52), that $\mathbf{A}=-(\mathbf{g}\times)$ and, by (31), that $(\mathbf{g}\times)\mathbf{q}=\mathbf{g}\times\mathbf{q}=-\mathbf{q}\times\mathbf{g}$, (53) takes the form

$$-\int_{\mathcal{S}}(\mathbf{g}\times)\colon\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}\,\mathrm{d}a = \int_{\mathcal{S}}(\operatorname{div}_{\mathcal{S}}(\mathbf{g}\times)-2H\mathbf{n}\times\mathbf{g})\cdot\boldsymbol{\phi}\,\mathrm{d}a\tag{57}$$

and therefore

$$\mathcal{W}_{\text{int}}(R,\boldsymbol{\phi}) = -\int_R (\operatorname{curl}\operatorname{div}\mathbf{G} + \operatorname{div}\mathbf{T})\cdot\boldsymbol{\phi}\,\mathrm{d}v + \int_{\mathcal{S}} \left((\mathbf{Tn} + \operatorname{div}_{\mathcal{S}}(\mathbf{g}\times) - 2H\mathbf{n}\times\mathbf{g} \quad + \mathbf{n}\times\operatorname{div}\mathbf{G})\cdot\boldsymbol{\phi} + (\mathbf{n}\times\mathbf{g})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\right)\mathrm{d}a. \tag{58}$$

We are now in a position to apply the virtual-power balance (47): by $(45)_2$ and (58),

$$\int_{\mathcal{S}}\left(\mathbf{t}_{\mathcal{S}}\cdot\boldsymbol{\phi} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\right)\mathrm{d}a + \int_R \mathbf{b}\cdot\boldsymbol{\phi}\,\mathrm{d}v = -\int_R (\operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G})\cdot\boldsymbol{\phi}\,\mathrm{d}v + \int_{\mathcal{S}}\left((\mathbf{Tn} + \operatorname{div}_{\mathcal{S}}(\mathbf{g}\times) -\ 2H\mathbf{n}\times\mathbf{g} + \mathbf{n}\times\operatorname{div}\mathbf{G})\cdot\boldsymbol{\phi} + (\mathbf{n}\times\mathbf{g})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\right)\mathrm{d}a; \tag{59}$$

using (50) and rearranging (59), we have the “only if" implication in the next result.

(#) *Given any virtual velocity $\boldsymbol{\phi}$ and any control volume R, the virtual balance*

$$\underbrace{\int_{\mathcal{S}}\left(\mathbf{t}_{\mathcal{S}}\cdot\boldsymbol{\phi} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\right)\mathrm{d}a + \int_R \mathbf{b}\cdot\boldsymbol{\phi}\,\mathrm{d}v}_{\mathcal{W}_{\text{ext}}(R,\boldsymbol{\phi})} = \underbrace{\int_R (\mathbf{T}:\operatorname{grad}\boldsymbol{\phi} + \mathbf{G}:\operatorname{grad}\operatorname{curl}\boldsymbol{\phi})\,\mathrm{d}v}_{\mathcal{W}_{\text{int}}(R,\boldsymbol{\phi})} \tag{60}$$

is satisfied if and only if

$$\int_{\mathcal{S}} (\mathbf{t}_{\mathcal{S}} - \mathbf{Tn} - \operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times) + 2H\mathbf{n}\times\mathbf{Gn} - \mathbf{n}\times\operatorname{div}\mathbf{G})\cdot\boldsymbol{\phi}\,\mathrm{d}a + \int_{\mathcal{S}} (\mathbf{m}_{\mathcal{S}} - \mathbf{n}\times\mathbf{Gn})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\,\mathrm{d}a = -\int_R (\operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G} + \mathbf{b})\cdot\boldsymbol{\phi}\,\mathrm{d}v. \tag{61}$$

The reverse implication, that (61) implies (60), follows upon reversing the argument leading to (60).

3.2 Local force balance and traction conditions

Since the control volume R and the virtual field $\boldsymbol{\phi}$ in (61) *may be arbitrarily chosen*, we may appeal to the fundamental lemma of the calculus of variations and arrive at the *local force balance*

$$\operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G} + \mathbf{b} = \mathbf{0} \quad (T_{ij,j} + \varepsilon_{ikr}G_{rj,jk} + b_i = 0) \tag{62}$$

and—bearing in mind that, since $\boldsymbol{\phi}$ is arbitrary, $\boldsymbol{\phi}$ and $\partial\boldsymbol{\phi}/\partial n$ may be arbitrarily chosen independent of one another on $\mathcal{S}$ (cf. the paragraph containing $(45)_2$)—the *traction conditions*

$$\left.\begin{aligned}\mathbf{t}_{\mathcal{S}} &= \mathbf{Tn} + \operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times) + \mathbf{n}\times(\operatorname{div}\mathbf{G} - 2H\mathbf{Gn}),\\ \mathbf{m}_{\mathcal{S}} &= \mathbf{n}\times\mathbf{Gn}.\end{aligned}\right\} \tag{63}$$

or in components

$$\left.\begin{aligned}(\mathbf{t}_{\mathcal{S}})_i &= T_{ij}n_j + \varepsilon_{irj}(G_{rs}n_s)_{,k}P_{kj} + \varepsilon_{ijk}n_j(G_{kr,r} - 2HG_{kr}n_r),\\ (\mathbf{m}_{\mathcal{S}})_i &= \varepsilon_{ijk}n_jG_{kr}n_r.\end{aligned}\right\} \tag{64}$$

In view of (6), the local force balance becomes the *local momentum balance*

$$\rho\dot{\mathbf{u}} = \operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G} \quad (\rho\dot{u}_i = T_{ij,j} + \varepsilon_{ikr}G_{rj,jk}). \tag{65}$$

Note that, as a consequence of (63)$_2$,

$$\text{the hypertraction } \mathbf{m}_{\mathcal{S}} \text{ is tangent to } \mathcal{S}. \tag{66}$$

Further, by (38) and (63)$_2$, we have the following relation between the hypercouple $\mathbf{n} \times \mathbf{m}_{\mathcal{S}}$ and the "traction" $\mathbf{Gn}$:

$$\mathbf{n} \times \mathbf{m}_{\mathcal{S}} = -\mathbf{PGn}. \tag{67}$$

Granted (66), the conditions (63)$_2$ and (67) are equivalent.

An important consequence of (63) is that the *tractions are local*: at any point $\mathbf{x}$ on $\mathcal{S}$, $\mathbf{t}_{\mathcal{S}}(\mathbf{x})$ depends on $\mathcal{S}$ through a dependence on the normal $\mathbf{n}(\mathbf{x})$ and curvature $\mathbf{K}(\mathbf{x})$ at $\mathbf{x}$, while $\mathbf{m}_{\mathcal{S}}(\mathbf{x})$ depends on $\mathcal{S}$ through $\mathbf{n}(\mathbf{x})$. (Here for convenience we have suppressed the argument t.) Next, letting $-\mathcal{S}$ denote for the surface $\mathcal{S}$ oriented by $-\mathbf{n}$ (which has curvature $-\mathbf{K}$), we see that, by (63) and (67),

$$\mathbf{t}_{\mathcal{S}} = -\mathbf{t}_{-\mathcal{S}}, \quad \mathbf{n} \times \mathbf{m}_{\mathcal{S}} = -\mathbf{n} \times \mathbf{m}_{-\mathcal{S}}, \quad \mathbf{m}_{\mathcal{S}} = \mathbf{m}_{-\mathcal{S}}; \tag{68}$$

the relations (68) represent an *action-reaction principle* for oppositely oriented surfaces that touch and are tangent at a point. Also, since $\partial\mathbf{u}/\partial n = (\operatorname{grad}\mathbf{u})\mathbf{n}$,

$$\mathbf{m}_{\mathcal{S}} \cdot \frac{\partial\mathbf{u}}{\partial n} = -\mathbf{m}_{-\mathcal{S}} \cdot \frac{\partial\mathbf{u}}{\partial(-n)}. \tag{69}$$

4 Frame-indifference

4.1 Transformation laws for the kinematical fields

The *principle of frame-indifference* makes precise the fundamental requirement that continuum theories be invariant under changes in the frame of reference—or, equivalently, invariant under changes in observer. For our purposes a *change of frame* is, at each *fixed time* t, defined by a rotation[13] $\mathbf{Q}(t)$ and a point $\mathbf{y}(t)$ and transforms points $\mathbf{x}$ to points

$$\mathbf{x}^* = \mathbf{y}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{o}), \tag{70}$$

with $\mathbf{o}$ the origin. Then, given a field Φ and a change in frame, we write $\Phi^*(\mathbf{x}^*, t)$ for $\Phi(\mathbf{x}, t)$ evaluated in the new frame.

Let

$$\mathbf{c}(t) = \dot{\mathbf{y}}(t).$$

Consequences of (70) are then the following transformation laws for the velocity and the velocity gradient under a change in frame:[14]

$$\mathbf{u}^* = \mathbf{c} + \mathbf{Qu} + \dot{\mathbf{Q}}(\mathbf{x} - \mathbf{o}), \quad \operatorname{grad}^* \mathbf{u}^* = \mathbf{Q}(\operatorname{grad}\mathbf{u})\mathbf{Q}^{\top} + \mathbf{Z}. \tag{71}$$

Here grad^* is the gradient with respect to $\mathbf{x}^*$, while $\mathbf{Z}$ is the *skew tensor*

$$\mathbf{Z} = \dot{\mathbf{Q}}\mathbf{Q}^{\top}. \tag{72}$$

At this point it is convenient to write

$$\mathbf{L} = \operatorname{grad}\mathbf{u} \quad \text{and} \quad \mathbf{J} = \operatorname{grad}\boldsymbol{\omega}, \tag{73}$$

so that $\mathbf{L}^* = \operatorname{grad}^*\mathbf{u}^*$ and $\mathbf{J}^* = \operatorname{grad}^*\boldsymbol{\omega}^*$. We then have the following transformation laws:

$$\mathbf{L}^* = \mathbf{QLQ}^{\top} + \mathbf{Z}, \quad \mathbf{J}^* = \mathbf{QJQ}^{\top}, \tag{74}$$

[13] There is some disagreement as to whether only rotations or all orthogonal tensors should be employed in the statement of the frame-indifference principle. This issue has been settled by Murdoch [28]: using a rigorous argument, Murdoch concludes that the statement of the principle should involve only rotations. In addition, Murdoch [28] notes that inclusion of the orthogonal tensor $\mathbf{Q} = -\mathbf{1}$ in this principle would preclude one from characterizing optically-active sugar solutions which rotate plane-polarized light in opposing senses.

[14] Cf., e.g., Chapter VII of Gurtin [29].

the first of which just repeats (71)$_2$. Since $\mathbf{Z}$ is skew, if we take the symmetric part of (74) we find that

$$\mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^\top \tag{75}$$

We now define, for any tensor field $\mathbf{A}$, the *corotational rate* $\mathring{\mathbf{A}}$ of $\mathbf{A}$ by

$$\mathring{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A}. \tag{76}$$

An important and well-known consequence of (75) is the following transformation law for the corotational rate of stretching:

$$\mathring{\overline{\mathbf{D}^*}} = \mathbf{Q}\mathring{\mathbf{D}}\mathbf{Q}^\top. \tag{77}$$

4.2 Symmetry of the stress $\mathbf{T}$ and transformation laws for the stresses $\mathbf{T}$ and $\mathbf{G}$

In accord with the frame-indifference principle we require that both *the internal virtual power and the external virtual power be invariant under changes in frame*—a requirement that renders the principle of virtual-power, itself, invariant.

With a minor abuse of notation let

$$\mathbf{L} = \operatorname{grad}\boldsymbol{\phi}, \quad \mathbf{J} = \operatorname{grad}\operatorname{curl}\boldsymbol{\phi},$$

and consider the internal power (45)$_1$. Invariance of the internal power under changes in frame requires that

$$\int\limits_R (\mathbf{T}:\mathbf{L} + \mathbf{G}:\mathbf{J})\,dv = \int\limits_R (\mathbf{T}^*:\mathbf{L}^* + \mathbf{G}^*:\mathbf{J}^*)\,dv. \tag{78}$$

Here, because the mapping (70) preserves distances, we have, without loss in generality, transformed the integral on the right from the transformed region R^* back to the original region R. By (74),

$$\begin{aligned}\int\limits_R (\mathbf{T}:\mathbf{L} + \mathbf{G}:\mathbf{J})\,dv &= \int\limits_R (\mathbf{T}^*:\mathbf{Q}\mathbf{L}\mathbf{Q}^\top + \mathbf{T}^*:\mathbf{Z} + \mathbf{G}^*:\mathbf{Q}\mathbf{J}\mathbf{Q}^\top)\,dv \\ &= \int\limits_R (\mathbf{Q}^\top\mathbf{T}^*\mathbf{Q}:\mathbf{L} + \mathbf{T}^*:\mathbf{Z} + \mathbf{Q}^\top\mathbf{G}^*\mathbf{Q}:\mathbf{J})\,dv;\end{aligned} \tag{79}$$

and, what is most important, (79) must hold for any change in frame, any choice of the region R, and any pair of virtual fields $\mathbf{L}$ and $\mathbf{J}$. Taking $\mathbf{L} = \mathbf{0}$ and $\mathbf{J} = \mathbf{0}$ we find that, since R is arbitrary, $\mathbf{T}^*:\mathbf{Z} = 0$, and hence, since the skew tensor $\mathbf{Z}$ is also arbitrary, $\mathbf{T}^*$ must be symmetric. On the other hand, returning to (79), but with $\mathbf{T}^*:\mathbf{Z} = \mathbf{0}$ and $\mathbf{L}$ and $\mathbf{J}$ arbitrary, we find that $\mathbf{Q}^\top\mathbf{T}^*\mathbf{Q} = \mathbf{T}$ and $\mathbf{Q}^\top\mathbf{J}^*\mathbf{Q} = \mathbf{J}$. Thus, we have the following results:

(i) the Cauchy stress is symmetric,

$$\mathbf{T} = \mathbf{T}^\top; \tag{80}$$

(ii) $\mathbf{T}$ and the hyperstress $\mathbf{G}$ transform according to

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^\top, \quad \mathbf{G}^* = \mathbf{Q}\mathbf{G}\mathbf{Q}^\top. \tag{81}$$

The results (i) and (ii) are basic to what follows. Traditionally, the symmetry of $\mathbf{T}$ follows from the local balance of angular momentum; here, interestingly, this result follows from frame-indifference.[15]

[15] Cf. Gurtin [7].

