Continuum Mech. Thermodyn. (1999) 4, 217–225

Application of High Moment Theory to the Plane Couette Flow

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Received November 30, 1998

In this paper we make use of 26 field equations derived by the moment method to describe the stationary plane Couette flow for dense and rarefied gases. In such high moment theories one needs more boundary conditions than can be controlled experimentally. We employ a new minimax principle for the entropy production to determine the remaining boundary conditions. There results a heat flux in the direction of the flow, proportional to the curvature of the velocity field. Such a heat flux confirms results from molecular dynamics, it vanishes in a dense gas.

1 Introduction

The stationary plane Couette flow is one of the standard problems in the kinetic theory of gases. It has been dealt with by many investigators [1]–[7]. Here we choose this problem to test a new minimax principle for the entropy production. With the help of this new principle it is possible to apply high moment theories to boundary value problems like the plane Couette flow.

We consider a steady flow of a monatomic ideal gas, generated by the relative motion of two infinite parallel plates shown in Fig. 1.

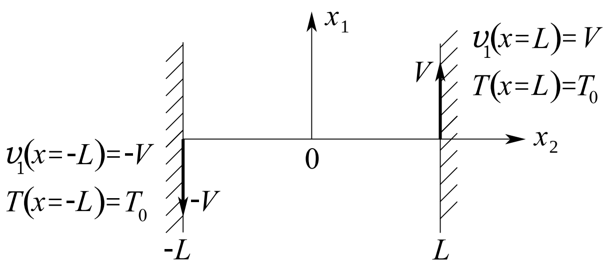

Fig. 1. The plane Couette flow.

The left plate moves with velocity $-V$ in its own plane at the space coordinate $x_2 = -L$, while the right plate at $x_2 = L$ moves parallel to the left plate with velocity $+V$. The flow is assumed to be laminar and no external forces are taken into account. In addition the temperatures of the plates are prescribed in the experiment. Furthermore we assume that all quantities depend only on the x_2 component of the space coordinate. The main objective is the determination of the fields mass density ϱ, velocity v_i, and temperature T. The results depend strongly on the density of the gas or more precisely on the Knudsen number Kn which is given by the ratio of the mean free path of the particles to a characteristic macroscopic length. We can roughly distinguish two regimes:

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Hydrodynamic regime: For a dense gas the Navier-Stokes-Fourier theory is sufficient to describe this problem satisfactorily. In this case the pressure deviator $p_{<ij>}$ and the heat flux q_i are given by

$$p_{<ij>} = -2\mu \frac{\partial v_{<i}}{\partial x_{j>}}, \tag{1}$$

$$q_i = -\kappa \frac{\partial T}{\partial x_i}. \tag{2}$$

The coefficients μ and κ are the viscosity and the heat conductivity, respectively. The angular brackets denote the symmetric and traceless part. In the linearized case, when μ and κ are constant, the velocity is a linear function connecting the values $-V_0$ and $+V_0$ at the left and the right plate.

Kinetic regime: For a rarefied gas the Navier-Stokes-Fourier theory fails. The velocity is not a linear function even in the linear case [1], and - one of the most interesting facts - the heat flux q_1 in the flow direction is not zero even though there is no temperature gradient in the flow direction! Therefore, Fourier's law is not valid in the kinetic regime. This result is obtained by the application of different kinetic theories [2],[5]-[8]. From the theory described below it follows for the plane Couette flow with $v_2 = v_3 = 0$, for the heat flux perpendicular to the plates

$$q_2 = -\kappa \frac{dT}{dx_2} - \frac{39}{14}\frac{\mu}{\varrho}\frac{dp_{<22>}}{dx_2}, \tag{3}$$

and for the heat flux parallel to the plates

$$q_1 = -\frac{909}{266}\frac{\mu^2}{\varrho}\frac{d^2 v_1}{dx_2^2}. \tag{4}$$

q_1 may be said to be induced by the shear flow and it can be observed in computer experiments. Figure 2 shows the heat fluxes q_1 and q_2, obtained by a molecular dynamic simulation[1] for hard disks made by Risso and Cordero [2]. This simulation refers to the large[2] Knudsen number $Kn = 0.1141$. Therefore, the gas is quite dilute and q_1 is different from zero, except in the center of the channel.

