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Multi-component convection-diffusion in a porous medium

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The stability of a triply diffusive fluid-saturated porous layer is investigated. A linear stability analysis similar to that of Pearlstein et al [1] is presented. This allows us to make a thorough investigation of the topology of the neutral curves. For some values of the thermal and solute diffusivities we obtain highly unusual neutral curves, in particular a heart-shaped, disconnected oscillatory curve. The effect of this is that three critical Rayleigh numbers are required to fully specify the linear stability criteria, a novel result in porous convection. The influence of nonlinear terms is likely to have important consequences for the experimental realisation of the linear results and so we investigate the nonlinear stability of the problem by making use of the energy method. This provides an unconditional nonlinear stability boundary and enables us to identify possible regions of subcritical instability.

1 Introduction

The subject of double diffusive convection of a viscous fluid, or of a fluid-saturated porous layer, has been an active area of research for many years. In the viscous fluid case in particular there is a considerable body of work. Reviews of this subject can be found in Turner [2] and Huppert and Turner [3]. In the area of porous media fluid mechanics there are many physical problems that can be modelled as fluid-saturated porous layers stratified by heat and salt concentration, such as the geothermal reservoirs found in the Imperial Valley in California (Cheng [4]), near Lake Kinnert in Israel (Rubin [5]) and the Wairakei geothermal system in New Zealand (Griffiths [6]).

Reviews of the theoretical treatment of the double diffusive porous problem can be found in Cheng [4] and Nield and Bejan [7]. The linear stability of a fluid-saturated porous layer stratified by heat and salt concentration was first studied by Nield [8] and Taunton et al [9] who considered the onset of salt fingers. The formulation of Nield [8] has been extended by Rubin [5] to introduce a nonlinear salinity profile, by Patil and Rudraiah [10] who included the effect of thermal diffusion (the Soret effect) and by Murray and Chen [11] to account for the effects of temperature dependent viscosity and volumetric expansion coefficients and a nonlinear basic-state salinity profile. There has been very little experimental work in double diffusive convection in porous media. Griffiths [6] obtained values of heat and salt flux through a thin “diffusive” interface between two layers of fluid with different temperatures and salt concentrations while Murray and Chen [11] incorporate a nonlinear time-dependent basic-state salinity profile in considering the onset of double diffusive convection in a finite box of porous medium.

In comparison there has been very little study of the effect of a third diffusing component on the onset of convection. Given the number of double diffusive problems there must be many examples where more than one salt concentration is present. There are many fluid system containing more than two components. For example, Degens at al [12] have described the waters of Lake Kivu in East Africa as having a salinity which is the sum of many salts and the oceans contain many salts with concentrations much less than the sodium chloride concentration. One particular application of triple diffusive convection can be found in experiments on double diffusive porous convection in which the effect of dyes or small temperature gradients should be considered. Further applications may be found in the area of contaminant transport (Celia et al [13], Chen et al [14]) where the chemical species that form the contaminants are non-reactive.

The linear stability of triple diffusive porous convection has been studied by Rudraiah and Vortmeyer [15] and Poulikakos [16]. They adapt the method of Griffiths [17] who considered the effect of a third component in the viscous fluid case. Since then the triple diffusive viscous flow problem has been studied by Pearlstein et al [1]. They found that the results of Griffiths [17] were not always true. In particular the conclusion of Griffiths [17] that “marginal stability of oscillatory modes occurs on a hyperboloid in Rayleigh number space but the surface is very closely approximated by its planar asymptotes for any diffusivity ratios” is shown to be incorrect. Pearlstein et al [1] show that for some fixed values of the diffusivity ratios, Prandtl number and two of the three Rayleigh numbers, three values of the third Rayleigh number may be required in order to specify the linear stability criteria. The effect of this is that the fluid is linearly unstable in two sections of the Rayleigh number domain and stable in the intermediate section. These novel results can be attributed to the existence of disconnected oscillatory neutral curves lying below the stationary neutral curve. In addition Pearlstein et al [1] find that the oscillatory neutral curve can be heart-shaped which those authors claim offers the possibility of simultaneous onset of instability at two different horizontal wavenumbers but the same Rayleigh number.

The results of Rudraiah and Vortmeyer [15] and Poulikakos [16] are similar to those of Griffiths [17]. Rudraiah and Vortmeyer [15] also claim that the stability boundary for oscillatory convection is a hyperboloid in Rayleigh number space that is closely approximated by its planar asymptotes. Motivated by the fact that Pearlstein et al [1] have shown the results of Griffiths [17] to be incomplete, we make a systematic investigation of the topology of the neutral curves and reconsider the results of Rudraiah and Vortmeyer [15] and Poulikakos [16].

The linear stability analysis presented here does not yield any information on the effect of nonlinear terms. In Sect. 5 we discuss the possible effect that nonlinear terms may have on the experimental realisation of the unusual results predicted by the heart-shaped oscillatory curves. Of particular relevance is the work of Proctor [18] and Hansen and Yuen [19]. Both consider subcritical instabilities in the double diffusive fluid problem and show that subcritical instability can occur at values of the thermal Rayleigh number much less than that predicted by the linear theory. Rudraiah, Srimani and Friedrich [20] consider the nonlinear stability of finite-amplitude convection of a two-component fluid-saturated porous layer using truncations of Fourier series. These authors also find that finite-amplitude instability is possible at subcritical values of the thermal Rayleigh number. Rudraiah, Shivakumara and Friedrich [21] use a similar method to investigate the effect of rotation on the double diffusive problem. Recently Guo and Kaloni [22] have applied the energy method to the double diffusive problem with rotation. There has been no work on the triple diffusive problem, however, and hence the need for the present analysis. One very important advantage of the application of the energy method given in the present work is that it provides unconditional results, i.e. nonlinear stability is guaranteed for initial perturbations of arbitrary sized amplitude.

The layout of this work is as follows. In Sect. 2 we present a linear stability analysis of the triple diffusive problem in a porous medium in the vein of Pearlstein et al [1]. The problem is formulated for heat and two salt concentrations as the three stratifying agencies. In Sect. 3 the energy method is applied to this problem for two distinct cases. Firstly, when all three stratifying agencies are destabilizing and secondly the case corresponding to heating from below with one salt field destabilizing and the other stabilizing. In Sect. 4 numerical results for both the linear and nonlinear analyses are presented and in the final section we discuss the difficulties of reproducing these results experimentally.

2 Linear stability analysis

2.1 Perturbation equations and dispersion relation

Consider a fluid-saturated porous layer lying in the infinite three-dimensional region $0 < z < d$. The boundaries $z = 0$ and $z = d$ are maintained at temperatures T_1 and T_2 respectively. Suppose further that the fluid has dissolved in it two different chemical components or "salts". Denote the concentration of component α by C^α $(\alpha = 1, 2)$. The concentration of component α at the lower and upper boundaries is held at C_l^α and C_u^α respectively.

The equation of state is given by

$$\rho = \rho_\circ \left(1 - \alpha(T - T_\circ) + A_1\left(C^1 - \hat{C^1}\right) + A_2\left(C^2 - \hat{C^2}\right)\right),$$

where $\rho_\circ, T_\circ$ and $\hat{C^\alpha}$ $(\alpha = 1, 2)$ are reference values of density, temperature and salt concentration respectively. The constants α and A_α $(\alpha = 1, 2)$ represent the thermal and solute expansion coefficients respectively.

The equations of motion which govern flow in a porous medium are largely based on a relation which is a generalisation of empirical observations (c.f. Joseph [23]). This relation is known as Darcy's Law and can be written

$$\nabla p = -\frac{\mu}{k}\mathbf{u} + \rho\mathbf{g},$$

where the variables $p, \mu, k, \mathbf{u}$ and $\mathbf{g}$ represent pressure, dynamic viscosity, porosity, velocity and gravity. In addition to Darcy's Law we have the incompressibility condition and the equations of conservation of temperature and solute. Combining these equations with the Darcy law and the equation of state we obtain the following system of governing equations:

$$\begin{aligned} p_{,i} &= -\frac{\mu}{k}v_i - g\rho_\circ(1 - \alpha(T - T_0) \\ &\quad + A_1(C^1 - \hat{C^1}) + A_2(C^2 - \hat{C^2}))k_i, && (2.1) \\ v_{i,i} &= 0, && (2.2) \\ T_{,t} + v_i T_{,i} &= \kappa \Delta T, && (2.3) \\ C^\alpha_{,t} + v_i C^\alpha_{,i} &= \kappa_\alpha \Delta C^\alpha, \quad (\alpha = 1, 2), && (2.4) \end{aligned}$$

where indicial notation and the Einstein summation convention have been employed. The vector $\mathbf{k}$ is the unit vector in the z-direction. The variables κ and κ_α $(\alpha = 1, 2)$ represent thermal and solute diffusivities respectively.

The boundary conditions we consider are

$$\begin{aligned} &T(0) = T_1, \;\; T(d) = T_2, \\ &C^\alpha(0) = C_l^\alpha, \;\; C^\alpha(d) = C_u^\alpha, \;\; (\alpha = 1, 2), \qquad (2.5) \\ &v_3 = 0 \;\text{ at }\; z = 0, d. \end{aligned}$$

The experimental realisation of prescribing these boundary conditions, especially those on the concentration fields, is discussed in Sect. 5.