4.3 Balance laws for forces and moments

Turning to the external power (45)$_2$, frame-indifference requires that

$$\int\limits_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}} \cdot \boldsymbol{\phi} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \boldsymbol{\phi}}{\partial n} \right) \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \boldsymbol{\phi} \, \mathrm{d}v = \int\limits_{\mathcal{S}} \left(\mathbf{t}^*_{\mathcal{S}} \cdot \boldsymbol{\phi}^* + \mathbf{m}^*_{\mathcal{S}} \cdot \frac{\partial \boldsymbol{\phi}^*}{\partial n} \right) \mathrm{d}a + \int\limits_{R} \mathbf{b}^* \cdot \boldsymbol{\phi}^* \, \mathrm{d}v \tag{82}$$

for all changes in frame, all virtual fields $\boldsymbol{\phi}$, and all regions R. In writing (82) we have adopted the convention expressed in the sentence following (78).

We apply (82) with $\boldsymbol{\phi} = \mathbf{0}$, in which case, by (71)$_1$ and (72),

$$\boldsymbol{\phi}^* = \mathbf{c} + \mathbf{Z}(\mathbf{x}^* - \mathbf{y}), \quad \operatorname{grad}^* \boldsymbol{\phi}^* = \mathbf{Z}. \tag{83}$$

Since $\mathbf{Z}$ is skew, there is a unique vector $\mathbf{z}$ such that $\mathbf{Z} = (\mathbf{z}\times)$; thus, letting $\mathbf{r} = \mathbf{x} - \mathbf{o}$, so that $\mathbf{r}^* = \mathbf{x}^* - \mathbf{y}$, we find that (82) becomes

$$\mathbf{c} \cdot \left[\int\limits_{\mathcal{S}} \mathbf{t}^*_{\mathcal{S}} \, \mathrm{d}a + \int\limits_{R} \mathbf{b}^* \, \mathrm{d}v \right] + \mathbf{z} \cdot \left[\int\limits_{\mathcal{S}} (\mathbf{r}^* \times \mathbf{t}_{\mathcal{S}} + \mathbf{n}^* \times \mathbf{m}^*_{\mathcal{S}}) \, \mathrm{d}a + \int\limits_{R} \mathbf{r}^* \times \mathbf{b}^* \, \mathrm{d}v \right] = 0;$$

since the vectors $\mathbf{c}$ and $\mathbf{z}$ are arbitrary, their coefficients must therefore vanish. We are thus led to balance laws for forces and moments in the starred frame—but since that frame is arbitrary, the corresponding balances expressed in the original frame must also vanish. Hence, we have the following expressions for balance of forces and moments for a region R:

$$\left.\begin{aligned} &\int_{\mathcal{S}} \mathbf{t}_{\mathcal{S}} \, \mathrm{d}a + \int_{R} \mathbf{b} \, \mathrm{d}v = \mathbf{0}, \\ &\int_{\mathcal{S}} (\mathbf{r} \times \mathbf{t}_{\mathcal{S}} + \mathbf{n} \times \mathbf{m}_{\mathcal{S}}) \, \mathrm{d}a + \int_{R} \mathbf{r} \times \mathbf{b} \, \mathrm{d}v = \mathbf{0}. \end{aligned}\right\} \tag{84}$$

These expressions bear comparison to their classical counterparts in which $\mathbf{t}_{\mathcal{S}} = \mathbf{T}\mathbf{n}$ and $\mathbf{m}_{\mathcal{S}} = \mathbf{0}$. When combined with (6), (84) represent balances for linear and angular momentum. The term $\mathbf{n} \times \mathbf{m}_{\mathcal{S}}$ represents a distribution of couples on $\mathcal{S}$. Our formulation of the virtual power principle ensures that the classical balances (84) are satisfied automatically.

We close this section by noting the substantial physical content encapsulated within the frame-indifference and virtual-power principles, considered together, as they yield:

- a local balance law for the Cauchy stress $\mathbf{T}$ and the hyperstress $\mathbf{G}$;
- expressions for the traction $\mathbf{t}_{\mathcal{S}}$ and hypertraction $\mathbf{m}_{\mathcal{S}}$;
- the symmetry of $\mathbf{T}$;
- transformation laws for $\mathbf{T}$ and $\mathbf{G}$ under a change in frame;
- integral balance laws for forces and moments.

5 Environmental tractions, surface tension and balance of forces and moments at the boundary

Let $B(t)$ denote the region of space occupied by the fluid at an arbitrarily chosen time and let $\mathbf{n}(\mathbf{x}, t)$ denote the outward unit normal to $\partial B(t)$. We assume that $\partial B(t)$ is smooth.

Guided by (84) we introduce *environmental tractions* $\mathbf{t}^{\text{env}}_{\partial B}$ and $\mathbf{m}^{\text{env}}_{\partial B}$ and assume that, given any subsurface $\mathcal{S}$ of ∂B,

$$\int\limits_{\mathcal{S}} \mathbf{t}^{\text{env}}_{\partial B} \, \mathrm{d}a \quad \text{and} \quad \int\limits_{\mathcal{S}} \mathbf{r} \times \mathbf{t}^{\text{env}}_{\partial B} \, \mathrm{d}a + \int\limits_{\mathcal{S}} \mathbf{n} \times \mathbf{m}^{\text{env}}_{\partial B} \, \mathrm{d}a \tag{85}$$

represent the net force and moment exerted on $\mathcal{S}$ by the environment. In addition, we let σ denote the *surface tension* of the fluid at the boundary and assume, for convenience, that σ is *constant*.

Consider an *arbitrary* evolving subsurface $\mathcal{S}(t)$ of $\partial B(t)$. We view $\mathcal{S}$ as a *boundary pillbox* of infinitesimal thickness containing a portion of the boundary, a view that allows us to isolate the physical processes in the material on the two sides of the boundary. The geometric boundary of $\mathcal{S}$ consists of its boundary curve $\partial\mathcal{S}$. But $\mathcal{S}$ viewed as pillbox has a *pillbox boundary* consisting of (Fig. 1):

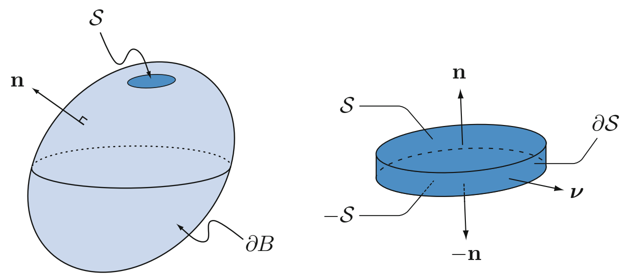

Fig. 1 Pillbox corresponding to a subsurface $\mathcal{S}$ of the boundary ∂B of the region B of space occupied by the body. Whereas $\mathbf{n}$ is directed into the environment, $-\mathbf{n}$ is directed into the fluid. The outward unit normal on the lateral face $\partial\mathcal{S}$ of the pillbox is denoted by $\boldsymbol{\nu}$

- a surface $\mathcal{S}$ with unit normal $\mathbf{n}$; $\mathcal{S}$ is viewed as lying in the *environment* at the interface of the fluid and the environment;
- a surface $-\mathcal{S}$ with unit normal $-\mathbf{n}$; $-\mathcal{S}$ is viewed as lying in the fluid adjacent to the boundary;
- a "lateral face" represented by $\partial\mathcal{S}$.

The outward unit normal on the lateral face $\partial\mathcal{S}$ of the pillbox is denoted by $\boldsymbol{\nu}$.

To derive force and moment balances for the boundary we first note that, by (37),

$$\int\limits_{\partial\mathcal{S}} \sigma\,\boldsymbol{\nu}\,\mathrm{d}s = \int\limits_{\partial\mathcal{S}} \sigma\mathbf{P}\boldsymbol{\nu}\,\mathrm{d}s \quad \text{and} \quad \int\limits_{\partial\mathcal{S}} \sigma\,\mathbf{r}\times\boldsymbol{\nu}\,\mathrm{d}s = \int\limits_{\partial\mathcal{S}} \sigma\,\mathbf{r}\times\mathbf{P}\boldsymbol{\nu}\,\mathrm{d}s.$$

represent the force and moment exerted by the fluid on the lateral face of the pillbox by surface tension. Further, by (68), the force and moment exerted by the fluid on the pillbox surface $-\mathcal{S}$ are

$$-\int\limits_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}\,\mathrm{d}a \quad \text{and} \quad -\int\limits_{\mathcal{S}} \mathbf{r}\times\mathbf{t}_{\mathcal{S}}\,\mathrm{d}a - \int\limits_{\mathcal{S}} \mathbf{n}\times\mathbf{m}_{\mathcal{S}}\,\mathrm{d}a, \tag{86}$$

while the equations $(85)_{1,2}$ represent the force and moment exerted by the environment on the pillbox surface $\mathcal{S}$. The force and moment balances for the pillbox therefore have the form

$$\left.\begin{aligned}
&\int_{\mathcal{S}} \mathbf{t}^{\mathrm{env}}_{\partial B}\,\mathrm{d}a - \int_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}\,\mathrm{d}a + \int_{\partial\mathcal{S}} \sigma\mathbf{P}\boldsymbol{\nu}\,\mathrm{d}s = \mathbf{0},\\
&\int_{\mathcal{S}} (\mathbf{r}\times\mathbf{t}^{\mathrm{env}}_{\partial B} + \mathbf{n}\times\mathbf{m}^{\mathrm{env}}_{\partial B})\,\mathrm{d}a - \int_{\mathcal{S}} (\mathbf{r}\times\mathbf{t}_{\mathcal{S}} + \mathbf{n}\times\mathbf{m}_{\mathcal{S}})\,\mathrm{d}a + \int_{\partial\mathcal{S}} \sigma\,\mathbf{r}\times\mathbf{P}\boldsymbol{\nu}\,\mathrm{d}s = \mathbf{0}.
\end{aligned}\right\} \tag{87}$$

We now localize these balances, starting with the force balance. The counterpart of (54) for tensor fields $\mathbf{A}$ that satisfy $\mathbf{A}\mathbf{n} = \mathbf{0}$ is

$$\int\limits_{\partial\mathcal{S}} \mathbf{A}\boldsymbol{\nu}\,\mathrm{d}s = \int\limits_{\mathcal{S}} \mathrm{div}_{\mathcal{S}}\mathbf{A}\,\mathrm{d}a. \tag{88}$$

Thus, since we have assumed that the surface tension σ is constant, we may use $(43)_1$ to conclude that

$$\int\limits_{\partial\mathcal{S}} \sigma\mathbf{P}\boldsymbol{\nu}\,\mathrm{d}s = \int\limits_{\mathcal{S}} 2\sigma H\mathbf{n}\,\mathrm{d}a$$

and $(87)_1$ becomes

$$\int\limits_{\mathcal{S}} (\mathbf{t}^{\mathrm{env}}_{\partial B} - \mathbf{t}_{\mathcal{S}} + 2\sigma H\mathbf{n})\,\mathrm{d}a = \mathbf{0};$$

thus, since $\mathcal{S}$ is an arbitrary subsurface of ∂B, we have the *local force balance for the boundary*:

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}^{\text{env}}_{\partial B} + 2\sigma H \mathbf{n}. \tag{89}$$

A slightly more complicated analysis results in the *local torque balance for the boundary*:[16]

$$\mathbf{n} \times \mathbf{m}_{\mathcal{S}} = \mathbf{n} \times \mathbf{m}^{\text{env}}_{\partial B}. \tag{90}$$

The hypertraction $\mathbf{m}^{\text{env}}_{\partial B}$ enters the theory through the torque balance (90) and since the normal part of $\mathbf{m}^{\text{env}}_{\partial B}$ is irrelevant to this balance, we assume without loss in generality that

$$\text{the hypertraction } \mathbf{m}^{\text{env}}_{\partial B} \text{ is tangent to } \partial B. \tag{91}$$

Thus, by (66), we may replace (90) by

$$\mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\text{env}}_{\partial B}. \tag{92}$$

Finally, by (63), the balances (89) and (92) expressed in terms of the stress $\mathbf{T}$ and hyperstress $\mathbf{G}$ have the form[17]

$$\left.\begin{aligned} &\mathbf{Tn} + \mathrm{div}_{\mathcal{S}}(\mathbf{Gn}\times) + \mathbf{n} \times (\mathrm{div}\,\mathbf{G} - 2H\mathbf{Gn}) = \mathbf{t}^{\text{env}}_{\partial B} + 2\sigma H \mathbf{n},\\ &\mathbf{n} \times \mathbf{Gn} = \mathbf{m}^{\text{env}}_{\partial B} \quad \text{or equivalently} \quad \mathbf{PGn} = -\mathbf{n} \times \mathbf{m}^{\text{env}}_{\partial B}; \end{aligned}\right\} \tag{93}$$

6 Free-energy imbalance

Let $\mathcal{R}(t)$ be an arbitrary region that convects with the body. We restrict attention to a purely mechanical theory based on the requirement that

(#) *the temporal increase in free energy of $\mathcal{R}(t)$ be less than or equal to the power expended on $\mathcal{R}(t)$.*

Precisely, letting ψ denote the *specific free-energy*, this requirement takes the form of a free-energy imbalance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\mathcal{R}(t)} \rho\psi \,\mathrm{d}v \le \mathcal{W}_{\text{ext}}(\mathcal{R}(t)). \tag{94}$$

The imbalance (94) is consistent with standard continuum thermodynamics based on balance of energy and an entropy imbalance (the Clausius–Duhem inequality): in that more general framework, granted isothermal conditions with temperature ϑ_0, (6) would be satisfied with right side minus left side equal to ϑ_0 times the entropy production.[18]

Balance of mass (36) implies that $(\mathrm{d}/\mathrm{d}t)\int_{\mathcal{R}(t)} \rho\psi \,\mathrm{d}v = \int_{R(t)} \rho\dot{\psi}\,\mathrm{d}v$. Thus since, by (47), $\mathcal{W}_{\text{ext}}(\mathcal{R}(t)) = \mathcal{W}_{\text{int}}(\mathcal{R}(t))$, we may use the expression $(45)_1$ defining the internal power $\mathcal{W}_{\text{int}}(\mathcal{R}(t))$ in conjunction with the symmetry of $\mathbf{T}$, to localize (94); the result is the *local free-energy imbalance*

$$\rho\dot{\psi} - \mathbf{T}:\mathbf{D} - \mathbf{G}:\mathrm{grad}\,\boldsymbol{\omega} \le 0 \quad (\rho\dot{\psi} - T_{ij}D_{ij} - G_{ij}\omega_{i,j} \le 0), \tag{95}$$

where $\mathbf{D}$ is the stretching defined in $(33)_1$. The difference

$$\Gamma \overset{\text{def}}{=} \mathbf{T}:\mathbf{D} + \mathbf{G}:\mathrm{grad}\,\boldsymbol{\omega} - \rho\dot{\psi} \ge 0 \tag{96}$$

represents the *bulk dissipation* and allows us to rewrite (94) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\mathcal{R}(t)} \rho\psi \,\mathrm{d}v - \mathcal{W}_{\text{ext}}(\mathcal{R}(t)) = -\int\limits_{\mathcal{R}(t)} \Gamma \,\mathrm{d}v \le 0. \tag{97}$$

The power expended, per unit volume, by the body force has the form

$$\mathbf{b} \cdot \mathbf{u} = -\tfrac{1}{2}\rho\overline{|\mathbf{u}|^2}^{\boldsymbol{\cdot}},$$

[16] Cf. the paragraph containing (5.12) in Anderson et al. [30].