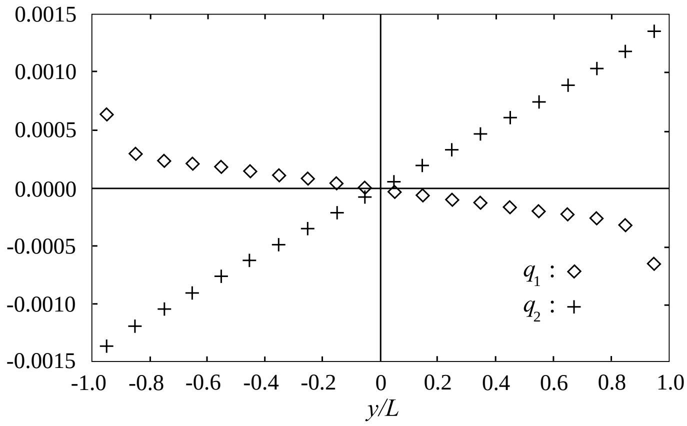

Fig. 2. Heat flux q_2 perpendicular to the plates and q_1 parallel to the plates from MD calculations [2].

2 Moment method

The moment method is based on the Boltzmann equation

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} = S(f, f), \tag{5}$$

[1] In the original paper by Risso and Cordero the heat flux q_1 is plotted for $v_1(-L) = V$ and $v_1(L) = -V$.

[2] For instance, for argon at $300K$, $1bar$ and a macroscopic length of $1m$ we have $Kn = 3.6 \cdot 10^{-8}$.

for the distribution function $f(\mathbf{x}, \mathbf{c}, \mathbf{t})$ of atoms. Here, c_i denotes the velocity of the atoms, t is the time and $S(f, f)$ is an abbreviation of the collision integral. It is possible to calculate all thermodynamic quantities of interest by the so called *moments* of the distribution function. The mass density ϱ and the momentum density ϱv_i are given by

$$\varrho = m \int f dc \qquad \text{and} \qquad \varrho v_i = m \int c_i f dc, \tag{6}$$

where m denotes the mass of an atom. With the peculiar velocity $C_i = c_i - v_i$ it is possible to calculate the internal energy density as

$$\varrho\varepsilon = \frac{m}{2} \int C^2 f dC, \tag{7}$$

the pressure deviator and the heat flux as

$$p_{<ij>} = m \int C_{<i} C_{j>} f dC \qquad \text{and} \qquad q_i = \frac{m}{2} \int C_i C^2 f dC. \tag{8}$$

This exhausts the moments with a physical meaning. But it is possible to proceee: We may define more quantities - *higher moments* - which have no immediate plausible physical interpretation. Thus

$$\varrho_{<ijk>} = m \int C_{<i} C_j C_{k>} f dC, \tag{9}$$

$$\Delta = m \int C^4 (f - f_M) f dC, \tag{10}$$

$$\varrho_{kk<ij>} = m \int C^2 C_{<i} C_{j>} f dC. \tag{11}$$

The distribution function in thermal equilibrium is the Maxwell distribution function f_M. From the Boltzmann equation (5) one may derive transfer equations for the moments. Such a set of transfer equations is not closed. The closure problem may be solved by the moment method of Grad [9] or by the entropy maximum method [10]. Thus it is possible to derive a specific system of field equations for the set of 26 moments given by (6)-(11).

In this work we are interested in the higher moment effects, not in the nonlinear effects. Therefore, we use the linearized system which has the advantage that we get analytical solutions and in addition no problems occur with non-unique solutions.

We linearize the system around an equilibrium state given by

$$\varrho = \varrho_0, \quad v_i = 0, \quad T = T_0, \quad p_{<ij>} = 0, \quad q_i = 0, \quad \varrho_{<ijk>} = 0, \quad \Delta = 0, \quad \varrho_{kk<ij>} = 0, \tag{12}$$

and obtain the following set of 26 equations

$$\frac{\partial \tilde{\varrho}}{\partial t} + \varrho_0 \frac{\partial \tilde{v}_k}{\partial x_k} = 0, \tag{13}$$

$$\frac{\partial \tilde{v}_i}{\partial t} + \frac{k}{m} T_0 \left(\frac{1}{\varrho_0} \frac{\partial \tilde{\varrho}}{\partial x_i} + \frac{1}{T_0} \frac{\partial \tilde{T}}{\partial x_i} \right) + \frac{1}{\varrho_0} \frac{\partial \tilde{p}_{<ik>}}{\partial x_k} = 0, \tag{14}$$

$$\frac{\partial \tilde{T}}{\partial t} + \frac{2}{3} T_0 \frac{\partial \tilde{v}_k}{\partial x_k} + \frac{2}{3} \frac{1}{\frac{k}{m} \varrho_0} \frac{\partial \tilde{q}_k}{\partial x_k} = 0, \tag{15}$$

