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Original Article

A modified Zener model of a viscoelastic body

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By using an internal variable method, we study a fractional derivative type model for uniaxial, isothermal deformation of a viscoelastic body. It is a generalization of a Zener model. Entropy inequality and minimum of free energy conditions are used to determine restrictions on coefficients in the constitutive equation. The asymptotic behavior in creep and stress relaxation tests are analysed in detail.

1 Introduction

Consider a standard viscoelastic body (Zener model), that in a uniaxial isothermal deformation has a stress strain relation of the form

$$\tau_\sigma \sigma^{(1)} + \sigma = E\tau_\varepsilon \varepsilon^{(1)} + E\varepsilon, \tag{1}$$

where σ and ε denote the stress and strain at time t, respectively. $(\cdot)^{(1)} = \frac{d}{dt}(\cdot)$ denotes the first derivative with respect to time, $\tau_\sigma, \tau_\varepsilon$ are constants called relaxation times, and E is the modulus of elasticity. The second law of thermodynamics, implies that in (1) the following restrictions on the constants must be satisfied

$$E > 0, \quad \tau_\sigma > 0, \quad \tau_\varepsilon > \tau_\sigma. \tag{2}$$

Equations of the type (1) are not always valid. Indeed, there is a class of viscoelastic materials which is better described by the constitutive equation (modified Zener model), see [3],[5]

$$\sigma + b\sigma^{(\beta)} = E_0\varepsilon + E_1\varepsilon^{(\alpha)}, \tag{3}$$

where $\sigma^{(\beta)}$ and $\varepsilon^{(\alpha)}$ are fractional derivatives; $0 < \alpha < 1, 0 < \beta < 1$. The α^{th} fractional derivative, $0 < \alpha < 1$, of a function $f(t)$ in the Riemann-Liouville form is defined as (see [1] or [2])

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha} f(t) &= f^{(\alpha)} \equiv \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f(\tau)\,d\tau}{(t-\tau)^\alpha} = \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f(t-\tau)\,d\tau}{\tau^\alpha} \\ &= \frac{f(0)\,t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f^{(1)}(t-\tau)\,d\tau}{\tau^\alpha} \\ &= \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} f^{(n)}(t), \end{aligned} \tag{4}$$

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where in $(4)_4$ the binomial coefficients are $\left(\begin{array}{c}\alpha\\ n\end{array}\right) = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)} = \frac{(-1)^n(-\alpha)_n}{n!}$ and $(z)_n = z(z+1)\ldots(z+n-1)$, $n = 1,2,\ldots,\ (z)_0 \equiv 1$.

From $(4)_4$ it is obvious that a constitutive equation of the form (3) takes into account all integer derivatives of σ and ε at a fixed time t, each one with the weighting factor equal to $\left(\begin{array}{c}\alpha\\ n\end{array}\right)\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}$.

By invoking the second law of thermodynamics for sinusoidal strain and sinusoidal stress the following restrictions on the constants α, β, b, E_0 and E_1 are obtained in [3]

$$E_0 \geq 0, \quad E_1 > 0, \quad b > 0, \quad \frac{E_1}{b} \geq E_0, \quad \alpha = \beta. \tag{5}$$

Except for the equality in $(5)_1$ these conditions are identical to (2), whenever a comparison is posible, i.e., for $\alpha = \beta = 1$.

There are other propositions for the constitutive relations as well. Thus in [4] the relation

$$\sigma(t) = D_*^{(\alpha)}\varepsilon = \frac{E\tau^\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{\varepsilon^{(1)}(u)}{(t-u)^\alpha}du, \tag{6}$$

was analyzed, where E and τ are constants. $D_*^{(\alpha)}\varepsilon$ is the *Caputo fractional derivative of the order* α. For the connection between $D_*^{(\alpha)}\varepsilon$ and $\varepsilon^{(\alpha)}$ see [8]. An element with the constitutive equation given by (6) is called a springpot. In the context of (6) it is possible to identify the inelastic part ε_{in} of the strain by

$$D_*^{(\alpha)}\varepsilon_{in} = \frac{1}{\tau^\alpha}(\varepsilon - \varepsilon_{in}). \tag{7}$$

and define a standard solid with the constitutive relation

$$\sigma(t) = E_{eq}\varepsilon + E(\varepsilon - \varepsilon_{in}). \tag{8}$$

By interpreting a fractional damping in terms of a continuous superposition of Maxwell elements in parallel it was shown in [4] that the condition $E > 0$ guarantees that the entropy inequality is satisfied (see [4] p. 89). As stated in [6] the parallel connection of springpot elements of the type (6) is exactly the rheological interpretation of the model (3) with $\alpha = \beta$. In [4] references to other works treating thermodynamic admissibility of viscoelastic models with fractional derivatives are also given.

Our intention in this note is to formulate a fractional derivatives type model for one dimensional deformations of a viscoelastic body. The constitutive equation that we construct is of the form (3) with $\alpha = \beta$ and with an additional term. It reads

$$\sigma + \tau_\sigma\sigma^{(\gamma)} = E\left[\varepsilon + \tau_\varepsilon\varepsilon^{(\gamma)}\right] + E\left(1 - \frac{\tau_\varepsilon}{\tau_\sigma}\right)\left[d^{(\gamma)} + \frac{1}{\tau_\sigma}d\right]. \tag{9}$$