[17] Cf. (38).

[18] Cf., e.g., (2.9) of Anderson et al. [30]; taking $\vartheta = \vartheta_0 =$ constant in $(2.9)_2$ of that reference and subtracting the resulting equation from $(2.9)_1$ yields (94).

and we may rewrite the external power expenditure as the sum of a non-inertial expenditure minus a kinetic-energy rate:

$$\mathcal{W}_{\rm ext}(\mathcal{R}(t)) = \int\limits_{\partial\mathcal{R}(t)} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{u} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{u}}{\partial n}\right) da - \frac{d}{dt}\int\limits_{\mathcal{R}(t)} \tfrac{1}{2}\rho|\mathbf{u}|^2\, dv. \tag{98}$$

By (98), the free energy imbalance (97)—for a control volume $\mathcal{R}(t)$ that convects with the fluid—takes the form of an imbalance of free and kinetic energy

$$\frac{d}{dt}\int\limits_{\mathcal{R}(t)} \rho\left(\psi + \tfrac{1}{2}|\mathbf{u}|^2\right) dv - \int\limits_{\partial\mathcal{R}(t)} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{u} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{u}}{\partial n}\right) da = -\int\limits_{\mathcal{R}(t)} \Gamma\, dv \le 0. \tag{99}$$

Further, using standard continuum mechanics, we may rewrite (99) as an imbalance for a control volume R; precisely, (99) is satisfied for all regions $\mathcal{R}(t)$ that convect with the body if and only if

$$\frac{d}{dt}\int\limits_{R} \rho(\psi + \tfrac{1}{2}|\mathbf{u}|^2)\, dv + \int\limits_{\mathcal{S}} \rho(\psi + \tfrac{1}{2}|\mathbf{u}|^2)\mathbf{u}\cdot\mathbf{n}\, da - \int\limits_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{u} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{u}}{\partial n}\right) da = -\int\limits_{R} \Gamma\, dv \le 0 \tag{100}$$

for all control volumes R.

7 Application to turbulent flow

Hereafter, we view the velocity $\mathbf{u}$ and the associated quantities

$$\mathbf{L} = \operatorname{grad}\mathbf{u}, \quad \mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^{\top}), \quad \mathbf{W} = \tfrac{1}{2}(\mathbf{L} - \mathbf{L}^{\top}), \quad \text{and} \quad \boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$$

as *filtered* fields.

We assume that the fluid is incompressible, so that

$$\rho = \text{constant} \quad \text{and} \quad \operatorname{div}\mathbf{u} = \operatorname{tr}\mathbf{D} = 0. \tag{101}$$

Without loss in generality, we may then suppose that

$$\mathbf{T} = \mathbf{S} - p\mathbf{1}, \quad \operatorname{tr}\mathbf{S} = 0, \tag{102}$$

where the *pressure* p is a constitutively indeterminate field that does not affect the internal power $(45)_1$;[19] the field $\mathbf{S}$ represents the *extra stress*. Then, by $(101)_2$,

$$\mathbf{T}:\mathbf{D} = \mathbf{S}:\mathbf{D}, \tag{103}$$

and the local free-energy imbalance (96) takes the form

$$\Gamma = \mathbf{S}:\mathbf{D} + \mathbf{G}:\operatorname{grad}\boldsymbol{\omega} - \rho\dot{\psi} \ge 0. \tag{104}$$

7.1 Basic assumptions

Let

$$\mathbf{L} = \operatorname{grad}\mathbf{u} \quad \text{and} \quad \mathbf{J} = \operatorname{grad}\boldsymbol{\omega} = \operatorname{grad}\operatorname{curl}\mathbf{u}. \tag{105}$$

In laying down relations for ψ, $\mathbf{S}$, and $\mathbf{G}$ relevant to the modeling of turbulent flow, we are guided by

(a) the Navier–Stokes-α model as discussed by Foias et al. [13];
(b) the theory of second-order fluids as developed in Sect. 3 of Dunn and Fosdick [18];[20]
(c) the local free-energy imbalance (104).

[19] Being associated with the constraint (101), p like $\mathbf{u}$ is to be viewed as a filtered field.
[20] Cf. Footnote 6.

An essential ingredient in (a) and (b) is a *specific free-energy* dependent on the stretching $\mathbf{D}$. But an energy of this form results in a term in the local free-energy imbalance (14) that is *linear in the rate* $\dot{\mathbf{D}}$, and experience with continuum theories tells us that this necessitates a dependence of the stress $\mathbf{S}$ on not only $\mathbf{D}$ but also $\dot{\mathbf{D}}$. But $\dot{\mathbf{D}}$ is *not*, by itself, frame-indifferent,[21] and thus, to allow for a dependence on any one of the frame-indifferent tensorial rates; i.e., the corotational rate of $\mathbf{D}$ defined by

$$\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{DW} - \mathbf{WD} \tag{106}$$

(or, among other possibilities, the convected rate $\dot{\mathbf{D}} + \mathbf{DL} + \mathbf{L}^{\top}\mathbf{D}$ or the contravariant rate $\dot{\mathbf{D}} - \mathbf{LD} - \mathbf{DL}^{\top}$), we allow for a dependence on $\mathbf{L} = \mathbf{D} + \mathbf{W}$. Finally, the term $\mathbf{G} : \operatorname{grad} \boldsymbol{\omega}$ in the local free-energy imbalance (14) would seem to indicate that the hyperstress $\mathbf{G}$ should be a dependent variable and that the vorticity gradient $\mathbf{J}$ should join the list of independent variables. Based on this discussion, we begin with relations for the specific free energy ψ, the extra stress $\mathbf{S}$, and the hyperstress $\mathbf{G}$ as functions of the form[22]

$$\left.\begin{aligned} \psi &= \hat{\psi}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J}), \\ \mathbf{S} &= \hat{\mathbf{S}}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J}), \\ \mathbf{G} &= \hat{\mathbf{G}}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J}). \end{aligned}\right\} \tag{107}$$

7.2 Consequences of frame-indifference and thermodynamic compatibility

We say that the relations (107) are *compatible with thermodynamics* if all flows related through (107) satisfy the local free-energy imbalance (104)—compatibility with thermodynamics is therefore equivalent to the requirement that the inequality

$$\rho\left(\frac{\partial \hat{\psi}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J})}{\partial \dot{\mathbf{D}}} : \ddot{\mathbf{D}} + \frac{\partial \hat{\psi}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J})}{\partial \mathbf{L}} : \dot{\mathbf{L}} + \frac{\partial \hat{\psi}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J})}{\partial \mathbf{J}} : \dot{\mathbf{J}}\right) - \hat{\mathbf{S}}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J}) : \mathbf{L} - \hat{\mathbf{G}}(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J}) : \mathbf{J} \le 0 \tag{108}$$

hold in all flows. The left side of this inequality is *linear* in the rates $\ddot{\mathbf{D}}$ and $\dot{\mathbf{J}}$; thus, emulating the argument of Dunn and Fosdick [18], we see that (108) can hold in all flows only if the coefficients $\partial \hat{\psi}/\partial \dot{\mathbf{D}}$ and $\partial \hat{\psi}/\partial \dot{\mathbf{J}}$ of the rates $\ddot{\mathbf{D}}$ and $\dot{\mathbf{J}}$ vanish.[23] Thus the specific free-energy must be independent of $\dot{\mathbf{D}}$ and $\mathbf{J}$ and hence given by a relation of the form

$$\psi = \hat{\psi}(\mathbf{L}). \tag{109}$$

Next, in view of the defining equation

$$\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{DW} - \mathbf{WD} \tag{110}$$

for the corotational rate of $\mathbf{D}$ and the identity $\mathbf{L} = \mathbf{D} + \mathbf{W}$ for the velocity gradient $\mathbf{L} = \operatorname{grad} \mathbf{u}$, it is clear that any function of $(\dot{\mathbf{D}}, \mathbf{L}, \mathbf{J})$ may be considered as a function of $(\overset{\circ}{\mathbf{D}}, \mathbf{L}, \mathbf{J})$, and vice versa. Thus, *without loss in generality*, we may replace the relations $(107)_{2,3}$ for the stresses by the relations[24]

$$\mathbf{S} = \hat{\mathbf{S}}(\overset{\circ}{\mathbf{D}}, \mathbf{L}, \mathbf{J}) \quad \text{and} \quad \mathbf{G} = \hat{\mathbf{G}}(\overset{\circ}{\mathbf{D}}, \mathbf{L}, \mathbf{J}). \tag{111}$$

Importantly, even though we have chosen a *particular* frame-indifferent rate for $\mathbf{D}$—namely the corotational rate (110)—the general relations (107) and (111) are equivalent.

Our initial hypothesis that the theory be invariant under changes in frame requires that the relations for ψ, $\mathbf{S}$, and $\mathbf{G}$ be so invariant. Specifically, denoting any one of these equations by the abstract relation $\boldsymbol{\Phi} = F(\boldsymbol{\Lambda})$, frame-indifference requires that

$$\boldsymbol{\Phi}^* = F(\boldsymbol{\Lambda}^*) \tag{112}$$

[21] That is, for some frame changes $\overline{\mathbf{D}^*}^{\,\cdot} \neq \mathbf{Q}\dot{\mathbf{D}}\mathbf{Q}^{\top}$.

[22] Cf. Footnote 7.

[23] Here, we rest content to simply sketch an argument analogous to that leading to (3.6) of Dunn and Fosdick [18].

[24] With a minor abuse of notation we use the same symbols $\hat{\mathbf{S}}$ and $\hat{\mathbf{G}}$ for the response functions in $(107)_{2,3}$ and (111).

for all changes in frame. To determine the consequences of this requirement it is useful to have at hand those transformation laws derived thus far, which are (74), (75), (77), and (81); viz.

$$\left.\begin{aligned} &\mathbf{L}^* = \mathbf{QLQ}^\top + \mathbf{Z}, \quad \mathbf{J}^* = \mathbf{QJQ}^\top, \quad \mathbf{D}^* = \mathbf{QDQ}^\top, \quad \overset{\circ}{\overline{\mathbf{D}^*}} = \mathbf{Q}\overset{\circ}{\mathbf{D}}\mathbf{Q}^\top, \\ &\mathbf{T}^* = \mathbf{QTQ}^\top, \quad \mathbf{G}^* = \mathbf{QGQ}^\top. \end{aligned}\right\} \tag{113}$$

Scalar fields are trivially invariant under changes in frame; thus applying (112) to the relations (109) and (111), we find that the functions $\hat{\psi}$, $\hat{\mathbf{S}}$, and $\hat{\mathbf{G}}$ must satisfy

$$\left.\begin{aligned} &\hat{\psi}(\mathbf{L}) = \hat{\psi}(\mathbf{QLQ}^\top + \mathbf{Z}), \\ &\mathbf{Q}\hat{\mathbf{S}}(\overset{\circ}{\mathbf{D}}, \mathbf{L}, \mathbf{J})\mathbf{Q}^\top = \hat{\mathbf{S}}(\mathbf{Q}\overset{\circ}{\mathbf{D}}\mathbf{Q}^\top, \mathbf{QLQ}^\top + \mathbf{Z}, \mathbf{QJQ}^\top), \\ &\mathbf{Q}\hat{\mathbf{G}}(\overset{\circ}{\mathbf{D}}, \mathbf{L}, \mathbf{J})\mathbf{Q}^\top = \hat{\mathbf{G}}(\mathbf{Q}\overset{\circ}{\mathbf{D}}\mathbf{Q}^\top, \mathbf{QLQ}^\top + \mathbf{Z}, \mathbf{QJQ}^\top), \end{aligned}\right\} \tag{114}$$

for all rotations $\mathbf{Q}$ and all skew tensors $\mathbf{Z}$. Taking $\mathbf{Q} = \mathbf{1}$ and $\mathbf{Z} = -\mathbf{W}$ we see that the dependencies on $\mathbf{L}$ must reduce to dependencies on $\mathbf{D}$, so that (109) and (111) reduce to

$$\psi = \hat{\psi}(\mathbf{D}), \quad \mathbf{S} = \hat{\mathbf{S}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, \mathbf{J}), \quad \mathbf{G} = \hat{\mathbf{G}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, \mathbf{J}). \tag{115}$$

Thus, by (114), the relations (115) must satisfy

$$\left.\begin{aligned} &\hat{\psi}(\mathbf{D}) = \hat{\psi}(\mathbf{QDQ}^\top), \\ &\mathbf{Q}\hat{\mathbf{S}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, \mathbf{J})\mathbf{Q}^\top = \hat{\mathbf{S}}(\mathbf{Q}\overset{\circ}{\mathbf{D}}\mathbf{Q}^\top, \mathbf{QDQ}^\top, \mathbf{QJQ}^\top), \\ &\mathbf{Q}\hat{\mathbf{G}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, \mathbf{J})\mathbf{Q}^\top = \hat{\mathbf{G}}(\mathbf{Q}\overset{\circ}{\mathbf{D}}\mathbf{Q}^\top, \mathbf{QDQ}^\top, \mathbf{QJQ}^\top), \end{aligned}\right\} \tag{116}$$

for all *rotations* $\mathbf{Q}$.

7.3 Isotropy

At this point it is tempting to view (116) as expressing isotropy of the turbulent flow, but that would not be correct: the standard view of isotropy requires invariance under the full orthogonal group, a group that contains—in addition to all rotations—the *reflection* $-\mathbf{1}$ and hence all tensors of the form $-\mathbf{Q}$ with $\mathbf{Q}$ a rotation. Basic to what follows is the assumption that

$$\text{the turbulent flow is } \textit{isotropic}, \tag{117}$$

and hence that the relations (115) are invariant under the full orthogonal group. Because frame-indifference renders the relations ψ, $\mathbf{S}$, and $\mathbf{G}$ be invariant under all rotations,[25] to establish isotropy we need only establish conditions that render these relations invariant under reflection.

Generally, tensors $\mathbf{A}$ transform to $\mathbf{A}^* = \mathbf{QAQ}^\top$ under the full orthogonal group, and among the kinematical fields this is true for the tensors $\overset{\circ}{\mathbf{D}}$ and $\mathbf{D}$,[26] *but it is not true for the tensor* $\mathbf{J}$. As is well known, the presence of the alternating symbol ε_{ijk} in the component form $J_{ip} = \varepsilon_{ijk} v_{k,jp}$ implies that the transformation law for $\mathbf{J}$ under an arbitrary orthogonal tensor $\mathbf{Q}$ has the form[27]

$$\mathbf{J}^* = q\mathbf{QJQ}^\top \quad \text{with } q = \det\mathbf{Q}; \tag{118}$$

thus, in particular, $\mathbf{J}^* = -\mathbf{J}$ under reflection. Further, arguing as in Sect. 4.2, we see that the stresses $\mathbf{T}$ and hence $\mathbf{S}$ are invariant under the full orthogonal group, but that $\mathbf{G}$ obeys the transformation law $\mathbf{G}^* = q\mathbf{QGQ}^\top$,

[25] Cf. (116).

