Continuum Mech. Thermodyn. (2023) 35:141–158
https://doi.org/10.1007/s00161-022-01174-9

ORIGINAL ARTICLE

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A note on dependence of the inertia tensor on the strain measures

Received: 12 September 2022 / Accepted: 14 November 2022 / Published online: 24 November 2022
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Abstract The paper focuses on the dependence of the micro-inertia tensor on continuum deformations. Modeling of the micropolar medium is based on the spatial description that opens the possibility to model situations and materials with a continuum point that on the microscale consists no longer of the same elementary units during a physical process. The tensor of inertia of the polar particle is obtained by averaging the inertia tensors of microparticles within a representative volume. Because of the medium deformation, the representative volume contains different micro-particles as the medium moves, and the inertia tensor of the volume will change due to the incoming or outgoing flux of inertia. A possibility of dependence of the tensor of inertia on the strain measures is demonstrated, and suitable forms of the dependence are suggested.

Keywords Micropolar continuum, Micro-inertia tensor, Strain measures, Spatial description

1 Introduction

In this paper, we discuss dependence of the inertia tensor of a continuum particle, $\boldsymbol{J}$, the so-called micro-inertia tensor, on deformations of the continuum. The micro-inertia tensor plays an important role in context with rotational degree of freedom, specifically in combination with the angular velocity vector, $\boldsymbol{\omega}$, assigned to the continuum point. The introduction of the additional degree of freedom, as is done in the theory of the micropolar continuum mechanics, gives a possibility to describe the physical behavior of a matter with an inner microstructure, such as polycrystalline and composite matter, porous media and foams, soil, granular- and powder-like materials [7,11,12,38]. The micropolar theory also has gained great importance in fluid mechanics reflecting a desire to treat such rheologically complex fluids as liquid crystals, melts of polymers, blood and biological solutions, etc., using the field theory approach to describe the macroscopic manifestations of microelement motions and deformations [1,8,10,26,35]. Since fluid particles usually move in a complex manner, keeping track of the motion of individual particles (their displacements and rotations) is not feasible. As a result, it is more preferable to use a spatial description in micropolar hydrodynamics, since it does not impose strict constraints on the motion of material particles. In fact, the spatial description is suitable for modeling not only continuous, but also granular materials. Although very promising from the practical point of view [2,16–23,43], micropolar theories based on the spatial description have not gained wide acceptance.

Communicated by Andreas Öchsner.

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Within the spatial description, it is customary to refer thermodynamic state quantities to an (open) control volume, fixed in space, containing an ensemble of particles. As the medium moves, different particles pass through the volume, with each of these particles having its own mass, tensor of inertia, angular and linear velocities. To account for these local structural aspects, some classical concepts require a reexamination. In particular, the question arises as to how the translation and angular velocities, displacement vector, rotation tensor, and tensor of inertia of the control volume can be introduced. An answer to that question was suggested in [25], where "extended" micropolar theory (EMT) was presented. It is based on the concept of spatial description and on homogenization of properties of discrete elementary particles. There are some precedent works based on the material description, which use a discrete approach, primarily [33,37] but also more recently [5,6,27,32,34,36], and the references cited therein. In [25], it was shown that while many principles of the classical micropolar theory still remain valid, the tensor of inertia of the control volume within the spatial description may change due to the inertia flux or internal structural transformations such as the consolidation or defragmentation of particles or changes of anisotropic properties. Thus, the tensor of inertia has to be treated as a variable rather than a parameter. Therefore, the system of equations has to be augmented with an additional balance law and constitutive relation for the tensor of inertia. The form of the constitutive equation depends on the problem under consideration and can be a function of many physical quantities. These new ideas have been illustrated by several examples in previous papers [13–15,29,30,41], where dependencies of the inertia tensor on the temperature field, internal and external stresses and an electric field were suggested.

In the present work, we treat the tensor of inertia of the control volume as a new variable characterizing the average size, shape and orientation of the particles located within the volume. We restrict our attention to the rigid elementary particles without internal structural transformation and demonstrate possible dependencies of the tensor of inertia on strain measures.

2 The micropolar continuum model within the spatial description

2.1 Outline of the peculiarities of the spatial description

The spatial description is a method of observing a motion that focuses on a specific location in space $\boldsymbol{x}(t)$, the so-called observation point, through which a matter moves as time passes. Since in that case the position vector $\boldsymbol{x}(t)$ is unrelated to the evolution of the matter, the velocity of the material particle located at the observation point at time t, $\boldsymbol{v}(\boldsymbol{x}, t)$, is an independent characteristic. There are two different time derivatives of a matter property (density of mass, temperature, etc.) in the spatial description [2,3,24]. The total derivative characterizes the rate of change of the considered property at the observation point

$$\frac{\mathrm{d}(\cdot)}{\mathrm{d}t} = \left.\frac{\partial(\cdot)}{\partial t}\right|_{\boldsymbol{x}=\mathrm{const}} + \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} \cdot \nabla(\cdot), \qquad \nabla = \frac{\partial}{\partial \boldsymbol{x}}. \tag{2.1}$$

The material derivative determines a rate of change of the property of the material particle located at the observation point at time t:

$$\frac{\delta(\cdot)}{\delta t} = \frac{\mathrm{d}(\cdot)}{\mathrm{d}t} + \left(\boldsymbol{v}(\boldsymbol{x}, t) - \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right) \cdot \nabla(\cdot). \tag{2.2}$$

In other words, the material derivative describes the property evolution w.r.t. the matter and the total derivative describes a temporal change of the property throughout the space. It should be noted that within the material description the observation point is the material point; thus, in that case the material derivative coincides with the total derivative.

In contrast to the material description that is based on the concept of a material particle, which is a closed indestructible system, within the spatial description, a continuum is understood to consist of a manifold of control volumes. The control volume is fixed in space or moving with a velocity that is independent of the motion of the medium passing through the volume. As a result, the spatial description opens a possibility to model materials with a continuum point that on the macroscale consists no longer of the same elementary units during a physical process.

Let the medium consists of elementary (material) particles, each having its own mass, tensor of inertia, angular and linear velocities. During its evolution, the control volume is occupied by different material particles. If the scale difference between the material particles (microscale) and the whole body (macroscale) is sufficiently large, the control volume can be treated as the Representative Volume Element (RVE). Since at different moments the RVE consists of different micro-particles each having its own inertial and kinematic characteristics, one has to introduce corresponding fields at the macrolevel as effective characteristics. Detailed information on the procedure can be found in [25,39,40], here we present just the main outlines.

Let us consider an open RVE, V. The geometric center of the volume is characterized by the vector $\boldsymbol{x}$ (for simplicity, we suppose that it is fixed). The RVE is filled with $i = 1, ..., N(\boldsymbol{x}, t)$ elementary particles of mass m_i and inertia tensor $\boldsymbol{J}_i$ (w.r.t. their center of mass). These particles possess linear velocities $\boldsymbol{v}_i$ and angular velocities $\boldsymbol{\omega}_i$. On the continuum level the matter traveling through the RVE can be characterized by average properties. Then, we assume that the inertial field quantities of the RVE (mass density and microinertia tensor field) coincide with the average properties of the matter traveling through the volume:

$$\rho(\boldsymbol{x}, t) = \frac{1}{V} \sum_{i=1}^{N(\boldsymbol{x},t)} m_i, \qquad \boldsymbol{J}(\boldsymbol{x}, t) = \frac{1}{N(\boldsymbol{x}, t)} \sum_{i=1}^{N(\boldsymbol{x},t)} \boldsymbol{J}_i. \tag{2.3}$$

Note that this microinertia tensor differs from the one introduced in material description. It is not obtained by application of Steiner's theorem w.r.t. the center mass of the control volume and characterizes the size, shape and orientation of the average particle rather than the mass distribution over the volume.

The linear and angular momenta of the RVE consisting of the original particles are required to be equal to those of the RVE consisting of average particles. The linear and angular velocities are obtained from this condition:

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{1}{\sum_{i=1}^{N(\boldsymbol{x},t)} m_i} \sum_{i=1}^{N(\boldsymbol{x},t)} m_i \boldsymbol{v}_i, \qquad \boldsymbol{\omega}(\boldsymbol{x}, t) = \left(\sum_{i=1}^{N(\boldsymbol{x},t)} \boldsymbol{J}_i \right)^{-1} \cdot \sum_{i=1}^{N(\boldsymbol{x},t)} \boldsymbol{J}_i \cdot \boldsymbol{\omega}_i \tag{2.4}$$

In contrast to the material description, the displacement vector, $\boldsymbol{u}$, and the rotation tensor, $\boldsymbol{Q}$, in the spatial description are not fundamental quantities. Instead, these are to be found as solutions from the following differential equations:

$$\frac{\delta \boldsymbol{u}(\boldsymbol{x}, t)}{\delta t} = \frac{d\boldsymbol{u}}{dt} + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{x}, t), \tag{2.5}$$

$$\frac{\delta \boldsymbol{Q}(\boldsymbol{x}, t)}{\delta t} = \frac{d\boldsymbol{Q}}{dt} + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{Q}(\boldsymbol{x}, t) = \boldsymbol{\omega}(\boldsymbol{x}, t) \times \boldsymbol{Q}(\boldsymbol{x}, t), \tag{2.6}$$

provided $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are known. Here $\boldsymbol{a} \times \boldsymbol{A} = (\boldsymbol{a} \times \boldsymbol{b}) \otimes \boldsymbol{c}$.