The last term, by which (9) is different from (3), results from the desire to make viscoelastic constitutive equation compatible with an internal variable theory which lends itself for a clear-cut exposition of the thermodynamic conditions imposed by the second law. In (9) the functional d is given as

$$\begin{aligned} d \;=\; & \int_0^t e^{-\frac{1}{\tau_\sigma}(t-\tau)}(\varepsilon(\tau) \\ & + \int_0^\tau e_{\gamma,\gamma}\left(u;\frac{1}{\tau_\sigma}\right)\left[-\frac{1}{\tau_\sigma}\varepsilon(\tau-u) - \varepsilon^{(1)}(\tau-u)\right]du)d\tau, \end{aligned} \tag{10}$$

where $e_{\gamma,\gamma}(t;\lambda)$ is defined by (35) below. The functional d is chosen so that (9) satisfies the entropy inequality for *all* deformations $\varepsilon(t)$. The explicit form (i.e. (9) solved for $\sigma(t)$) of the constitutive equation is given by (62) below. The functional $d(\varepsilon)$ is equal to zero when $\gamma = 1$ and decreases strongly with time. Its influence is important only at the beginning of the motion. We shall examine the restrictions which the entropy inequality *and* equilibrium stability conditions imply on a constitutive equation of the type (9) and we shall conclude that the presence of $d(\varepsilon)$ in (9) is essential, if one wants to satisfy the entropy inequality for all deformations $\varepsilon(t)$ (see Remark 1).

2 The internal variable theory

We recall the description of the constitutive relation (1) and of the thermodynamic stability conditions (2) in the context of an internal variable theory.

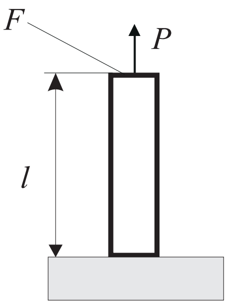

Fig. 1. Load configuration

Consider a rod in the uniaxial isothermal deformation (see Fig. 1). The length is L in the undeformed state and $l\,(t)$ during the deformation. The rod is loaded by the force P and F is the cross-sectional area in the undeformed state. Thus stress[1] and strain are given by

$$\sigma\,(t) = \frac{P}{F} \quad \text{and} \quad \varepsilon\,(t) = \frac{l}{L} - 1, \tag{11}$$

We describe a state of the body by two variables: the strain $\varepsilon\,(t)$ and an internal variable $\xi\,(t)$. The equilibrium state of the unloaded body corresponds to

$$\varepsilon = 0, \quad \xi = 0. \tag{12}$$

Thus the internal energy U, the entropy S and the free energy $U - TS$ are all functions of ε, or l and ξ and we may write the Gibbs equation for the free energy $U - TS$ in the form

$$\frac{d\,(U - TS)}{dt} = \sigma V \frac{d\varepsilon}{dt} - \Theta \frac{d\xi}{dt}. \tag{13}$$

T is the temperature, assumed to be constant. Θ is the "force" associated with the internal variable ξ so that $\Theta \frac{d\xi}{dt}$ is the power of the force Θ, and V is the volume of the body which is assumed to be constant. We assume that both σ and Θ are linear in the variables ε and ξ so that

$$\sigma = E_\infty \varepsilon + \beta\xi, \quad \Theta = \gamma\varepsilon + \delta\xi, \tag{14}$$

where E_∞, β, γ and δ are constants. From (14) we conclude that ξ is proportional to the difference between the instantaneous and equilibrium stress $E_\infty\varepsilon$. Note that with (14) the force P is given as

$$P = F\,(E_\infty\varepsilon + \beta\xi)\,. \tag{15}$$

The integrability for the free energy requires

$$V\beta = -\gamma. \tag{16}$$

Therefore, with (16) the equation (14) becomes

$$\sigma = E_\infty\varepsilon + \beta\xi, \quad \Theta = -V\beta\varepsilon + \delta\xi. \tag{17}$$

We return to the Gibbs equation (13) in which we replace $\frac{dU}{dt}$ and $\frac{d\xi}{dt}$ by the equations of balance of energy and of internal variable, viz.

$$\frac{dU}{dt} = \dot{Q} + \sigma V \frac{d\varepsilon}{dt} \quad \text{and} \quad \frac{d\xi}{dt} = P_\xi. \tag{18}$$

[1] The stress σ is the load referred to the cross-sectional area of the unloaded rod.

$\dot{Q}$ is the heating and P_ξ is the production of ξ. Thus we obtain an equation of balance of entropy in the form

$$\frac{dS}{dt} - \frac{\dot{Q}}{T} = \frac{\Theta}{T} P_\xi \geq 0, \tag{19}$$

where we have indicated that the entropy production is non-negative. The inequality (19) may be satisfied by setting $P_\xi = \alpha\Theta$ with a non-negative coefficient α. Therefore

$$\xi^{(1)} = \alpha\Theta \quad \alpha \geq 0. \tag{20}$$

Elimination of ξ and Θ between the three equations (17) and (20) provides

$$\sigma + \frac{1}{-\delta\alpha}\sigma^{(1)} = \left(E_\infty + V\frac{\beta^2}{\delta}\right)\varepsilon + \frac{E_\infty}{-\alpha\delta}\varepsilon^{(1)}. \tag{21}$$

which is of the form (1), if we identify the coefficients in (1) with the coefficients of the internal variable theory as follows

$$\tau_\sigma = \frac{1}{-\delta\alpha}, \quad E = \left(E_\infty + V\frac{\beta^2}{\delta}\right), \quad \tau_\varepsilon = \frac{E_\infty}{-\alpha\delta}\frac{1}{E_\infty + \frac{V\beta^2}{\delta}}. \tag{22}$$

Thus the viscoelastic constitutive equation (1) is a consequence of the internal variable theory. It results upon elimination of all explicit reference to the internal variable field.