[26] The relation $(116)_1$ for the specific free-energy is therefore isotropic.

[27] We could eliminate the use of the alternating symbol by developing the theory based on the *third-order tensor* $\mathbb{J} = \operatorname{grad}\mathbf{W}$; cf. $(33)_2$. Such a theory would be equivalent to the present theory (e.g., the transformation law for the third-order tensor $\mathbb{J}$ would remain $\mathbb{J}^* = -\mathbb{J}$ under reflection, but the resulting theory would be more complicated).

so that $\mathbf{G}^* = -\mathbf{G}$ under reflection. Summarizing, the fields involved in the relations (115) transform as follows under reflection:

$$\overset{\circ}{\mathbf{D}}, \mathbf{D}, \text{ and } \mathbf{S} \text{ are invariant}, \quad \mathbf{J}^* = -\mathbf{J}, \quad \mathbf{G}^* = -\mathbf{G}. \tag{119}$$

Thus the assumption of isotropy requires that the relations for $\mathbf{S}$ and $\mathbf{G}$ in (115) satisfy

$$\hat{\mathbf{S}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, \mathbf{J}) = \hat{\mathbf{S}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, -\mathbf{J}) \quad \text{and} \quad \hat{\mathbf{G}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, \mathbf{J}) = -\hat{\mathbf{G}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, -\mathbf{J}). \tag{120}$$

7.4 Linearity

As a final hypothesis we assume that the relations

$$\mathbf{S} = \hat{\mathbf{S}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, \mathbf{J}) \quad \text{and} \quad \mathbf{G} = \hat{\mathbf{G}}(\overset{\circ}{\mathbf{D}}, \mathbf{D}, \mathbf{J}) \quad \text{are linear.}$$

Then (120) implies that $\mathbf{S}$ is independent of $\mathbf{J}$, while $\mathbf{G}$ is independent of $\overset{\circ}{\mathbf{D}}$ and $\mathbf{D}$. We are therefore led to relations for $\mathbf{S}$ and $\mathbf{G}$ of the form

$$\mathbf{S} = \mathfrak{S}_1(\mathbf{D}) + \mathfrak{S}_2(\overset{\circ}{\mathbf{D}}) \quad \text{and} \quad \mathbf{G} = \mathfrak{H}(\mathbf{J}). \tag{121}$$

with $\mathfrak{S}_1$, $\mathfrak{S}_2$, and $\mathfrak{H}$ linear transformations. By (119) these transformations are automatically invariant under the central inversion. Thus we need only require that the relations (121) be invariant under arbitrary rotations.[28] We now use this invariance in conjunction with (a)–(c) of the *Representation Theorem* in Appendix A.1 to drastically reduce the relations (121).

To begin with, (a) applied separately to $\mathfrak{S}_1$ and $\mathfrak{S}_2$ yields a simple relation for the extra stress:

$$\mathbf{S} = 2\mu\mathbf{D} + 2\lambda\overset{\circ}{\mathbf{D}}, \tag{122}$$

with μ and λ constant scalar moduli.

Consider next the relation $\mathbf{G} = \mathfrak{H}(\mathbf{J})$. Let $\mathbf{J}_{\rm sy}$ and $\mathbf{J}_{\rm sk}$ denote the symmetric and skew parts of $\mathbf{J}$,

$$\mathbf{J} = \mathbf{J}_{\rm sy} + \mathbf{J}_{\rm sk}, \quad \mathbf{J}_{\rm sy} = \tfrac{1}{2}\left(\mathbf{J} + \mathbf{J}^{\top}\right), \quad \mathbf{J}_{\rm sk} = \tfrac{1}{2}\left(\mathbf{J} - \mathbf{J}^{\top}\right), \tag{123}$$

so that

$$\mathbf{G} = \mathfrak{H}(\mathbf{J}_{\rm sy}) + \mathfrak{H}(\mathbf{J}_{\rm sk}). \tag{124}$$

Consider first $\mathfrak{H}(\mathbf{J}_{\rm sy})$ as a mapping of symmetric and traceless tensors into (arbitrary) tensors. Then by (a) there is a scalar modulus ζ such that

$$\mathfrak{H}(\mathbf{J}_{\rm sy}) = \zeta\mathbf{J}_{\rm sy}. \tag{125}$$

To determine the form of the remaining function $\mathfrak{H}(\mathbf{J}_{\rm sk})$ we first decompose the *function* $\mathfrak{H}$ into symmetric and skew parts, so that

$$\mathfrak{H}(\mathbf{J}_{\rm sk}) = \mathfrak{H}_{\rm sy}(\mathbf{J}_{\rm sk}) + \mathfrak{H}_{\rm sk}(\mathbf{J}_{\rm sk}).$$

Then by (b) there is a scalar modulus ξ such that $\mathfrak{H}_{\rm sk}(\mathbf{J}_{\rm sk}) = \xi\mathbf{J}_{\rm sk}$. On the other hand, (c) implies that $\mathfrak{H}_{\rm sy}(\mathbf{J}_{\rm sk}) \equiv \mathbf{0}$, and hence that

$$\mathfrak{H}(\mathbf{J}_{\rm sk}) = \xi\mathbf{J}_{\rm sk}. \tag{126}$$

Thus by (124)–(126) the relation for the hyperstress has the form

$$\mathbf{G} = \zeta(\operatorname{grad}\boldsymbol{\omega})_{\rm sy} + \xi(\operatorname{grad}\boldsymbol{\omega})_{\rm sk}, \tag{127}$$

or, equivalently,

$$\mathbf{G} = \tfrac{1}{2}(\zeta + \xi)\operatorname{grad}\boldsymbol{\omega} + \tfrac{1}{2}(\zeta - \xi)(\operatorname{grad}\boldsymbol{\omega})^{\top}. \tag{128}$$

[28] Cf. the sentence containing (116).

Our final step is to derive those restrictions placed on the relations for ψ, $\mathbf{S}$, and $\mathbf{G}$ by the local free-energy imbalance (96). With this in mind we first note that $\mathbf{D}:(\mathbf{DW}) = \mathbf{D}^2:\mathbf{W} = 0$ and, similarly, that $\mathbf{D}:(\mathbf{WD}) = 0$; hence, by (110),

$$\mathbf{D}:\overset{\circ}{\mathbf{D}} = \mathbf{D}:\dot{\mathbf{D}}.$$

Thus substituting (122) and (127) into the augmented local free-energy imbalance (108), we arrive at the inequality

$$\left(2\lambda\mathbf{D} - \rho\frac{\partial\hat{\psi}(\mathbf{D})}{\partial\mathbf{D}}\right):\dot{\mathbf{D}} + 2\mu|\mathbf{D}|^2 + \zeta|(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sy}}|^2 + \xi|(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sk}}|^2 \geq 0. \tag{129}$$

Thus arguing as in the paragraph leading to (109), we note that we can always find a flow such that at any given point and time the fields $\mathbf{D}$, $\dot{\mathbf{D}}$, $(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sy}}$, and $(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sk}}$ have arbitrary values. Thus, assuming that

$$\hat{\psi}(\mathbf{0}) = 0, \tag{130}$$

we conclude that the inequality (129) is satisfied in all flows only if:

(i) the free energy per unit volume has the form

$$\rho\hat{\psi}(\mathbf{D}) = \lambda|\mathbf{D}|^2; \tag{131}$$

(ii) the moduli satisfy

$$\mu \geq 0, \quad \zeta \geq 0, \quad \xi \geq 0. \tag{132}$$

Conversely, granted (132), relations (122), (128), and (131) satisfy the inequality (108) and hence these relations are compatible with thermodynamics.

Finally, we assume that

$$\lambda > 0 \tag{133}$$

so that the free energy is a minimum when and only when the fluid is at rest, and we strengthen the inequalities (132) to the extent that

$$\mu > 0 \quad \text{and} \quad \zeta + \xi > 0. \tag{134}$$

Consistent with (134), we introduce length scales $\alpha > 0$ and $\beta > 0$ such that

$$\rho\alpha^2 = \lambda, \quad \mu\beta^2 = \tfrac{1}{2}(\zeta + \xi). \tag{135}$$

Further, we introduce a dimensionless parameter γ via

$$\gamma = \frac{\zeta - \xi}{\zeta + \xi} \tag{136}$$

and note, as a consequence of $(132)_{2,3}$, that

$$|\gamma| \leq 1. \tag{137}$$

Recalling from $(101)_1$ that ρ is constant, the assumption that μ, λ, ζ, and ξ be constant and the definitions (135) and (136) imply that α, β, and γ must also be constant.

In view of (135) and (136), the relations (122), (128), and (131) determining $\mathbf{S}$, $\mathbf{G}$, and ψ become

$$\left.\begin{aligned} \mathbf{S} &= 2(\mu\mathbf{D} + \rho\alpha^2\overset{\circ}{\mathbf{D}}),\\ \mathbf{G} &= \mu\beta^2(\operatorname{grad}\boldsymbol{\omega} + \gamma(\operatorname{grad}\boldsymbol{\omega})^{\top}),\\ \psi &= \alpha^2|\mathbf{D}|^2. \end{aligned}\right\} \tag{138}$$

It then follows from (96) that the bulk dissipation Γ is given by

$$\Gamma = 2\mu|\mathbf{D}|^2 + \mu\beta^2(1+\gamma)|(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sy}}|^2 + \mu\beta^2(1-\gamma)|(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sk}}|^2 \geq 0. \tag{139}$$

7.5 Flow equation

Bearing in mind (101), $(102)_1$, $(138)_1$, and the stipulation that α and μ are constant, we find that

$$\operatorname{div}\mathbf{T} = \operatorname{div}\mathbf{S} - \operatorname{grad}p = \mu\Delta\mathbf{u} + 2\rho\alpha^2\operatorname{div}\mathring{\mathbf{D}} - \operatorname{grad}p; \tag{140}$$

similarly, in view of $(101)_2$, $(138)_2$, and the stipulation that μ, β, and γ are constant, we find that

$$\begin{aligned}\operatorname{curl}\operatorname{div}\mathbf{G} &= \mu\beta^2\left[\operatorname{curl}(\operatorname{div}\operatorname{grad}\boldsymbol{\omega}) + \gamma\operatorname{curl}(\operatorname{div}(\operatorname{grad}\boldsymbol{\omega})^\top)\right]\\ &= \mu\beta^2\left[\operatorname{curl}(\Delta\boldsymbol{\omega}) + \gamma\operatorname{curl}(\operatorname{grad}\underbrace{\operatorname{div}\boldsymbol{\omega}}_{=0})\right]\\ &= \mu\beta^2\Delta(\operatorname{curl}\operatorname{curl}\mathbf{u})\\ &= \mu\beta^2\Delta(\operatorname{grad}\underbrace{\operatorname{div}\mathbf{u}}_{=0} - \Delta\mathbf{u}) = -\mu\beta^2\Delta\Delta\mathbf{u}.\end{aligned} \tag{141}$$

Using (140) and (141) in the local momentum balance (65), we arrive at the *flow equation*

$$\rho\dot{\mathbf{u}} = -\operatorname{grad}p + \mu(1-\beta^2\Delta)\Delta\mathbf{u} + 2\rho\alpha^2\operatorname{div}\mathring{\mathbf{D}}. \tag{142}$$

We refer to (142) as the *Navier–Stokes-$\alpha\beta$ equation*. The special choice $\beta = \alpha$ reduces (142) to the Navier–Stokes-α equation (1).

7.6 Navier–Stokes-$\alpha\beta$ equation as a system

A direct but lengthy calculation shows that the Navier–Stokes-$\alpha\beta$ equation (142) can be written alternatively as a system

$$\left.\begin{aligned}\rho\left(\frac{\partial\mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{u} + (\operatorname{grad}\mathbf{u})^\top\mathbf{v}\right) &= -\operatorname{grad}P + \mu(1-\beta^2\Delta)\Delta\mathbf{u},\\ \mathbf{v} &= (1-\alpha^2\Delta)\mathbf{u},\end{aligned}\right\} \tag{143}$$

where $\mathbf{v}$ and P can be interpreted as *unfiltered* velocity and pressure fields, with

$$P = p - \tfrac{1}{2}\rho(|\mathbf{u}|^2 + \alpha^2|\mathbf{D}|^2). \tag{144}$$

For $\beta = \alpha$, (143) reduces to the form of the Navier–Stokes-α equation most commonly encountered in the literature.[29]

7.7 Free-energy imbalance revisited

Next, we may use (139) and (135) to write the free-energy imbalance (100) (for a control volume R) in the form

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_R \rho(\tfrac{1}{2}|\mathbf{u}|^2 + \alpha^2|\mathbf{D}|^2)\,\mathrm{d}v + \int_{\mathcal{S}} \rho(\tfrac{1}{2}|\mathbf{u}|^2 + \alpha^2|\mathbf{D}|^2)\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}a - \int_{\mathcal{S}}\left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{u} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\right)\mathrm{d}a\\ &= -\int_R \mu(2|\mathbf{D}|^2 + \beta^2(1+\gamma)|(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sy}}|^2 + \beta^2(1-\gamma)|(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sk}}|^2)\,\mathrm{d}v \le 0.\end{aligned} \tag{145}$$

Turbulence is often studied assuming spatial periodicity and restricting attention to a control volume R consisting of a single cubic cell. We now derive the form of the free-energy imbalance (145) for a cubic cell in a spatial periodic flow. Since each face of such a cell must have bulk fields $\mathbf{u}$, $\operatorname{grad}\mathbf{u}$, …each equal to its value

[29] Cf. Footnote 2.

on the opposing face, while the outward normals on the two faces are equal and opposite, we may conclude, using (63), that[30]

$$\int\limits_{\mathcal{S}} \rho(\tfrac{1}{2}|\mathbf{u}|^2 + \alpha^2|\mathbf{D}|^2)\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}a = 0, \qquad \int\limits_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{u} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{u}}{\partial n}\right)\mathrm{d}a = 0. \tag{146}$$

Further, since $\operatorname{div}\mathbf{u} = 0$,

$$\int\limits_R \operatorname{grad}\mathbf{u} : (\operatorname{grad}\mathbf{u})^{\top}\,\mathrm{d}v = \int\limits_R \operatorname{div}\,((\operatorname{grad}\mathbf{u})\mathbf{u})\,\mathrm{d}v = \int\limits_{\mathcal{S}} \mathbf{n}\cdot(\operatorname{grad}\mathbf{u})\mathbf{u}\,\mathrm{d}a = 0;$$

hence

$$2\int\limits_R |\mathbf{D}|^2\,\mathrm{d}v = 2\int\limits_R |\mathbf{W}|^2\,\mathrm{d}v\,\mathrm{d}v = \int\limits_R |\boldsymbol{\omega}|^2\,\mathrm{d}v, \tag{147}$$

and a similar argument yields

$$2\int\limits_R |(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sy}}|^2\,\mathrm{d}v = 2\int\limits_R |(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sk}}|^2\,\mathrm{d}v = \int\limits_R |\operatorname{grad}\boldsymbol{\omega}|^2\,\mathrm{d}v. \tag{148}$$

Thus for R for a cubic cell in a spatially periodic flow the free-energy imbalance (100) has the simple form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_R \tfrac{1}{2}\rho(|\mathbf{u}|^2 + \alpha^2|\boldsymbol{\omega}|^2)\,\mathrm{d}v = -\int\limits_R \mu(|\boldsymbol{\omega}|^2 + \beta^2|\operatorname{grad}\boldsymbol{\omega}|^2)\,\mathrm{d}v \le 0 \tag{149}$$

and yields the conclusion that $\int_R \frac{1}{2}\rho(|\mathbf{u}|^2 + \alpha^2|\boldsymbol{\omega}|^2)\,\mathrm{d}v$ decreases with time.