Consider a steady solution $(\bar{v_i}, \bar{p}, \bar{T}, \bar{C^\alpha})$ of (2.1)–(2.5) where $\bar{v_i} = 0$ and $\bar{T}$ and $\bar{C^\alpha}$ are functions of z. Equations (2.3) and (2.4) show that, utilising (2.5),

$$\begin{aligned} \bar{T} &= T_1 - \beta z, \;\; (\beta = \frac{T_1 - T_2}{d}), \\ \bar{C^\alpha} &= C_l^\alpha - \frac{\triangle C^\alpha}{d}z, \;\; (\triangle C^\alpha = C_l^\alpha - C_u^\alpha). \end{aligned}$$

The steady state pressure $\bar{p}$ can be obtained from (2.1) which shows that

$$\frac{d\overline{p}}{dz} = -g\rho_\circ\left[1-\alpha(T_1-\beta z-T_\circ)+A_1\left(C_l^1-\frac{\triangle C^1}{d}z-\hat{C}^1\right)\right.$$
$$\left.+A_2\left(C_l^2-\frac{\triangle C^2}{d}z-\hat{C}^2\right)\right],$$

and so,

$$\begin{aligned}\overline{p} &= \rho_\circ g z^2\left[-\frac{\alpha\beta}{2}+A_1\frac{\triangle C^1}{2d}+A_2\frac{\triangle C^2}{2d}\right]\\ &\quad -\rho_\circ g z\left[1-\alpha(T_1-T_\circ)+A_1(C_l^1-\hat{C}^1)+A_2(C_l^2-\hat{C}^2)\right]+p_\circ,\end{aligned}$$

where $p_\circ$ is constant.

In order to investigate the linear stability of this basic solution we introduce perturbations $(u_i, \pi, \theta, \phi^\alpha)$ to $(\overline{v_i}, \overline{p}, \overline{T}, \overline{C^\alpha})$ via

$$v_i = \overline{v_i}+u_i,\quad p=\overline{p}+\pi,\quad T=\overline{T}+\theta,\quad C^\alpha=\overline{C^\alpha}+\phi^\alpha.$$

The resultant perturbation equations are non-dimensionalized using the following scalings:

$$t=t^*\frac{d^2}{\kappa},\ \mathbf{u}=\mathbf{u}^*\frac{\kappa}{d},\ \pi=\pi^*\frac{\mu\kappa}{k},\ \mathbf{x}=\mathbf{x}^*d,$$

$$\theta=\theta^*T^\#,\ \phi^\alpha=(\phi^\alpha)^*\Phi^\alpha,$$

$$T^\#=\left(\frac{\mu\kappa|\delta T|}{\alpha\rho_o gkd}\right)^{1/2},\ \Phi^\alpha=\left(\frac{\mu\kappa P_\alpha|\triangle C^\alpha|}{A_\alpha\rho_o gkd}\right)^{1/2},$$

$$R=\left(\frac{\alpha\rho_o gkd|\delta T|}{\mu\kappa}\right)^{1/2},\ R_\alpha=\left(\frac{A_\alpha\rho_o gkdP_\alpha|\triangle C^\alpha|}{\mu\kappa}\right)^{1/2},$$

$$\delta T=T_1-T_2, H=\mathrm{sgn}(\delta T),\ H_\alpha=\mathrm{sgn}(\triangle C^\alpha),\ P_\alpha=\frac{\kappa}{\kappa_\alpha}.$$

Here R and R_α are the thermal and salt Rayleigh numbers and P_α are salt Prandtl numbers.

The nonlinear perturbation equations are then, in non-dimensional form (dropping the asterisks),

$$\pi_{,i} = -u_i+\left[R\theta-R_1\phi^1-R_2\phi^2\right]k_i, \tag{2.6}$$
$$u_{i,i} = 0, \tag{2.7}$$
$$\theta_{,t}+u_i\theta_{,i} = HRw+\Delta\theta, \tag{2.8}$$
$$P_1(\phi^1_{,t}+u_i\phi^1_{,i}) = H_1R_1w+\Delta\phi^1, \tag{2.9}$$
$$P_2(\phi^2_{,t}+u_i\phi^2_{,i}) = H_2R_2w+\Delta\phi^2. \tag{2.10}$$

where $w=u_3$. The boundary conditions which follow from (2.5) for the perturbed quantities are

$$w=\theta=\phi^1=\phi^2=0 \ \text{ on } \ z=0,1. \tag{2.11}$$

We wish to perform a linearized stability analysis on (2.6)–(2.11) in the vein of Pearlstein et al [1]. Firstly equations (2.6)–(2.10) are linearized by neglecting terms containing products of the perturbed quantities. A time dependence of $e^{\sigma t}$ is introduced by substituting

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) &= \mathbf{u}(\mathbf{x})e^{\sigma t},\\ \theta(\mathbf{x},t) &= \theta(\mathbf{x})e^{\sigma t},\\ \phi^\alpha(\mathbf{x},t) &= \phi^\alpha(\mathbf{x})e^{\sigma t}\quad(\alpha=1,2).\end{aligned}$$

The pressure term is eliminated by taking curlcurl of equations (2.6) and then taking the third component. This gives

$$\begin{aligned} \Delta w &= R\Delta^*\theta - R_1\Delta^*\phi^1 - R_2\Delta^*\phi^2, & (2.12)\\ \sigma\theta &= HRw + \Delta\theta, & (2.13)\\ P_1\sigma\phi^1 &= H_1R_1w + \Delta\phi^1, & (2.14)\\ P_2\sigma\phi^2 &= H_2R_2w + \Delta\phi^2, & (2.15) \end{aligned}$$

where $\Delta^* = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the horizontal Laplacian.

In order to obtain an equation in w alone we operate on (2.12) with $(\sigma-\Delta)(P_1\sigma-\Delta)(P_2\sigma-\Delta)$ and use (2.13) – (2.15) to eliminate θ, ϕ^1 and ϕ^2. The resultant equation for w is found to be

$$\begin{aligned} (\sigma-\Delta)(P_1\sigma-\Delta)(P_2\sigma-\Delta)\Delta w &= HR^2(P_1\sigma-\Delta)(P_2\sigma-\Delta)\Delta^*w\\ &\quad -H_1R_1^2(\sigma-\Delta)(P_2\sigma-\Delta)\Delta^*w\\ &\quad -H_2R_2^2(\sigma-\Delta)(P_1\sigma-\Delta)\Delta^*w. \end{aligned} \tag{2.16}$$

We introduce normal modes, i.e.

$$w = W(z)e^{i(mx+ny)}.$$

In order to put our equation into a similar form to equation (2.3) of Pearlstein et al [1] the following transformations are introduced:

$$HR^2 \to R_1,\ H_1R_1^2 \to -R_2,\ H_2R_2^2 \to -R_3,\ P_1 \to P_2,\ P_2 \to P_3.$$

Equation (2.16) becomes

$$\begin{aligned} &(\sigma-(D^2-k^2))(P_2\sigma-(D^2-k^2))(P_3\sigma-(D^2-k^2))(D^2-k^2)W\\ &\quad = -k^2\left\{R_1(P_2\sigma-(D^2-k^2))(P_3\sigma-(D^2-k^2))\right.\\ &\qquad +R_2(\sigma-(D^2-k^2))(P_3\sigma-(D^2-k^2))\\ &\qquad \left.+R_3(\sigma-(D^2-k^2))(P_2\sigma-(D^2-k^2))\right\}W, \end{aligned}$$

where $k^2 = m^2+n^2$ and $D = \dfrac{d}{dz}$.

The boundary conditions on w imply that $W(z) = \sin n\pi z$. Putting $y_n = n^2\pi^2 + k^2$, we have

$$\begin{aligned} (\sigma+y_n)(P_2\sigma+y_n)(P_3\sigma+y_n)y_n = k^2\{&R_1(P_2\sigma+y_n)(P_3\sigma+y_n)\\ &+R_2(\sigma+y_n)(P_3\sigma+y_n)\\ &+R_3(\sigma+y_n)(P_2\sigma+y_n)\}. \end{aligned} \tag{2.17}$$

We wish to use this equation to obtain information about the stability of the basic solution. To do this we can consider y_n, P_2, P_3, R_2 and R_3 to be known while R_1 can be varied until a neutral solution (i.e. $Re(\sigma) = 0$) is obtained. Rewrite (2.17) as

$$R_1 = \left(\frac{\sigma+y_n}{k^2}\right)y_n - R_2\frac{\sigma+y_n}{P_2\sigma+y_n} - R_3\frac{\sigma+y_n}{P_3\sigma+y_n}. \tag{2.18}$$

We are looking for neutral solutions so set the real part of σ equal to zero, i.e. let $\sigma = 0 + i\omega$. Then (2.18) becomes, upon removing complex quantities from the denominators,

$$\begin{aligned} R_1 &= \frac{y_n^2}{k^2} - R_2\frac{P_2\omega^2+y_n^2}{P_2^2\omega^2+y_n^2} - R_3\frac{P_3\omega^2+y_n^2}{P_3^2\omega^2+y_n^2}\\ &\quad +i\omega y_n\left[\frac{1}{k^2} - R_2\frac{1-P_2}{P_2^2\omega^2+y_n^2} - R_3\frac{1-P_3}{P_3^2\omega^2+y_n^2}\right], \end{aligned} \tag{2.19}$$

which we rewrite as

$$R_1 = f_1(k,\omega,R_2,R_3,P_2,P_3) + i\omega y_n f_2(k,\omega,R_2,R_3,P_2,P_3).$$

The quantity R_1 is real so equation (2.19) implies that either $\omega = 0$ or $f_2 = 0$.

The case $\omega = 0$ corresponds to stationary onset of convection. Setting $\omega = 0$ in (2.19) yields

$$R_1 = R_1^{\mathrm{s}} = \frac{y_n^2}{k^2} - R_2 - R_3. \tag{2.20}$$

So, for stationary neutral solutions R_1 is a single-valued function of the wavenumber. The critical value of k which gives the minimum value of R_1^{s} can be found by setting the derivative of (2.20) with respect to k equal to zero, to find $k = n\pi$. So the critical Rayleigh number for steady onset is

$$R_1^{\mathrm{s,crit}} = 4\pi^2 - R_2 - R_3, \tag{2.21}$$

since $n = 1$ clearly gives the minimum value.