Note that the thus obtained displacement vector is a formally introduced characteristic; in general, it does not correspond to the actual displacement of an elementary particle. It is also evident that the rotation tensor $\boldsymbol{Q}$ expressed in terms of the average angular velocity is different from the rotation tensor of the RVE as a whole. It is a formally introduced effective characteristic like the displacement vector.

In summary, within that approach, a continuum is understood as a manifold of RVEs. The RVE, in turn, is constructed as a manifold of micro-particles and links the micro- and mesoscales. Obviously, the macroscopic scale L of the matter is much larger than the characteristic size of the RVE l, that in turn is much larger than the characteristic size of elementary particles. Now let us introduce the displacement of elementary particle i on the different scale levels. We set on the RVE scale (see Fig. 1a)

$$\boldsymbol{x}_i^* = \boldsymbol{x} + \boldsymbol{\xi}_i, \tag{2.7}$$

where $\boldsymbol{\xi}_i$ is the position vector of the point within the RVE where the particle is located w.r.t. the geometric center of the RVE. It is assumed that the position vectors $\boldsymbol{x}_i^*$ can be understood as field quantities in the RVE. Then, we define $\boldsymbol{u}_i^*$ as a solution of the differential equation

$$\frac{\delta_i \boldsymbol{u}_i^*(\boldsymbol{x}_i^*, t)}{\delta t} = \boldsymbol{v}_i^*(\boldsymbol{x}_i^*, t), \qquad \frac{\delta_i\,(\cdot)}{\delta t} = \frac{d\,(\cdot)}{dt} + \boldsymbol{v}_i^*(\boldsymbol{x}_i^*, t) \cdot \nabla^*(\cdot), \qquad \nabla^* = \frac{\partial}{\partial \boldsymbol{x}_i^*}, \tag{2.8}$$

with the initial condition $\boldsymbol{u}_i^*(\boldsymbol{x}_i, t_0) = 0$.

Generally, the spatial description considers the current configuration. Nevertheless, one could introduce the concept of a reference position vector of the elementary particle in the same way as in the case of the material description

$$\boldsymbol{X}_i^*(\boldsymbol{x}_i^*, t) = \boldsymbol{x}_i^* - \boldsymbol{u}_i^*(\boldsymbol{x}_i^*, t). \tag{2.9}$$

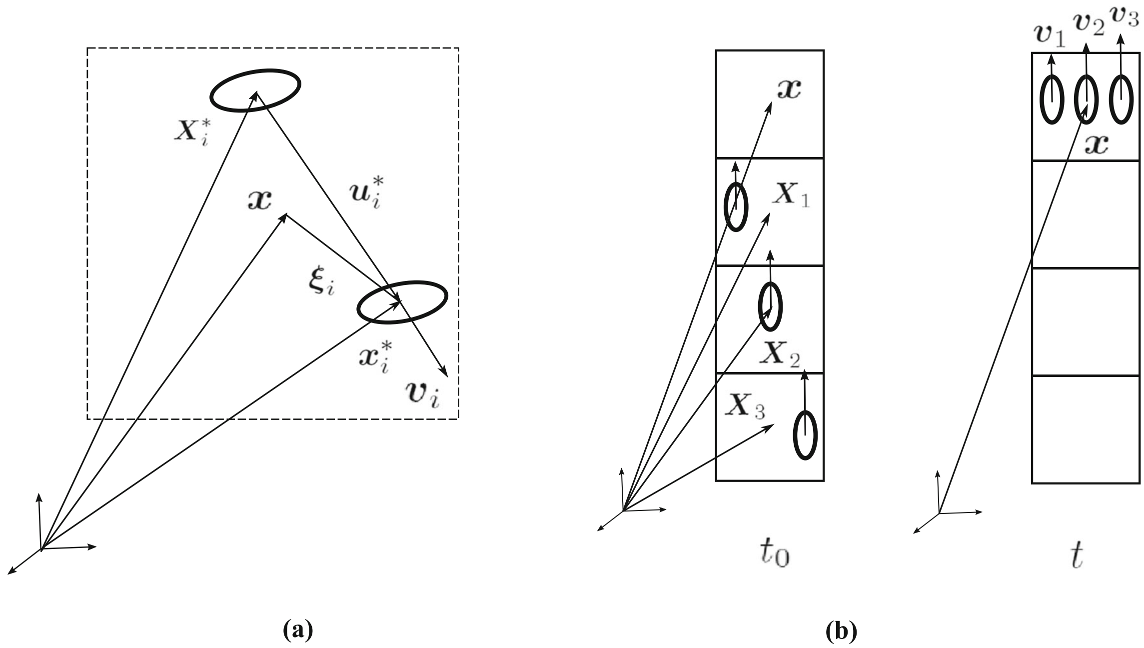

Fig. 1 Displacements and reference positions of elementary particles. **a** On the RVE scale; **b** on the macro scale

It is necessary to point out that the reference position vector is time-dependent since the position $\boldsymbol{x}_i^*$ is occupied by different elementary particle at different points in time.

Since $\delta_i \boldsymbol{x}_i^*/\delta t = \boldsymbol{v}_i^*$, from (2.8) and (2.9) follows that

$$\frac{\delta_i \boldsymbol{X}_i^*(\boldsymbol{x}_i^*, t)}{\delta t} = 0, \qquad \Longrightarrow \qquad \frac{d\boldsymbol{X}_i^*(\boldsymbol{x}_i^*, t)}{dt} = -\boldsymbol{v}_i^*(\boldsymbol{x}_i^*, t) \cdot \nabla^* \boldsymbol{X}_i^*(\boldsymbol{x}_i^*, t). \tag{2.10}$$

This differential equation determines the relation between the reference position of the elementary particle and its velocity. Thus, $\boldsymbol{u}_i^*(\boldsymbol{x}_i^*, t)$ and $\boldsymbol{X}_i^*(\boldsymbol{x}_i^*, t)$ are geometrical characteristics of the matter on the RVE level.

To introduce the displacement and the reference position vectors of the elementary particle on the macro scale, one has to express them in terms of the macroscopic quantities $\boldsymbol{x}$ and $\boldsymbol{v}$. We can write

$$\boldsymbol{v}_i(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{x}, t) + \bar{\boldsymbol{v}}_i. \tag{2.11}$$

Note that vectors $\boldsymbol{v}_i$ and $\boldsymbol{v}_i^*$ are the same vector (this is the velocity of the i-th particle at time t), written as functions of different arguments. Next we define $\boldsymbol{u}_i$ and $\boldsymbol{X}_i$ on the macro scale as solutions from the corresponding differential equations

$$\frac{\delta_i \boldsymbol{u}_i(\boldsymbol{x}, t)}{\delta t} = \frac{d\boldsymbol{u}_i(\boldsymbol{x}, t)}{dt} + \boldsymbol{v}_i(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{u}_i(\boldsymbol{x}, t) = \boldsymbol{v}_i(\boldsymbol{x}, t), \quad \boldsymbol{u}_i(\boldsymbol{x}, t_0) = 0, \tag{2.12}$$

$$\frac{\delta_i \boldsymbol{X}_i(\boldsymbol{x}, t)}{\delta t} = \frac{d\boldsymbol{X}_i(\boldsymbol{x}, t)}{dt} + \boldsymbol{v}_i(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{X}_i(\boldsymbol{x}, t) = 0, \quad \boldsymbol{X}_i(\boldsymbol{x}, t_0) = \boldsymbol{x}, \tag{2.13}$$

provided $\boldsymbol{v}_i$ is known. In addition to dependence on different arguments, the functions $\boldsymbol{u}_i(\boldsymbol{x}, t)$ and $\boldsymbol{u}_i^*(\boldsymbol{x}_i^*, t)$ (as well as $\boldsymbol{X}_i(\boldsymbol{x}, t)$ and $\boldsymbol{X}_i^*(\boldsymbol{x}_i^*, t)$) are determined by different initial conditions. Note that on the macrolevel all particles included in the RVE are denoted by the same position vector $\boldsymbol{x}$ and indistinguishable. Therefore, the initial conditions of Eqs. (2.12) and (2.13) are approximate ones with errors of no more than l/L. It follows that the magnitude of the norms of $\boldsymbol{u}_i(\boldsymbol{x}, t) - \boldsymbol{u}_i^*(\boldsymbol{x}_i^*, t)$ and $\boldsymbol{X}_i(\boldsymbol{x}, t) - \boldsymbol{X}_i^*(\boldsymbol{x}_i^*, t)$ are of RVE size

$$\begin{aligned} |\boldsymbol{u}_i(\boldsymbol{x}, t) - \boldsymbol{u}_i^*(\boldsymbol{x}_i^*, t)| &\le \frac{1}{2}\sqrt{(\triangle x)^2 + (\triangle y)^2 + (\triangle z)^2}, \\ |\boldsymbol{X}_i(\boldsymbol{x}, t) - \boldsymbol{X}_i^*(\boldsymbol{x}_i^*, t)| &\le \frac{1}{2}\sqrt{(\triangle x)^2 + (\triangle y)^2 + (\triangle z)^2}. \end{aligned} \tag{2.14}$$

It should be noted that even if the velocities of the elementary particles within the RVE are almost the same, their reference position vectors can differ significantly as time goes (Fig. 1b).

