Continuum Mech. Thermodyn. (1999) 11: 107–111 © Springer-Verlag 1999

Maximum wave velocity in the moments system of a relativistic gas

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Received September 28, 1998

We consider the system of moments associated with the relativistic Boltzmann-Chernikov equation. Using the particular symmetric form obtained by the closure procedure of Extended Thermodynamics we deduce a lower bound for the maximum velocity of wave propagation in terms of the number of moments for a non-degenerate gas. When the number of moments increases this velocity tends to the speed of light. We also give the lower bound estimate in the limit cases of ultrarelativistic fluids and in the non relativistic approximation.

In the relativistic kinetic theory of a rarefied gas the phase density $f(x^\alpha, p^\alpha)$ $(\alpha = 0, 1, 2, 3)$ satisfies the Boltzmann-Chernikov equation

$$p^\alpha \partial_\alpha f = Q, \qquad \partial_\alpha = \partial / \partial x^\alpha \tag{1}$$

in which x^α and p^α are the space-time coordinates and the four-momentum of an atom respectively. We have $p_\alpha p^\alpha = (p^0)^2 - \mathbf{p}^2 = m^2c^2$, $\mathbf{p}^2 = (p^1)^2 + (p^2)^2 + (p^3)^2$, where m is the atomic rest mass and c the speed of light. The right-hand side of (1) is due to collision between the atoms.

Upon multiplication by $p^{\alpha_1 \cdot\cdot} p^{\alpha_k}$, $(k = 1, \ldots)$ and integration the Boltzmann-Chernikov equation provides an infinite system of balance equations

$$\partial_\alpha F^{\alpha A} = g^A, \quad A = 0, \ldots \tag{2}$$

for the moments $F^{\alpha A}$ and productions g^A given by

$$F^{\alpha A}(x^\beta) = \int p^\alpha p^A f \, dP, \qquad g^A(x^\beta) = \int Q p^A f \, dP, \tag{3}$$

where the index A is a multindex. Specifically we have

$$p^A = \begin{cases} 1 \\ p^{\alpha_1} p^{\alpha_2} \ldots p^{\alpha_A} \end{cases}, \quad F^{\alpha A} = \begin{cases} F^\alpha \\ F^{\alpha\alpha_1 \ldots \alpha_A} \end{cases}, \quad g^A = \begin{cases} 0 & \text{for } A = 0 \\ g^{\alpha_1 \ldots \alpha_A} & \text{for } A \geq 1. \end{cases}$$

We recall that the first five equations of (2) are the conservation laws of mass, momentum and energy; according by the first five productions vanish. The volume element of momentum space is given by $dP = \sqrt{-g}\, dp^1 dp^2 dp^3 / p^0$ and the integrals - supposed convergent - are taken over the whole of $\mathbf{p}$-space. We consider a finite number of moments equations with the tensorial index $A = 0, \ldots, n$.

In a previous paper [1] the closure procedure of *Extended Thermodynamics* [2] was applied and, in particular, it was proved that f depends on the single variable

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$$\chi = \sum_{A=0}^{n} u'_A(x^\beta) p^A, \tag{4}$$

if the truncated system (2) is compatible with the entropy principle, i.e. any solution of (2) is also solution of the entropy equation

$$\partial_\alpha h^\alpha = g \leq 0$$

where $-h^\alpha$ and $-g$ are the entropy four vector and the entropy production respectively. The fields

$$u'_A = \begin{cases} u' & \text{for } A = 0 \\ u'_{\alpha_1\alpha_2\cdots\alpha_A} & \text{for } 1 \leq A \leq n \end{cases} \tag{5}$$

represent the components of the *main field* for which the truncated original system (2) may be written in the symmetric hyperbolic form as follows

$$\sum_{B=0}^{n} H^{\alpha AB}(u'_C)\, \partial_\alpha u'_B = g^A(u'_C), \qquad A = 0, \ldots, n. \tag{6}$$