For oscillatory onset $\omega \neq 0$, so equation (2.19) requires that $f_2 = 0$, i.e.

$$\frac{1}{k^2} - R_2 \frac{1 - P_2}{P_2^2\omega^2 + y_n^2} - R_3 \frac{1 - P_3}{P_3^2\omega^2 + y_n^2} = 0.$$

Rewrite this as a quadratic dispersion relation in ω^2,

$$\omega^4 P_2^2 P_3^2 + \omega^2 \left[y_n^2(P_2^2 + P_3^2) + k^2(R_2(P_2 - 1)P_3^2 + R_3(P_3 - 1)P_2^2)\right] + y_n^2 \left[y_n^2 + k^2(R_2(P_2 - 1) + R_3(P_3 - 1))\right] = 0,$$

or as

$$\alpha(P_2, P_3)\omega^4 + \beta(k, R_2, R_3, P_2, P_3)\omega^2 + \gamma(k, R_2, R_3, P_2, P_3) = 0. \tag{2.22}$$

The fact that this is a quadratic in ω^2 means that it may give rise to solutions with more than one positive value of ω^2 for fixed P_2, P_3, R_2, R_3, k. This has important consequences for the linear stability of our basic solution and we now concentrate on finding such solutions.

Firstly, necessary conditions for the existence of multiple oscillatory neutral solutions are obtained.

If two real positive roots of (2.22) exist then $\beta < 0$ and $\gamma > 0$, i.e.,

$$y_n^2(P_2^2 + P_3^2) + k^2(R_2(P_2 - 1)P_3^2 + R_3(P_3 - 1)P_2^2) < 0, \tag{2.23}$$

and

$$y_n^2 + k^2(R_2(P_2 - 1) + R_3(P_3 - 1)) > 0. \tag{2.24}$$

Multiplying (2.24) by P_2^2 and adding to $-1\times$(2.23) gives

$$y_n^2(P_2^2 - P_2^2 - P_3^2) + k^2 \left[R_2(P_2 - 1)(P_2^2 - P_3^2) + R_3(P_3 - 1)(P_2^2 - P_2^2)\right] > 0,$$

i.e.

$$R_2(P_2 - 1)(P_2 - P_3) > \frac{y_n^2 P_3^2}{k^2(P_2 + P_3)} > 0.$$

Similarly, multiplying (2.24) by P_3^2 and adding to $-1\times$(2.23) yields

$$R_3(P_3 - 1)(P_3 - P_2) > \frac{y_n^2 P_2^2}{k^2(P_2 + P_3)} > 0.$$

So, necessary conditions for the existence of two frequencies on the oscillatory curve are

$$R_2(P_2 - 1)(P_2 - P_3) > 0, \tag{2.25}$$

$$R_3(P_3 - 1)(P_3 - P_2) > 0. \tag{2.26}$$

For fixed P_2, P_3 satisfying $(P_2 - 1)(P_3 - 1)(P_2 - P_3) \neq 0$, i.e. the three diffusivities being distinct from each other, (2.25) and (2.26) are satisfied in exactly one quadrant of the (R_2, R_3)-plane. If we consider the case of $R_2 < 0$ and $R_3 < 0$ (i.e. both salt fields stabilizing) then (2.25) and (2.26) imply that we cannot have two onset frequencies at one wavenumber, so in order to have two onset frequencies at one wavenumber one of these stratifying agencies must be destabilizing. This is in contrast with Pearlstein et al [1] who claim that two destabilizing effects cannot give rise to two onset frequencies at one wavenumber. However, examination of the sign convention used by Griffiths [17] and subsequently by Pearlstein et al [1] shows that the effects that Pearlstein et al [1] claim are destabilizing are actually stabilizing so our results are, in fact, in agreement with those of Pearlstein et al [1] .

We want to find the values of R_1, R_1°, on the oscillatory (R_1, k) neutral curve corresponding to two different onset frequencies at one wavenumber. To do this, rewrite the real and imaginary parts of (2.17) as

$$-\omega^2\left[y_n^2 f_1 - k^2(R_1^o f_2 + f_3)\right] + y_n^2\left[y_n^2 - k^2(R_1^\circ + f_4)\right] = 0, \tag{2.27}$$

$$-\omega^3 f_2 + \omega\left[y_n^2 f_5 - k^2(R_1^\circ f_6 + f_7)\right] = 0, \tag{2.28}$$

where

$$f_1 = P_2 + P_3 + P_2P_3, f_2 = P_2P_3, f_3 = R_2P_3 + R_3P_2, f_4 = R_2 + R_3,$$

$$f_5 = 1 + P_2 + P_3, f_6 = P_2 + P_3, f_7 = R_2(1 + P_3) + R_3(1 + P_2).$$

On the oscillatory (R_1, k) neutral curve $\omega = 0$ only at the bifurcation points with the stationary (R_1, k) neutral curve. Here a bifurcation point is one at which the oscillatory and stationary neutral curves intersect *and* the frequency ω tends to zero as the bifurcation point is approached. So, away from the bifurcation points we can divide (2.28) by ω to get

$$\omega^2 = \frac{y_n^2 f_5 - k^2(R_1^\circ f_6 + f_7)}{f_2}. \tag{2.29}$$

Substitute (2.29) into (2.27) to get

$$\frac{y_n^4}{k^4} f_8 + \frac{y_n^2}{k^2}(R_1^\circ f_9 + f_{10}) - (R_1^\circ f_6 + f_7)(R_1^\circ f_2 + f_3) = 0, \tag{2.30}$$

where

$$f_8 = -f_1 f_5 + f_2,\ f_9 = f_1 f_6 + f_2 f_5 - f_2,\ f_{10} = f_1 f_7 + f_3 f_5 - f_2 f_4.$$

Equation (2.30) is satisfied on the oscillatory neutral curve. We can use it to locate extremal points on the oscillatory curve in the (R_1, k)-plane. To find these points differentiate (2.30) with respect to k and set $\frac{\partial R_1^\circ}{\partial k} = 0$ to get

$$2y_n(2k^2 - y_n)(2y_n^2 f_8 + k^2(R_1^\circ f_9 + f_{10})) = 0.$$

So the extremal values of R_1° occur at either

$$2k^2 - y_n = 0, \tag{2.31}$$

or

$$2y_n^2 f_8 + k^2(R_1^\circ f_9 + f_{10}) = 0. \tag{2.32}$$

Equation (2.31) corresponds to the case $k = n\pi$. From (2.31), $\frac{y_n^2}{k^2} = 4n^2\pi^2$ which is substituted into (2.30) to obtain

$$(R_1^\circ)^2 f_2 f_6 + R_1^\circ(f_2 f_7 + f_3 f_6 - 4n^2\pi^2 f_9) + (f_3 f_7 - 16n^4\pi^4 f_8 - 4n^2\pi^2 f_{10}) = 0. \tag{2.33}$$

For fixed P_2, P_3, R_2, R_3 this is a quadratic in R_1° which has zero, one or two real solutions. For each solution the sign of ω^2 in (2.29) must be checked. So there may be zero, one or two physically meaningful extremal values of R_1° on the oscillatory neutral curve corresponding to $k = n\pi$.

In the other case substitution of (2.32) into (2.30) yields

$$(R_1^\circ)^2(f_9^2 + 4f_2f_6f_8) + R_1^\circ(2f_9f_{10} + 4f_2f_7f_8 + 4f_3f_6f_8) + (f_{10}^2 + 4f_3f_7f_8) = 0. \tag{2.34}$$

Again this is a quadratic in R_1° which may have zero, one or two physically meaningful ($\omega^2 \geq 0$) solutions at wavenumbers other than $k = n\pi$. In this case we can obtain some more information. If we define

$$f_{11} = -\frac{R_1^\circ f_9 + f_{10}}{2f_8},$$

then (2.32) may be rewritten as

$$(n^2\pi^2 + k^2)^2 - f_{11}k^2 = 0,$$

i.e.

$$k^4 + (2n^2\pi^2 - f_{11})k^2 + n^4\pi^4 = 0.$$

This has zero or two positive real roots. So, for each physically meaningful value of R_1° satisfying (2.32) there are two extrema on the oscillatory neutral curve with $k \neq n\pi$. So there may be two extrema at one R_1° (from (2.32)) and two extrema at $k = n\pi$ (from (2.31)) in which case the oscillatory neutral curve is heart-shaped.

2.2 Locating the oscillatory neutral curves

The existence of the closed oscillatory curves can be decided by locating any bifurcation points and points of infinite slope on the oscillatory neutral curves. The advantage of this approach is that it eliminates the need to search for the oscillatory curves in the (R_1, k)-plane.

Bifurcation points are the only points on the oscillatory curve at which $\omega = 0$. They can be located by setting $\omega = 0$ in (2.22). This yields $\gamma = 0$, i.e.,

$$y_n^2 + k^2(R_2(P_2 - 1) + R_3(P_3 - 1)) = 0.$$

Set $\delta = R_2(P_2 - 1) + R_3(P_3 - 1)$ and rewrite this as a quadratic in k^2,

$$k^4 + k^2(2n^2\pi^2 + \delta) + n^4\pi^4 = 0\ .$$

This has zero or two real positive solutions, corresponding to zero or two bifurcation points. The value of R_1° corresponding to k can be found by setting $\omega = 0$ in (2.27). This gives us

$$R_1^\circ = \frac{y_n^2}{k^2} - f_4.$$

The resultant pair (R_1°, k) must be substituted into (2.29) to ensure that $\omega^2 \geq 0$.

At points of infinite slope the number of branches on the oscillatory curve changes from zero to two or vice-versa. Consequently at these points the number of positive roots of (2.22) changes from zero to two, so points of infinite slope may be determined by solving

$$\beta^2 - 4\alpha\gamma = 0$$

from (2.22). Rearranging this yields a quartic in k^2

$$Ak^8 + k^6(4n^2\pi^2 A + B) + k^4(6n^4\pi^4 A + 2n^2\pi^2 B + C) + k^2(4n^6\pi^6 A + n^4\pi^4 B) + An^8\pi^8 = 0,$$

where

$$A = \left(P_2^2 - P_3^2\right)^2 \geq 0,$$

$$B = -2(P_2 + P_3)\left[P_3^2 R_2(P_2 - 1)(P_2 - P_3) + P_2^2 R_3(P_3 - 1)(P_3 - P_2)\right],$$

$$C = \left[R_2(P_2 - 1)P_3^2 + R_3(P_3 - 1)P_2^2\right]^2 \geq 0.$$

This has four possible sign changes and so, by Descartes' rule of signs, four possible positive real roots. However, it can be shown that there are at most two physically meaningful positive roots and consequently at most two points of infinite slope.