Finally, one could introduce the concept of a reference position vector on the macro level as follows

$$\boldsymbol{X}(\boldsymbol{x},t)=\boldsymbol{x}-\boldsymbol{u}(\boldsymbol{x},t), \quad \Longrightarrow \quad \frac{\delta \boldsymbol{X}(\boldsymbol{x},t)}{\delta t}=\frac{d\boldsymbol{X}(\boldsymbol{x},t)}{dt}+\boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{X}(\boldsymbol{x},t)=0. \tag{2.15}$$

Analogously, the rotation tensors of the elementary particle $\boldsymbol{Q}_i$ on different scale levels can be introduced as solutions from the corresponding differential equations. On the RVE scale:

$$\frac{\delta_i\,\boldsymbol{Q}_i^*(\boldsymbol{x}_i^*,t)}{\delta t}=\frac{d\,\boldsymbol{Q}_i^*(\boldsymbol{x}_i^*,t)}{dt}+\boldsymbol{v}_i(\boldsymbol{x}_i^*,t)\cdot\nabla^*\boldsymbol{Q}_i^*(\boldsymbol{x}_i^*,t)=\boldsymbol{\omega}_i(\boldsymbol{x}_i^*,t)\times\boldsymbol{Q}_i^*(\boldsymbol{x}_i^*,t), \tag{2.16}$$

with the initial condition $\boldsymbol{Q}_i^*(\boldsymbol{x}_i^*,t_0)=\boldsymbol{I}$.

On the macro level:

$$\frac{\delta_i\,\boldsymbol{Q}_i(\boldsymbol{x},t)}{\delta t}=\frac{d\,\boldsymbol{Q}_i(\boldsymbol{x},t)}{dt}+\boldsymbol{v}_i(\boldsymbol{x},t)\cdot\nabla\boldsymbol{Q}_i(\boldsymbol{x},t)=\boldsymbol{\omega}_i(\boldsymbol{x},t)\times\boldsymbol{Q}_i(\boldsymbol{x},t), \quad \boldsymbol{\omega}_i(\boldsymbol{x},t)=\boldsymbol{\omega}(\boldsymbol{x},t)+\bar{\boldsymbol{\omega}}_i. \tag{2.17}$$

with the initial condition $\boldsymbol{Q}_i(\boldsymbol{x}_i,t_0)=\boldsymbol{I}$.

Similar to the displacement vectors, besides being functions of different variables tensors $\boldsymbol{Q}_i(\boldsymbol{x},t)$ and $\boldsymbol{Q}_i^*(\boldsymbol{x}_i^*,t)$ differ due to the initial conditions. However, since the motion of the elementary particles takes place on the RVE scale, the rotation tensor $\boldsymbol{Q}_i(\boldsymbol{x},t)^T\cdot\boldsymbol{Q}_i^*(\boldsymbol{x}_i^*,t)$ defines a small rotation.

2.2 The inertia tensor as a new constitutive quantity

Traditionally, the microinertia tensor of a continuum particle has properties of a rigid body and the tensor of inertia in the actual configuration is given by the algebraic equation [9,11,12,28,38]

$$\boldsymbol{J}(\boldsymbol{x},t)=\boldsymbol{Q}(\boldsymbol{x},t)\cdot\boldsymbol{J}^0(\boldsymbol{X})\cdot\boldsymbol{Q}^T(\boldsymbol{x},t), \tag{2.18}$$

where $\boldsymbol{J}^0(\boldsymbol{X})$ is the tensor of inertia in the reference configuration. By virtue of Eq. (2.6), this algebraic relation can be rewritten as a differential equation (see Appendix A):

$$\frac{\delta\boldsymbol{J}(\boldsymbol{x},t)}{\delta t}=\boldsymbol{\omega}(\boldsymbol{x},t)\times\boldsymbol{J}(\boldsymbol{x},t)-\boldsymbol{J}(\boldsymbol{x},t)\times\boldsymbol{\omega}(\boldsymbol{x},t). \tag{2.19}$$

This rate equation characterizes the change of the inertia tensor, which is exclusively due to the rigid body rotation.

Definition $(2.3)_2$ introduces the microinertia tensor as the effective inertial quantity that characterizes the average size, shape, and orientation of material particles located in the RVE at the given time. This means that in-flux and out-flux of matter in the RVE can be taken into account. Moreover, internal structural transformations leading to changes in size, shape, or preferred orientation of the elementary particles are also possible. These can be due to the combination or fragmentation of the particles, chemical reactions, or changes of the anisotropy of the material, for example by applying external electromagnetic fields. In that case, the rate equation for the microinertia tensor (2.19) is no longer a kinematic identity but should be extended by including an additional production term, $\boldsymbol{\chi}$ on the right hand side.

$$\frac{\delta\boldsymbol{J}(\boldsymbol{x},t)}{\delta t}=\boldsymbol{\omega}(\boldsymbol{x},t)\times\boldsymbol{J}(\boldsymbol{x},t)-\boldsymbol{J}(\boldsymbol{x},t)\times\boldsymbol{\omega}(\boldsymbol{x},t)+\boldsymbol{\chi}(\boldsymbol{x},t). \tag{2.20}$$

On the continuum level, this production term must be considered as a new constitutive quantity for which an additional constitutive equation has to be formulated. The form of the constitutive equation depends on the problem under consideration and can be a function of many physical quantities. A suitable form of the production term could be specified by following the rules of constitutive theory or being motivated by physics and using micromechanics, such that fundamental principles are not violated. These new ideas have been illustrated by several examples in previous papers [13–15,29,30,41], where dependencies of the production term and, as a result, the inertia tensor on the temperature field, internal and external stresses and an electric field were suggested. In what follows, we concentrate on the dependence of the tensor of inertia on the strain measures.

3 Dependence of the microinertia tensor on the strain measures

Generally, there are three main methods of introducing the strain measures into the micropolar continuum: by a geometrical approach, by applying the principle of material frame-indifference to the strain energy density, and by defining the strain measures as the fields work-conjugate to the respective internal stress and couple-stress tensor fields (see details in [7,31]). Different authors introduce the strain measures in various forms using, for example, different representations of the rotation group, Lagrangian or Eulerian descriptions, etc. Following Zhilin [42,43] and Altenbach et al. [2] we will use Eulerian stretch and wryness tensors, $\boldsymbol{g}$ and $\boldsymbol{\Theta}$, that are given by the formulae

$$\boldsymbol{g}(\boldsymbol{x},t) = \boldsymbol{I} - \nabla \boldsymbol{u}(\boldsymbol{x},t) = \nabla \boldsymbol{X}(\boldsymbol{x},t), \qquad \nabla \boldsymbol{Q}(\boldsymbol{x},t) = \boldsymbol{\Theta}(\boldsymbol{x},t) \times \boldsymbol{Q}(\boldsymbol{x},t) \tag{3.1}$$

here $\boldsymbol{I}$ is the identity tensor.

As the first step, let us suppose that the elementary particles are rigid and cannot consolidate. Then, the tensor of inertia of an individual particle in its current state can be written in the form:

$$\boldsymbol{J}_i(\boldsymbol{x},t) = \boldsymbol{Q}_i(\boldsymbol{x},t) \cdot \boldsymbol{J}_i^0(\boldsymbol{X}_i(\boldsymbol{x},t)) \cdot \boldsymbol{Q}_i^T(\boldsymbol{x},t), \tag{3.2}$$

here $\boldsymbol{J}_i^0(\boldsymbol{X}_i(\boldsymbol{x},t))$ is the tensor of inertia of the particle in the reference configuration, $\boldsymbol{X}_i(\boldsymbol{x},t)$ is the reference position vector of the particle defined by Eq. (2.10) and tensor $\boldsymbol{Q}_i$ defined by Eq. (2.6) rotates the particle to its current state, as illustrated in Fig. 2.

In accordance with definition $(2.3)_2$, the tensor of inertia of the RVE is

$$\boldsymbol{J}(\boldsymbol{x},t) = \frac{1}{N(\boldsymbol{x},t)} \sum_{i=1}^{N(\boldsymbol{x},t)} \boldsymbol{Q}_i(\boldsymbol{x},t) \cdot \boldsymbol{J}_i^0(\boldsymbol{X}_i(\boldsymbol{x},t)) \cdot \boldsymbol{Q}_i^T(\boldsymbol{x},t). \tag{3.3}$$

Dot multiplying from the left and from the right by $\boldsymbol{Q}(\boldsymbol{x},t) \cdot \boldsymbol{Q}^T(\boldsymbol{x},t)$ gives

$$\boldsymbol{J}(\boldsymbol{x},t) = \boldsymbol{Q}(\boldsymbol{x},t) \cdot \left(\frac{1}{N(\boldsymbol{x},t)} \sum_{i=1}^{N(\boldsymbol{x},t)} \bar{\boldsymbol{Q}}_i \cdot \boldsymbol{J}_i^0(\boldsymbol{X}_i(\boldsymbol{x},t)) \cdot \bar{\boldsymbol{Q}}_i^T \right) \cdot \boldsymbol{Q}^T(\boldsymbol{x},t). \tag{3.4}$$

Tensor $\bar{\boldsymbol{Q}}_i = \boldsymbol{Q}^T \cdot \boldsymbol{Q}_i$ determines the relative rotation of the particle w.r.t. the effective rotational tensor of the RVE.