$$\frac{\partial \tilde{p}_{<ij>}}{\partial t} + \frac{4}{5} \frac{\partial \tilde{q}_{<i}}{\partial x_{j>}} + \frac{\partial \tilde{\varrho}_{<ijk>}}{\partial x_k} + \underline{2 \varrho_0 \frac{k}{m} T_0 \frac{\partial \tilde{v}_{<i}}{\partial x_{j>}}} = \underline{-\alpha \varrho_0 \tilde{p}_{<ij>}}, \tag{16}$$

$$\frac{\partial \tilde{q}_i}{\partial t} - \frac{5}{2} \frac{k}{m} T_0 \frac{\partial \tilde{p}_{<ik>}}{\partial x_k} + \frac{1}{6} \frac{\partial \tilde{\Delta}}{\partial x_i} + \frac{1}{2} \frac{\partial \tilde{\varrho}_{nn<ik>}}{\partial x_k} + \underline{\frac{5}{2} \frac{k^2}{m^2} \varrho_0 T_0 \frac{\partial \tilde{T}}{\partial x_i}} = \underline{-\frac{2}{3} \alpha \varrho_0 \tilde{q}_i}, \tag{17}$$

$$\frac{\partial \tilde{\varrho}_{<ijk>}}{\partial t} + \frac{1}{7}\left\{\frac{\partial \tilde{\varrho}_{rr<ij>}}{\partial x_k} + \frac{\partial \tilde{\varrho}_{rr<jk>}}{\partial x_i} + \frac{\partial \tilde{\varrho}_{rr<ik>}}{\partial x_j} - \frac{2}{5}\left(\frac{\partial \tilde{\varrho}_{rr<in>}}{\partial x_n}\delta_{jk} + \frac{\partial \tilde{\varrho}_{rr<jn>}}{\partial x_n}\delta_{ik} + \frac{\partial \tilde{\varrho}_{rr<kn>}}{\partial x_n}\delta_{ij}\right)\right\} = -\frac{3}{2}\alpha\varrho_0\tilde{\varrho}_{<ijk>} \tag{18}$$

$$\frac{\partial \tilde{\Delta}}{\partial t} + 8\frac{k}{m}T_0\frac{\partial \tilde{q}_k}{\partial x_k} = -\frac{2}{3}\alpha\varrho_0\tilde{\Delta}, \tag{19}$$

$$\frac{\partial \tilde{\varrho}_{rr<ij>}}{\partial t} + 14\varrho_0\left(\frac{k}{m}T_0\right)^2\frac{\partial \tilde{v}_{<i}}{\partial x_{j>}} + \frac{56}{5}\frac{k}{m}T_0\frac{\partial \tilde{q}_{<i}}{\partial x_{j>}} + 9\frac{k}{m}\frac{\partial \tilde{\varrho}_{<ijk>}}{\partial x_k} = -\frac{7}{6}\alpha\varrho_0\left(\tilde{\varrho}_{rr<ij>} - \frac{k}{m}T_0\tilde{p}_{<ij>}\right), \tag{20}$$

for the 26 unknowns

$$\tilde{\varrho}, \quad \tilde{v}_i, \quad \tilde{T}, \quad \tilde{p}_{<ij>}, \quad \tilde{q}_i, \quad \tilde{\varrho}_{<ijk>}, \quad \tilde{\Delta}, \quad \tilde{\varrho}_{rr<ij>}. \tag{21}$$

The quantities with the tilde denote deviations from the equilibrium state given by (12). k denotes the Boltzmann constant. The parameter α is a number and follows from the interaction potential of the Maxwell molecules for which the right hand side of the system is calculated [11].

In addition we need the entropy production Σ for the 26 moment case. The entropy production contains no linear parts, rather it is quadratic in the nonequilibrium quantities. Thus we obtain the following expression

$$\tilde{\Sigma} = \frac{\alpha}{\frac{k}{m}T_0^2}\left(\frac{1}{2}\tilde{p}_{<ij>}\tilde{p}_{<ij>} + \frac{1}{24\left(\frac{k}{m}T_0\right)^2}\left(\tilde{\varrho}_{kk<ij>} - 7\frac{k}{m}T_0\tilde{p}_{<ij>}\right)\left(\tilde{\varrho}_{ll<ij>} - 7\frac{k}{m}T_0\tilde{p}_{<ij>}\right) + \frac{1}{4\frac{k}{m}T_0}\tilde{\varrho}_{<ijk>}\tilde{\varrho}_{<ijk>} + \frac{4}{15\frac{k}{m}T_0}\tilde{q}_i\tilde{q}_i + \frac{1}{180\left(\frac{k}{m}T_0\right)^2}\tilde{\Delta}^2\right). \tag{22}$$

For details concerning the explicit derivation of such systems and of the entropy production see [11].