The internal variable theory lends itself for an easy evaluation of thermodynamic stability conditions. Indeed, elimination of $\dot{Q}$ between (18) and (19) provides the inequality

$$\frac{d\left(U - TS - Pl\right)}{dt} \leq l\frac{dP}{dt}. \tag{23}$$

Thus for a constant force the Gibbs free energy $(U - TS - Pl)$ tends to a minimum and it assumes that minimum in equilibrium. Therefore the matrix of the second derivatives of the Gibbs free energy must be positive definite. Thus by (13) an (17) we must have

$$\begin{bmatrix} E_\infty & \beta V \\ \beta V & -\delta \end{bmatrix} \text{ pos.def.} \tag{24}$$

or

$$E_\infty > 0, \quad -E_\infty\delta - \beta^2 V > 0, \tag{25}$$

so that $E_\infty > 0$ and $\delta < 0$ or $E_\infty + \frac{\beta^2 V}{\delta} > 0$. By using this in (22) we obtain

$$\tau_\sigma > 0, \quad E > 0, \quad \tau_\varepsilon > \tau_\sigma, \tag{26}$$

i.e. the conditions (2).

3 The fractional derivative model

Consider now the constitutive equation of the type (9). We write it as

$$\sigma + \tau_\sigma\sigma^{(\gamma)} = E\left[\varepsilon + \tau_\varepsilon\varepsilon^{(\gamma)}\right] + g\left(\varepsilon\right). \tag{27}$$

where $g\left(\varepsilon\right)$ is given functional. If $g\left(\varepsilon\right) \equiv 0$ equation (27) reduces to (3) with $\alpha = \beta$. We want to include this type of equation into the internal variable framework of the previous section. We repeat the relevant system of equations (17),(18),(19)

$$\begin{aligned} \sigma &= E_\infty\varepsilon + \beta\xi, \quad \Theta = -V\beta\varepsilon + \delta\xi, \\ \xi^{(1)} &= P_\xi, \quad \frac{dS}{dt} - \frac{1}{T}\frac{dQ}{dt} = \frac{\Theta}{T}P_\xi \geq 0, \end{aligned} \tag{28}$$

that must be satisfied by $\sigma\left(t\right)$ and $\varepsilon\left(t\right)$ satisfying (27). Suppose that the γ^{th} derivative of ξ is given by

$$\xi^{(\gamma)} = \alpha\Theta + X, \tag{29}$$

where X is a functional that will be specified later. Equation (29) is a generalization of (20) and is of central importance in the analysis that follows. Note that with $X = 0$ the equation (29) leads to a constitutive relation of the type (3) with $\alpha = \beta$. Equation (20) is obtained if we take $\gamma = 1$ and $X = 0$. The special case of (29) with $X = 0, \gamma \neq 1$ was used for an internal variable description of the fractional derivative model in [6][2]. Our intention is to choose X so that $(28)_4$ is satisfied. Now from $(17)_2$ and (29) it follows

$$\xi^{(\gamma)} + (-\alpha\delta)\,\xi = K\varepsilon + X, \tag{30}$$

where $K = -\alpha V\beta$. Suppose that X is given as

$$X = x^{(\gamma)} + (-\alpha\delta)\,x, \tag{31}$$

where x is another function. With (31) equation (30) could be written as

$$(\xi - x)^{(\gamma)} + (-\alpha\delta)(\xi - x) = K\varepsilon. \tag{32}$$

The solution to the fractional order differential equation (32) reads (see [1] p. 837 or [7])

$$\begin{aligned} \xi - x &= K\frac{E_{\gamma,\gamma}(\alpha\delta t^{\gamma})}{t^{1-\gamma}}\frac{1}{\Gamma(1-\alpha)}\left(\int_0^t \frac{(\xi(\tau)+x(\tau))\,d\tau}{(t-\tau)^{\alpha}}\right)_{t=0} \\ &\quad + K\int_0^t \frac{E_{\gamma,\gamma}\left(\alpha\delta(t-\tau)^{\gamma}\right)}{(t-\tau)^{1-\gamma}}\varepsilon(\tau)\,d\tau, \end{aligned} \tag{33}$$

where $E_{\alpha,\beta}(t)$ is the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n + \beta)}, \tag{34}$$

(see [9] vol. III p.210, or [10]). Note that $E_{1,1}(t) = e^t$. With $E_{\alpha,\beta}(t)$ thus given, a new function $e_{\alpha,\beta}$ can be defined by

$$e_{\alpha,\beta}(t;\lambda) \equiv \frac{E_{\alpha,\beta}(-\lambda t^{\alpha})}{t^{1-\beta}}, \tag{35}$$

(see [8] p.267). This function possesses the following properties

$$\begin{aligned} e_{1,1}(t;\lambda) &= e^{-\lambda t}, \quad (-1)^n\frac{d^n}{dt^n}\left(e_{\alpha,\beta}(t;\lambda)\right) \geq 0, \quad n = 0,1,2,.., \\ e_{\alpha,\beta}(t;\lambda) &= \int_0^t e^{-rt}\frac{1}{\pi}\frac{\lambda\sin[(\beta-\alpha)] + r^{\alpha}\sin(\beta r)}{r^{2\alpha} + 2\lambda r^{\alpha}\cos(\alpha\pi) + \lambda^2}r^{\alpha-\beta}dr. \end{aligned} \tag{36}$$

With (35) and by use of the fact that at $t = 0$ both ξ and x are equal to zero (33) becomes

$$\begin{aligned} \xi &= K\int_0^t e_{\gamma,\gamma}(t-\tau;-\alpha\delta)\,\varepsilon(\tau)\,d\tau + x \\ &= K\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon(t-\tau)\,d\tau + x. \end{aligned} \tag{37}$$

We now specify x in the form

$$\begin{aligned} x &= K\left\{\int_0^t e^{-\lambda(t-\tau)}(\varepsilon(\tau)\right. \\ &\quad \left. + \int_0^{\tau} e_{\gamma,\gamma}(u;-\alpha\delta)\,[\Lambda\varepsilon(\tau-u) - \varepsilon^{(1)}(\tau-u)]du)d\tau\right\}, \end{aligned} \tag{38}$$