One of the ways to obtain an approximate expression for $\boldsymbol{J}_i^0(\boldsymbol{X}_i(\boldsymbol{x},t))$ is to perform the Taylor series expansion about reference position vector $\boldsymbol{X}$

$$\boldsymbol{J}_i^0(\boldsymbol{X}_i) = \boldsymbol{J}_i^0(\boldsymbol{X}) + (\boldsymbol{X}_i - \boldsymbol{X}) \cdot \left. \frac{\partial \boldsymbol{J}_i^0}{\partial \boldsymbol{X}_i} \right|_{\boldsymbol{X}_i = \boldsymbol{X}} + \frac{1}{2} (\boldsymbol{X}_i - \boldsymbol{X})(\boldsymbol{X}_i - \boldsymbol{X}) \cdot\cdot \left. \frac{\partial^2 \boldsymbol{J}_i^0}{\partial \boldsymbol{X}_i^2} \right|_{\boldsymbol{X}_i = \boldsymbol{X}} + \cdots \tag{3.5}$$

Taking into account (2.9) and (2.10), we obtain the partial differential equation for the difference of the reference position vectors

$$\frac{d\,(\boldsymbol{X}_i - \boldsymbol{X})}{dt} + \boldsymbol{v}_i \cdot \nabla\,(\boldsymbol{X}_i - \boldsymbol{X}) = (\boldsymbol{v} - \boldsymbol{v}_i) \cdot \boldsymbol{g}. \tag{3.6}$$

Since the right-hand side of this equation does not depend on $\boldsymbol{X}_i - \boldsymbol{X}$ and is a linear function of the strain measure, the solution is also a linear function with respect to $\boldsymbol{g}$. Hence, from (3.5) follows that $\boldsymbol{J}_i^0(\boldsymbol{X}_i)$ depends on tensor $\boldsymbol{g}$ raised to various powers.

We now proceed to obtain a differential equation for the tensor of relative rotation $\bar{\boldsymbol{Q}}_i$. The local rate of change of the tensor of relative rotation can be obtained from the explicit form of (2.17). A lengthy calculation (see details in Appendix B) leads to the following partial differential equation for $\bar{\boldsymbol{Q}}_i$:

$$\frac{d\,\bar{\boldsymbol{Q}}_i}{dt} + \boldsymbol{v}_i \cdot \nabla \bar{\boldsymbol{Q}}_i = (\boldsymbol{\omega}_i - \boldsymbol{\omega} + (\boldsymbol{v} - \boldsymbol{v}_i) \cdot \boldsymbol{\Theta}) \cdot \boldsymbol{Q} \times \bar{\boldsymbol{Q}}_i. \tag{3.7}$$

It is seen that the solution of this equation can be an arbitrary function of $\boldsymbol{\Theta}$.

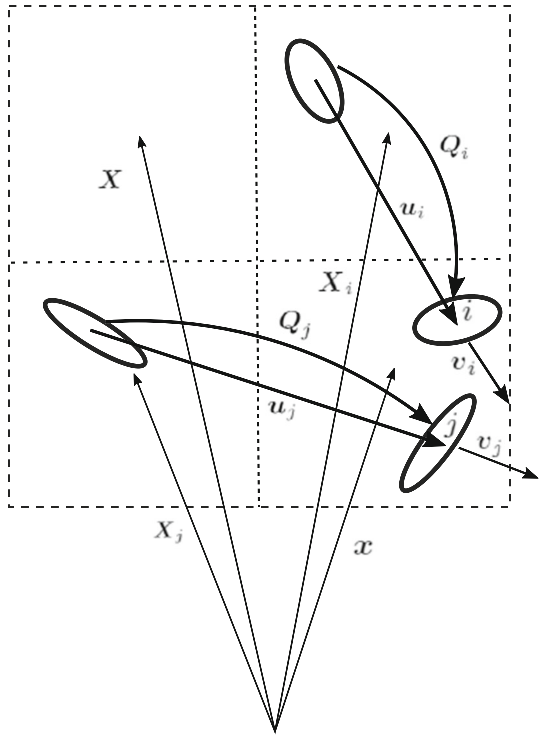

Fig. 2 Current and reference positions and orientations of the elementary particles

It is necessary to point out that the solutions of Eqs. (3.6) and (3.7) depend on the strain measures only if the velocities of the elemental particles differ from the averaged macroscopic velocity. In other words, within the suggested approach the dependence of the inertia tensor on the strain measures is due to elementary particles mixing.

As a result, the tensor of inertia can depend on both translation and rotation strain measures

$$\begin{aligned} &\boldsymbol{J}(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)=\boldsymbol{Q}(\boldsymbol{x},t)\cdot\boldsymbol{J}^0(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)\cdot\boldsymbol{Q}^T(\boldsymbol{x},t)\\ &\boldsymbol{J}^0(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)=\frac{1}{N(\boldsymbol{x},t)}\sum_{i=1}^{N(\boldsymbol{x},t)}\bar{\boldsymbol{Q}}_i(\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)\cdot\boldsymbol{J}_i^0(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{x},t)\cdot\bar{\boldsymbol{Q}}_i^T(\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t). \end{aligned} \tag{3.8}$$

The key difference between expressions (3.8) and (2.18) is non-constancy of the "reference" inertia tensor $\boldsymbol{J}^0$ within the spatial description. It reflects an internal inhomogeneity of the matter and mixing of elementary particles during the matter motion.

In view of Eq. (2.6), the material derivative of the tensor of inertia is

$$\begin{aligned} \frac{\delta\boldsymbol{J}(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)}{\delta t}&=\boldsymbol{\omega}(\boldsymbol{x},t)\times\boldsymbol{J}(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)-\boldsymbol{J}(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)\times\boldsymbol{\omega}(\boldsymbol{x},t)\\ &+\boldsymbol{Q}(\boldsymbol{x},t)\cdot\frac{\delta\boldsymbol{J}^0(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)}{\delta t}\cdot\boldsymbol{Q}^T(\boldsymbol{x},t). \end{aligned} \tag{3.9}$$

The last term on the right hand side can be treated as the production term [cf. Eq. (2.20)] and reflects changes of the tensor of inertia of the averaged particle due to elementary particle movements as the continuum deforms. Thus, it follows that the temporal change of the tensor of inertia in the observation point is

$$\begin{aligned} \frac{d\boldsymbol{J}(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)}{dt}&=\boldsymbol{\omega}(\boldsymbol{x},t)\times\boldsymbol{J}(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)-\boldsymbol{J}(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)\times\boldsymbol{\omega}(\boldsymbol{x},t)\\ &-\boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{J}(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t)+\boldsymbol{\chi}(\boldsymbol{g}(\boldsymbol{x},t),\boldsymbol{\Theta}(\boldsymbol{x},t),\boldsymbol{x},t). \end{aligned} \tag{3.10}$$

The first two terms on the right-hand side characterize the change of the inertia tensor, which is exclusively due to rigid body rotation. The third term corresponds to the change of the inertia tensor due to in-flux and out- flux of matter in the RVE. And, finally, the last one describes the change of the inertia tensor due to mixing of the elementary particles as the matter deforms. In fact, it models situations and materials with a continuum particle that on the microscale consists no longer of the same elementary units during a physical process.

If in addition to that the elementary particles can deform, we have to replace Eq. (3.2) by

$$\boldsymbol{J}_i(\boldsymbol{x},t) = \boldsymbol{Q}_i(\boldsymbol{x},t)\cdot \boldsymbol{J}_i^0(\boldsymbol{X}_i(\boldsymbol{x},t),t)\cdot \boldsymbol{Q}_i^T(\boldsymbol{x},t). \tag{3.11}$$

The dependence of the tensor of inertia in the reference configuration on time reflects a change of the elementary particle size or shape due to internal structural transformations such as consolidation or fragmentation of particles during mechanical crushing, chemical reactions, or deformation of the matter. The form of Eq. (3.10) remains the same, but the constitutive equation for the production term has to be adjusted to describe the corresponding internal transformations.

Finally, it should be emphasized once more that the macroscopic quantities of the RVE are introduced by homogenization of the micro particles within the RVE [Eqs. (2.4), (2.3)]. It follows that the tensor of inertia is a new constitutive quantity for which an additional constitutive equation has to be formulated. The constitutive equation depends on the problem under consideration and can be a function of many physical quantities. Recall that a constitutive theory of materials on the macrolevel is a formal one that is based on representation theorems leading to suitable reduction of constitutive equations in form and content. On the other hand, a mesoscopic line of reasoning can be very helpful during the reduction process of possible forms of the constitutive equations. In other words, it can give a clue what the macroterm looks like, which processes or quantities it depends on, etc. For example, by looking at Eqs. (3.5)–(3.8) we realize that the macroscopic tensor of inertia may depend on the strain measures. But of course that line of reasoning cannot lead to as an all-purpose equations in the macrotheory.

Suitable forms of the constitutive equation for the modeling of dependence of the tensor of inertia on the strain measures will be discussed in the next section.

4 Examples

4.1 Dependence on the stretch tensor

As the first example, let us consider a matter consisting of spherical particles of various radius, that occupies the region $0 \le x_2 \le x_{2max}$. During the period $t_0 \le t \le t_1$, the matter undergoes a simple shear deformation in the $\boldsymbol{e}_2$-direction with the shear parameter $k > 0$, that determines the velocity field $\boldsymbol{v}(\boldsymbol{x},t) = -kx_1\boldsymbol{e}_2$. Then, the matter starts to move in the $\boldsymbol{e}_1$-direction with the speed depending on the x_2-coordinate: $\boldsymbol{v} = v(x_2)\boldsymbol{e}_1$.