The underlined terms in (16) and (17) represent the Navier-Stokes law

$$p_{<ij>} = -2\mu\frac{\partial v_{<i}}{\partial x_{j>}}, \qquad \text{with} \qquad \mu = \frac{\frac{k}{m}T_0}{\alpha}, \tag{23}$$

and the Fourier law

$$q_i = -\kappa\frac{\partial T}{\partial x_i} \qquad \text{with} \qquad \kappa = \frac{15}{4}\frac{\frac{k^2}{m^2}T_0}{\alpha}. \tag{24}$$

Therefore, we may interpret (16) and (17) as generalizations of the laws of Navier-Stokes and Fourier, respectively. Furthermore we obtain explicit expressions for the viscosity μ and the heat conductivity κ for the Maxwell molecules considered.

It is possible to derive systems with differently many moments, for instance Grad's well-known 13 moment system [9], or systems with 14, 20, 26, 35 and more moments [11]. Here we consider the 26 moment case because this is the system with a minimal number of moments which differs in the linear case and for the plane Couette flow from the Navier-Stokes-Fourier system.

The high moment theories were applied to several physical phenomena like plane harmonic waves [11], light scattering [11, 12], the shock structure problem [11, 13] and the thermodynamics of radiation [11, 14]. From all these applications one may conclude that the moment method is capable of describing the phenomena satisfactorily, if the number of moments is sufficiently high. For the circumstances in this paper this means: if the gas becomes more and more dilute, systems with more and more moments must be considered in order to obtain satisfactory results. Therefore, we may expect the 26 moment system to be good in the hydrodynamic regime. In the kinetic regime, at least up to a certain Knudsen number.

3 Application to the plane Couette flow

The following dimensionless quantities are introduced

$$\hat{y}=\frac{x_2}{L},\quad B=\frac{\sqrt{\frac{k}{m}T_0}}{\alpha\varrho_0 L},\quad \hat{\varrho}=\frac{\tilde{\varrho}}{\varrho_0},\quad \hat{v}_i=\frac{\tilde{v}_i}{\sqrt{\frac{k}{m}T_0}},\quad \hat{T}=\frac{\tilde{T}}{T_0},\quad \hat{p}_{<ij>}=\frac{\tilde{p}_{<ij>}}{\varrho_0\frac{k}{m}T_0},\tag{25}$$

$$\hat{q}_i=\frac{\tilde{q}_i}{\varrho_0\left(\frac{k}{m}T_0\right)^{\frac{3}{2}}},\quad \hat{\Delta}=\frac{\tilde{\Delta}}{\varrho_0\left(\frac{k}{m}T_0\right)^{2}},\quad \hat{\varrho}_{<ijk>}=\frac{\tilde{\varrho}_{<ijk>}}{\varrho_0\left(\frac{k}{m}T_0\right)^{\frac{3}{2}}},\quad \hat{\varrho}_{rr<ij>}=\frac{\tilde{\varrho}_{rr<ij>}}{\varrho_0\left(\frac{k}{m}T_0\right)^{2}}.\tag{26}$$

We assume that all quantities depend only on the space coordinate x_2. With this ansatz some equations of the system (13) up to (20) become trivial[3] and we obtain at last eleven equations for the eleven unknowns

$$\hat{\varrho},\quad \hat{v}_1,\quad \hat{T},\quad \hat{p}_{<11>},\quad \hat{p}_{<12>},\quad \hat{p}_{<22>},\quad \hat{q}_1,\quad \hat{q}_2,\quad \hat{\varrho}_{<112>},\quad \hat{\varrho}_{<222>},\quad \hat{\varrho}_{kk<12>}.\tag{27}$$