The value of R_1 on the oscillatory curve at the point of infinite slope can be found by differentiating (2.30) and setting $\dfrac{\partial k}{\partial R_1^\circ} = 0$. This yields

$$R_1^\circ = -\frac{(f_2 f_7 + f_3 f_6)k^2 - y_n^2 f_9}{2k^2 f_2 f_6}.$$

Again for each pair (R_1°, k) the sign of ω^2 in (2.29) must be checked.

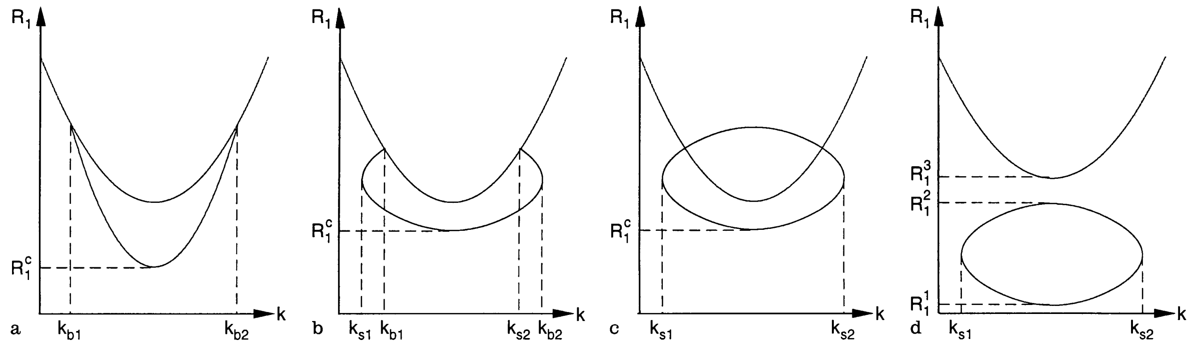

Fig. 1. Topology of the neutral curves. **a** Two bifurcation points, no points of infinite slope. **b** Two bifurcation points, two points of infinite slope. **c** No bifurcation points, two points of infinite slope. **d** No bifurcation points, two points of infinite slope

2.3 Topology of the neutral curves

The possible combinations of bifurcation points and points of infinite slope allow us to determine the shape of the neutral curves. If there are no bifurcation points and no points of infinite slope then there is no oscillatory curve, since the oscillatory curve must be either connected to the the stationary curve (and so there are two bifurcation points), or disconnected from the stationary curve and closed (and so there are two points of infinite slope).

If there are two bifurcation points and no points of infinite slope then the neutral curves look like Fig. 1a. The oscillatory curve is single-valued between k_{b1} and k_{b2} and does not exist for any other values of k.

If there are two points of infinite slope and two bifurcation points then the neutral curves look like Fig. 1b. The oscillatory curve is double-valued between k_{s1} and k_{b1} and between k_{b2} and k_{s2} and single-valued between k_{b1} and k_{b2}. This has no bearing on stability as the single critical Rayleigh number still occurs at the minimum of the oscillatory curve.

If there are two points of infinite slope and no bifurcation points then the neutral curves will look like Fig. 1c or Fig. 1d. In Fig. 1c the oscillatory neutral curve does not lie entirely below the stationary curve and so still only one critical Rayleigh number is required to describe linear instability. The points where the stationary and oscillatory curves meet are not true bifurcation points. As the point of intersection is approached along the oscillatory curve the frequency, ω, does not tend to zero. In Fig. 1d the oscillatory curve lies wholly below the stationary curve and now three values of R_1 are required to specify the linear instability criteria. The fluid is linearly unstable for $R_1^1 < R_1 < R_1^2$ and for $R_1 > R_1^3$ and stable for $R_1^2 < R_1 < R_1^3$.

3 Nonlinear stability analysis

We now present an analysis of the nonlinear stability of the basic solution by use of the energy method. The nonlinear perturbation equations are, from (2.6)–(2.10),

$$\pi_{,i} = -u_i + [R\theta - S\phi - T\psi]\, k_i\,, \tag{3.1}$$

$$u_{i,i} = 0, \tag{3.2}$$

$$\theta_{,t} + u_i\theta_{,i} = HRw + \Delta\theta, \tag{3.3}$$

$$P_1\left(\phi_{,t} + u_i\phi_{,i}\right) = H_1 S w + \Delta\phi, \tag{3.4}$$

$$P_2\left(\psi_{,t} + u_i\psi_{,i}\right) = H_2 T w + \Delta\psi. \tag{3.5}$$

where, for later convenience, the following transformations have been used

$$\phi^1 \to \phi, \quad \phi^2 \to \psi, \quad R_1 \to S\,, \quad R_2 \to T\,.$$

Let V denote a period cell for the solution. The boundary conditions we consider are

$$w = \theta = \phi = \psi = 0 \quad \text{on}\, z = 0, 1,$$

and further that u_i, θ, ϕ, ψ and π are periodic on the lateral boundaries of V.

To commence multiply (3.1) by u_i, (3.3) by θ, (3.4) by ϕ and (3.5) by ψ and integrate over V. Integration by parts and use of the boundary conditions yields

$$0 = -\|\mathbf{u}\|^2 + R(\theta, w) - S(\phi, w) - T(\psi, w), \tag{3.6}$$

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 = HR(w, \theta) - \|\nabla\theta\|^2, \tag{3.7}$$

$$\frac{d}{dt}\frac{P_1}{2}\|\phi\|^2 = H_1 S(w, \phi) - \|\nabla\phi\|^2, \tag{3.8}$$

$$\frac{d}{dt}\frac{P_2}{2}\|\psi\|^2 = H_2 T(w, \psi) - \|\nabla\psi\|^2, \tag{3.9}$$

where $\|\cdot\|$ denotes the $L^2(V)$ norm and $(f, g) = \int_V f g \, dV$.

If we now form (3.6) + λ (3.7) + ξ (3.8) + μ (3.9), where λ, ξ and μ are positive coupling parameters to be selected at our discretion, then

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\lambda}{2}\|\theta\|^2 + \frac{\xi P_1}{2}\|\phi\|^2 + \frac{\mu P_2}{2}\|\psi\|^2\right) \\ = \; &(\lambda H + 1)R(w, \theta) + (\xi H_1 - 1)S(w, \phi) + (\mu H_2 - 1)T(w, \psi) \\ &- \left(\|\mathbf{u}\|^2 + \lambda\|\nabla\theta\|^2 + \xi\|\nabla\phi\|^2 + \mu\|\nabla\psi\|^2\right). \end{aligned} \tag{3.10}$$

If we now define an energy

$$E(t) = \frac{\lambda}{2}\|\theta\|^2 + \frac{\xi P_1}{2}\|\phi\|^2 + \frac{\mu P_2}{2}\|\psi\|^2,$$

then (3.10) shows that

$$\frac{dE}{dt} = \mathscr{I} - \mathscr{D},$$

where

$$\mathscr{I} = (\lambda H + 1)R(w, \theta) + (\xi H_1 - 1)S(w, \phi) + (\mu H_2 - 1)T(w, \psi),$$

$$\mathscr{D} = \|\mathbf{u}\|^2 + \lambda\|\nabla\theta\|^2 + \xi\|\nabla\phi\|^2 + \mu\|\nabla\psi\|^2.$$

By rearrangement,

$$\frac{dE}{dt} = \mathscr{I} - \mathscr{D} = -\mathscr{D}\left(1 - \frac{\mathscr{I}}{\mathscr{D}}\right).$$

If we now define

$$\frac{1}{\Lambda} = \max_{\mathscr{H}} \frac{\mathscr{I}}{\mathscr{D}}, \tag{3.11}$$

where $\mathscr{H}$ is the space of admissible functions, then

$$\frac{dE}{dt} \le -\mathscr{D}\left(1 - \frac{1}{\Lambda}\right).$$

If now

$$\Lambda > 1, \tag{3.12}$$

then

$$1 - \frac{1}{\Lambda} = a > 0,$$

and so

$$\frac{dE}{dt} \le -a\mathscr{D}.$$

The Poincaré inequality is (see e.g. Straughan [24])

$$\|\nabla\theta\|^2 \ge \lambda_* \|\theta\|^2,$$

where $\lambda_* > 0$, with similar results for ϕ and ψ. Application of this gives that $\exists \lambda_1 > 0$ such that

$$\mathscr{D} \geq \lambda_1 E.$$

Therefore,

$$\frac{dE}{dt} \leq -a\lambda_1 E, \tag{3.13}$$

which can be integrated to yield

$$E(t) \leq E(0)e^{-a\lambda_1 t}.$$

So, if $\Lambda > 1$ then $E(t) \to 0$ as $t \to \infty$ at least exponentially fast and so our steady solution is stable.

The problem remains to find the maximum in (3.11).