[2] In [6] equation (29) with $X = 0$ is used in the context of finite deformations of a viscoelastic body.

where λ and Λ are constants. Combining (37) and (38) we obtain

$$\begin{aligned}\xi &= K\left\{\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon(t-\tau)\,d\tau+\int_0^t e^{-\lambda(t-\tau)}(\varepsilon(\tau)\right.\\ &\quad\left.+\int_0^\tau e_{\gamma,\gamma}(u;-\alpha\delta)\,[\Lambda\varepsilon(\tau-u)-\varepsilon^{(1)}(\tau-u)]du)d\tau\right\}. \qquad (39)\end{aligned}$$

Next we use (39) to determine $\xi^{(1)}$, and thus P_ξ (see $(28)_3$). The first derivative of ξ becomes

$$\begin{aligned}\xi^{(1)} &= K\left\{e_{\gamma,\gamma}(t;-\alpha\delta)\,\varepsilon(0)+\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon^{(1)}(t-\tau)\,d\tau\right.\\ &\quad+\varepsilon(t)+\int_0^t e_{\gamma,\gamma}(u;-\alpha\delta)\,[\Lambda\varepsilon(t-u)-\varepsilon^{(1)}(t-u)]du\\ &\quad-\lambda\int_0^t e^{-\lambda(t-\tau)}(\varepsilon(\tau)\\ &\quad\left.+\int_0^\tau e_{\gamma,\gamma}(u;-\alpha\delta)\,[\Lambda\varepsilon(\tau-u)-\varepsilon^{(1)}(\tau-u)]du)d\tau\right\}, \qquad (40)\end{aligned}$$

and since $\varepsilon(0)=0$

$$\begin{aligned}\xi^{(1)} &= K\left\{\varepsilon(t)+\Lambda\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon(t-\tau)\,d\tau-\lambda\int_0^t e^{-\lambda(t-\tau)}(\varepsilon(\tau)\right.\\ &\quad\left.+\int_0^\tau e_{\gamma,\gamma}(u;-\alpha\delta)\,[\Lambda\varepsilon(\tau-u)-\varepsilon^{(1)}(\tau-u)]du)d\tau\right\}\\ &= \alpha\left\{-\beta V\varepsilon(t)-\beta V\Lambda\int_0^t e_{\gamma,\gamma}(t-\tau;\alpha\delta)\,\varepsilon(\tau)\,d\tau\right.\\ &\quad+\beta V\lambda\int_0^t e^{-\lambda(t-\tau)}(\varepsilon(\tau)\\ &\quad\left.+\int_0^\tau e_{\gamma,\gamma}(u;-\alpha\delta)\,[\Lambda\varepsilon(\tau-u)-\varepsilon^{(1)}(\tau-u)]du)d\tau\right\}. \qquad (41)\end{aligned}$$

Suppose that Λ and λ are selected so that

$$-\beta V\Lambda=K\delta=(-\alpha V\beta)\,\delta,\quad \beta V\lambda=K\delta=-\alpha V\beta\delta, \qquad (42)$$

or

$$\Lambda=(\alpha\delta),\quad \lambda=-(\alpha\delta). \qquad (43)$$

Then (41) becomes (use $\Theta=-V\beta\varepsilon+\delta\xi$ and (39))

$$\begin{aligned}\xi^{(1)} &= \alpha\left\{-\beta V\varepsilon(t)+\delta K\left[\int_0^t e_{\gamma,\gamma}(\tau;\alpha\delta)\,\varepsilon(t-\tau)\,d\tau+\int_0^t e^{-\lambda(t-\tau)}\right.\right.\\ &\quad\left.\left.\times(\varepsilon(\tau)+\int_0^\tau e_{\gamma,\gamma}(u;-\alpha\delta)\,[\Lambda\varepsilon(\tau-u)-\varepsilon^{(1)}(\tau-u)]du)d\tau\right]\right\}\\ &= \alpha\left\{-\beta V\varepsilon(t)+\delta\xi(t)\right\}=\alpha\Theta. \qquad (44)\end{aligned}$$

Thus, with

$$\alpha>0, \qquad (45)$$

the condition $(28)_4$ is satisfied. Note also that (44) guarantees that the constitutive equation for the internal variable ξ is local in time, since the right hand side of (44) is *a function* of $\varepsilon(t)$ and $\xi(t)$.

We determine now the form of the constitutive equation ($\sigma-\varepsilon$ relation) that follows from (30). Combining $(28)_1$,(30),(31) and (38) we have

$$\left(\frac{\sigma - E_\infty \varepsilon}{\beta}\right)^\gamma + (-\alpha\delta)\left(\frac{\sigma - E_\infty \varepsilon}{\beta}\right) = (-\alpha V \beta)\,\varepsilon + x^{(\gamma)} + (-\alpha\delta)\,x, \tag{46}$$

where

$$\begin{aligned} x &= -\alpha V \beta \left\{ \int_0^t e^{\alpha\delta(t-\tau)}(\varepsilon(\tau) \right. \\ &\left. + \int_0^\tau e_{\gamma,\gamma}(u;-\delta)\,[\alpha\delta\varepsilon(\tau-u) - \varepsilon^{(1)}(\tau-u)]du)d\tau \right\}. \end{aligned} \tag{47}$$

From (46) we obtain

$$\sigma^{(\gamma)} - E_\infty \varepsilon^{(\gamma)} - \alpha\delta\sigma + \alpha\delta E_\infty \varepsilon = -\alpha V \beta^2 \varepsilon + \beta x^{(\gamma)} - \alpha\delta\beta x, \tag{48}$$

or

$$\frac{1}{-\alpha\delta}\sigma^{(\gamma)} + \sigma = \frac{E_\infty}{-\alpha\delta}\varepsilon^{(\gamma)} + \left[E_\infty + \frac{V\beta^2}{\delta}\right]\varepsilon + \frac{\beta}{-\alpha\delta}x^{(\gamma)} + \beta x. \tag{49}$$