The reference position vector during the first stage is found from Eq. $(2.15)_2$

$$\frac{\partial \boldsymbol{X}(\boldsymbol{x},t)}{\partial t} - kx_1\frac{\partial \boldsymbol{X}(\boldsymbol{x},t)}{\partial x_2} = 0, \qquad t_0 \le t \le t_1. \tag{4.1}$$

The general solution of Eq. (4.1) is

$$\boldsymbol{X} = (f_1\,(x_2 + kx_1(t-t_0))) + c_1(x_1) + a_1)\,\boldsymbol{e}_1 + (f_2\,(x_2 + kx_1(t-t_0))) + c_2(x_1) + a_2)\,\boldsymbol{e}_2 + x_3\boldsymbol{e}_3. \tag{4.2}$$

The functions f_1, f_2 and c_1, c_2 as well as constants a_1 and a_2 are to be determined from the initial and boundary conditions

$$\boldsymbol{X}(\boldsymbol{x},\ t_0) = \boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3, \qquad \boldsymbol{X}(\boldsymbol{x},\ t)|_{x_1=0} = x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3. \tag{4.3}$$

Thus, the reference position vector and the stretch tensor are

$$\boldsymbol{X}(\boldsymbol{x},\ t) = x_1\boldsymbol{e}_1 + (x_2 + kx_1(t-t_0))\,\boldsymbol{e}_2 + x_3\boldsymbol{e}_3, \qquad \boldsymbol{g}(t) = \boldsymbol{I} + k(t-t_0)\boldsymbol{e}_1\boldsymbol{e}_2. \tag{4.4}$$

Similarly, for times $t \ge t_1$ the reference position vector is found from the differential equation

$$\begin{gathered}\frac{\partial \boldsymbol{X}(\boldsymbol{x},t)}{\partial t} + v(x_2)\frac{\partial \boldsymbol{X}(\boldsymbol{x},t)}{\partial x_1} = 0,\\ \boldsymbol{X}(\boldsymbol{x},\ t_1) = x_1\boldsymbol{e}_1 + (x_2 + kx_1(t_1-t_0))\,\boldsymbol{e}_2 + x_3\boldsymbol{e}_3, \quad \boldsymbol{X}(\boldsymbol{x},\ t)|_{x_2=0} = x_1\boldsymbol{e}_1 + x_3\boldsymbol{e}_3.\end{gathered} \tag{4.5}$$

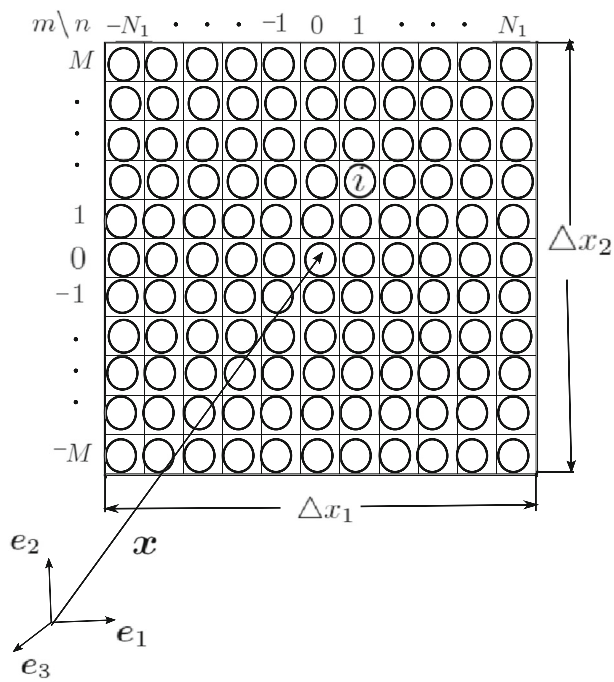

Fig. 3 Numeration of particles

As the result

$$\boldsymbol{X}(\boldsymbol{x},\ t) = (x_1 - v(x_2)(t - t_1))\,\boldsymbol{e}_1 + (k(t_1 - t_0)\,(x_1 - v(x_2)(t - t_1)) + x_2)\,\boldsymbol{e}_2 + x_3\boldsymbol{e}_3. \tag{4.6}$$

The stretch tensor is

$$\boldsymbol{g}(\boldsymbol{x}, t) = \boldsymbol{I} + g\boldsymbol{e}_1\boldsymbol{e}_2 - (t - t_1)\frac{dv(x_2)}{dx_2}\boldsymbol{e}_2\boldsymbol{e}_1 - g(t - t_1)\frac{dv(x_2)}{dx_2}\boldsymbol{e}_2\boldsymbol{e}_1, \qquad g = k(t_1 - t_0). \tag{4.7}$$

To obtain $\boldsymbol{X}_i$, we identify particle "i" by two numbers n and m (see Fig. 3[1]) and set for the first stage

$$\boldsymbol{v}_{nm} = -k\left(x_1 + \frac{n}{2N_1}\triangle x_1\right)\boldsymbol{e}_2 = \boldsymbol{v} + \bar{v}_n\boldsymbol{e}_2. \tag{4.8}$$

For simplicity sake, it is assumed that the particles are homogeneously distributed through the RVE and have the same mass. Note that the macroscopic velocity obtained from the averaging procedure (2.4) coincides with the above introduced shear velocity $\boldsymbol{v} = -kx_1\boldsymbol{e}_2$.

In order to describe the temporal and space change of the reference position vector of i-th particle during the simple shear, we turn to Eq. (3.6), which we specialize to

$$\frac{\partial\,(\boldsymbol{X}_i - \boldsymbol{X})}{\partial t} - (kx_1 - \bar{v}_n)\,\frac{\partial\,(\boldsymbol{X}_i - \boldsymbol{X})}{\partial x_2} = \bar{v}_n\boldsymbol{e}_2, \qquad (\boldsymbol{X}_i - \boldsymbol{X})\,(\boldsymbol{x}, t_0) = 0. \tag{4.9}$$

From this differential equation and Eq. (4.4) follows

$$\boldsymbol{X}_i - \boldsymbol{X} = \bar{v}_n\,(t - t_0)\,\boldsymbol{e}_2, \quad \boldsymbol{X}_i = (x_1 + \bar{v}_n\,(t - t_0))\,\boldsymbol{e}_1 + (x_2 + kx_1(t - t_0))\,\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 \tag{4.10}$$

[1] Figures 5 and 4 are to illustrate the introduced numeration and the particle movement. For a better visualization, the particles have the same size.

For the second stage, let us consider the cubic dependence of the velocity on the x_2-coordinate. We have

$$\begin{aligned} \boldsymbol{v}_{nm} &= k_2 \left(x_2 + \frac{m}{2M} \triangle x_2 \right)^3 \boldsymbol{e}_1, \\ \boldsymbol{v} &= \frac{1}{2M+1} \sum_{m=-M}^{M} \boldsymbol{v}_{nm} = k_2 \left(x_2^3 + \frac{(M+1)\triangle x_2^2 x_2}{4M} \right) \boldsymbol{e}_1 \approx k_2 x_2^3 \boldsymbol{e}_1, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \boldsymbol{v}_{nm} &= \boldsymbol{v} + k_2 \left(3x_2^2 \frac{m}{2M} \triangle x_2 + 3x_2 \left(\frac{m}{2M} \triangle x_2 \right)^2 + \left(\frac{m}{2M} \triangle x_2 \right)^3 \right) \boldsymbol{e}_1 \\ &\approx \boldsymbol{v} + 3k_2 x_2^2 \frac{m}{2M} \triangle x_2 \boldsymbol{e}_1 = \boldsymbol{v} + \bar{v}_m \boldsymbol{e}_1. \end{aligned} \tag{4.12}$$

Thus, in view of Eq. (4.7) the differential equation for $\boldsymbol{X}_i - \boldsymbol{X}$ is

$$\frac{\partial\,(\boldsymbol{X}_i - \boldsymbol{X})}{\partial t} + \left(k_2 x_2^3 + \bar{v}_m \right) \frac{\partial\,(\boldsymbol{X}_i - \boldsymbol{X})}{\partial x_1} = -\bar{v}_m \left(\boldsymbol{e}_1 + g \boldsymbol{e}_2 \right), \quad (\boldsymbol{X}_i - \boldsymbol{X})\,(\boldsymbol{x}, t_1) = \bar{v}_n\,(t_1 - t_0)\,\boldsymbol{e}_2. \tag{4.13}$$

The solution is linear w.r.t. g

$$\begin{aligned} \boldsymbol{X}_i - \boldsymbol{X} &= -\bar{v}_m\,(t - t_1)\,\boldsymbol{e}_1 + (\bar{v}_n\,(t_1 - t_0) - g\bar{v}_m\,(t - t_1))\,\boldsymbol{e}_2, \\ \boldsymbol{X}_i &= \left(x_1 - \left(k_2 x_2^3 + \bar{v}_m \right)(t - t_1) \right) \boldsymbol{e}_1 + \left(g \left(x_1 - \left(k_2 x_2^3 + \bar{v}_m \right)(t - t_1) \right) + \bar{v}_n\,(t_1 - t_0) + x_2 \right) \boldsymbol{e}_2 + x_3 \boldsymbol{e}_3. \end{aligned} \tag{4.14}$$

The matter deformations and reference position vectors are visualized in Fig. 4.