If rather than periodic flow we have, instead, flow in a fixed container B with $\mathbf{u} = \mathbf{0}$ on ∂B, then (147) remains valid (with $R = B$), as does $(146)_1$, but, by $(44)_2$ and (66), $(146)_2$ is replaced by

$$\int\limits_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{u} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{u}}{\partial n}\right)\mathrm{d}a = \int\limits_{\partial B} (\mathbf{n}\times\mathbf{m}_{\mathcal{S}})\cdot\boldsymbol{\omega}\,\mathrm{d}a. \tag{150}$$

Thus the free-energy imbalance (100) takes a form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_B \tfrac{1}{2}\rho(|\mathbf{u}|^2 + \alpha^2|\boldsymbol{\omega}|^2)\,\mathrm{d}v &= \int\limits_{\partial B} (\mathbf{n}\times\mathbf{m}_{\partial B})\cdot\boldsymbol{\omega}\,\mathrm{d}a \\ &\quad - \int\limits_R \mu(2|\boldsymbol{\omega}|^2 + \beta^2(1+\gamma)|(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sy}}|^2 + \beta^2(1-\gamma)|(\operatorname{grad}\boldsymbol{\omega})_{\mathrm{sk}}|^2)\,\mathrm{d}v \le 0,\end{aligned}$$

in which all terms except that with integrand $\frac{1}{2}\rho|\mathbf{u}|^2$ are due solely to vorticity.

8 Boundary conditions

In this section we develop counterparts of the classical notions of a free surface and a fixed, impermeable surface without slip; that is, an impermeable surface at which the fluid abuts and adheres to a motionless, nondeformable environment. For convenience, when discussing free surfaces we neglect the pressure of the environment.[31]

[30] The fact that H is undefined at each corner is not a problem: simply replace each corner with a spherical cap of radius ϵ; then, since the area of each cap is $\mathrm{O}(\epsilon^2)$, while $H = \epsilon^{-1}$, the integral over each cap tends to zero as $\epsilon \to 0$.

[31] That is, we tacitly impose a normalization in which the pressure of the environment is taken to vanish.

8.1 General conditions

We begin by focusing on that portion

$$\int_{\partial B} \left(\mathbf{t}_{\mathcal{S}} \cdot \mathbf{u} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{u}}{\partial n} \right) \mathrm{d}a$$

of the external power expenditure (45)$_2$ associated with the boundary ∂B. Using the boundary force and moment balances[32]

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma H \mathbf{n} \quad \text{and} \quad \mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\mathrm{env}}_{\partial B} \tag{151}$$

and the identity (91) we can express the power expended on any subsurface $\mathcal{S}$ of ∂B as follows:

$$\mathcal{W}^{\mathrm{env}}_{\partial B}(\mathcal{S}) \stackrel{\mathrm{def}}{=} \int_{\mathcal{S}} \left((\mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma H \mathbf{n}) \cdot \mathbf{u} + \mathbf{m}^{\mathrm{env}}_{\partial B} \cdot \mathbf{P} \frac{\partial \mathbf{u}}{\partial n} \right) \mathrm{d}a, \tag{152}$$

with $\mathbf{P}$ the projection onto the plane tangent to ∂B defined by (37).[33] Further, in view of (151), (152), and the sentence containing (10), when $\mathbf{u} = \mathbf{0}$ on $\mathcal{S}$ the power expended by the environment on $\mathcal{S}_{\mathrm{nslp}}$ has the simple form

$$\mathcal{W}^{\mathrm{env}}_{\partial B}(\mathcal{S}) = \int_{\mathcal{S}} (\mathbf{n} \times \mathbf{m}^{\mathrm{env}}_{\partial B}) \cdot \boldsymbol{\omega} \, \mathrm{d}a. \tag{153}$$

Consider—in the integrands of (152) and (153)—each pairing $\mathfrak{f} \cdot \mathfrak{v}$ of a generalized force $\mathfrak{f}$ and a generalized velocity $\mathfrak{v}$. Experience with the principle of virtual power suggests that each such pairing leads to a possible boundary condition consisting of a prescription of either $\mathfrak{f}$, or $\mathfrak{v}$, or a relation between $\mathfrak{f}$ and $\mathfrak{v}$. Consistent with this, based on (93) we consider specific *boundary conditions* in which a portion $\mathcal{S}_{\mathrm{free}}$ of ∂B is a *free surface* and the remainder $\mathcal{S}_{\mathrm{nslp}}$ is a *fixed, impermeable surface without slip*:

(I) On $\mathcal{S}_{\mathrm{free}}$ the environmental tractions vanish ($\mathbf{t}^{\mathrm{env}}_{\partial B} = \mathbf{m}^{\mathrm{env}}_{\partial B} = \mathbf{0}$), so that[34]

$$\mathbf{T}\mathbf{n} + \mathrm{div}_{\mathcal{S}}(\mathbf{G}\mathbf{n}\times) + \mathbf{n} \times \mathrm{div}\,\mathbf{G} = 2\sigma H \mathbf{n} \quad \text{and} \quad \mathbf{n} \times \mathbf{G}\mathbf{n} = \mathbf{0} \quad \text{on } \mathcal{S}_{\mathrm{free}}. \tag{154}$$

(II) On $\mathcal{S}_{\mathrm{nslp}}$ the fluid velocity vanishes and the hypercouple $\mathbf{n} \times \mathbf{m}^{\mathrm{env}}_{\partial B}$ is prescribed, so that[35]

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad \mathbf{P}\mathbf{G}\mathbf{n} = -\mathbf{n} \times \mathbf{m}^{\mathrm{env}}_{\partial B} \quad \text{on } \mathcal{S}_{\mathrm{nslp}}. \tag{155}$$

8.2 The wall-eddy condition on $\mathcal{S}_{\mathrm{nslp}}$

Bearing in mind (153), the field $\mathbf{n} \times \mathbf{m}^{\mathrm{env}}_{\partial B}$ in (II) may be prescribed as a function of $\boldsymbol{\omega}$ (and possibly other fields). Here, we consider the simple relation[36]

$$\mathbf{n} \times \mathbf{m}^{\mathrm{env}}_{\partial B} = \mu \ell \boldsymbol{\omega} \tag{156}$$

or equivalently, by (155)$_2$,

$$\mathbf{P}\mathbf{G}\mathbf{n} = -\mu \ell \boldsymbol{\omega}. \tag{157}$$

We refer to (156) as the *wall-eddy condition* and to ℓ (which carries dimensions of length) as the *wall-eddy length*. In the wall-eddy condition $\mathbf{n} \times \mathbf{m}^{\mathrm{env}}_{\partial B}$ represents a hypercouple induced by the formation of eddies at a fixed, slip-free surface and hence arises in response to the shedding of vortices at the boundary. Further, the

[32] Cf. (89) and (92).
[33] Cf. (91).
[34] Cf. (93). In particular, (154)$_2$ follows directly from (93)$_2$ on requiring that $\mathbf{m}^{\mathrm{env}}_{\partial B} = \mathbf{0}$ and (154)$_1$ follows from (93)$_1$ and (154)$_2$ on requiring that $\mathbf{t}^{\mathrm{env}}_{\partial B} = \mathbf{0}$.
[35] The boundary condition (155)$_1$ precludes specifying a second boundary condition for $\mathbf{t}^{\mathrm{env}}_{\partial B}$. Under these circumstances, (93)$_1$ is a tautology determining the *effective* value of $\mathbf{t}^{\mathrm{env}}_{\partial B}$ needed to ensure satisfaction of force balance on $\mathcal{S}_{\mathrm{nslp}}$.
[36] When $\mathbf{u} = \mathbf{0}$, consistent with (156), $\boldsymbol{\omega} \cdot \mathbf{n} = 0$; cf. (44)$_1$ and the subsequent discussion.

wall-eddy condition requires that this hypercouple be parallel to the vorticity, which is itself tangent to the boundary. Note that (157) combines with $(138)_2$ to yield the wall-eddy condition in the form

$$\beta^2\mathbf{P}(\operatorname{grad}\boldsymbol{\omega} + \gamma(\operatorname{grad}\boldsymbol{\omega})^{\top})\mathbf{n} = -\ell\,\boldsymbol{\omega}. \tag{158}$$

By (156) the power expenditure by the hypercouple $\mathbf{n}\times\mathbf{m}^{\mathrm{env}}_{\partial B}$ in (153) has the form

$$(\mathbf{n}\times\mathbf{m}^{\mathrm{env}}_{\partial B})\cdot\boldsymbol{\omega} = \mu\ell\,|\boldsymbol{\omega}|^2. \tag{159}$$

The quantity $|\boldsymbol{\omega}|^2$ is known as the *enstrophy*; (159) therefore asserts that, at a wall, the power expended is proportional to the enstrophy associated with the shedding of vortices.

9 Weak formulation of the flow equation and boundary conditions

Because we work within a framework based on the principle of virtual power, it is fairly straightforward to derive a weak (variational) formulation of the flow equation and the boundary conditions discussed in Sect. 8. We begin by rewriting the virtual balance (60) with $R = B$ and with the tractions $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ specified, via the boundary force and moment balances (151):

$$\int\limits_{\partial B}\left((\mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma H\mathbf{n})\cdot\boldsymbol{\phi} + \mathbf{m}^{\mathrm{env}}_{\partial B}\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\right)\mathrm{d}a + \int\limits_{B}\mathbf{b}\cdot\boldsymbol{\phi}\,\mathrm{d}v = \int\limits_{B}(\mathbf{T}:\operatorname{grad}\boldsymbol{\phi} + \mathbf{G}:\operatorname{grad}\operatorname{curl}\boldsymbol{\phi})\,\mathrm{d}v. \tag{160}$$

As is customary when discussing boundary conditions of the form $(155)_1$, we restrict attention to virtual velocity fields $\boldsymbol{\phi}$ that are *kinematically admissible* in the sense that

$$\boldsymbol{\phi} = \mathbf{0} \quad \text{on } \mathcal{S}_{\mathrm{nslp}}. \tag{161}$$

Given such a field, granted the boundary conditions (154) and (155), (160) yields the *virtual balance*:

$$\int\limits_{\mathcal{S}_{\mathrm{free}}}\sigma H\mathbf{n}\cdot\boldsymbol{\phi}\,\mathrm{d}a + \int\limits_{\mathcal{S}_{\mathrm{nslp}}}\mathbf{m}^{\mathrm{env}}_{\partial B}\cdot\mathbf{P}\frac{\partial\boldsymbol{\phi}}{\partial n}\,\mathrm{d}a + \int\limits_{B}\mathbf{b}\cdot\boldsymbol{\phi}\,\mathrm{d}v = \int\limits_{B}(\mathbf{T}:\operatorname{grad}\boldsymbol{\phi} + \mathbf{G}:\operatorname{grad}\operatorname{curl}\boldsymbol{\phi})\,\mathrm{d}v. \tag{162}$$

The result (#) at the end of Sect. 3.1 then implies that, given any kinematically admissible $\boldsymbol{\phi}$, (162) *is equivalent to* (61) and hence to

$$\begin{aligned}&\int\limits_{\mathcal{S}_{\mathrm{free}}}\left((\sigma H\mathbf{n} - (\mathbf{T}\mathbf{n} + \operatorname{div}_{\mathcal{S}}(\mathbf{G}\mathbf{n}\times) - 2H\mathbf{n}\times\mathbf{G}\mathbf{n}))\cdot\boldsymbol{\phi} - (\mathbf{n}\times\mathbf{G}\mathbf{n})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\right)\mathrm{d}a\\ &\quad+\int\limits_{\mathcal{S}_{\mathrm{nslp}}}(\mathbf{m}^{\mathrm{env}}_{\partial B} - \mathbf{n}\times\mathbf{G}\mathbf{n})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\,\mathrm{d}a = -\int\limits_{B}(\operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G} + \mathbf{b})\cdot\boldsymbol{\phi}\,\mathrm{d}v\end{aligned} \tag{163}$$

Thus, arguing as in the steps leading to (62) and (63), we see that, by (6), the momentum balance (65) is satisfied in B, while the condition (154) and—in view of the equivalence relation in (93)—the condition $(155)_2$ is satisfied on ∂B.

Conversely, (65), (154), and (155) imply that (163) and (hence) (162) are satisfied for all kinematically admissible $\boldsymbol{\phi}$. Finally, as is clear from the discussion in Sect. 7, granted the relations (138), the momentum balance is equivalent to the flow equation (142). We have therefore established a *weak formulation* of the flow equation and the boundary conditions (154) and $(155)_2$:

- Granted (6) and the relations (138), the virtual balance (162) is satisfied for all kinematically admissible virtual fields $\boldsymbol{\phi}$ if and only if:
  - (i) the flow equation (142) is satisfied within the fluid;
  - (ii) the conditions (154) and $(155)_2$ are satisfied on the boundary of the fluid.

10 Free-energy imbalance at a wall

Recently, Fried and Gurtin [31] provided a general discussion of the use of a free-energy imbalance for a boundary pillbox to develop relations describing the interaction of a fluid with its environment. We now sketch the corresponding analysis, but only as it applies to the boundary conditions (155) for a fixed, impermeable surface without slip. Thus, let $\mathcal{S}$ denote a fixed (i.e. time-independent) subsurface of $\mathcal{S}_{\rm nslp}$. We find it useful to once again view $\mathcal{S}$ as a *boundary pillbox* of infinitesimal thickness as shown schematically in Fig. 1.

Let $\psi^{\rm x}$ denote the *excess free-energy*, measured per unit area, of the fluid at the surface $\mathcal{S}_{\rm nslp}$, so that

$$\int\limits_{\mathcal{S}} \psi^{\rm x}\,{\rm d}a \tag{164}$$

represents the net free-energy of the pillbox.