With

$$\tau_\sigma = \frac{1}{-\alpha\delta}, \quad E = E_\infty + \frac{V\beta^2}{\delta}, \quad \tau_\varepsilon = \frac{E_\infty}{-\alpha\delta}\frac{1}{E} = \tau_\sigma \frac{E_\infty}{E} \tag{50}$$

(49) becomes

$$\tau_\sigma \sigma^{(\gamma)} + \sigma = E[\tau_\varepsilon \varepsilon^{(\gamma)} + \varepsilon] + E\left(1 - \frac{\tau_\varepsilon}{\tau_\sigma}\right)\left[d^{(\gamma)} + \frac{1}{\tau_\sigma}d\right], \tag{51}$$

where

$$\begin{aligned} d &= \int_0^t e^{-\frac{1}{\tau_\sigma}(t-\tau)} \left(\varepsilon(\tau) \right. \\ &\left. + \int_0^\tau e_{\gamma,\gamma}\left(u;\frac{1}{\tau_\sigma}\right)\left[-\frac{1}{\tau_\sigma}\varepsilon(\tau-u) - \varepsilon^{(1)}(\tau-u)\right]du\right)d\tau. \end{aligned} \tag{52}$$

Note that for $\gamma = 1$ we have

$$e_{1,1}\left(t;\frac{1}{\tau_\sigma}\right) = e^{-\frac{1}{\tau_\sigma}t}, \tag{53}$$

so that

$$\begin{aligned} d &= \int_0^t e^{-\frac{1}{\tau_\sigma}(t-\tau)}\left(\varepsilon(\tau) + \int_0^\tau e^{-\frac{1}{\tau_\sigma}u}\left[-\frac{1}{\tau_\sigma}\varepsilon(\tau-u) - \varepsilon^{(1)}(\tau-u)\right]du\right)d\tau \\ &= \int_0^t e^{-\frac{1}{\tau_\sigma}(t-\tau)}(\varepsilon(\tau) + \int_0^\tau \frac{d}{du}\left(e^{-\frac{1}{\tau_\sigma}u}\varepsilon(\tau-u)\right)du)d\tau \\ &= \int_0^t e^{-\frac{1}{\tau_\sigma}(t-\tau)}(\varepsilon(\tau) - \varepsilon(\tau))d\tau = 0, \end{aligned} \tag{54}$$

and (51) reduces to (1).

Remark 1. The constitutive equation (51) reduces to (3), if we take $d = 0$. We show that in this case there is a deformation process $\varepsilon(t)$ such that $(28)_4$ is violated. If $d = 0$ then, $x = 0$ so that (39) becomes

$$\xi = K\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon(t-\tau)\,d\tau. \tag{55}$$

Also, instead of (40) we get

$$\begin{aligned} \xi^{(1)} &= K\left(e_{\gamma,\gamma}(t;-\alpha\delta)\,\varepsilon(0) + \int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon^{(1)}(t-\tau)\,d\tau\right) \\ &= K\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon^{(1)}(t-\tau)\,d\tau. \end{aligned} \tag{56}$$

Therefore $P_\xi = K\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon^{(1)}(t-\tau)\,d\tau$. Also, with $d = x = 0$ the function Θ reads

$$\begin{aligned}\Theta(t) &= -V\beta\varepsilon + \delta\xi \\ &= -V\beta\varepsilon(t) - \alpha V\beta\delta\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon(t-\tau)\,d\tau, \end{aligned}\tag{57}$$

so that

$$\begin{aligned}\frac{\Theta}{T}P_\xi &= \frac{1}{T}\left(-V\beta\varepsilon(t) - \alpha V\beta\delta\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon(t-\tau)\,d\tau\right)\times \\ &\quad (-\alpha V\beta)\int_0^t e_{\gamma,\gamma}(\tau;-\alpha\delta)\,\varepsilon^{(1)}(t-\tau)\,d\tau. \end{aligned}\tag{58}$$

It is obvious from (58) that we can always choose $\varepsilon(t)$ so that $(28)_4$ is violated. □

The restrictions that follow from the stability of the equilibrium states are satisfied since we started with (28) and since $(28)_1$ guarantees the integrability conditions for $U - TS$. Therefore the convexity of $U - TS$ as a function of the variables ε, ξ leads, again, to (25),(26).

To analyze (51) we apply the Laplace transform ($\mathscr{L}f = \int_0^\infty e^{-st}f(t)\,dt = \bar{f}(s)$) so that

$$\tau_\sigma s^\gamma\bar{\sigma} + \bar{\sigma} = E\bar{\varepsilon}[\tau_\varepsilon s^\gamma + 1] + \left[E\left(1-\frac{\tau_\varepsilon}{\tau_\sigma}\right)\left(s^\gamma + \frac{1}{\tau_\sigma}\right)\right]\bar{d}, \tag{59}$$

where it is assumed that $\sigma(t)$ and $\varepsilon(t)$ are bounded[3] for $t \to +0$. From (52) it follows