The eleven equations split into two systems. For $\hat{v}_1$, $\hat{p}_{<12>}$, $\hat{q}_1$, $\hat{\varrho}_{kk<12>}$ we obtain the four equations

$$\frac{d\hat{p}_{<12>}}{d\hat{y}}=0,\qquad \frac{38}{63}\frac{d\hat{q}_1}{d\hat{y}}+\frac{d\hat{v}_1}{d\hat{y}}=-\frac{1}{B}\hat{p}_{<12>},\tag{28}$$

$$\frac{d\hat{\varrho}_{kk<12>}}{d\hat{y}}=-\frac{4}{3}\frac{1}{B}\hat{q}_1,\qquad 7\frac{d\hat{v}_1}{d\hat{y}}+\frac{52}{7}\frac{d\hat{q}_1}{d\hat{y}}=-\frac{7}{6}\frac{1}{B}\left(\hat{\varrho}_{kk<12>}-\hat{p}_{<12>}\right),\tag{29}$$

and for $\hat{\varrho}$, $\hat{T}$, $\hat{p}_{<11>}$, $\hat{p}_{<22>}$, $\hat{q}_2$, $\hat{\varrho}_{<112>}$, $\hat{\varrho}_{<222>}$ we obtain the seven equations

$$\frac{d\hat{\varrho}}{d\hat{y}}+\frac{d\hat{T}}{d\hat{y}}+\frac{d\hat{p}_{<22>}}{d\hat{y}}=0,\qquad \frac{d\hat{q}_2}{d\hat{y}}=0,\qquad \frac{d\hat{\varrho}_{<222>}}{d\hat{y}}=-\frac{1}{B}\hat{p}_{<22>},\tag{30}$$

$$\frac{26}{7}\frac{d\hat{p}_{<22>}}{d\hat{y}}+5\frac{d\hat{T}}{d\hat{y}}=-\frac{4}{3}\frac{1}{B}\hat{q}_2,\qquad \frac{366}{245}\frac{d\hat{p}_{<22>}}{d\hat{y}}=-\frac{1}{B}\hat{\varrho}_{<222>}\tag{31}$$

$$\frac{d\hat{\varrho}_{<112>}}{d\hat{y}}=-\frac{1}{B}\hat{p}_{<11>},\qquad \frac{61}{49}\frac{d\hat{p}_{<11>}}{d\hat{y}}-\frac{122}{245}\frac{d\hat{p}_{<22>}}{d\hat{y}}=-\frac{3}{2}\frac{1}{B}\hat{\varrho}_{<112>}.\tag{32}$$

We may say that the equations (28) and (29) describe the shear flow problem and that the equations (30), (31) and (32) are relevant to the heat flow problem. This is a result of the linearization.

From the equations (28) and (29) follows a simple expression for the heat flux $\hat{q}_1$. By use of $(23)_2$ its dimensional form is given by (4). Also from $(31)_1$ follows an expression for q_2 which is given by (3).

In this work we are not interested in the details of accommodation of the particles at the walls. Therefore, we forbid velocity slip and temperature jump at the boundaries.

In order to solve the complete system (28) through (32) of 11 equations for the 11 unknowns (27), we need 11 boundary conditions to determine the 11 constants that arise from integration. However, in an experiment we can control only 4 boundary conditions! Here we prescribe the velocity $\tilde{v}_1(-L)=-V$, $\tilde{v}_1(+L)=+V$ and the temperature $\tilde{T}(-L)=\tilde{T}(+L)=0$ at the plates. For the dimensionless quantities we obtain

$$\hat{v}_1\,(\hat{y}=-1)=\frac{-V}{\sqrt{\frac{k}{m}T_0}}=-\hat{V},\qquad \hat{v}_1\,(\hat{y}=+1)=\frac{+V}{\sqrt{\frac{k}{m}T_0}}=+\hat{V},\tag{33}$$

$$\hat{T}\,(\hat{y}=-1)=0,\qquad \hat{T}\,(\hat{y}=+1)=0.\tag{34}$$

It is impossible in an experiment to control more quantities. For instance, if we prescribe the temperature at the plate we cannot also prescribe the heat flux. Further, if we prescribe the velocity at the wall we cannot also prescribe the pressure deviator. Of course, it is possible to use boundary conditions different from (33) and (34). For instance, we may prescribe the heat flux at the plate, but then we cannot also control the temperature.

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[3] For instance from (13) we get $v_2 = const$ and then the no-slip boundary condition implies $v_2 = 0$.

Sometimes one has additional conditions which are not boundary conditions. For instance, in an experiment it is possible to fix the mass in the channel. Then the deviation $\hat{\varrho}$ must satisfy the condition

$$\int_{-1}^{1} \hat{\varrho} d\hat{y} = 0. \tag{35}$$

However, even with this additional condition it is impossible to determine all integration constants. This problem gets worse in high moment theories where the number of integration constants becomes much greater than the number of the physical controllable boundary conditions. Therefore, we need additional physical information to determine the remaining boundary conditions.