In order to clear the denominator of the maximisation problem of coupling parameters we make the following transformations

$$\sqrt{\lambda}\theta \to \theta,\ \sqrt{\xi}\phi \to \phi,\ \sqrt{\mu}\psi \to \psi.$$

The resultant maximisation problem to be considered is

$$\frac{1}{\Lambda} = \max_{\mathscr{H}} \frac{\left(\frac{\lambda H + 1}{\sqrt{\lambda}}\right) R(w,\theta) + \left(\frac{\xi H_1 - 1}{\sqrt{\xi}}\right) S(w,\phi) + \left(\frac{\mu H_2 - 1}{\sqrt{\mu}}\right) T(w,\psi)}{\|\mathbf{u}\|^2 + |\nabla\theta\|^2 + \|\nabla\phi\|^2 + \|\nabla\psi\|^2}.$$

The Euler-Lagrange equations for this maximum are derived as follows

$$\begin{aligned} \delta\left(\frac{\mathscr{I}}{\mathscr{D}}\right) &= \frac{\delta\mathscr{I}}{\mathscr{D}} - \mathscr{I}\frac{1}{\mathscr{D}^2}\delta\mathscr{D} \\ &= \frac{1}{\mathscr{D}}\left(\delta\mathscr{I} - \frac{\mathscr{I}}{\mathscr{D}}\bigg|_{\max}\delta\mathscr{D}\right) \\ &= \frac{1}{\mathscr{D}}\left(\delta\mathscr{I} - \frac{1}{\Lambda}\delta\mathscr{D}\right) = 0, \end{aligned}$$

where

$$\delta\mathscr{F} = \frac{\partial\mathscr{F}}{\partial w_i} - \frac{\partial}{\partial x_j}\left(\frac{\partial\mathscr{F}}{\partial w_{i,j}}\right) = 0\ ,$$

with w_i standing for u_i, θ, ϕ or ψ. Therefore

$$\delta\mathscr{I} - \frac{1}{\Lambda}\delta\mathscr{D} = 0. \tag{3.14}$$

Since $\mathscr{H}$ is restricted to those functions that are divergence free, the solenoidal condition $u_{i,i} = 0$ must be added into the maximisation problem by means of a Lagrange multiplier. This is done by adding a term

$$\int_V \pi(\mathbf{x})u_{i,i}\, dV = 0$$

in the maximisation. With the above condition included, the Euler-Lagrange equations are

$$\Lambda\left[\left(\frac{\lambda H + 1}{2\sqrt{\lambda}}\right) R\theta - \left(\frac{1 - H_1\xi}{2\sqrt{\xi}}\right) S\phi - \left(\frac{1 - H_2\mu}{2\sqrt{\mu}}\right) T\psi\right] k_i - u_i = \pi_{,i} \tag{3.15}$$

$$\Lambda\left(\frac{\lambda H + 1}{2\sqrt{\lambda}}\right) Rw + \Delta\theta = 0, \tag{3.16}$$

$$-\Lambda\left(\frac{1 - H_1\xi}{2\sqrt{\xi}}\right) Sw + \Delta\phi = 0, \tag{3.17}$$

$$-\Lambda\left(\frac{1 - H_2\mu}{2\sqrt{\mu}}\right) Tw + \Delta\psi = 0. \tag{3.18}$$

At the stability limit $\Lambda \to 1$. Setting $\Lambda = 1$ in the Euler-Lagrange equations (3.15)–(3.18) will then yield the optimum results. The equations to be solved are now

$$\left[\left(\frac{\lambda H + 1}{2\sqrt{\lambda}}\right) R\theta - \left(\frac{1 - H_1\xi}{2\sqrt{\xi}}\right) S\phi - \left(\frac{1 - H_2\mu}{2\sqrt{\mu}}\right) T\psi\right] k_i - u_i = \pi_{,i} \tag{3.19}$$

$$\left(\frac{\lambda H + 1}{2\sqrt{\lambda}}\right) Rw + \Delta\theta = 0, \tag{3.20}$$

$$-\left(\frac{1 - H_1\xi}{2\sqrt{\xi}}\right) Sw + \Delta\phi = 0, \tag{3.21}$$

$$-\left(\frac{1 - H_2\mu}{2\sqrt{\mu}}\right) Tw + \Delta\psi = 0. \tag{3.22}$$

We now consider S and T to be fixed and investigate the variation of R, where now

$$R = R(\lambda,\ \xi,\ \mu,\ a^2),$$

where a is a wavenumber.

We will consider two special cases. Firstly,

$$H = 1, H_1 = H_2 = -1.$$

This corresponds to heating from below and salting from above with both salt fields. In this case all three effects are destabilizing. Equations (3.19)–(3.22) become

$$\left[\left(\frac{\lambda + 1}{2\sqrt{\lambda}}\right) R\theta - \left(\frac{\xi + 1}{2\sqrt{\xi}}\right) S\phi - \left(\frac{\mu + 1}{2\sqrt{\mu}}\right) T\psi\right] k_i - u_i = \pi_{,i} \tag{3.23}$$

$$\left(\frac{\lambda + 1}{2\sqrt{\lambda}}\right) Rw + \Delta\theta = 0, \tag{3.24}$$

$$-\left(\frac{\xi + 1}{2\sqrt{\xi}}\right) Sw + \Delta\phi = 0, \tag{3.25}$$

$$-\left(\frac{\mu + 1}{2\sqrt{\mu}}\right) Tw + \Delta\psi = 0. \tag{3.26}$$

We will now vary each of λ, ξ and μ in turn and find the optimum values of these coupling parameters. Firstly, we consider ξ, μ, S and T to be fixed and investigate the optimum value of λ by using parametric differentiation. Let now superscripts 1 and 2 refer to a solution of (3.23)–(3.26) corresponding to parameters λ_1 and λ_2 respectively. We now form the inner products $((3.23)^1, \mathbf{u}^2), ((3.23)^2, \mathbf{u}^1), ((3.24)^1, \theta^2), ((3.24)^2, \theta^1), ((3.25)^1, \phi^2), ((3.25)^2, \phi^1), ((3.26)^1, \psi^2)$ and $((3.26)^2, \psi^1)$. Putting

$$f = \frac{\lambda + 1}{2\sqrt{\lambda}}, g = \frac{\xi + 1}{2\sqrt{\xi}}, h = \frac{\mu + 1}{2\sqrt{\mu}}, \tag{3.27}$$

we derive the equations

$$R^1 f^1(\theta^1, w^2) - Sg(\phi^1, w^2) - Th(\psi^1, w^2) = (\mathbf{u}^1, \mathbf{u}^2), \tag{3.28}$$

$$R^2 f^2(\theta^2, w^1) - Sg(\phi^2, w^1) - Th(\psi^2, w^1) = (\mathbf{u}^2, \mathbf{u}^1), \tag{3.29}$$

$$f^1 R^1(w^1, \theta^2) = D(\theta^1, \theta^2), \tag{3.30}$$

$$f^2 R^2(w^2, \theta^1) = D(\theta^2, \theta^1), \tag{3.31}$$

$$-Sg(w^1, \phi^2) = D(\phi^1, \phi^2), \tag{3.32}$$

$$-Sg(w^2, \phi^1) = D(\phi^2, \phi^1), \tag{3.33}$$

$$-Th(w^1, \psi^2) = D(\psi^1, \psi^2), \tag{3.34}$$

$$- Th(w^2, \psi^1) = D(\psi^2, \psi^1), \tag{3.35}$$

where $D(\alpha, \beta) = (\nabla\alpha, \nabla\beta)$.

To proceed, form (3.28) + (3.30) − (3.29) − (3.31) + (3.32) − (3.33) + (3.34) − (3.35) to find

$$\left(R^2 f^2 - R^1 f^1\right)\left[(\theta^2, w^1) + (\theta^1, w^2)\right] = 0.$$

Divide this by $\lambda_2 - \lambda_1$ and rearrange to find

$$\left[R^2 \frac{f^2 - f^1}{\lambda^2 - \lambda^1} + f^1 \frac{R^2 - R^1}{\lambda^2 - \lambda^1}\right]\left[(\theta^2, w^1) + (\theta^1, w^2)\right] = 0.$$

If we now let $\lambda_2 \to \lambda_1$, then

$$\left(R\frac{\partial f}{\partial\lambda} + f\frac{\partial R}{\partial\lambda}\right)(\theta, w) = 0. \tag{3.36}$$

However, from (3.20)

$$Rf(\theta, w) = D(\theta, \theta) = \|\nabla\theta\|^2. \tag{3.37}$$

At the optimum value of λ, $\dfrac{\partial R}{\partial\lambda} = 0$ and so, from (3.36) and (3.37),

$$R\frac{\partial f}{\partial\lambda}\frac{\|\nabla\theta\|^2}{Rf} = 0,$$

and since $f > 0$, we have $\dfrac{\partial f}{\partial\lambda} = 0$. From (3.27),

$$\frac{\partial f}{\partial\lambda} = \frac{\lambda - 1}{4\lambda^{\frac{3}{2}}}.$$

So,

$$\frac{\partial R}{\partial\lambda} = 0 \Rightarrow \lambda = 1. \tag{3.38}$$

If we now fix λ and μ and vary ξ then a similar argument to the above will show that

$$\frac{\partial R}{\partial\xi} = 0 \Rightarrow \xi = 1,$$

and similarly, for fixed λ and ξ the optimum value of μ is 1.

With $\lambda = \xi = \mu = 1$ the equations (3.23)–(3.26) are identical to the linearized versions of (3.1)–(3.5) with a time dependence $e^{\sigma t}$ assumed and σ set equal to zero. So, if we can show that σ is real, the linear instability and nonlinear stability boundaries will coincide.

The linearized perturbation equations are, from (3.1)–(3.4) with a time dependence $e^{\sigma t}$ assumed,

$$\pi_{,i} = -u_i + [R\theta - S\phi - T\psi]k_i, \tag{3.39}$$

$$\sigma\theta = Rw + \Delta\theta, \tag{3.40}$$

$$P_1\sigma\phi = -Sw + \Delta\phi, \tag{3.41}$$

$$P_2\sigma\psi = -Tw + \Delta\psi. \tag{3.42}$$

If we now multiply (3.39) by u_i^* (the complex conjugate of u_i), (3.40) by θ^*, (3.41) by ϕ^*, (3.42) by ψ^*, integrate each over V and make use of the boundary conditions we find

$$0 = -\|\mathbf{u}\|^2 + R(\theta, w^*) - S(\phi, w^*) - T(\psi, w^*), \tag{3.43}$$

$$\sigma\|\theta\|^2 = R(w, \theta^*) - (\theta_{,j}, \theta^*_{,j}), \tag{3.44}$$

$$P_1\sigma\|\phi\|^2 = -S(w, \phi^*) - (\phi_{,j}, \phi^*_{,j}), \tag{3.45}$$

$$P_2\sigma\|\psi\|^2 = -T(w, \psi^*) - (\psi_{,j}, \psi^*_{,j}), \tag{3.46}$$

where now $\|a\| = (a, a^*)$. Adding (3.43) + (3.44) + (3.45) + (3.46) yields

$$\begin{aligned}
&\sigma\left(\|\theta\|^2 + P_1\|\phi\|^2 + P_2\|\psi\|^2\right) \\
&= R\left[(\theta, w^*) + (\theta^*, w)\right] - S\left[(\phi, w^*) + (\phi^*, w)\right] \\
&-T\left[(\psi, w^*) + (\psi^*, w)\right] \\
&-\left(\|\mathbf{u}\|^2 + \|\nabla\theta\|^2 + \|\nabla\phi\|^2 + \|\nabla\psi\|^2\right),
\end{aligned} \tag{3.47}$$

The right hand side of (3.47) is real and so if we let $\sigma = \sigma_r + i\sigma_i$, then taking the imaginary part of (3.47) yields

$$\sigma_i\left(\|\theta\|^2 + P_1\|\phi\|^2 + P_2\|\psi\|^2\right) = 0.$$

Hence,

$$\sigma_i = 0.$$

Therefore the growth rate is real and so the linear instability and nonlinear stability boundaries coincide in this case. This is an important result and demonstrates that when $H = 1, H_1 = H_2 = -1$ there can be no subcritical instabilities.