The four matrices $H^{\alpha AB}(\alpha = 0, 1, 2, 3)$ are Hessian matrices. Indeed we have

$$H^{\alpha AB}(u'_C) = \frac{\partial^2 h'^\alpha}{\partial u'_A \partial u'_B} = \int F''(\chi)\, p^\alpha p^A p^B \, dP, \quad \text{with} \quad h'^\alpha(u'_C) = \int F(\chi) p^\alpha \, dP,$$

and $F(\chi)$ is an antiderivative of $f(\chi)$ with $dF/d\chi = f > 0$, $d^2F/d\chi^2 > 0$. Moreover h^α must be equal to $\int p^\alpha \left(\chi F'(\chi) - F(\chi)\right) dP$. Comparing with the expression of the entropy four-vector for a non degenerate gas $h^\alpha = k \int p^\alpha f \ \log f \, dP$ provides

$$f = e^{\chi/k - 1}. \tag{7}$$

In [1] it was proved that this closure procedure is equivalent to the one obtained by the *Maximum Entropy Principle* by which the entropy density is a maximum under the constraint that the moments are assigned functions [3]. Also in [1] it was shown that the maximum characteristic velocity of (6) does not exceed the speed of light. However, according to the arguments of [1] the maximum speed might still be smaller than c. This gap is closed in the present paper. Indeed here we show that a lower bound for the maximum velocity tends to c when the number of moments increases.

In the observer frame the characteristic polynomial which determines the wave speeds λ of the system (6) is given by

$$\det\left(\frac{\partial^2 h'^i}{\partial u'_A \partial u'_B} n_i - \frac{\lambda}{c} \frac{\partial^2 h'^0}{\partial u'_A \partial u'_B}\right) = 0,$$

where n_i are the spatial components of the normal of the wave front ($i = 1, 2, 3$). The system is symmetric hyperbolic since h'^0 is a convex function of u'_A. Therefore a theorem of linear algebra provides the result that

$$\frac{\partial^2 h'^i}{\partial u'_A \partial u'_B} n_i - \frac{\lambda_{\max}}{c} \frac{\partial^2 h'^0}{\partial u'_A \partial u'_B} \quad \text{is negative semi-definite} \tag{8}$$

where $\lambda_{\max}$ is the largest characteristic velocity. Since we must consider only the independent components of the main field $u'_{\alpha_1\ldots\alpha_k}$, which are symmetric tensors, we may choose $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k$. All of the indices may assume the values $0, 1, 2, 3$ and therefore, - following [1], - the components $u'_{\alpha_1\ldots\alpha_k}$ may be mapped into the variables u'_{pqrs} with $p + q + r + s = k$ ($k = 1, \ldots, n$), where p, q, r, s are the numbers of indices among $\alpha_1 \ldots \alpha_k$ which are equal to $0, 1, 2$ or 3 respectively. With this notation we have

$$\chi = \sum_{k=0}^{n} u'_{pqrs} \left(p^0\right)^p \left(p^1\right)^q \left(p^2\right)^r \left(p^3\right)^s, \qquad p + q + r + s = k$$

and because $(p^0)^2$ is equal to $m^2c^2 + \mathbf{p}^2$, the first index p can take only two values: 0 and 1. Now, since the number of elements with the sum of three integers $q + r + s = k$ with $k = 0, \ldots, m$ is $(m+1)(m+2)(m+3)/6$, the number $N(n)$ of independent components of u'_{pqrs} - and equations - up to order n is $N(n) = (n + 1)(n +$

$2)(n+3)/6+n(n+1)(n+2)/6 = (n+1)(n+2)(2n+3)/6$ so that $N(n+1) = N(n)+(n+2)^2$. Hence the previous condition (8) may be written in the form

$$n_i \frac{\partial^2 h'^i}{\partial u'_{pqrs} \partial u'_{tuvw}} - \frac{\lambda_{\max}}{c} \frac{\partial^2 h'^0}{\partial u'_{pqrs} \partial u'_{tuvw}} =$$
$$(1/k) \int e^{\chi/k-1} \left(p^i n_i - \lambda_{\max} p^0/c\right) \left(p^0\right)^{p+t} \left(p^1\right)^{q+u} \left(p^2\right)^{r+v} \left(p^3\right)^{s+w} dP$$

and this matrix must be negative semi-definite. The elements a_{ij} of such a matrix satisfy the inequality $a_{ii} a_{jj} \geq a_{ij}^2$ and therefore we have the inequalities

$$\begin{aligned} & \int e^{\chi/k} \left(p^i n_i - \lambda_{\max} p^0/c\right) \left(p^0\right)^{2p} \left(p^1\right)^{2q} \left(p^2\right)^{2r} \left(p^3\right)^{2s} dP \cdot \\ \cdot & \int e^{\chi/k} \left(p^i n_i - \lambda_{\max} p^0/c\right) \left(p^0\right)^{2t} \left(p^1\right)^{2u} \left(p^2\right)^{2v} \left(p^3\right)^{2w} dP \geq \\ & \left(\int e^{\chi/k} \left(p^i n_i - \lambda_{\max} p^0/c\right) \left(p^0\right)^{p+t} \left(p^1\right)^{q+u} \left(p^2\right)^{r+v} \left(p^3\right)^{s+w} dP \right)^2 . \end{aligned} \tag{9}$$