From Eqs.(4.14) to (3.5), it follows that in the case of a linear distribution of the moment of inertia of elementary particles along the vertical axis the inertia tensor on the macro level also will be a linear function of g. Indeed, from

$$\boldsymbol{J}_i^0(\boldsymbol{X}_i(g, \boldsymbol{x}, t)) = (J_{min} + \alpha \boldsymbol{X}_i(g, \boldsymbol{x}, t) \cdot \boldsymbol{e}_2)\,\boldsymbol{I} = J_i^0(\boldsymbol{X}_i(g, \boldsymbol{x}, t))\boldsymbol{I}, \qquad \alpha = \frac{J_{max} - J_{min}}{x_{2max}}, \tag{4.15}$$

where J_{min} and J_{max} are the minimal and maximal moments of inertia, follows that

$$J^0(g, \boldsymbol{x}, t) = \frac{1}{N} \sum_{i=1}^{N} J_i^0(\boldsymbol{X}_i(g, \boldsymbol{x}, t)) = J_{min} + \alpha \left(g \left(x_1 - k_2 x_2^3\,(t - t_1) \right) + x_2 \right) \tag{4.16}$$

However, under the assumption that the radius of the particles, R, is a linear function of the vertical coordinate we can write

$$J_i^0(\boldsymbol{X}_i(g, \boldsymbol{x}, t)) = \frac{2}{5} m_0 \left(R_{min} + \boldsymbol{X}_i(g, \boldsymbol{x}, t) \cdot \boldsymbol{e}_2 \right)^2, \tag{4.17}$$

where m_0 is the mass of the elementary particle and R_{min} is its minimal radius. The averaging procedure gives

$$J^0(g, \boldsymbol{x}, t) \approx \frac{2}{5} m_0 \left(g^2 \left(x_1 - k_2^2 x_2^3\,(t - t_1) \right)^2 + 2g \left(x_1 - k_2^2 x_2^3\,(t - t_1) \right) (R_{min} + x_2) + (R_{min} + x_2)^2 \right) \tag{4.18}$$

Clearly, since the tensor of inertia of a spherical particle does not depend on its orientation, we have

$$\boldsymbol{J}(g, \boldsymbol{x}, t) = J^0(g, \boldsymbol{x}, t)\boldsymbol{I}. \tag{4.19}$$

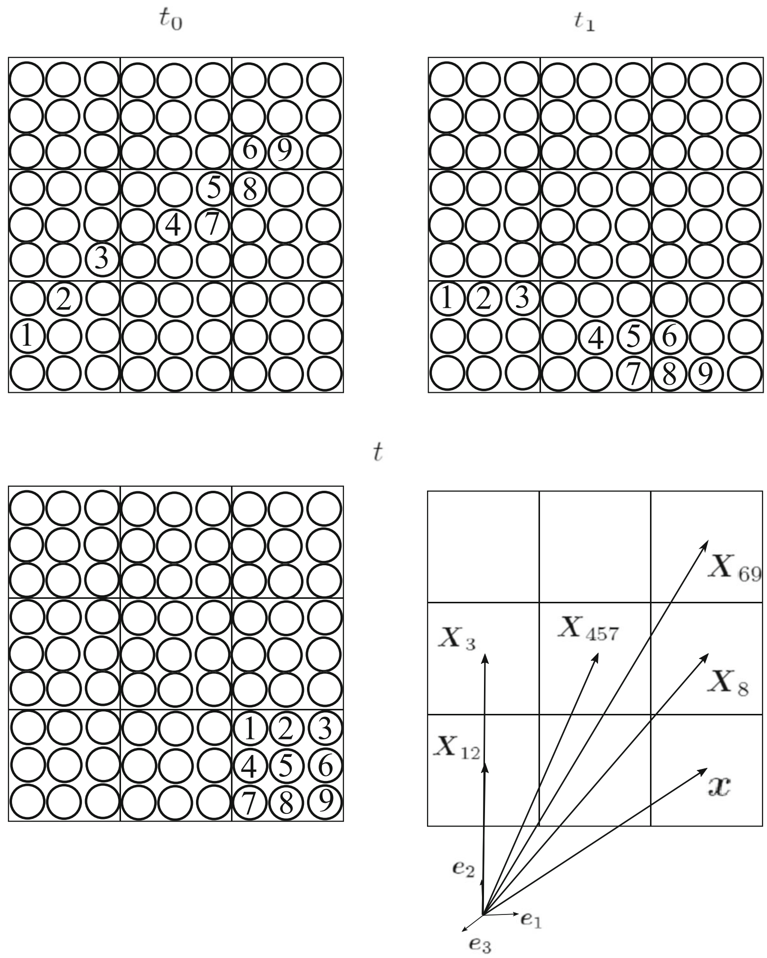

Fig. 4 Matter deformations and reference positions of the elementary particles

4.2 Dependence on the wryness tensor

To demonstrate a dependence of the tensor of inertia on wryness tensor, let us suppose that

$$\boldsymbol{v}_i = v_i \boldsymbol{e}_1, \qquad \boldsymbol{\omega}_i = \omega_i \boldsymbol{e}_3. \tag{4.20}$$

In that case we are dealing with spinning around the fixed axis and the angular velocity of i-th particle expressed through its angle of the rotation about $\boldsymbol{e}_3$, ϕ_i, by simple formula (see Appendix C)

$$\boldsymbol{\omega}_i = \frac{\delta \phi_i}{\delta t} \boldsymbol{e}_3. \tag{4.21}$$

In order to determine the rotations of particles, a reference directors $\boldsymbol{D}_k(\boldsymbol{x})$ must be locally introduced at each point of the medium. These directors may coincide with the basis vectors of the reference coordinate system. The rotation tensor $\boldsymbol{Q}_i(\phi_i \boldsymbol{e}_3)$ characterizes the orientation of particle i at a given moment in time and a given point of space with respect to the indicating frame of reference associated with this point (see Fig. 5) and has the representation via the Euler' formula

$$\boldsymbol{Q}_i(\phi_i \boldsymbol{e}_3) = \cos\phi_i \boldsymbol{I} + (1 - \cos\phi_i)\, \boldsymbol{e}_3 \boldsymbol{e}_3 + \sin\phi_i \boldsymbol{e}_3 \times \boldsymbol{I}. \tag{4.22}$$

For simplicity, we assume that all particles have the same mass and transversely isotropic tensors of inertia

$$\boldsymbol{J}_i = J_{1i} \boldsymbol{e}_i \boldsymbol{e}_i + J_2 (\boldsymbol{I} - \boldsymbol{e}_i \boldsymbol{e}_i), \qquad \boldsymbol{e}_i = \boldsymbol{Q}_i(\phi_i \boldsymbol{e}_3) \cdot \boldsymbol{e}_1, \qquad (\boldsymbol{e}_3 \cdot \boldsymbol{e}_i = 0). \tag{4.23}$$

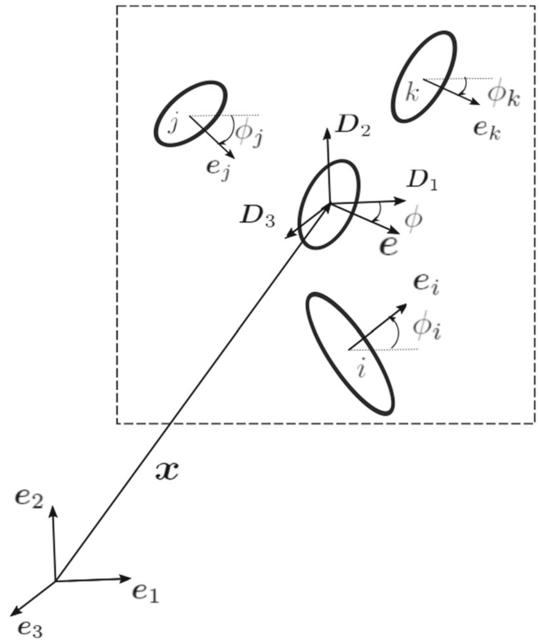

Fig. 5 Orientation of the elementary particles

The intrinsic angular momentum of the particle is

$$\boldsymbol{S}_i = \boldsymbol{J}_i \cdot \boldsymbol{\omega}_i = \boldsymbol{Q}_i(\phi_i \boldsymbol{e}_3) \cdot \boldsymbol{J}_i^0 \cdot \boldsymbol{Q}_i(\phi_i \boldsymbol{e}_3) \cdot \omega_i \boldsymbol{e}_3 = J_2 \omega_i \boldsymbol{e}_3. \tag{4.24}$$

The averaging procedure yields

$$\boldsymbol{J} = \frac{1}{N} \sum_i^N (J_{1i} - J_2)\, \boldsymbol{e}_i \boldsymbol{e}_i + J_2 \boldsymbol{I} \tag{4.25}$$

$$\boldsymbol{v} = \frac{1}{N} \sum_i^N \boldsymbol{v}_i = v \boldsymbol{e}_1, \qquad \boldsymbol{\omega} = \boldsymbol{J}^{-1} \cdot \frac{1}{N} \sum_i^N \boldsymbol{S}_i = \frac{1}{N} \sum_i^N \boldsymbol{\omega}_i = \omega \boldsymbol{e}_3 \tag{4.26}$$

Note that even in that simplified case the average angular velocity does not coincide with the material derivative of the average angle of rotation. However, we can introduce the macroscopic angle of rotation as a solution of the following differential equation

$$\frac{\delta \phi(\boldsymbol{x},\, t)}{\delta t} = \frac{\partial \phi(\boldsymbol{x},\, t)}{\partial t} + v(\boldsymbol{x},\, t) \frac{\partial \phi(\boldsymbol{x},\, t)}{\partial x_1} = \omega(\boldsymbol{x},\, t), \qquad \phi(\boldsymbol{x},\, t_0) = \phi_0(\boldsymbol{x}), \tag{4.27}$$

and write the rotation tensor and wryness tensor obtained as a solution of (3.1)$_2$ in the forms (see Appendix C):

$$\boldsymbol{Q}(\boldsymbol{x},\, t) = \boldsymbol{Q}(\phi(\boldsymbol{x},\, t) \boldsymbol{e}_3), \qquad \boldsymbol{\Theta} = \nabla\left(\phi(\boldsymbol{x},\, t)\right) \boldsymbol{e}_3. \tag{4.28}$$

Let the space distribution of the elementary particles at moment $t_0 = 0$ be a function of the horizontal coordinate. Similar to the previous example, we identify particle "i" by numbers n and m and write

$$J_{1i}(\boldsymbol{X}, 0) = J\left(X_1 + \frac{n}{2N_1} \triangle x_1\right), \qquad \phi_i(\boldsymbol{X}, 0) = \phi_i^0(\boldsymbol{X}) = \varphi\left(X_1 + \frac{n}{2N_1} \triangle x_1\right), \tag{4.29}$$

where $J(X_1)$ and $\varphi(X_1)$ are arbitrary functions.