The next case we consider is

$$H = H_1 = 1, H_2 = -1.$$

This corresponds to the situation studied in Sect. 2 where the layer is heated from below, salted from below in component 1 but salted from above in component 2. This means that heat and component 2 are destabilizing but component 1 is in competition and acts as a stabilizing agent. Since the differential equations (2.6)–(2.10) do not form a symmetric system we do not expect agreement between the linear and nonlinear stability results. In this case the Euler-Lagrange equations (3.19)–(3.22) become

$$\left[\left(\frac{\lambda+1}{2\sqrt{\lambda}}\right)R\theta - \left(\frac{1-\xi}{2\sqrt{\xi}}\right)S\phi - \left(\frac{\mu+1}{2\sqrt{\mu}}\right)T\psi\right]k_i - u_i = \pi_{,i} \tag{3.48}$$

$$\left(\frac{\lambda+1}{2\sqrt{\lambda}}\right)Rw + \Delta\theta = 0, \tag{3.49}$$

$$-\left(\frac{1-\xi}{2\sqrt{\xi}}\right)Sw + \Delta\phi = 0, \tag{3.50}$$

$$-\left(\frac{\mu+1}{2\sqrt{\mu}}\right)Tw + \Delta\psi = 0. \tag{3.51}$$

Now set

$$f = \frac{\lambda+1}{2\sqrt{\lambda}}, g = \frac{1-\xi}{2\sqrt{\xi}}, h = \frac{\mu+1}{2\sqrt{\mu}}. \tag{3.52}$$

If we now use parametric differentiation to find the optimum values of λ, ξ and μ, then a similar argument to that leading to (3.38) will show that

$$\frac{\partial R}{\partial\lambda} = 0 \Rightarrow \lambda = 1, \quad \frac{\partial R}{\partial\mu} = 0 \Rightarrow \mu = 1.$$

However, when we apply this argument for λ and μ fixed and consider the variation in ξ we find, with $\lambda = \mu = 1$,

$$\frac{\partial R}{\partial\xi}(\theta, w) - S\frac{\partial g}{\partial\xi}(\phi, w) = 0.$$

From (3.49), (3.50) and (3.52)

$$\frac{\|\nabla\theta\|^2}{R}\frac{\partial R}{\partial\xi} = -S\frac{\partial g}{\partial\xi}\frac{\|\nabla\phi\|^2}{gS} = \frac{\xi+1}{2\xi(1-\xi)}\|\nabla\phi\|^2,$$

where $\|\cdot\|$ once again denotes the $L^2(V)$ norm. So the system is singular at $\xi = 1$. While we cannot find a solution for $\frac{\partial R}{\partial \xi} = 0$ note that

$$\xi < 1 \Rightarrow \frac{\partial R}{\partial \xi} > 0, \quad \xi > 1 \Rightarrow \frac{\partial R}{\partial \xi} < 0.$$

This suggests that the best value of ξ is one. With this value of ξ, equations (3.48)–(3.51) become (the ϕ equation dropping out),

$$R\theta k_i - T\psi k_i - u_i = \pi_{,i} \tag{3.53}$$

$$Rw + \Delta\theta = 0, \tag{3.54}$$

$$-Tw + \Delta\psi = 0. \tag{3.55}$$

We now solve these equations for T fixed. We eliminate the $\pi_{,i}$ term by taking curlcurl of equations (3.53) and taking the third component to leave

$$\begin{aligned} R\Delta^*\theta - T\Delta^*\psi - \Delta w &= 0, \\ \Delta\theta &= -Rw, \\ \Delta\psi &= Tw, \end{aligned}$$

where Δ^* is once again the horizontal Laplacian. If we now eliminate θ and ψ we have a single equation in w

$$R^2\Delta^* w + T^2\Delta^* w + \Delta^2 w = 0.$$

Now assume $w = \sin n\pi z\, e^{i(m_1 x + m_2 y)}$ to find

$$R^2 + T^2 = \frac{\left(n^2\pi^2 + a^2\right)^2}{a^2},$$

where $a^2 = m_1^2 + m_2^2$ is a wavenumber. If we now minimise the right-hand side over n and a we find the least value is $4\pi^2$. So, our nonlinear energy boundary is

$$\left(R^2 + T^2\right)_{\min} = 4\pi^2. \tag{3.56}$$

4 Results

4.1 Linear instability

All results are for the case $n = 1$. Although there is no proof that $n = 1$ yields the minimum critical Rayleigh number, both the present work and the results of Pearlstein et al [1] suggest this to be so.

The case where $P_2 = 4.545454$ and $P_3 = 4.761904$ yields similar results to those of Pearlstein et al [1]. These values for P_2 and P_3 correspond to the values for κ, κ_1 and κ_2 chosen by Pearlstein et al [1]. Figure 2 shows the stability boundary R_1^{crit} as a function of R_2 for fixed $R_3 = 261.0$. There are three regions of interest. To the left of the cusp ($R_2 < -285.28$) there is a region of oscillatory onset. Here oscillatory instability first occurs at a smaller value of R_1 than does stationary instability and there is a single critical value of R_1. To the right of the point of infinite slope ($R_2 > -284.92$) instability occurs with real growth rate. Here oscillatory instability does not occur and again there is one critical value of R_1. The intervening region is the most interesting and is shown in the right-hand graph of Fig. 2. Here three values of R_1^{crit} are required to fully specify the linear instability criteria. Oscillatory instability sets in first at the lowest critical Rayleigh number. Then there is a region of oscillatory instability until the middle critical Rayleigh number is reached. At this point the system becomes linearly stable again until the third critical Rayleigh number is reached. Here stationary instability sets in and the system remains linearly unstable for all higher values of R_1. These stability boundaries are identical to those of Pearlstein et al [1] in that each of R_1^{crit} and R_2 can be a multi-valued function of the other for fixed R_3, P_2 and P_3.

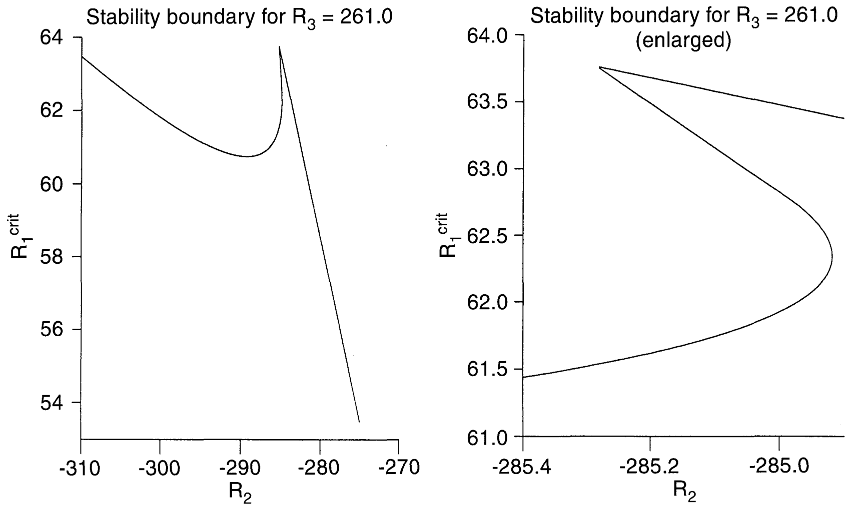

Fig. 2. (R_1^{crit}, R_2) stability boundary for R_3 = 261.0, P_2 = 4.545454, P_3 = 4.761904. The right-hand graph shows the multi-valued region in more detail

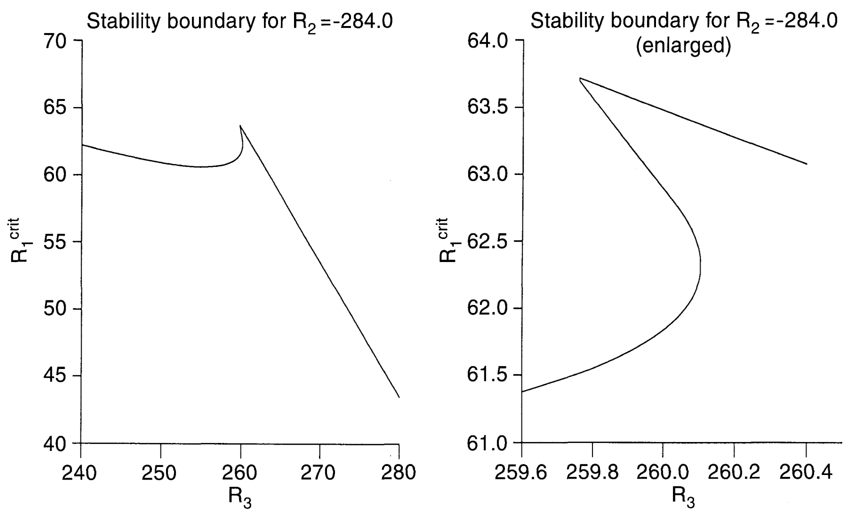

Fig. 3. (R_1^{crit}, R_3) stability boundary for $R_2 = -284.0$, P_2 = 4.545454, P_3 = 4.761904. The right-hand graph shows the multi-valued region in more detail

Figure 3 shows the (R_1^{crit}, R_3) stability boundary for the same values of P_2 and P_3 with $R_2 = -284.0$. Clearly R_1^{crit} and R_3 can be multi-valued functions of each other. Again three values of R_1^{crit} may be required to fully describe the linear stability criteria.