Now we consider a wave propagating into an equilibrium state. In this case the first five components of the main field are: $u' = G/T$, $u'_\alpha = -u_\alpha/T$, where G, T and u_α are respectively the chemical potential, the absolute temperature and the four-velocity [4]. The remaining main fields are zero [2]. Therefore we have $\chi_e/k = (G - u_\alpha p^\alpha)/(kT)$. In the rest frame, $u^i = 0$, $u^0 = c$, $p^0 = \sqrt{m^2c^2 + \mathbf{p}^2}$, the previous expression becomes

$$\frac{\chi_e}{k} = \frac{G}{kT} - \gamma \sqrt{1 + \frac{\mathbf{p}^2}{m^2c^2}}, \qquad \gamma = \frac{mc^2}{kT}.$$

Inserting this into (7) we obtain the well known Jüttner equilibrium distribution function.

With the special choice of indices values $p = t = r = v = s = w = 0$, $q = n$, $u = n-1$, and $n_i \equiv (1, 0, 0)$ the inequality (9) reduces to

$$\left(\lambda_{\max}^2/c^2\right) \int e^{\chi_e/k} \left(p^1\right)^{2n} d^3p \int e^{\chi_e/k} \left(p^1\right)^{2(n-1)} d^3p \geq \left(\int e^{\chi_e/k} \left(p^1\right)^{2n} d^3p/p^0 \right)^2 \tag{10}$$

since the integrals of the odd functions vanish ($d^3p = dp^1 dp^2 dp^3$). With the polar representation $p^1 = mcr \sin\theta \cos\varphi$, $p^2 = mcr \sin\theta \sin\varphi$, $p^3 = mcr \cos\theta$ the inequality (10) becomes

$$\begin{aligned} & \frac{\lambda_{\max}^2}{c^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{+\infty} \exp\left(-\gamma\sqrt{1+r^2}\right) r^{2n+2} (\sin\theta)^{2n+1} (\cos\varphi)^{2n} \, dr d\theta d\varphi \cdot \\ & \cdot \int_0^{2\pi} \int_0^{\pi} \int_0^{+\infty} \exp\left(-\gamma\sqrt{1+r^2}\right) r^{2n} (\sin\theta)^{2n-1} (\cos\varphi)^{2n-2} \, dr d\theta d\varphi \geq \\ & \left[\int_0^{2\pi} \int_0^{\pi} \int_0^{+\infty} \exp\left(-\gamma\sqrt{1+r^2}\right) r^{2n+2} (\sin\theta)^{2n+1} (\cos\varphi)^{2n} \, dr d\theta d\varphi / \sqrt{1+r^2} \right]^2 . \end{aligned}$$

Let us denote the integrals as

$$I_n = \int_0^\infty \exp\left(-\gamma\sqrt{1+r^2}\right) r^{2n} dr, \quad J_n = \int_0^\infty \exp\left(-\gamma\sqrt{1+r^2}\right) r^{2n} dr/\sqrt{1+r^2}. \tag{11}$$

Integration by part provides

$$I_n = \gamma J_{n+1}/\left(2n+1\right). \tag{12}$$

On the other hand we have

$$\begin{aligned} A_n &= \int_0^\pi (\sin\theta)^{2n} d\theta = \sqrt{\pi}\,\Gamma\left(n + \frac{1}{2}\right) / \Gamma\left(n+1\right), \\ B_n &= \int_0^{2\pi} (\cos\theta)^{2n} d\theta = 2^{2n+1}\pi^2 / \Gamma\left(2n+1\right) \left[\Gamma\left(\frac{1}{2} - n\right)\right]^2 . \end{aligned}$$

Γ is the classical Euler's function. Therefore we arrive at

$$\frac{\lambda_{\max}^2}{c^2} \geq \frac{(2n+1)^2\, I_n A_{n+\frac{1}{2}} B_n}{\gamma^2 I_{n+1} A_{n-\frac{1}{2}} B_{n-1}}. \tag{13}$$

Now I_n may be written in terms of the Bessel function of the second kind

$$I_n = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\gamma}\right)^n K_{n+1}(\gamma)\, \Gamma\left(n + \frac{1}{2}\right)$$

while $A_n/A_{n-1} = B_n/B_{n-1} = 1 - 1/2n$ and (13) gives

$$\frac{\lambda_{\max}^2}{c^2} \geq \frac{2n-1}{2n+1} \psi_{n+1}, \tag{14}$$

with

$$\psi_{n+1} = \frac{(2n+1)^2}{\gamma^2} \frac{I_n}{I_{n+1}} = \frac{(2n+1)}{\gamma} \frac{K_{n+1}(\gamma)}{K_{n+2}(\gamma)}. \tag{15}$$