$$\begin{aligned}\bar{d} &= \frac{1}{s+\frac{1}{\tau_\sigma}}\mathscr{L}\left(\varepsilon(\tau) + \int_0^\tau e_{\gamma,\gamma}\left(u;\frac{1}{\tau_\sigma}\right)\left[-\frac{1}{\tau_\sigma}\varepsilon(\tau-u) - \varepsilon^{(1)}(\tau-u)\right]du\right) \\ &= \frac{1}{s+\frac{1}{\tau_\sigma}}\left(\bar{\varepsilon} + \mathscr{L}\left(\int_0^\tau e_{\gamma,\gamma}\left(u;\frac{1}{\tau_\sigma}\right)\left[-\frac{1}{\tau_\sigma}\varepsilon(\tau-u) - \varepsilon^{(1)}(\tau-u)\right]du\right)\right) \\ &= \frac{1}{s+\frac{1}{\tau_\sigma}}\left(\bar{\varepsilon} + \mathscr{L}\left(e_{\gamma,\gamma}\left(t;\frac{1}{\tau_\sigma}\right)\right)\mathscr{L}\left(\left[-\frac{1}{\tau_\sigma}\varepsilon(t) - \varepsilon^{(1)}(t)\right]\right)\right) \\ &= \frac{1}{s+\frac{1}{\tau_\sigma}}\left(1 + \frac{1}{s^\gamma+\frac{1}{\tau_\sigma}}\left(-\frac{1}{\tau_\sigma} - s\right)\right)\bar{\varepsilon} \\ &= \left(\frac{1}{s+\frac{1}{\tau_\sigma}} - \frac{1}{s^\gamma+\frac{1}{\tau_\sigma}}\right)\bar{\varepsilon} = \left(\frac{s^\gamma - s}{\left(s+\frac{1}{\tau_\sigma}\right)\left(s^\gamma+\frac{1}{\tau_\sigma}\right)}\right)\bar{\varepsilon}, \end{aligned}\tag{60}$$

where $\mathscr{L}\left(e_{\gamma,\gamma}\left(t;1/\tau_\sigma\right)\right) = 1/\left(s^\gamma + \frac{1}{\tau_\sigma}\right)$ (see [8] p.267), and $\varepsilon(0) = 0$ was used. From (60) and (59) we get

$$\begin{aligned}\bar{\sigma} &= \bar{\varepsilon}\frac{E[\tau_\varepsilon s^\gamma + 1] + E\left(1-\frac{\tau_\varepsilon}{\tau_\sigma}\right)[s^\gamma + \frac{1}{\tau_\sigma}]\frac{s^\gamma - s}{\left(s+\frac{1}{\tau_\sigma}\right)\left(s^\gamma+\frac{1}{\tau_\sigma}\right)}}{\tau_\sigma s^\gamma + 1} \\ &= E\bar{\varepsilon}\frac{\tau_\varepsilon s^\gamma + 1 + \left(1-\frac{\tau_\varepsilon}{\tau_\sigma}\right)\frac{s^\gamma - s}{\left(s+\frac{1}{\tau_\sigma}\right)}}{\tau_\sigma s^\gamma + 1}. \end{aligned}\tag{61}$$

The solution $\sigma(t)$ can be obtained by finding the inverse Laplace transform of (61) or from $(28)_1$. By using (39),(43) and (50) we may write the solution of (51) for σ as

[3] The Laplace transform of $f^{(\gamma)}$ is given as $\mathscr{L}\left(f^{(\gamma)}\right) = s^\gamma\bar{f} - \left[\frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{f(\tau)}{(t-\tau)^\gamma}\right]_{t=0}$. The term in brackets vanishes if $\lim_{t\to+0} f(t)$ is bounded (see [2]).

$$\sigma(t) = E\frac{\tau_\varepsilon}{\tau_\sigma}\varepsilon + \frac{1}{\tau_\sigma}E\left(1-\frac{\tau_\varepsilon}{\tau_\sigma}\right)\left\{\int_0^t e_{\gamma,\gamma}\left(\tau;\frac{1}{\tau_\sigma}\right)\varepsilon(t-\tau)\,d\tau + \int_0^t e^{-\frac{1}{\tau_\sigma}(t-\tau)} \right.$$
$$\left.\times\left(\varepsilon(\tau) + \int_0^\tau e_{\gamma,\gamma}\left(u;\frac{1}{\tau_\sigma}\right)\left[-\frac{1}{\tau_\sigma}\varepsilon(\tau-u) - \varepsilon^{(1)}(\tau-u)\right]du\right)d\tau\right\}. \tag{62}$$

Note that the constitutive equation (51) with d given by (52) is equivalent to (1). This can be concluded from (44) and (20) or from (61).

4 Creep and stress relaxation

To examine the asymptotic behavior of the solution to (51) or (62) and to compare it to the asymptotic behavior of the solutions to (3) for $\alpha = \beta$ we consider the special kind of applied stress (strain). Thus, suppose that

$$\sigma(t) = \begin{cases} 0 & t \le 0 \\ \sigma_0 & t > 0 \end{cases}. \tag{63}$$

Then $\mathscr{L}(\sigma) = \sigma_0/s$, so that (61) leads to

$$\bar{\varepsilon} = \frac{\sigma_0}{E}\frac{1}{s}\frac{\tau_\sigma s^\gamma + 1}{[\tau_\varepsilon s^\gamma + 1] + \left(1-\frac{\tau_\varepsilon}{\tau_\sigma}\right)\frac{s^\gamma - s}{\left(s+\frac{1}{\tau_\sigma}\right)}}. \tag{64}$$

Now, if $\lim_{t\to\infty}\varepsilon(t)$ exists, it is given as $\lim_{t\to\infty}\varepsilon(t) = \lim_{s\to 0} s\bar{\varepsilon}(s)$ (see [11]). By use of this in (64) it follows that

$$\lim_{t\to\infty}\varepsilon(t) = \frac{\sigma_0}{E}. \tag{65}$$

Also, if $\lim_{t\to 0}\varepsilon(t) = \varepsilon(+0)$ exists, it is given as $\varepsilon(+0) = \lim_{s\to\infty} s\bar{\varepsilon}(s)$. Therefore

$$\lim_{t\to 0}\varepsilon(t) = \varepsilon(+0) = \frac{\sigma_0}{E}\frac{\tau_\sigma}{\tau_\varepsilon} < \frac{\sigma_0}{E}. \tag{66}$$