Here we use the new minimax principle proposed by Struchtrup and Weiss [15]. This principle states:

In a stationary process the boundary values which are not controlled in the experiment assume values such that the global maximum of the local entropy production becomes minimal.

Therefore, we have to adjust the remaining boundary values so that $\max(\Sigma(x_2))$ becomes minimal. Due to the fact that we have a linear system we can do this analytically.

It is clear that the part of the entropy production depending on the variables of the system (30), (31) and (32) is minimal for

$$\hat{\varrho} = 0, \quad \hat{T} = 0, \quad \hat{p}_{<11>} = 0, \quad \hat{p}_{<22>} = 0, \quad \hat{q}_2 = 0, \quad \hat{\varrho}_{<112>} = 0, \quad \hat{\varrho}_{<222>} = 0. \tag{36}$$

And indeed this is a solution of the system which satisfies the boundary conditions (34) and the integral condition (35).

Now we consider the system (28) and (29). Here we have four equations and the two boundary conditions (33). The solution is given by

$$\hat{v}_1 = C_1 - BC_2\hat{y} - \frac{38}{63}\left(C_3 \sinh\left(\sqrt{\frac{49}{101}}\frac{\hat{y}}{B}\right) + C_4 \cosh\left(\sqrt{\frac{49}{101}}\frac{\hat{y}}{B}\right)\right), \tag{37}$$

$$p_{<12>} = C_2, \tag{38}$$

$$\hat{q}_1 = C_3 \sinh\left(\sqrt{\frac{49}{101}}\frac{\hat{y}}{B}\right) + C_4 \cosh\left(\sqrt{\frac{49}{101}}\frac{\hat{y}}{B}\right), \tag{39}$$

$$\hat{\varrho}_{kk<12>} = 7C_2 - \frac{4}{3}\sqrt{\frac{101}{49}}\left(C_3 \cosh\left(\sqrt{\frac{49}{101}}\frac{\hat{y}}{B}\right) + C_4 \sinh\left(\sqrt{\frac{49}{101}}\frac{\hat{y}}{B}\right)\right). \tag{40}$$

We have to determine the four integration constants $C_1,\ C_2,\ C_3$ and C_4. For the velocity we have the two boundary conditions (33). Further we assume[4] a skewsymmetric velocity profile $\hat{v}_1(\hat{y}) = -\hat{v}_1(-\hat{y})$. From these three conditions we obtain

$$C_1 = 0, \qquad C_3 = \frac{-\hat{V} - \frac{1}{B}C_2}{\frac{38}{63}\sinh\left(\sqrt{\frac{49}{101}}\frac{1}{B}\right)}, \qquad C_4 = 0. \tag{41}$$

Therefore, only C_2 has to be determined by the minimax principle. From (22) and the above results the following expression results for the dimensionless entropy production $\hat{\Sigma}$

$$\hat{\Sigma}(\hat{y}, C_2, B) = \frac{\Sigma(\hat{y}, C_2)}{\varrho_0^2 \frac{k}{m}\alpha} = C_2^2 + \frac{303}{361}\left(\hat{V} + \frac{1}{B}C_2\right)^2 \frac{1 + 2\sinh^2\left(\sqrt{\frac{49}{101}}\frac{\hat{y}}{B}\right)}{\sinh^2\left(\sqrt{\frac{49}{101}}\frac{1}{B}\right)}. \tag{42}$$

Now we are able to evaluate the minimax principle. For a given velocity $\hat{V}$ of the plates and a fixed parameter B we look for the value of C_2, for which $\max(\hat{\Sigma}(\hat{y}), -1 \leq \hat{y} \leq +1)$ becomes minimal. This may be done analytically or numerically.

[4] This assumption is not necessary. The minimax principle leads to the same result, irrespective of whether symmetry is assumed or not. However, without the symmetry assumption we have to determine two constants.