The averaged tensor of inertia at t_0 is

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{X}, t_0) = &\left(J_2 \sin^2 \varphi(X_1) + J(X_1) \cos^2 \varphi(X_1)\right) \boldsymbol{e}_1 \boldsymbol{e}_1 + \left(J_2 \cos^2 \varphi(X_1) + J(X_1) \sin^2 \varphi(X_1)\right) \boldsymbol{e}_2 \boldsymbol{e}_2 \\ &+ \frac{1}{2} \left(J(X_1) - J_2\right) \sin\left(2\varphi(X_1)\right) (\boldsymbol{e}_1 \boldsymbol{e}_2 + \boldsymbol{e}_2 \boldsymbol{e}_1) + J_2 \boldsymbol{e}_3 \boldsymbol{e}_3 \end{aligned} \tag{4.30}$$

Or in its principal axes

$$\boldsymbol{J}(\boldsymbol{X}, t_0) = J(X_1)\boldsymbol{e}(\boldsymbol{X})\boldsymbol{e}(\boldsymbol{X}) + J_2\left(\boldsymbol{I} - \boldsymbol{e}(\boldsymbol{X})\boldsymbol{e}(\boldsymbol{X})\right), \quad \boldsymbol{e}(\boldsymbol{X}) = \boldsymbol{Q}(\varphi(\boldsymbol{X})\boldsymbol{e}_3)\cdot\boldsymbol{e}_1, \quad \phi_0(\boldsymbol{X}) = \varphi(X_1). \tag{4.31}$$

Under assumptions that $\omega_i = \text{const}$ and v_i are functions of x_2, the solutions of Eqs. (4.27) and $(2.15)_2$ are

$$\phi(\boldsymbol{x}, t) = \omega t + \varphi(\zeta), \quad \boldsymbol{X}(\boldsymbol{x}, t) = \zeta\boldsymbol{e}_1 + x_2\boldsymbol{e}_1 + x_3\boldsymbol{e}_3, \quad \zeta = x_1 - v(x_2)t \tag{4.32}$$

Therefore, the strain measures are

$$\boldsymbol{\Theta} = \frac{d\varphi}{d\zeta}\boldsymbol{e}_1\boldsymbol{e}_3 - \frac{d\varphi}{d\zeta}\frac{dv}{dx_2}t\,\boldsymbol{e}_2\boldsymbol{e}_3, \qquad \boldsymbol{g} = \boldsymbol{I} - \frac{dv}{dx_2}t\,\boldsymbol{e}_2\boldsymbol{e}_1. \tag{4.33}$$

In this case, the right hand side of Eq. (3.6) is zero and as a result $\boldsymbol{X} - \boldsymbol{X}_i$ as well as $\boldsymbol{J}_i^0(\boldsymbol{X}_i)$ are independent on $\boldsymbol{g}$.

$$\boldsymbol{J}_i^0(\boldsymbol{X}(\boldsymbol{x}, t)) = J\left(x_1 - v_i(x_2)t + \frac{n}{2N_1}\triangle x_1\right)\boldsymbol{e}_1\boldsymbol{e}_1 + J_2\left(\boldsymbol{I} - \boldsymbol{e}_1\boldsymbol{e}_1\right). \tag{4.34}$$

Since all rotations are around the same axis, we can write

$$\bar{\boldsymbol{Q}}_i(\bar{\phi}_i\boldsymbol{e}_3) = \cos\bar{\phi}_i\boldsymbol{I} + \left(1 - \cos\bar{\phi}_i\right)\boldsymbol{e}_3\boldsymbol{e}_3 + \sin\bar{\phi}_i\boldsymbol{e}_3\times\boldsymbol{I}, \qquad \bar{\phi}_i = \phi_i - \phi. \tag{4.35}$$

In view of Eq. $(4.33)_1$, we may write Eq. (3.7) in the form

$$\frac{\partial\bar{\phi}_i}{\partial t}\boldsymbol{e}_3\times\bar{\boldsymbol{Q}}_i + v_i\frac{\partial\bar{\phi}_i}{\partial x_1}\boldsymbol{e}_3\times\bar{\boldsymbol{Q}}_i = \left(\omega_i - \omega + (v - v_i)\,\Theta\right)\boldsymbol{e}_3\cdot\boldsymbol{Q}(\phi\boldsymbol{e}_3)\times\bar{\boldsymbol{Q}}_i, \quad \Theta = \frac{d\varphi}{d\zeta}. \tag{4.36}$$

It leads to the differential equation for $\bar{\phi}_i$

$$\begin{aligned}&\frac{\partial\bar{\phi}_i(\boldsymbol{x}, t)}{\partial t} + v_i(x_2)\frac{\partial\bar{\phi}_i(\boldsymbol{x}, t)}{\partial x_1} = \omega_i - \omega + \left(v(x_2) - v_i(x_2)\right)\Theta(\boldsymbol{x}, t),\\ &\bar{\phi}_i(\boldsymbol{x},\, 0) = \bar{\phi}_i^0(\boldsymbol{x}) = \phi_i^0(\boldsymbol{x}) - \phi_0(\boldsymbol{x}).\end{aligned} \tag{4.37}$$

Note that in the considered case the right-hand side of the above equation does not depend on the angle of the relative rotation anymore and therefore the solution of Eq. (4.37) will be a linear function with respect to Θ.

The velocity an orientation of particle "i" can be written as

$$\begin{aligned}&v_{mn}\left(x_2 + \frac{m}{2M}\triangle x_2\right) = v_{mn}(\zeta_2) \approx v(x_2) + \left.\frac{dv_{mn}}{d\zeta_2}\right|_{x_2}\frac{m}{2M}\triangle x_2 = v(x_2) + \bar{v}_m(x_2),\\ &\phi_{mn}^0\left(x_1 + \frac{n}{2N_1}\triangle x_1\right) = \phi_{mn}^0(\zeta_1) \approx \phi_0(x_1) + \left.\frac{d\phi_{mn}^0}{d\zeta_1}\right|_{x_1}\frac{n}{2N_1}\triangle x_1 = \phi_0(x_1) + \bar{\phi}_n^0(x_1).\end{aligned} \tag{4.38}$$

The solution of Eq. (4.37) is

$$\bar{\phi}_i = \bar{\phi}_i^0\left(x_1 - v_{nm}(x_2)t\right) + \left(\omega_i - \omega - \bar{v}_m(x_2)\Theta(\boldsymbol{x}, t)\right)t. \tag{4.39}$$

Therefore, if all particles have the same angular velocity, in the case of the linear function $\varphi(\zeta) = 2\pi/k\,\zeta$, from Eqs.(3.8), (4.34), (4.37) and $(4.32)_1$ follows:

$$\begin{aligned}&\boldsymbol{J}(\boldsymbol{x}, t) = J\left(x_1 - v(x_2)t\right)\boldsymbol{e}(\boldsymbol{x}, t)\boldsymbol{e}(\boldsymbol{x}, t) + J_2\left(\boldsymbol{I} - \boldsymbol{e}(\boldsymbol{x}, t)\boldsymbol{e}(\boldsymbol{x}, t)\right)\\ &\boldsymbol{e}(\boldsymbol{x}, t) = \boldsymbol{Q}\left(\omega t + \left(\left(x_1 - v(x_2)t\right)\Theta\right)\boldsymbol{e}_3\right)\cdot\boldsymbol{e}_1, \qquad \Theta = \frac{2\pi}{k}.\end{aligned} \tag{4.40}$$

It is seen that the eigenvalue of the inertia tensor in a co-rotating frame may change in time due to particles traveling and mixing. In addition, the orientation of the averaged particle may change in time not only because of the rotation of the particles but also due to their movements and wryness deformation.

5 Concluding remarks

In this paper, we discussed peculiarities of the suggested extended micropolar theory. While the classical approach is based on the concept of a subcontinuum within a material particle, EMT fields are introduced by homogenization from the physical properties of discrete microparticles. Specifically, the averages of microscopic physical properties within an RVE are determined such that the total physical property within the original RVE and the homogenized replacement stay the same. The main idea is to relate information on the mesoscale by taking the intrinsic microstructure within RVE into account with the macroscopic world, i.e., with the balances of micropolar continua in combination with suitable constitutive equations. This new approach enables us to study the temporal change of rotational inertial characteristics. In this context, the tensor of inertia is an additional internal variable that may depend on various external and internal factors. In the present paper, we focused on the dependence of the inertia tensor on the strain measures and illustrated the approach on examples.