In Fig. 2 the function[4] $\varepsilon(t)$ is shown, schematically.
Next suppose that

$$\varepsilon(t) = \begin{cases} 0 & t \le 0 \\ \varepsilon_0 & t > 0 \end{cases}. \tag{67}$$

From (61) it follows that

$$\bar{\sigma} = E\varepsilon_0\frac{1}{s}\frac{\tau_\varepsilon s^\gamma + 1 + \left(1-\frac{\tau_\varepsilon}{\tau_\sigma}\right)\frac{s^\gamma - s}{\left(s+\frac{1}{\tau_\sigma}\right)}}{\tau_\sigma s^\gamma + 1}. \tag{68}$$

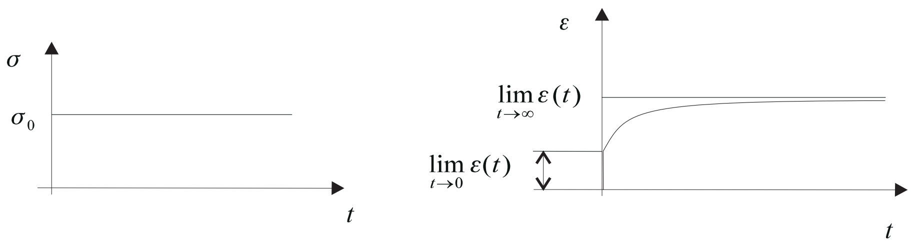

Fig. 2. Strain as a function of time for suddenly applied stress

Again $\lim_{t\to\infty}\sigma(t) = \lim_{s\to 0} s\bar{\sigma}(s)$ so that

[4] Note that $\varepsilon(t = 0) = 0$. Thus, $\varepsilon(t)$ has a discontinuity at $t = 0$. Also $\varepsilon(t)$ is drawn as a monotone increasing function. That this property holds can be shown by analyzing the inversion of (64). We present such an analysis later.

$$\lim_{t\to\infty} \sigma(t) = E\varepsilon_0. \tag{69}$$

Also, for $t \to 0$ we have

$$\lim_{t\to 0} \sigma(t) = \sigma(+0) = \lim_{s\to\infty} s\bar{\sigma}(s) = E\varepsilon_0 \frac{\tau_\varepsilon}{\tau_\sigma} > E\varepsilon_0. \tag{70}$$

In Fig. 3 the function $\sigma(t)$ is shown schematically.

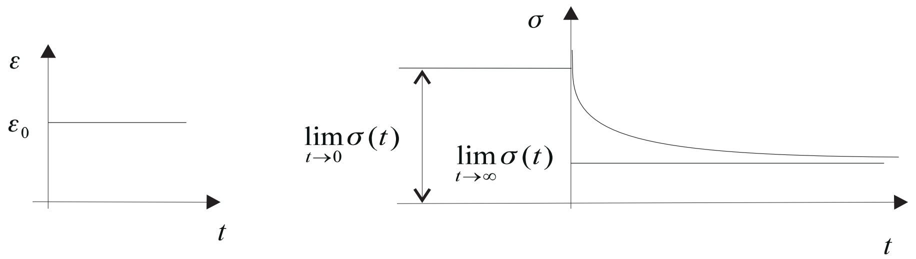

Fig. 3. Stress as a funcion of time for suddenly applied strain

We prove next that $\sigma(t)$ is a decreasing function, as shown in Fig. 3. First we decompose $\sigma(t)$ as

$$\sigma(t) = E\varepsilon_0 \frac{\tau_\varepsilon}{\tau_\sigma} + \sigma_R(t). \tag{71}$$

Then from (68) it follows that

$$\begin{aligned} \bar{\sigma}_R &= -E\varepsilon_0 \frac{\tau_\varepsilon}{\tau_\sigma}\left(\frac{1}{\tau_\sigma} - \frac{1}{\tau_\varepsilon}\right)\left\{\frac{1}{s}\frac{1}{s^\gamma + \frac{1}{\tau_\sigma}} + \frac{s^{\gamma-1}}{s^\gamma + \frac{1}{\tau_\sigma}}\frac{1}{s + \frac{1}{\tau_\sigma}}\right. \\ &\quad \left. - \frac{1}{s^\gamma + \frac{1}{\tau_\sigma}}\frac{1}{s + \frac{1}{\tau_\sigma}}\right\}. \end{aligned} \tag{72}$$

Inversion of (72) leads to

$$\begin{aligned} \sigma_R(t) &= -E\varepsilon_0 \frac{\tau_\varepsilon}{\tau_\sigma}\left(\frac{1}{\tau_\sigma} - \frac{1}{\tau_\varepsilon}\right)\left\{\int_0^t e_{\gamma,\gamma}\left(\tau; \frac{1}{\tau_\sigma}\right) d\tau\right. \\ &\quad \left. + \int_0^t e^{-\frac{1}{\tau_\sigma}(t-\tau)}\left[E_\gamma\left(-\frac{1}{\tau_\sigma}\tau^\gamma\right) - e_{\gamma,\gamma}\left(\tau; \frac{1}{\tau_\sigma}\right)\right] d\tau\right\}, \end{aligned} \tag{73}$$

where $E_\gamma(t) = E_{\gamma,1}(t)$ (see [8] p. 267). With (71) and (73) we have

$$\begin{aligned} \sigma(t) &= E\varepsilon_0 \frac{\tau_\varepsilon}{\tau_\sigma} - E\varepsilon_0 \frac{\tau_\varepsilon}{\tau_\sigma}\left(\frac{1}{\tau_\sigma} - \frac{1}{\tau_\varepsilon}\right)\left\{\int_0^t \left(1 - e^{-\frac{1}{\tau_\sigma}(t-\tau)}\right) e_{\gamma,\gamma}\left(\tau; \frac{1}{\tau_\sigma}\right) d\tau\right. \\ &\quad \left. + \int_0^t e^{-\frac{1}{\tau_\sigma}(t-\tau)} E_\gamma\left(-\frac{1}{\tau_\sigma}\tau^\gamma\right) d\tau\right\}. \end{aligned} \tag{74}$$

Since $E_\gamma(-\frac{1}{\tau_\sigma}t^\gamma) = e_{\gamma,1}(t; \frac{1}{\tau_\sigma})$, there follows that $E_\gamma(-\frac{1}{\tau_\sigma}t^\gamma)$ is a monotone (positive) function. Therefore from (74) we conclude that $\sigma^{(1)}(t) < 0$, i.e. $\sigma(t)$ is a decreasing function.