Consider next the power expended on the pillbox surface $-\mathcal{S}$ by the fluid. By $(68)_3$ and $(151)_2$, $\mathbf{m}_{-\mathcal{S}} = \mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\rm env}_{\partial B}$, the power expended *by the fluid* on the pillbox surface $-\mathcal{S}$ has the form

$$-\int\limits_{\mathcal{S}} (\mathbf{n}\times\mathbf{m}^{\rm env}_{\partial B})\cdot\boldsymbol{\omega}\,{\rm d}a. \tag{165}$$

We assume that the power expended by the environment on the pillbox surface $\mathcal{S}$ vanishes and, hence, that the environment is *passive*. Further, we neglect *hyperstresses within* the fluid-environment interface. Hence there is no expenditure of power on the lateral face of the pillbox. Thus, if we parallel the development in bulk with the requirement that the temporal increase in free energy of $\mathcal{S}$ be less than or equal to the power expended on $\mathcal{S}$, we arrive at the *free-energy imbalance*

$$\underbrace{\frac{\rm d}{{\rm d}t}\int\limits_{\mathcal{S}} \psi^{\rm x}\,{\rm d}a}_{\text{free-energy rate}} - \underbrace{\int\limits_{\mathcal{S}} (-(\mathbf{n}\times\mathbf{m}^{\rm env}_{\partial B})\cdot\boldsymbol{\omega})\,{\rm d}a}_{\text{power expenditure}} \le 0, \tag{166}$$

which should be compared with the bulk free-energy imbalance (94).

Since $\mathcal{S}$ is fixed, we may interchange the operations of integration and time differentiation in (166); thus, since $\mathcal{S}$ is arbitrary, if we appeal to (156), we are led to the inequality

$$\mu\ell\,|\boldsymbol{\omega}|^2 \le -\dot{\psi}^{\rm x}. \tag{167}$$

In our discussion of channel flow in Sect. 11.2 we note that the velocity field as characterized by our theory captures the observed features of turbulent channel flow *only if* the wall-eddy length obeys[37]

$$\ell > 0; \tag{168}$$

This conclusion is underlined by the fact that, for plane channel flow, the theory with $\ell > 0$ agrees well with the DNS simulations. On the other hand, for channel flow $\dot{\psi}^{\rm x} \equiv 0$, because the flow is steady and $\mathbf{u} \equiv \mathbf{0}$ at the wall. The free-energy imbalance (167) therefore becomes

$$\mu\ell\,|\boldsymbol{\omega}|^2 \le 0 \tag{169}$$

and is *violated* when $\ell > 0$. This observation would seem to indicate a conceptual error in the free-energy imbalance (166). In fact there is such an error!

Indeed, the field $\mathbf{u}$ is not the *actual* fluid velocity $\mathbf{v}$, but instead is a *filtered* velocity representing an average of $\mathbf{v}$; consequently, $\boldsymbol{\omega} = {\rm curl}\,\mathbf{u}$ represents a filtered vorticity. Thus—in terms more suggestive than precise—the left side of the free-energy imbalance (166) represents, for a pillbox $\mathcal{S}$, a difference of the form

$$\frac{\rm d}{{\rm d}t}\left\{\begin{matrix}\text{filtered free}\\ \text{energy of } \mathcal{S}\end{matrix}\right\} - \left\{\begin{matrix}\text{power expended on } \mathcal{S}\\ \text{over the filtered vorticity}\end{matrix}\right\} \tag{170}$$

[37] Cf. (184).

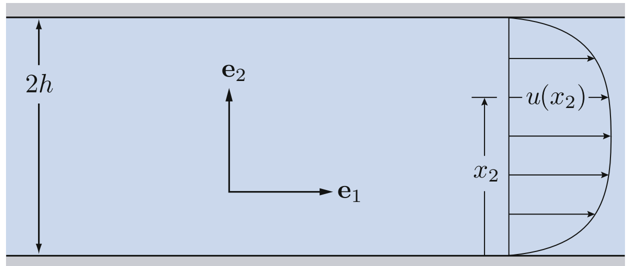

Fig. 2 Schematic for the problem of flow in a channel of gap $2h$. The coordinates in the directions downstream and out of the plane are, respectively, x_1 and x_3

and, hence, does not account for power and energy associated with the actual motion of the fluid at the small scales (i.e., those scales which have been filtered and are not included). While it is to be expected that a free-energy imbalance should be satisfied in any flow, laminar or turbulent, it would seem unreasonable to require that filtered variables obey a free-energy imbalance at a wall.

In principle, we may account for the power expenditures and energy rates at the small scales via a "supply term"

$$-\left\{\begin{array}{c}\text{effective supply of energy to } \mathcal{S} \text{ due}\\ \text{to behavior at the filtered scales}\end{array}\right\}; \tag{171}$$

in this manner we are led to consider a generalization of (166) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}} \psi^{\mathrm{x}}\,\mathrm{d}a - \int_{\mathcal{S}} (-(\mathbf{n}\times\mathbf{m}^{\mathrm{env}}_{\partial B})\cdot\boldsymbol{\omega})\,\mathrm{d}a - \left\{\begin{array}{c}\text{effective supply of energy to } \mathcal{S} \text{ due}\\ \text{to behavior at the filtered scales}\end{array}\right\} \le 0. \tag{172}$$

A theory accounting for behavior at the fine scales would be needed to determine a specific form of the effective energy supply. Based on this observation we do not consider (166) to be a viable free-energy imbalance and *consider the theory as complete without* (166).

Finally, if we assume that the flow at the wall is *dissipationless*, then the inequality in (172) becomes an equality and, granted that the effective supply of energy has a local form measured per unit area on $\mathcal{S}_{\mathrm{nslp}}$, we can trivially compute the local effective supply for channel flow:

$$\left\{\begin{array}{c}\textit{local} \text{ effective supply of energy to } \mathcal{S}\\ \text{due to behavior at the filtered scales}\end{array}\right\} = \mu\ell\,|\boldsymbol{\omega}|^2.$$

11 Flow in a rectangular channel

We now consider the problem of a steady, turbulent flow through an infinite, rectangular channel formed by two parallel walls separated by a gap $2h$ (Fig. 2). We suppose that the channel walls are fixed, impermeable, and without slip in the sense that the boundary conditions $(155)_1$ and (158) hold. This simple model problem allows us to investigate the effects of the parameters α, β, and ℓ and to make comparisons with numerical results.

11.1 Explicit solution of the channel problem

Employing the notation of Fig. 2, we assume that the filtered velocity $\mathbf{u}$ has the form

$$\mathbf{u}(\mathbf{x}) = u(x_2)\mathbf{e}_1; \tag{173}$$

$\mathbf{u}$ is therefore consistent with the constraint (101) of incompressibility and obeys $\dot{\mathbf{u}} = \mathbf{0}$. In view of (173), the Navier–Stokes-$\alpha\beta$ equation (142) gives

$$\mu(u - \beta^2 u'')'' = \frac{\partial p}{\partial x_1}, \quad 2\rho\alpha^2 u' u'' = \frac{\partial p}{\partial x_2}, \quad 0 = \frac{\partial p}{\partial x_3}, \tag{174}$$

while the no-slip and wall-eddy conditions (155)$_1$ and (158) give

$$u(0) = u(2h) = 0, \quad \beta^2 u''(0) = -\ell u'(0), \quad \beta^2 u''(2h) = \ell u'(2h). \tag{175}$$

In (174)–(175) and what follows a prime is used to denote differentiation with respect to the spanwise coordinate x_2.

Since u depends only on x_2, (174) implies that

$$p(x_1, x_2) = -Ax_1 + \rho\alpha^2 |u'(x_2)|^2, \tag{176}$$

with $A =$ constant. We assume, without loss of generality, that the pressure decreases with increasing x_2. It then follows that

$$A > 0. \tag{177}$$

Further, in view of (174), (175), and (176), u can be expressed as

$$u(x_2) = \frac{Ah^2}{2\mu}\left[1 - \left(1 - \frac{x_2}{h}\right)^2 + \frac{2B}{\frac{h}{\beta}}\left(1 - \frac{\cosh\frac{h}{\beta}\left(1 - \frac{x_2}{h}\right)}{\cosh\frac{h}{\beta}}\right)\right], \tag{178}$$

with

$$B = \frac{\frac{\ell}{\beta} - \frac{\beta}{h}}{1 - \frac{\ell}{\beta}\tanh\frac{h}{\beta}}. \tag{179}$$

To ensure that (178) is nonsingular, the wall-eddy length ℓ is assumed consistent with

$$\frac{\ell}{\beta}\tanh\frac{h}{\beta} \neq 1. \tag{180}$$

11.2 Behavior at the wall

Experiments and DNS simulations of channel flow show that, for suitably normalized laminar and turbulent velocity profiles, the slopes of the turbulent profiles at the channel walls have magnitudes greater than their laminar counterparts.[38] Consistent with this observation, we normalize u by its maximum value to yield

$$U(x_2) = \frac{u(x_2)}{u(h)}. \tag{181}$$

For comparison, we introduce

$$U_c(x_2) = 1 - \left(1 - \frac{x_2}{h}\right)^2, \tag{182}$$

which is the analogous normalization of the laminar solution to the plane channel problem. Then, $U'(0) > U_c'(0)$ if and only if

$$B = \frac{\frac{\ell}{\beta} - \frac{\beta}{h}}{1 - \frac{\ell}{\beta}\tanh\frac{h}{\beta}} > 0. \tag{183}$$

Since $\beta > 0$ and $h > 0$ it follows that u as defined by (178) captures the observed features of turbulent channel flow only if the wall-eddy length obeys

$$\ell > \frac{\beta^2}{h} > 0. \tag{184}$$

[38] Cf., e.g., Pope [19].

Table 1 Values of h/β, ℓ/β, and θ determined by fitting u^+ to the DNS data of Kim et al. [20] and Moser et al. [21] for the nominal values $Re_\tau = 180$, $Re_\tau = 395$, and $Re_\tau = 590$ of the friction Reynolds number

Re_τ	h/β	ℓ/β	θ
180	16.6	0.957	0.0583
395	34.8	0.974	0.0336
590	48.1	0.980	0.0239

11.3 Comparison with DNS data for the velocity profile

Assuming that wall-eddy length ℓ obeys (184), we now compare the analytical solution u to the problem for channel flow to the mean downstream velocity for turbulent channel flow as predicted by the DNS data of Kim et al. [20] and Moser et al. [21].

To facilitate comparisons, we employ standard definitions for the *friction velocity* u_τ, *friction Reynolds number* Re_τ, and the *viscous length* y^+:

$$u_\tau = \sqrt{\frac{\tau_\mathrm{w}}{\rho}}, \quad Re_\tau = \frac{\rho h u_\tau}{\mu}, \quad \text{and} \quad y^+ = \frac{Re_\tau}{h} x_2, \tag{185}$$

with $\tau_\mathrm{w} > 0$ being the *wall shear stress*.[39] In addition, we introduce a dimensionless velocity u^+ via

$$u^+(y^+) = \frac{1}{u_\tau} u\left(\frac{h}{Re_\tau} y^+\right), \tag{186}$$

with u as given by (178). In view of (185) and (186),

$$u^+(y^+) = \frac{Re_\tau \theta}{2}\left[1 - \left(1 - \frac{y^+}{Re_\tau}\right)^2 + \frac{2B}{\frac{h}{\beta}}\left(1 - \frac{\cosh \frac{h}{\beta}\left(1 - \frac{y^+}{Re_\tau}\right)}{\cosh \frac{h}{\beta}}\right)\right], \tag{187}$$

where θ is defined by the pressure drop A, the channel half-gap h, and the wall shear stress τ_w by

$$\theta = \frac{Ah}{\tau_\mathrm{w}}. \tag{188}$$

We use the nonlinear least-squares method to fit u^+ as defined by (186) to the average downstream velocity profile determined by the DNS simulations of Kim et al. [20] and Moser et al. [21] for the nominal values $Re_\tau = 180$, $Re_\tau = 395$, and $Re_\tau = 590$ of the friction Reynolds number. The values of the parameters h/β, ℓ/β (note from (179) that B as defined in (179) depends on both h/β and ℓ/β), and θ determined by these fits are listed in Table 1 and plots of u^+ corresponding to these fits are shown, along with the DNS data, in Fig. 3. These data show that the ratios ℓ/h and β/h are on the order of 10^{-2}. These ratios therefore correspond to dimensionless lengths in the lower half of the buffer layer.

The second and third columns of Table 1 combine to yield data relating ℓ/h to Re_τ. A power-law fit then shows that $\ell/h \sim Re_\tau^{-0.882}$ (Fig. 4). If we invoke Blasius' [22] empirical resistance law $Re_\tau \sim Re^{7/8}$, we find that

$$\frac{\ell}{h} \sim Re^{-0.772}, \tag{189}$$

where Re denotes the Reynolds number. If we identify the channel half-gap h with the integral length L and ℓ with the Kolmogorov microscale η the result (189) is then strikingly reminiscent of Kolmogorov's [23–25] classical scaling relation

$$\frac{\eta}{L} \sim Re^{-3/4} \tag{190}$$

for the ratio of the smallest to largest length scales present in a turbulent flow. Conversely, supposing that $\ell/h \sim Re^{-3/4}$ and using the relation $\ell/h \sim Re_\tau^{-0.882}$, we find that $Re_\tau \sim Re^{0.850}$ in close agreement with Blasius' [22] resistance law.

[39] Throughout this section, we employ the terminology and notation of Pope [19].

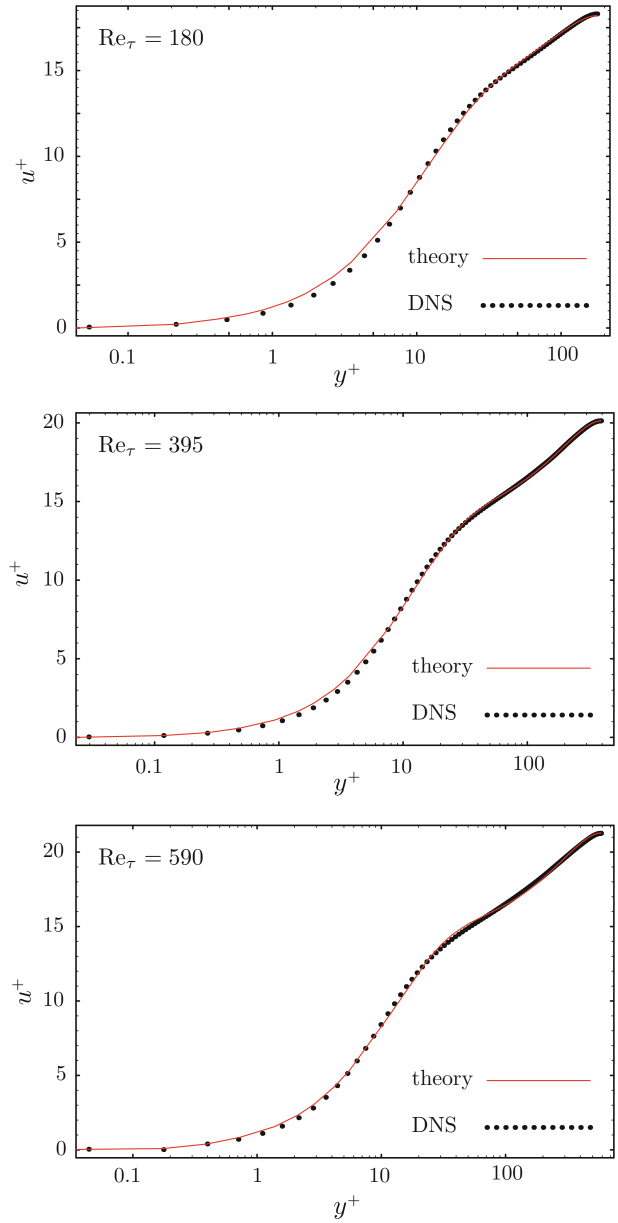

Fig. 3 Comparison of the dimensionless velocity u^+ with the downstream velocity determined by the DNS simulations of Kim et al. [20] and Moser et al. [21] for the nominal values $Re_\tau = 180$, $Re_\tau = 395$, and $Re_\tau = 590$ of the friction Reynolds number

Another interesting feature of the data in Table 1 is that it suggests that ℓ increases monotonically with Re_τ and should most likely obey the limit

$$\lim_{Re_\tau \to \infty} \ell = \beta. \tag{191}$$

Granted (191), the wall-eddy length ℓ would be less than or equal to the dissipative length scale β:

$$\ell \leq \beta. \tag{192}$$

This is consistent with the view that the distribution of eddy scales represented near the boundary of a flow domain should be dominated by the smallest scales present in the flow, as seen for example in extensive statistical studies of DNS data recently reported by Das et al. [26].