The parameter B is related to the Knudsen number. Between the viscosity μ and the mean free path λ of the particles we have [8]

$$\mu = 0.499\varrho_0 \sqrt{\frac{8}{\pi}\frac{k}{m}T_0}\ \lambda. \tag{43}$$

As the characteristic macroscopic length of the problem the distance between the plates is chosen. Therefore, with the viscosity given by (23) and the definition of B given by $(25)_2$ we obtain the relation

$$Kn = \frac{\lambda}{2L} = 0.628B. \tag{44}$$

Since we are dealing with a linear system we choose a small value V viz. $V = 0.1$ for the velocity of the plates. The following figures show the results for several Knudsen numbers. The dependence of the integration constant C_2 as a function of Kn is shown in Fig. 4a. In Fig. 3a the velocity $\hat{v}_1$ is shown as a function of the space coordinate $\hat{y}$. It seems to be a straight line for every Knudsen number, but it is not: In Fig. 3b the deviation of the velocity profile from the straight line is plotted. For small Knudsen numbers $Kn \leq 0.001$ the velocity $\hat{v}_1$ is a straight line as it should be. For a moderately rarefied gas $\hat{v}_1$ is not a straight line, even though we are considering a linear system. The curve has an s-shape (Fig. 3b) which is in qualitative agreement with other kinetic calculations [1]. The maximal deviation from the straight line is reached for $Kn = 0.266$. For $Kn > 0.266$ the s-shape behavior becomes smaller, see Fig. 3b, $Kn = 0.4$ and $Kn = 0.5$. In a very rarefied gas the result for the velocity profile is again a straight line. This is also in agreement with another kinetic theory [3].

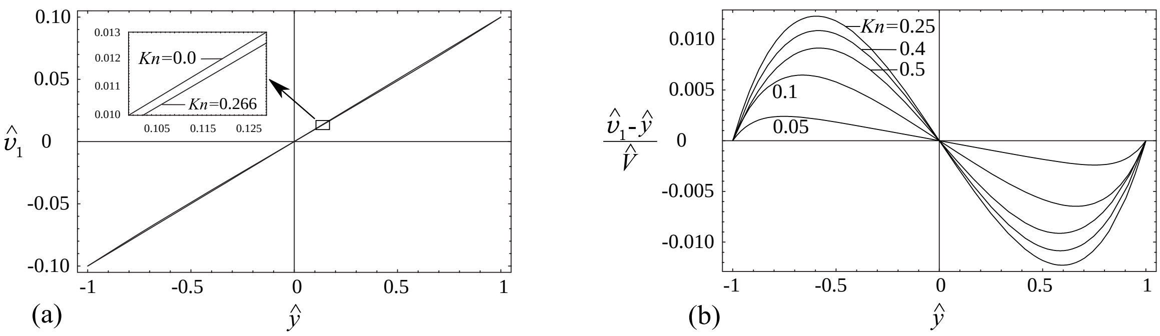

Fig. 3. Velocity profile for several Knudsen numbers.

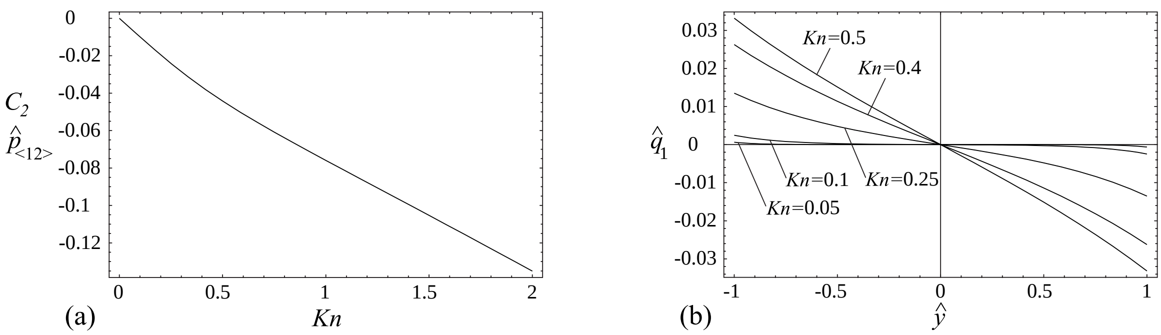

Fig. 4. Pressure deviator as a function of the Knudsen number (a) and the profile of the heat flux parallel to the plates (b).