Figures 4 and 5 show the (R_1, k) neutral curves for R_3 = 261.0, P_2 = 4.545454, P_3 = 4.761904 and various R_2. For $R_2 = -320.0$ the oscillatory curve is connected to the stationary curve at two bifurcation points and the single critical Rayleigh number occurs at the minimum on the oscillatory curve. As R_3 is increased to $-310.0, -305.0$ and -300.0 the bifurcation points move closer together. At $R_2 = -288.5$ the curve has lost its single-valued nature and there are two points of infinite slope. However, still only one critical Rayleigh number occurs. At a value of R_2 lying between -288.5 and -287.0 the bifurcation points move together and coallesce and a closed oscillatory neutral curve is formed. At $R_2 = -287.0$ the oscillatory curve has become detached from the stationary curve. The graphs of $R_2 = -286.0$ show the heart-shaped curve more clearly. As R_2 is increased the oscillatory curve moves entirely below the stationary curve and now three critical values of R_1 are required. For the larger values of R_2 shown the oscillatory curve becomes increasingly smaller until at a value of R_2 between -285.1 and -284.9 the oscillatory curve collapses to a point and disappears. At

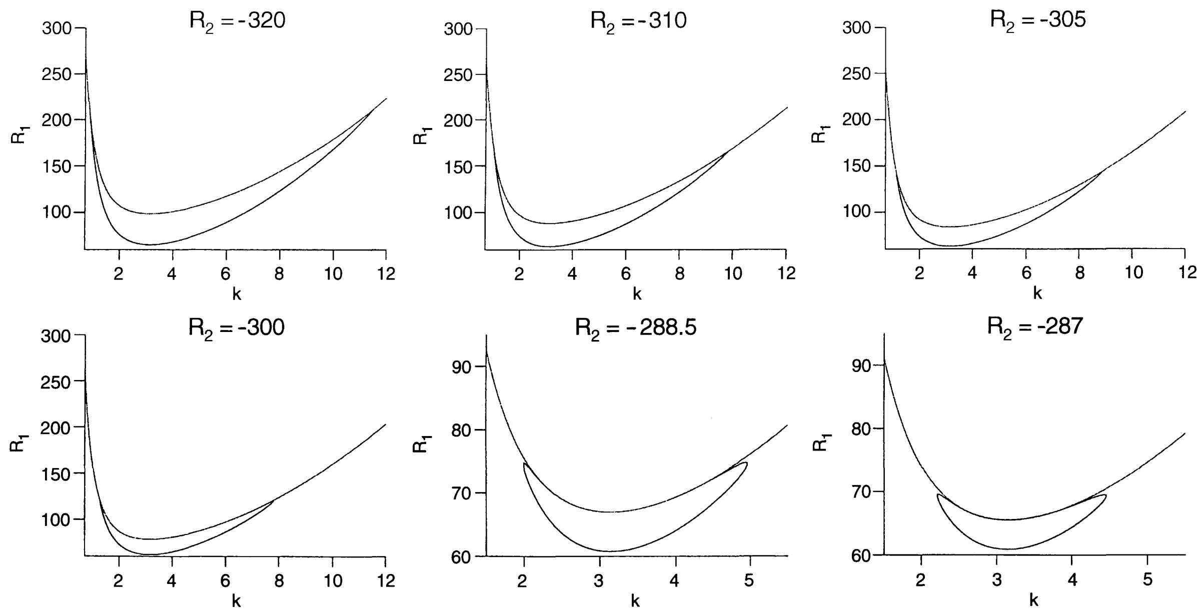

Fig. 4. Neutral curves for R_3 = 261.0, P_2 = 4.545454, P_3 = 4.761904

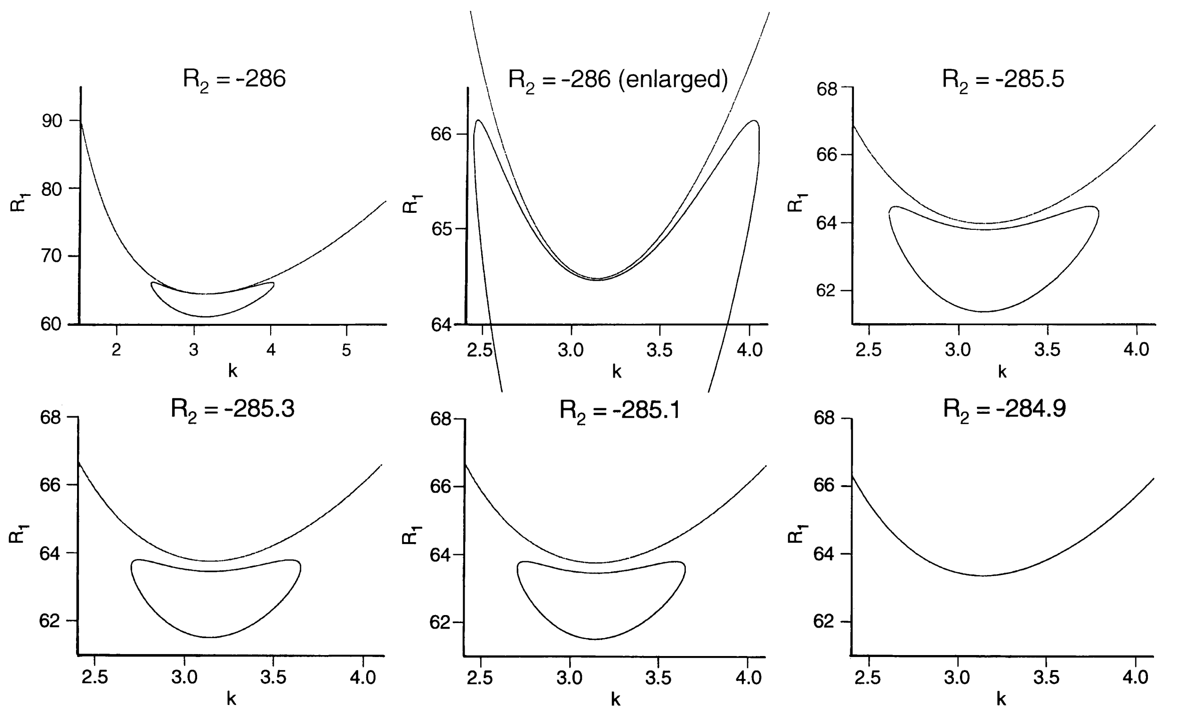

Fig. 5. Further neutral curves for R_3 = 261.0, P_2 = 4.545454, P_3 = 4.761904

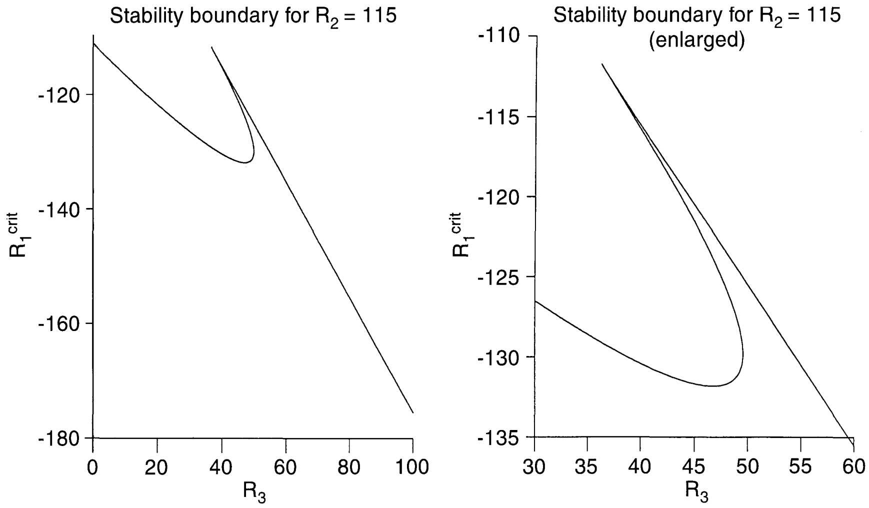

Fig. 6. (R_1^{crit}, R_3) stability boundary for R_2 = 115.0, P_2 = 0.5, P_3 = 1.5. The right-hand graph shows the multi-valued region in more detail

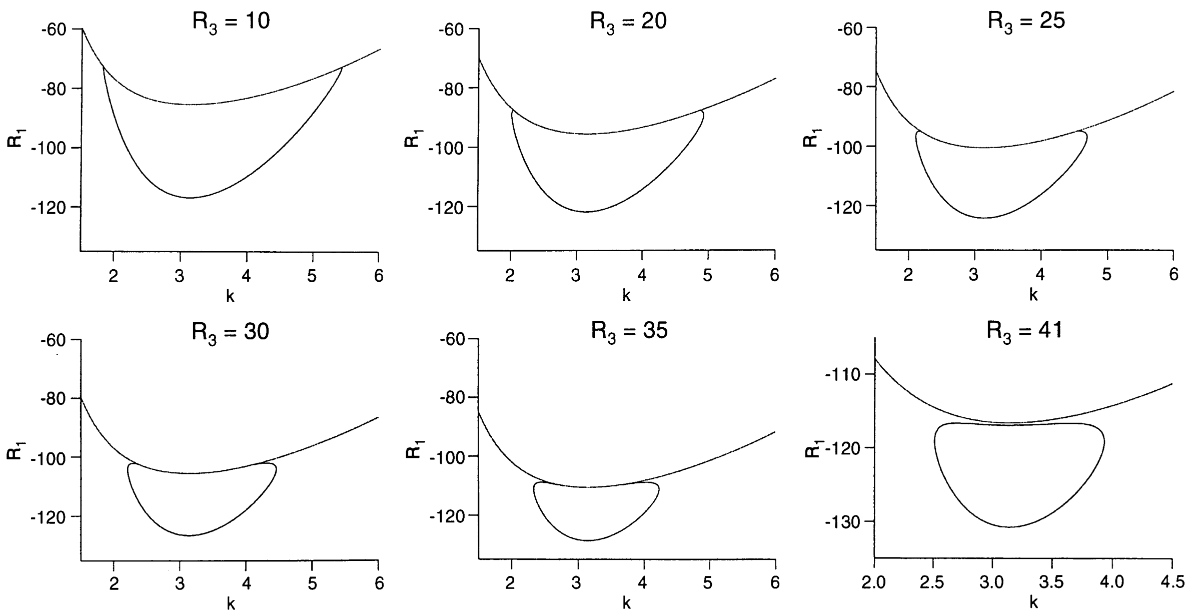

Fig. 7. Neutral curves for R_2 = 115.0, P_2 = 0.5, P_3 = 1.5

$R_2 = -284.9$ the oscillatory curve is no longer found and the single critical Rayleigh number occurs at the minimum on the stationary curve.