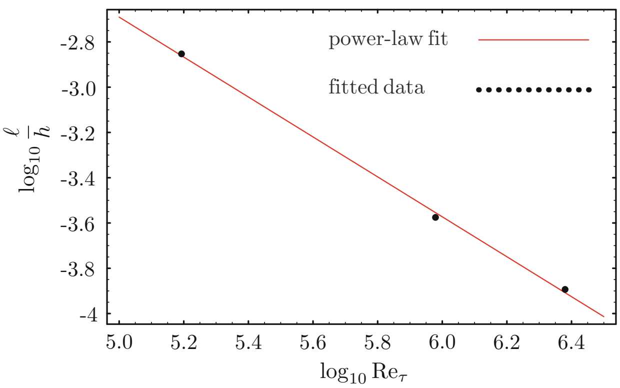

Fig. 4 Plot of $\log_{10}(\ell/h)$ versus $\log Re_\tau$, as determined by the fitted data in Table 1. The straight line shows a power-law fit of the form $\ell/h \sim Re_\tau^{-0.882}$, with $\chi^2 = 1.11 \times 10^{-4}$

Granted (192), it then follows from (184) that the wall-eddy length must obey

$$\ell < h. \tag{193}$$

This inequality is certainly consistent with the scaling relation (189).

11.4 Comparison with DNS data for the Reynolds shear stress in the plane of the channel

Like Chen et al. [4], we consider the Reynolds shear stress in the downstream plane of the channel. On identifying u with the mean downstream velocity in turbulent channel flow and writing the velocity field as $u\mathbf{e}_1 + \mathbf{w}$, with $\mathbf{w}$ the *fluctuating velocity*, the downstream component of the Reynolds-averaged Navier–Stokes equations is

$$\mu u'' - \rho \langle w_1 w_2 \rangle' = \frac{\partial p}{\partial x_1}, \tag{194}$$

where

$$\langle w_1 w_2 \rangle(0) = \langle w_1 w_2 \rangle(2h) = 0 \tag{195}$$

and $\partial p/\partial x_1$ is constant. Integration of (194) yields

$$-\rho \langle w_1 w_2 \rangle(x_2) = \mu u'(0)\left(1 - \frac{x_2}{h}\right) - \mu u'(x_2). \tag{196}$$

To make appropriate comparisons, we introduce the dimensionless Reynolds shear stress

$$\begin{aligned} \tau^+(y^+) &= -\frac{1}{\tau_{\mathrm{w}}} \langle w_1 w_2 \rangle \left(\frac{h}{Re_\tau} y^+\right) \\ &= \frac{\mathrm{d}u^+(y^+)}{\mathrm{d}y^+}\bigg|_{y^+=0} \left(1 - \frac{y^+}{Re_\tau}\right) - \frac{\mathrm{d}u^+(y^+)}{\mathrm{d}y^+}. \end{aligned} \tag{197}$$

Plots of τ^+ corresponding to these fits are shown, along with the DNS data, in Fig. 5. In contrast to the plots of u^+, which exhibit very close agreement with the data, the plots of τ^+ agree with the data only for $y^+ \gtrsim 70$—that is, for y^+ outside the viscous wall region. As Chen et al. [4] note in similar work concerning the Navier–Stokes-α equation, this discrepancy might be attributed to the presence of statistically inhomogeneous and/or anisotropic fluctuations within the viscous wall region.

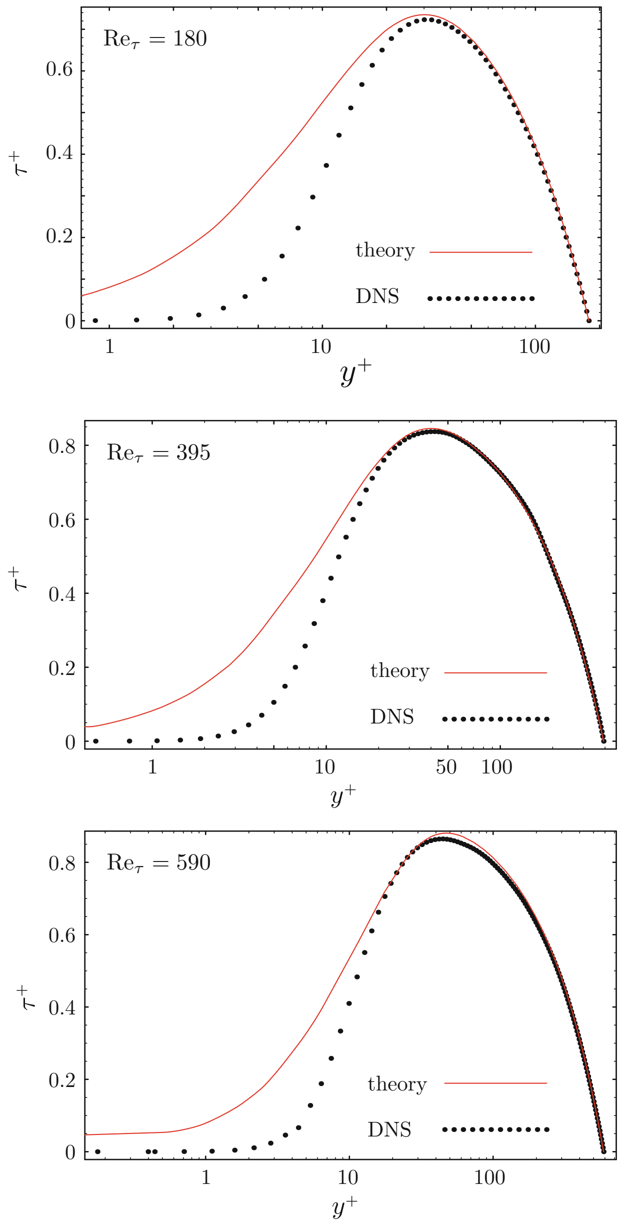

Fig. 5 Comparison of the dimensionless Reynolds shear stress τ^+ with the in-plane Reynolds shear stress determined by the DNS simulations of Kim et al. [20] and Moser et al. [21] for the nominal values $Re_\tau = 180$, $Re_\tau = 395$, and $Re_\tau = 590$ of the friction Reynolds number

11.5 Comparison with DNS data for the turbulent kinetic-energy profile

Due to the idealized kinematics of plane channel flow, the velocity field as determined by (173) and (178) is independent of the energetic length scale α. Information concerning that scale can nonetheless be obtained by identifying the specific free-energy $\psi = \alpha^2|\mathbf{D}|^2$ with the specific turbulent kinetic-energy. With this identification, we will find that agreement with the DNS data requires that the energetic length scale α be substantially larger than the dissipative length scale β.

By (173) and (178), $\psi = \alpha^2|\mathbf{D}|^2 = \alpha^2|u'|^2/2$. Thus introducing the dimensionless specific free-energy $k^+ = 2\psi/u_\tau^2$ and using the nondimensionalization (185)–(186), we find that

$$k^+(y^+) = \frac{\alpha^2 Re_\tau^2}{2h^2}\left|\frac{\mathrm{d}u^+(y^+)}{\mathrm{d}y^+}\right|^2. \tag{198}$$

Table 2 Values of α/h determined by fitting k^+ the turbulent kinetic-energy determined by the DNS simulations of Kim et al. [20] and Moser et al. [21] for the nominal values $Re_\tau = 180$, $Re_\tau = 395$, and $Re_\tau = 590$ of the friction Reynolds number

Re_τ	α/h
180	0.359
395	0.258
590	0.237

Taking the previously obtained values of h/β, ℓ/β, and θ, we use the nonlinear least-squares method to fit k^+ to the specific turbulent kinetic-energy (i.e., one-half the trace of the Reynolds stress tensor) determined by the DNS simulations of Kim et al. [20] and Moser et al. [21]. The values of α/h determined by these fits are listed in Table 2 and plots of k^+ corresponding to these fits are shown, along with the DNS data, in Fig. 6. Quite interestingly, the values of α/h coincide approximately with the upper bound of the log-law region.

Although the overall trend of the fits agrees with the data, their detailed features show deviations. In particular, the fitted peak values of the turbulent kinetic-energy occur too far from the channel walls and are too low. As a consequence, the turbulent kinetic-energy is too low in most of the buffer layer and too high in the log-law region. Also, the fitted turbulent kinetic-energy vanishes, incorrectly, at the center of the channel. These shortcomings might be attributed to the one-dimensional nature of the analytical model. Bearing in mind that the DNS simulations are three-dimensional, the fits are unexpectedly good.

Combining the second columns of Tables 1 and 2, we arrive at data relating α/β and Re_τ. A power-law fit then shows that $\alpha/\beta \sim Re_\tau^{0.538}$ (Fig. 7) and if we again invoke Blasius' [22] resistance law, we find that

$$\frac{\alpha}{\beta} \sim Re^{0.471}. \tag{199}$$

For turbulent flow ($Re \gg 1$), this result suggests that dissipative length scale β should be less than the energetic length scale α, viz.,

$$\beta < \alpha. \tag{200}$$

When discussing turbulence, it is conventional to divide the range of eddy scales into *integral*, *inertial*, and *dissipation subranges*.[40] The integral scales are the largest and are associated with external driving forces. The dissipative scales are the smallest and are associated with the conversion of kinetic energy into heat. The intermediate, inertial, scales are commonly thought to be dissipationless. It seems reasonable to expect that the energetic length α should represent a characteristic average of the eddy scales within the inertial subrange whereas β should represent a characteristic average of the eddy scales within the dissipation subrange, in which case α and β would obey (200).

The importance of allowing the energetic and dissipative length scales to differ is underscored by the foregoing results. For the Navier–Stokes-α model, $\alpha = \beta$ is determined by fitting the velocity profile. Since the values of β/h are less than those of α/h by two orders-of-magnitude, the corresponding peak values of the dimensionless specific free-energy for the Navier–Stokes-α model must be lower by four orders of magnitude than those obtained for the Navier–Stokes-$\alpha\beta$ model. In this sense, it would be unphysical to identify the specific free-energy with the specific turbulent kinetic-energy in the Navier–Stokes-α model.

12 Discussion

The generalization of the Navier–Stokes-α model discussed here involves the Navier–Stokes-$\alpha\beta$ equation

$$\rho\dot{\mathbf{u}} = -\mathrm{grad}\, p + \mu(1 - \beta^2\Delta)\Delta\mathbf{u} + 2\rho\alpha^2 \mathrm{div}\, \mathring{\mathbf{D}} \tag{201}$$

for the filtered velocity $\mathbf{u}$, with $\mathbf{D} = \frac{1}{2}(\mathrm{grad}\,\mathbf{u} + (\mathrm{grad}\,\mathbf{u})^\top)$, and, for a confined flow, the no-slip boundary condition

$$\mathbf{u} = \mathbf{0} \tag{202}$$

and the wall-eddy condition

$$\beta^2\mathbf{P}(\mathrm{grad}\,\boldsymbol{\omega} + \gamma(\mathrm{grad}\,\boldsymbol{\omega})^\top)\mathbf{n} = -\ell\boldsymbol{\omega}, \tag{203}$$

[40] Aside from the classical contributions of Richardson [32] and Kolmogorov [23–25], see Pope [19].

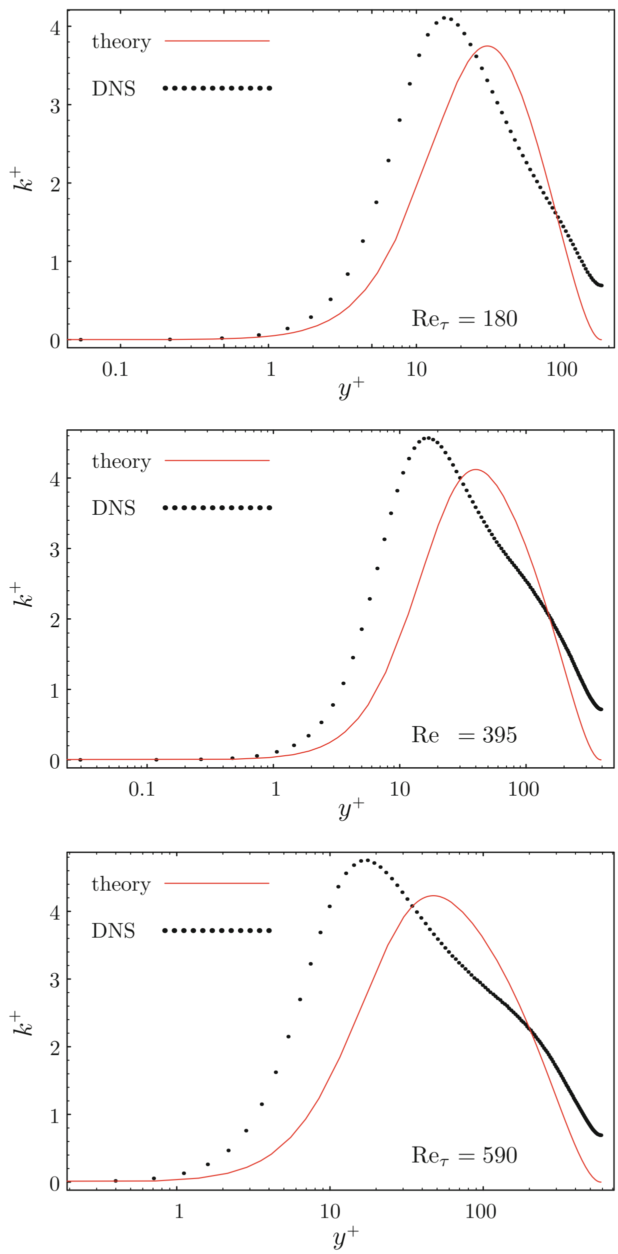

Fig. 6 Comparison of the dimensionless specific free-energy k^+ with the turbulent kinetic-energy determined by the DNS simulations of Kim et al. [20] and Moser et al. [21] for the nominal values $Re_\tau = 180$, $Re_\tau = 395$, and $Re_\tau = 590$ of the friction Reynolds number

where $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$ is the filtered vorticity, $\mathbf{n}$ denotes the outward unit normal to the boundary, and γ is dimensionless and consistent with $|\gamma| \le 1$.