We conclude from (38) that the pressure deviator $p_{<12>}$ is equal to C_2. Therefore, $p_{<12>}$ is constant with respect to $\hat{y}$ and its dependence of the Knudsen number may be read off from Fig. 4a. The heat flux q_1 parallel to the plates is shown in Fig. 4b. In the rarefied gas q_1 is not zero except in the center of the channel. For small Knudsen numbers $\hat{q}_1$ vanishes except in a small boundary layer near the plates. This boundary layer vanishes in the limit $Kn \to 0$ as may be expected. In order to show the boundary layer for small Knudsen numbers, q_1 is plotted in Fig. 5a, normalized by its value on the left boundary. The qualitative behavior is

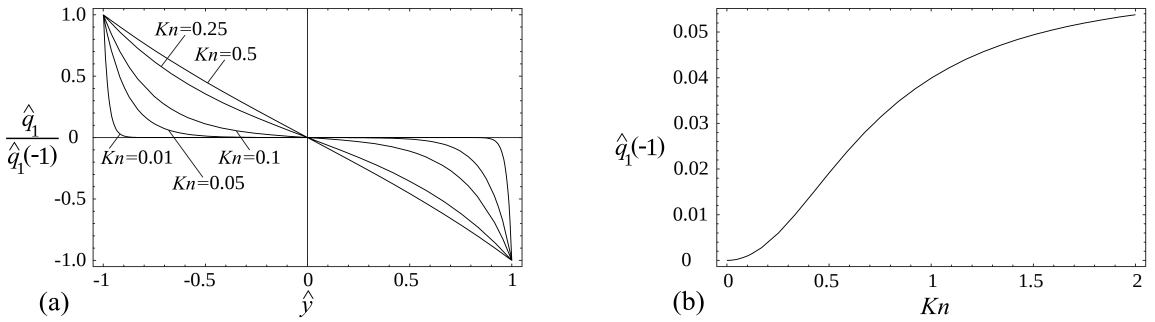

Fig. 5. Heat flux parallel to the plates.

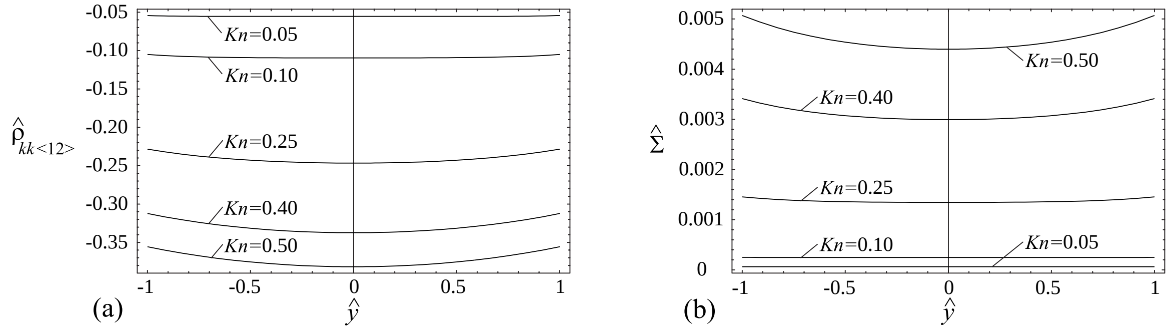

Fig. 6. The profiles of the fourth moment (a) and the entropy production (b).

the same as in Fig 2 which was obtained by molecular-dynamics. The value of q_1 at the left boundary as a function of the Knudsen number is shown in Fig. 5b. The dependence of $q_1(-1)$ for $Kn \to 0$ is quadratic in Kn, while for $Kn \to \infty$ we get $q_1(-1) \to 0.061$. The quantity $\hat{\varrho}_{kk<12>}$ is shown in Fig. 6a, but only for completeness, because we have no physical interpretation for it. More interesting is the entropy production $\hat{\Sigma}$, represented in Fig. 6b. We see that the maximum of $\hat{\Sigma}$ is always at the boundary.

From the results for the velocity $\hat{v}_1$ and the heat flux $\hat{q}_1$ we may conclude that the linearized 26 moment system together with the minimax principle is able to describe the plane Couette flow qualitatively correctly for dense and rarefied gases. However, for better results which deserve direct comparison with MD calculations the nonlinear system must be solved. This remains to be done in a future work.

4 Conclusion about the minimax principle

We want to point out an additional interesting fact concerning the minimax principle. In order to solve the whole system (13) up to (20) we have assumed that all quantities depend only on the space coordinate x_2, - nothing more. In addition it is possible to assume $\hat{q}_1 = 0$. If we do that, we obtain a much simpler system than (28) through (32) which is identical with the linearized Navier-Stokes-Fourier system. There are enough boundary conditions to solve that system and no minimax principle is necessary. It is remarkable that the system (13) through (18) together with the minimax principle does not lead to this solution, although it is a possible one. Instead it leads to the solution we have presented above in which $\hat{q}_1$ is different from zero for the dilute gas and which is supported by molecular dynamics. Therefore, the plane Couette flow is an example in which the minimax principle shows its relevance in the hydrodynamic *and* the kinetic regime.

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