The case where P_2 = 0.5, P_3 = 1.5 and R_2 = 115.0 is of interest as the results correspond to heating the fluid from above while the two salts are gravitationally unstable. Similar results to the case where R_3 = 261.0, P_2 = 4.545454, P_3 = 4.761904 are found. The stability boundary is shown in Fig. 6. As before there is a multivalued region where three critical Rayleigh numbers occur. The neutral curves in Figs. 7 and 8 show similar behaviour to the previous numerical example. At R_3 = 10.0 the oscillatory curve is single-valued and connected to the stationary curve at two bifurcation points. The single critical Rayleigh number occurs at the minimum on the oscillatory curve. As R_3 is increased to 20.0 the bifurcation points move closer together and the oscillatory curve loses its single-valued nature. As R_3 is increased further the bifurcation points move closer together until at a value of R_3 between 35.0 and 41.0 they coallesce and a closed oscillatory curve is formed. This closed curve then moves below the stationary curve and three critical Rayleigh numbers are required. The closed curve is heart-shaped over a small range of values (approximately R_3 = 41.0 – 43.0).

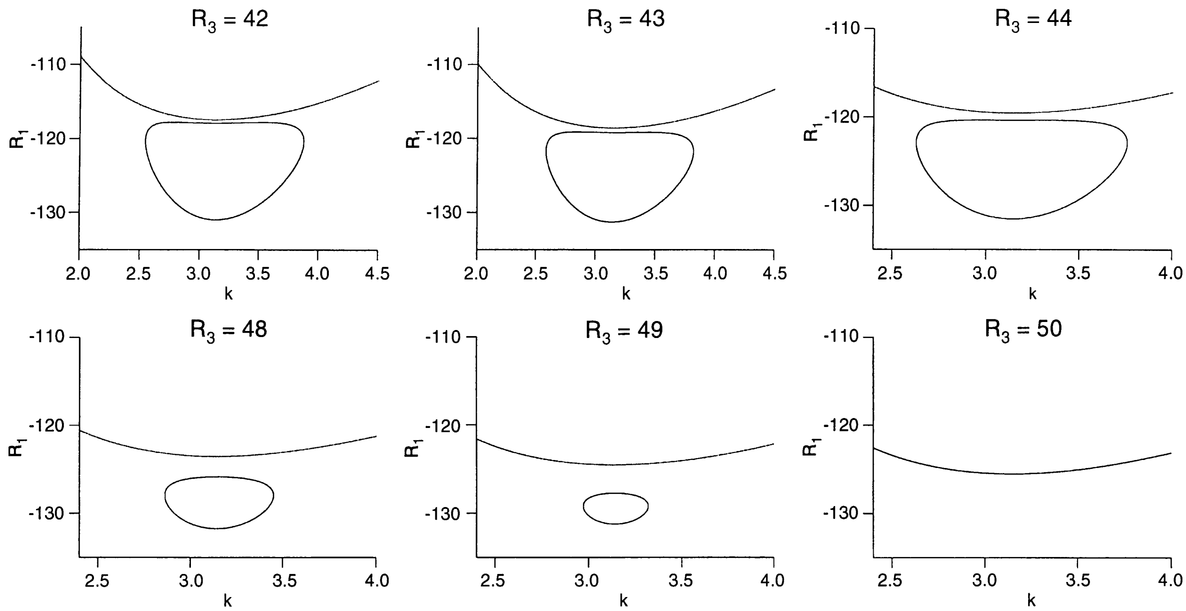

Fig. 8. Further neutral curves for R_2 = 115.0, P_2 = 0.5, P_3 = 1.5

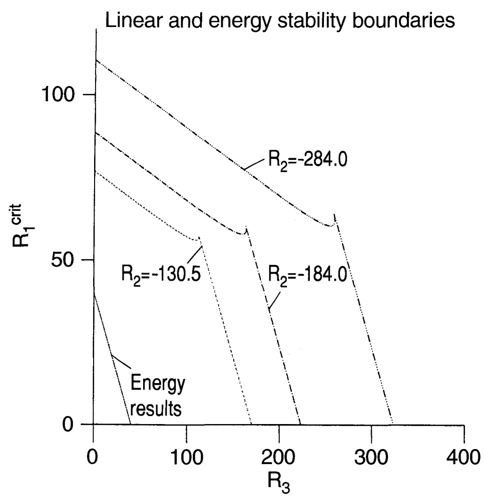

Fig. 9. Linear instability and nonlinear stability boundaries. The linear instability boundaries are shown for three values of R_2

At R_3 = 44.0 the curve has lost its heart shape and is now a convex curve. For increasing values of R_3 this convex curve decreases in size until it eventually collapses to a point and disappears. At R_3 = 50.0 only the stationary neutral curve is found and the single critical Rayleigh number occurs at the minimum of this curve.

4.2 Nonlinear stability

Figure 9 shows the nonlinear stability boundary for the case $H = H_1 = 1, H_2 = -1$ from equation (3.56) plotted with the linear instability boundary for fixed values of $R_2 = -284.0, P_2 = 4.545454, P_3 = 4.761904$. In addition we show the linear instability boundaries for $R_2 = -184.0$ and $R_2 = -130.5$. For larger values of R_2 than -130.5 we do not find closed oscillatory neutral curves. As these closed curves are the main focus of this work we do not consider any larger values of R_2. As can be seen from the figure, the larger the value of R_2 the closer the nonlinear energy boundary is to the linear instability boundary and the smaller the region of possible subcritical instabilities. This is not surprising as the equation corresponding to R_2 (the ϕ equation) drops out of the analysis in Sect. 3 and so does not provide any information. However, these results do have the advantage that they are unconditional, i.e. nonlinear stability is guaranteed for initial perturbations of arbitrary sized amplitude.

5 Conclusions and discussion

In the case where $P_2 = 4.545454, P_3 = 4.761904$ we find that the important results of Pearlstein et al [1] have carried over to the porous case. In particular the existence of stability boundaries in the (R_1, R_3) (or (R_1, R_2))-plane that are multivalued functions of both R_1 and R_3 (or both R_1 and R_2) shows that the conclusion of Rudraiah and Vortmeyer [15] that "marginal stability of oscillatory modes occurs on a hyperboloid in Rayleigh number space but the surface is very closely approximated by its planar asymptotes for any diffusivity ratios" is incorrect. In addition the existence of heart-shaped oscillatory neutral curves resulting in the onset of oscillatory instability at a given value of R_1 for two different horizontal wavenumbers is a feature not seen before in multi-component porous convection.

In the case where $P_2 = 1.5, P_3 = 0.5$ we find positive values of R_2 and R_3 that give rise to heart-shaped oscillatory curves. These values correspond to the case of having both salt fields destabilizing. In the work of Pearlstein et al [1] they claim (erroneously) that when the stratifying agencies corresponding to R_2 and R_3 are destabilizing then it is impossible to have two onset frequencies at the same wavenumber. As explained in Sect. 2 the necessary conditions (2.25) and (2.26) that we derive show that having both salt fields stabilizing rules out the possibility of having a multi-valued oscillatory curve.

In equations (2.5) we regarded temperature and the normal component of velocity as being prescribed at the boundaries. These boundary conditions are discussed by Joseph (1976). In a porous medium the fluid will stick to a solid wall but this effect is confined to a boundary layer whose size is measured in pore diameters. As the wall friction does not overtly affect the motion in the interior it is reasonable to replace the true wall with a frictionless wall in our analysis.

Pearlstein et al [1] discuss the experimental problems of prescribing constant concentration at the boundaries. They suggest the use of semi-permeable membranes as boundaries through which solute can pass into the working fluid volume. If the fluid outside the membrane is maintained at a constant concentration then the solute boundary condition could be realised to within a good approximation.

There is some doubt as to whether the onset of instability at two wavenumbers and the same Rayleigh number would be seen experimentally. In a situation where we have a heart-shaped oscillatory neutral curve the initial onset of instability occurs at the minimum on the oscillatory curve. The value of R_1 corresponding to onset at two different frequencies is, however, not a minimum and this instability lies in the range where nonlinear effects are likely to be important. Work by Proctor [18] and Hansen and Yuen [19] on finite amplitude double diffusive convection and by Rudraiah, Srimani and Friedrich [20] on the equivalent problem in a porous medium have shown that subcritical convection could occur at values of the thermal Rayleigh number much less than that predicted by the linear stability theory and hence the need for the nonlinear analysis presented here.

In addition there is the physical relevance of the equation of state. McKay and Straughan [25] argue that the density of a fluid is never a linear function of temperature. Straughan and Walker [26] consider the results of Pearlstein et al [1] in the case where the density is a quadratic function of temperature. They find that the closed oscillatory neutral curves are no longer perfectly heart-shaped but instead are slightly skewed. The phenomenon of onset of instability at two different wavenumbers and the same Rayleigh number is no longer seen. In the porous media case work in progress by this author with a quadratic temperature law shows similar results to Straughan and Walker [26]. The oscillatory neutral curves are no longer perfectly heart shaped and the feature described above is no longer observed. In order to improve the energy results given here it may be possible to adapt the generalized energy method of Mulone [27]. Again, work to this end is in progress.

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