Written as a system for the filtered and unfiltered velocities $\mathbf{u}$ and $\mathbf{v}$, (201) takes the form

$$\left.\begin{aligned} \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{u} + (\operatorname{grad}\mathbf{u})^{\top}\mathbf{v}\right) &= -\operatorname{grad} P + \mu(1-\beta^2\Delta)\Delta\mathbf{u}, \\ \mathbf{v} &= (1-\alpha^2\Delta)\mathbf{u}, \end{aligned}\right\} \tag{204}$$

where $P = p - \frac{1}{2}\rho(|\mathbf{u}|^2 + \alpha^2|\mathbf{D}|^2)$ is the unfiltered pressure.

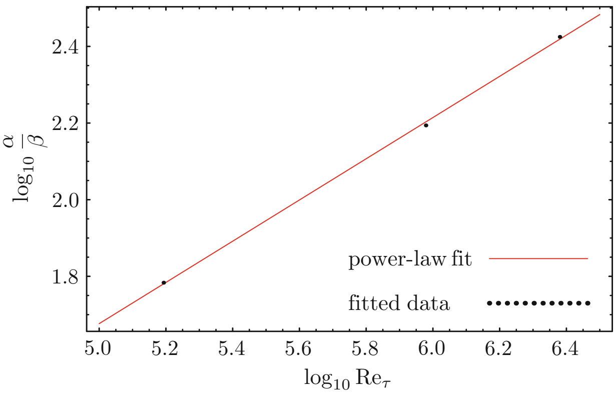

Fig. 7 Plot of $\log_{10}(\alpha/\beta)$ versus $\log Re_\tau$, as determined by the fitted data in Tables 1 and 2. The straight line shows a power-law fit of the form $\alpha/\beta \sim Re_\tau^{0.538}$, with $\chi^2 = 2.43 \times 10^{-4}$

For the particular choice $\beta = \alpha$, (201) and its equivalent (204) reduce to the Navier–Stokes-α equation. The wall-eddy condition (203) is a special case of our general condition $(155)_2$ for a fixed, impermeable surface without slip. Conditions for a free surface are given in (154).

Our consideration of the channel-flow problem demonstrates that the energetic length scale α, the dissipative length scale β, and the wall-eddy length ℓ should be viewed as problem-dependent flow parameters rather than as constitutive moduli that characterize different fluids. Conventional views concerning the distribution of eddy scales in the inertial and dissipative subranges suggest that $\beta < \alpha$. Consistent with this view, we find—based on an identification between the specific free-energy $\psi = \alpha^2|\mathbf{D}|^2$ with the specific turbulent kinetic-energy—that

$$\frac{\alpha}{\beta} \sim Re^{0.471} \tag{205}$$

and, thus, since $Re \gg 1$ for turbulent flows, that

$$\beta < \alpha. \tag{206}$$

Even for low Reynolds number turbulent flows, α must therefore be substantially larger than β. Furthermore, consideration of the velocity profile indicates that

$$\frac{\ell}{h} \sim Re^{-0.772} \tag{207}$$

and, thus, since $Re \gg 1$ for turbulent flows, that

$$\ell \leq \beta. \tag{208}$$

In combination, the hierarchy

$$\ell \leq \beta < \alpha \tag{209}$$

of scales implied by (206) and (208) should apply under more generic flow conditions remains a matter for further investigation. We are currently using numerical methods to explore this important issue.

Acknowledgments This work was supported by the U.S. Department of Energy. We thank Bob Moser for providing us with the data from the DNS simulations of Kim et al. [20] and Moser et al. [21] and Xuemei Chen for her careful reading and suggested improvements to various drafts of this paper.

Appendix A

A.1 Some representations for linear tensor functions that are invariant under the group of all rotations

Let

$$\text{Lin}, \quad \text{Sym}, \quad \text{Sym}_0, \quad \text{Skw},$$

respectively, denote the following spaces of (second-order) tensors: all tensors; all symmetric tensors; all symmetric tensors that are traceless; all skew tensors. We now list some results that are basic in our discussion of relations for the specific free-energy ψ, the extra stress $\mathbf{S}$, and the hyperstress $\mathbf{G}$.

Representation Theorem In (a)–(c) below $\mathfrak{L}$ represents a linear transformation of tensors into tensors such that, for each $\mathbf{A}$ in the domain of $\mathcal{L}$,

$$\mathbf{Q}(\mathcal{L}\mathbf{A})\mathbf{Q}^{\top} = \mathcal{L}(\mathbf{Q}\mathbf{A}\mathbf{Q}^{\top}) \tag{210}$$

for all *rotations* $\mathbf{Q}$.

(a) Let $\mathfrak{L} : \text{Sym}_0 \to \text{Lin}$. Then there is a scalar constant c such that

$$\mathfrak{L}(\mathbf{A}) = c\mathbf{A} \quad \text{for all } \mathbf{A} \in \text{Sym}_0.$$

(b) Let $\mathfrak{L} : \text{Skw} \to \text{Skw}$. Then there is a scalar constant d such that

$$\mathfrak{L}(\mathbf{A}) = d\mathbf{A} \quad \text{for all } \mathbf{A} \in \text{Skw}. \tag{211}$$

(c) Let $\mathfrak{L} : \text{Skw} \to \text{Sym}_0$. Then $\mathfrak{L} \equiv \mathbf{0}$.

Proof (a) Here we refer to Gurtin [29]; in particular, to the corollary on p. 236 of the representation theorem for linear isotropic functions on p. 235. This representation theorem is based on the transfer theorem on p. 231 of Gurtin [29] and, consequently, the range Sym of the mapping $\mathbf{G}$ in its statement may be replaced by Lin.

(b) By (32), given any skew tensor $\mathbf{A}$, there is a unique vector $\mathbf{a}$ such that $\mathbf{A} = (\mathbf{a}\times)$. Moreover, for any rotation $\mathbf{Q}$,[41]

$$\mathbf{Q}\mathbf{A}\mathbf{Q}^{\top} = [(\mathbf{Q}\mathbf{a})\times]. \tag{212}$$

Thus, by linearity, $\mathfrak{L}$ can be considered as a linear mapping $\mathfrak{l}$ of vectors into vectors consistent with $\mathbf{Q}\mathfrak{l}(\mathbf{a}) = \mathfrak{l}(\mathbf{Q}\mathbf{a})$ for all rotations $\mathbf{Q}$. Hence, as is well known, there is a constant e such that $\mathfrak{l}(\mathbf{a}) = e\mathbf{a}$ for every vector $\mathbf{a}$, which implies (211).

(c) Use (32) to convert $\mathfrak{L}$ to a mapping from the underlying vector space to Sym_0, a mapping that satisfies $\mathbf{Q}\mathfrak{L}(\mathbf{a})\mathbf{Q}^{\top} = \mathfrak{L}(\mathbf{Q}\mathbf{a})$ for all rotations $\mathbf{Q}$. Then

$$\mathbf{H} = \mathcal{L}\mathbf{a} \quad \text{has the component representation} \quad H_{ij} = L_{ijk}a_k$$

relative to a "right-handed" orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and these components satisfy

$$L_{ijk} = \mathbf{e}_i \otimes \mathbf{e}_j : \mathcal{L}(\mathbf{e}_k), \quad L_{ijk} = L_{jik}. \tag{213}$$

Then using the rotation $\mathbf{Q}$ that satisfies

$$\mathbf{e}_2 = \mathbf{Q}\mathbf{e}_1, \quad \mathbf{e}_3 = \mathbf{Q}\mathbf{e}_2, \quad \mathbf{e}_1 = \mathbf{Q}\mathbf{e}_3, \tag{214}$$

we see that

$$L_{123} = L_{231} = L_{312}. \tag{215}$$

Consider next a right-handed rotation of 180° about $\mathbf{e}_1$:

$$\mathbf{Q}\mathbf{e}_1 = \mathbf{e}_1, \quad \mathbf{Q}\mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{Q}\mathbf{e}_3 = -\mathbf{e}_2. \tag{216}$$

This yields $L_{123} = -L_{132}$, while (213) and (215) imply that $-L_{132} = -L_{312} = -L_{123}$. Thus L_{123} and L_{132} both vanish, and proceeding in this manner we see that $L_{ijk} = 0$ whenever ijk is an odd or even permutation of 123. Next, applying (216) twice we find that $L_{322} = -L_{233} = -L_{322}$ and repeating this computation using (216) with the integers 123 permuted evenly we see that $L_{ijk} = 0$ whenever exactly

[41] Cf., e.g., Eq. (2.11) of Cermelli and Gurtin [33].

two of its subscripts are equal (no sum intended). Finally, by (216) applied twice, $L_{222} = -L_{333} = -L_{222}$ and arguing as above we see that $L_{ijk} = 0$ when all three of its subscripts are the same. Thus $\mathcal{L} = \mathbf{0}$ and the proof of the Representation Theorem is complete.

A.2 The virtual power balance within the Cosserat framework; Cosserat stress and couple balances and the free-energy imbalance

Assume that the stress $\mathbf{T}$, hyperstress $\mathbf{G}$, and body force $\mathbf{b}$ are consistent with the local force balance (62) (or equivalently (65)) as well as the symmetry condition

$$\mathbf{T} = \mathbf{T}^{\top}$$

implied by frame-indifference. Then, by (49), whose derivation is based on the explicit form (45) of the internal power $\mathcal{W}_{\text{int}}(R, \boldsymbol{\phi})$,

$$\mathcal{W}_{\text{int}}(R, \boldsymbol{\phi}) = \int_{\mathcal{S}} (\mathbf{Gn} \cdot \operatorname{curl} \boldsymbol{\phi} + (\mathbf{Tn} - (\operatorname{div} \mathbf{G}) \times \mathbf{n}) \cdot \boldsymbol{\phi})\, \mathrm{d}a + \int_R \mathbf{b} \cdot \boldsymbol{\phi}\, \mathrm{d}v; \tag{217}$$

thus appealing to (11) we arrive at the identity

$$\int_{\mathcal{S}} (\mathbf{Gn} \cdot \operatorname{curl} \boldsymbol{\phi} + (\mathbf{Tn} - (\operatorname{div} \mathbf{G}) \times \mathbf{n}) \cdot \boldsymbol{\phi})\, \mathrm{d}a + \int_R \mathbf{b} \cdot \boldsymbol{\phi}\, \mathrm{d}v = \int_R (\mathbf{T} \colon \operatorname{grad} \boldsymbol{\phi} + \mathbf{G} \colon \operatorname{grad} \operatorname{curl} \boldsymbol{\phi})\, \mathrm{d}v. \tag{218}$$

Thus if we define the *Cosserat stress* $\mathbf{T}^{\mathrm{C}}$ by

$$\mathbf{T}^{\mathrm{C}} = \mathbf{T} - [(\operatorname{div} \mathbf{G})\times] \quad (T^{\mathrm{C}}_{ij} = T_{ij} - \varepsilon_{irj} G_{rk,k}), \tag{219}$$

then (218) becomes

$$\underbrace{\int_{\mathcal{S}} (\mathbf{T}^{\mathrm{C}}\mathbf{n} \cdot \boldsymbol{\phi} + \mathbf{Gn} \cdot \operatorname{curl} \boldsymbol{\phi})\, \mathrm{d}a + \int_R \mathbf{b} \cdot \boldsymbol{\phi}\, \mathrm{d}v}_{\mathcal{W}_{\text{ext}}(R,\boldsymbol{\phi})} = \underbrace{\int_R (\mathbf{T} \colon \operatorname{grad} \boldsymbol{\phi} + \mathbf{G} \colon \operatorname{grad} \operatorname{curl} \boldsymbol{\phi})\, \mathrm{d}v}_{\mathcal{W}_{\text{int}}(R,\boldsymbol{\phi})}, \tag{220}$$

which is the form the virtual-power balance would take within the Cosserat framework, a form that identifies $\mathbf{G}$ as the *Cosserat couple stress*.

Next, by (65) and (219),

$$T^{\mathrm{C}}_{ij,j} = T_{ij,j} - \varepsilon_{irj} G_{rk,kj} = (\operatorname{div} \mathbf{T} + \operatorname{curl} \operatorname{div} \mathbf{G})_i = \rho \ddot{u}_i$$

and we have the well-known balance law for the Cosserat stress:

$$\operatorname{div} \mathbf{T}^{\mathrm{C}} = \rho \ddot{\mathbf{u}}. \tag{221}$$

Further, since $\mathbf{T}$ is symmetric and, by (31), $(\operatorname{div} \mathbf{G})\times$ is skew, (219) yields

$$-(\operatorname{div} \mathbf{G})\times = \operatorname{skw} \mathbf{T}^{\mathrm{C}} \tag{222}$$

or equivalently

$$G_{ij,j} = \tfrac{1}{2} \varepsilon_{ilm} T^{\mathrm{C}}_{lm}, \tag{223}$$

an equation that represents a local balance law for the couple stress.

Note that, by (217) and the power balance $\mathcal{W}_{\text{ext}}(\mathcal{R}(t)) = \mathcal{W}_{\text{int}}(\mathcal{R}(t))$ we can rewrite the free energy imbalance (97) equivalently as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{R(t)} \rho \psi\, \mathrm{d}v - \underbrace{\left[\int_{\mathcal{S}(t)} (\mathbf{T}^{\mathrm{C}}\mathbf{n} \cdot \boldsymbol{\phi} + \mathbf{Gn} \cdot \operatorname{curl} \boldsymbol{\phi})\, \mathrm{d}a + \int_R \mathbf{b} \cdot \mathbf{v}\, \mathrm{d}v \right]}_{\text{internal power expenditure in Cosserat form}} = -\int_{\mathcal{R}(t)} \Gamma\, \mathrm{d}v \le 0. \tag{224}$$

The virtual balance (220) gives a sense in which the "traction" **Gn** is power-conjugate to the rotation rate. Unfortunately, this balance would be of little use in virtual power arguments because a knowledge of $\boldsymbol{\phi}$ on $\mathcal{S}$ implies a knowledge of the tangential derivatives of $\boldsymbol{\phi}$ on $\mathcal{S}$, and so $\boldsymbol{\phi}$ and curl$\,\boldsymbol{\phi}$ could not generally be varied independently. For that reason (220) would seem of little use in developing weak formulations of the Cosserat theory and associated boundary conditions. The formulation presented in Sects. 3 and 4, which is based on the virtual balance (47), has no such drawback.

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