Thus finally we obtain the lower bound

$$\frac{\lambda_{\max}^2}{c^2} \geq \frac{(2n-1)}{\gamma} \frac{K_{n+1}(\gamma)}{K_{n+2}(\gamma)}. \tag{16}$$

Therefore we conclude: *For any truncated moment system - with tensorial index n - compatible with an entropy principle, the maximum velocity of a disturbance propagating into an equilibrium state satisfies the lower-bound* (16).

Now our goal is to prove that this lower-bound increases with n to the light velocity c: In fact it is obvious from the definitions (11) that $J_n < I_n$ so that we get from (12)

$$\frac{I_n}{I_{n+1}} < \frac{\gamma}{2n+1}, \quad \text{and from (15),} \quad 0 < \frac{\gamma\psi_n}{2n-1} < 1. \tag{17}$$

From the recurrence relation for Bessel functions $K_{n+2} - K_n = 2(n+1) K_{n+1}/\gamma$ we obtain finally

$$\frac{1}{\psi_{n+1}} - \frac{\gamma\psi_n}{2n-1} \frac{\gamma}{2n+1} = \frac{2(n+1)}{2n+1}.$$

With this and $(17)_2$ we conclude that $\psi_{n+1} \to 1$ for $n \to \infty$ and therefore by (14) the limit value of $\lambda_{\max}$ is c, since it has already been proved that it cannot be larger.

Thus *when the number of moments tends to infinity the maximum velocity in equilibrium tends to the light velocity.*

This last result improves a previous one [5] in which it was shown that the maximum velocity does not decrease when the number of moments increases and it is in agreement with the one (e.g. [6]) obtained directly from the linear Boltzmann-Chernikov equation in which the maximum phase velocity in equilibrium is c. We observe that in contrast to [6] our proof is completely independent of the interaction term Q of (1).

In the *ultrarelativistic case* corresponding to small γ, taking the properties of the Bessel functions into account when $\gamma \to 0$: $K_{n+1}/(\gamma K_{n+2}) \to 1/(2(1+n))$, we obtain for the inequality (16)

$$\frac{\lambda_{\max}^2}{c^2} \geq \frac{(2n-1)}{2(n+1)}.$$

On the other hand when $\gamma \to \infty$ in *the non relativistic limit* $K_n/K_{n+1} \to 1$, and (16) yields:

$$\frac{\lambda_{\max}^2}{c^2} \geq \frac{2n-1}{\gamma} = \frac{2n-1}{c^2} \frac{kT}{m}.$$

In this classic case we introduce the sound velocity $c_S = \sqrt{\frac{5}{3}\frac{kT}{m}}$, and obtain

$$\frac{\lambda_{\max}^2}{c_S^2} \geq \frac{6}{5}\left(n - \frac{1}{2}\right),$$

which coincides with the inequality already obtained in [1].

Acknowledgements. This paper was supported by Consiglio Nazionale delle Ricerche (CNR-GNFM), Ministero della Università e della Ricerca Scientifica e Tecnologica (MURST - fondi ex 40% and 60%), by Istituto Nazionale di Fisica Nucleare (INFN). We thank Ingo Müller and Alberto Strumia for interesting discussions.

References

1. Boillat G, Ruggeri T (1997) Moment equations in the kinetic theory of gases and wave velocities. Continuum Mech. Thermodyn. 9, 205
2. Müller I, Ruggeri T (1998) Rational Extended Thermodynamics, Springer Tracts in Natural Philosophy 37 (II Edition), Springer-Verlag (New York)
3. Dreyer, W (1987) Maximisation of the Entropy in Non-Equilibrium. J. Phys. A: Math. Gen. 20, 6505
4. Ruggeri T, Strumia A (1981) Main field and convex covariant density for quasi-linear hyperbolic systems. Relativistic fluid dynamics. Ann. Inst. H. Poincaré, 34A, 65
5. Boillat G, Ruggeri T (1997) Hyperbolic Principal Subsystems: Entropy Convexity and Subcharacteristic Conditions. Arch. Rat. Mech. Anal. 137 (4), 305
6. Cercignani, C (1983) Speed of Propagation of Infinitesimal Disturbances in a Relativistic Gas. Phys. Rev. Lett. 50, (15), 1122