Remark 2. If the term d in (51) is neglected we obtain a solution to (3) with $\alpha = \beta$. By setting the corresponding terms to zero we have, instead of (74)

$$\sigma_{d=0} = E\varepsilon_0\left(\frac{\tau_\varepsilon}{\tau_\sigma} - \frac{\tau_\varepsilon}{\tau_\sigma}\left(\frac{1}{\tau_\sigma} - \frac{1}{\tau_\varepsilon}\right)\int_0^t e_{\gamma,\gamma}(\tau; \frac{1}{\tau_\sigma})d\tau\right). \tag{75}$$

Let $\Delta = \sigma - \sigma_{d=0}$. From (74),(75) it follows that $\Delta(0) = 0$. To determine $\lim_{t\to\infty}\Delta(t)$ note that from (68) it follows that

$$\bar{\Delta} = E\varepsilon_0 \left(1 - \frac{\tau_\varepsilon}{\tau_\sigma}\right) \frac{1}{s} \frac{s^\gamma - s}{(\tau_\sigma s^\gamma + 1)\left(s + \frac{1}{\tau_\sigma}\right)}. \tag{76}$$

By using (76) we conclude that $\lim_{s\to 0} s\bar{\Delta}(s) = 0$. Therefore $\lim_{t\to\infty} \Delta(t) = 0$ and thus, the difference between $\sigma(t)$ determined by (51) and $\sigma_{d=0}$ is zero at $t = 0$ and tends to zero when $t \to \infty$. □

5 Conclusions

1. By using the internal variable approach we formulate a constitutive equation for a viscoelastic body in isothermal uniaxial deformation. It is assumed that the state of the body is described by two variables, strain (ε), and an internal variable (ξ). The time evolution (balance equation) of the internal variable is assumed to be in the form

$$\xi^{(\gamma)} = \alpha\Theta + X, \tag{77}$$

and X is chosen so that

$$\xi^{(1)} = P_\xi = \alpha\Theta. \tag{78}$$

The functional X is given by (31). Note that $\xi^{(1)}$ is local in time when expressed in terms of $\varepsilon(t)$ and $\xi(t)$. However when $\xi^{(1)}$ is expressed in terms of $\varepsilon(t)$ *only*, it is not local in time (see (39)).

2. The constitutive equation that follows from (77) is of fractional derivative type, see (51),(52). In the form solved for stress the constitutive equation is given by (62). Equation (62) satisfies the entropy inequality for all deformations $\varepsilon(t)$ and the stability condition of the equilibrium state (minimum of Gibbs free energy in dead loading). The model shows creep and stress relaxation shown in Fig. 2 and Fig. 3.

3. To compare (62) with the generalized Zener model (in an arbitrary deformation process) we solve (3) with $\alpha = \gamma, \beta = \gamma, b = \tau_\sigma, E = E_0, E_1 = E\tau_\varepsilon$ for stress. The Laplace transform of (3) leads to

$$\begin{aligned} \bar{\sigma}_z &= \bar{\varepsilon}\frac{E[\tau_\varepsilon s^\gamma + 1]}{\tau_\sigma s^\gamma + 1} = E\frac{\tau_\varepsilon}{\tau_\sigma}\frac{s^\gamma + \frac{1}{\tau_\varepsilon}}{s^\gamma + \frac{1}{\tau_\sigma}}\bar{\varepsilon} \\ &= E\frac{\tau_\varepsilon}{\tau_\sigma}\left[\bar{\varepsilon} + \left(\frac{1}{\tau_\varepsilon} - \frac{1}{\tau_s}\right)\frac{1}{s^\gamma + \frac{1}{\tau_\sigma}}\bar{\varepsilon}\right]. \end{aligned} \tag{79}$$

By use of $\mathscr{L}^{-1}\left(1/\left(s^\gamma + \frac{1}{\tau_\sigma}\right)\right) = e_{\gamma,\gamma}(t; \frac{1}{\tau_\sigma})$ (see [8]) and of the convolution theorem it follows that

$$\sigma_z(t) = E\frac{\tau_\varepsilon}{\tau_\sigma}\varepsilon + \frac{1}{\tau_\sigma}E\left(1 - \frac{\tau_\varepsilon}{\tau_\sigma}\right)\int_0^t e_{\gamma,\gamma}\left(\tau; \frac{1}{\tau_\sigma}\right)\varepsilon(t-\tau)\,d\tau. \tag{80}$$

Equation (80) may be obtained from (62), if we neglect the terms containing the factor $e^{-\frac{1}{\tau_\sigma}(t-\tau)}$ under the integral sign. The difference between $\sigma(t)$ and $\sigma_z(t)$ in a stress relaxation test is zero for $t = 0$ and when $t \to \infty$ (see Remark 2). Thus (62) has the same asymptotic behavior as the generalized Zener model (3) or (80). The most important difference between (80) and (62), with the restriction on the coefficients (2), is that (62) satisfies the entropy inequality and the condition of stability of equilibrium for all deformation processes $\varepsilon(t)$.

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