It is necessary to point out that one should consider the dependence of the inertia tensor on the strain measures (3.8) as a constitutive equation that reflects an internal inhomogeneity of the matter and mixing of microelements during the matter motion. At the same time, the elementary particles may deform together with the matter deformation. As a result, the averaged tensor of inertia of RVE may change even in the case of RVE being a closed indestructible system consisting of the same particles. Such situation was considered in [4], where a dependence of the tensor of inertia of a polar particle on the strain measures was assumed. Within the framework of the present paper, it means that the representation (3.11) has to be used and the constitutive equation for the production term should reflect the polar particle deformation as the matter deforms.

Acknowledgements The research is partially funded by the Ministry of Science and Higher Education of the Russian Federation as part of World-class Research Center program: Advanced Digital Technologies (Contract No. 075-15-2022-311 dated 20.04.2022).

Declarations

Conflict of interest The authors of this manuscript have no conflict of interest.

Appendix A: Derivation of Eq. (2.19)

The tensor of inertia of a rigid body changes only due to the body's rotation

$$\boldsymbol{J}(t) = \boldsymbol{Q}(t) \cdot \boldsymbol{J}^0 \cdot \boldsymbol{Q}^T(t), \tag{5.1}$$

where $\boldsymbol{J}$ is the tensor of inertia in the reference position and $\boldsymbol{Q}(t)$ is the proper orthogonal tensor ($\boldsymbol{Q} \cdot \boldsymbol{Q}^T = \boldsymbol{I}$, $\det \boldsymbol{Q} = 1$) connecting the eigenvectors of the tensor of inertia in the reference position with those of the current one.
Let us calculate the material derivative of (5.1). Since $\boldsymbol{J}^0$ does not depend on time, we get

$$\frac{\delta \boldsymbol{J}(t)}{\delta t} = \frac{\delta \boldsymbol{Q}(t)}{\delta t} \cdot \boldsymbol{J}^0 \cdot \boldsymbol{Q}^T(t) + \boldsymbol{Q}(t) \cdot \boldsymbol{J}^0 \cdot \frac{\delta \boldsymbol{Q}^T(t)}{\delta t}. \tag{5.2}$$

To calculate the material derivative of $\boldsymbol{Q}$ note that

$$\frac{\delta \boldsymbol{Q}(t) \cdot \boldsymbol{Q}^T(t)}{\delta t} = \frac{\delta \boldsymbol{Q}(t)}{\delta t} \cdot \boldsymbol{Q}^T(t) + \boldsymbol{Q}(t) \cdot \frac{\delta \boldsymbol{Q}^T(t)}{\delta t} = \boldsymbol{0}. \tag{5.3}$$

Taking into account that the operations of differentiation and transpose are permutable from (5.3)l follows that

$$\left(\frac{\delta \boldsymbol{Q}(t)}{\delta t} \cdot \boldsymbol{Q}^T(t) \right)^T = -\frac{\delta \boldsymbol{Q}(t)}{\delta t} \cdot \boldsymbol{Q}^T(t). \tag{5.4}$$

Thus, being an antisymmetric tensor $\frac{\delta \boldsymbol{Q}(t)}{\delta t} \cdot \boldsymbol{Q}^T(t)$ has a representation

$$\frac{\delta \boldsymbol{Q}(t)}{\delta t} \cdot \boldsymbol{Q}^T(t) = \boldsymbol{\omega}(t) \times \boldsymbol{I}. \tag{5.5}$$

Dot multiplying (5.5) from the right by $\boldsymbol{Q}$ gives

$$\frac{\delta\boldsymbol{Q}(t)}{\delta t} = \boldsymbol{\omega}(t)\times\boldsymbol{Q}(t). \tag{5.6}$$

Insertion of this relation into (5.2) yields

$$\frac{\delta\boldsymbol{J}(t)}{\delta t} = \boldsymbol{\omega}(t)\times\boldsymbol{Q}(t)\cdot\boldsymbol{J}^0\cdot\boldsymbol{Q}^T(t)+\boldsymbol{Q}(t)\cdot\boldsymbol{J}^0\cdot(\boldsymbol{\omega}(t)\times\boldsymbol{Q}(t))^T. \tag{5.7}$$

For any second order tensor $\boldsymbol{A}$

$$(\boldsymbol{c}\times\boldsymbol{A})^T = ((\boldsymbol{c}\times\boldsymbol{a}_k)\,\boldsymbol{b}_k)^T = \boldsymbol{b}_k\,(\boldsymbol{c}\times\boldsymbol{a}_k) = -\boldsymbol{b}_k\,(\boldsymbol{a}_k\times\boldsymbol{c}) = -\boldsymbol{A}^T\times\boldsymbol{c}. \tag{5.8}$$

Finally, taking into account (5.1)| and (5.8) we have

$$\frac{\delta\boldsymbol{J}(t)}{\delta t} = \boldsymbol{\omega}(t)\times\boldsymbol{J}(t)-\boldsymbol{J}(t)\times\boldsymbol{\omega}(t). \tag{5.9}$$

Appendix B: Derivation of Eq. (3.7)

Let us consider the tensor of relative rotation which can be written as a composition of the rotational tensors $\boldsymbol{Q}^T$ and $\boldsymbol{Q}_i$. These tensors are related to the corresponding angular velocities by the formulae

$$\frac{d\boldsymbol{Q}}{dt}+\boldsymbol{v}\cdot\nabla\boldsymbol{Q} = \boldsymbol{\omega}\times\boldsymbol{Q}, \qquad \frac{d\boldsymbol{Q}_i}{dt}+\boldsymbol{v}_i\cdot\nabla\boldsymbol{Q}_i = \boldsymbol{\omega}_i\times\boldsymbol{Q}_i. \tag{5.10}$$

The local rate of change of the tensor of relative rotation $\bar{\boldsymbol{Q}}_i = \boldsymbol{Q}^T\cdot\boldsymbol{Q}_i$ is

$$\frac{d}{dt}\left(\boldsymbol{Q}^T\cdot\boldsymbol{Q}_i\right) = (\boldsymbol{\omega}\times\boldsymbol{Q}-\boldsymbol{v}\cdot(\nabla\boldsymbol{Q}))^T\cdot\boldsymbol{Q}_i+\boldsymbol{Q}^T\cdot\left(\boldsymbol{\omega}_i\times\boldsymbol{Q}_i-\boldsymbol{v}_i\cdot\nabla\boldsymbol{Q}_i\right) \tag{5.11}$$

To transform the terms with angular velocities, let us first develop a useful identity.
For any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and the proper orthogonal tensor $\boldsymbol{Q}$, the following equality is valid

$$\left(\boldsymbol{Q}^T\cdot\boldsymbol{a}\right)\times\left(\boldsymbol{Q}^T\cdot\boldsymbol{b}\right) = \boldsymbol{Q}^T\cdot(\boldsymbol{a}\times\boldsymbol{b}). \tag{5.12}$$

Rewrite it in the form

$$\left(\left(\boldsymbol{Q}^T\cdot\boldsymbol{a}\right)\times\boldsymbol{I}\right)\cdot\boldsymbol{Q}^T\cdot\boldsymbol{b} = \boldsymbol{Q}^T\cdot(\boldsymbol{a}\times\boldsymbol{I})\cdot\boldsymbol{b}. \tag{5.13}$$

Arbitrariness of $\boldsymbol{b}$ yields the equation

$$\boldsymbol{Q}^T\cdot(\boldsymbol{a}\times\boldsymbol{I}) = \left(\left(\boldsymbol{Q}^T\cdot\boldsymbol{a}\right)\times\boldsymbol{E}\right)\cdot\boldsymbol{Q}^T. \tag{5.14}$$

Dot multiplying (5.14) from the right by $\boldsymbol{Q}$, we get the needed identity:

$$\boldsymbol{Q}^T\cdot(\boldsymbol{a}\times\boldsymbol{I})\cdot\boldsymbol{Q} = \left(\boldsymbol{Q}^T\cdot\boldsymbol{a}\right)\times\boldsymbol{I}. \tag{5.15}$$

Now we can write

$$(\boldsymbol{\omega}\times\boldsymbol{Q})^T = -\boldsymbol{Q}^T\times\boldsymbol{\omega} = -\boldsymbol{Q}^T\cdot(\boldsymbol{I}\times\boldsymbol{\omega})\cdot\boldsymbol{Q}\cdot\boldsymbol{Q}^T = -\left(\boldsymbol{Q}^T\cdot\boldsymbol{\omega}\right)\times\boldsymbol{I}\cdot\boldsymbol{Q}^T = -\left(\boldsymbol{Q}^T\cdot\boldsymbol{\omega}\right)\times\boldsymbol{Q}^T \tag{5.16}$$

$$\boldsymbol{Q}^T\cdot\left(\boldsymbol{\omega}_i\times\boldsymbol{Q}_i\right) = \boldsymbol{Q}^T\cdot(\boldsymbol{\omega}_i\times\boldsymbol{I})\cdot\boldsymbol{Q}_i = \boldsymbol{Q}^T\cdot(\boldsymbol{\omega}_i\times\boldsymbol{I})\cdot\boldsymbol{Q}\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}_i = \left(\left(\boldsymbol{Q}^T\cdot\boldsymbol{\omega}_i\right)\times\boldsymbol{I}\right)\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}_i \tag{5.17}$$

Thus,

$$-\left(\boldsymbol{Q}^T\cdot\boldsymbol{\omega}\right)\times\bar{\boldsymbol{Q}}_i+\left(\boldsymbol{Q}^T\cdot\boldsymbol{\omega}_i\right)\times\bar{\boldsymbol{Q}}_i = \left(\boldsymbol{Q}^T\cdot(\boldsymbol{\omega}_i-\boldsymbol{\omega})\right)\times\bar{\boldsymbol{Q}}_i = (\boldsymbol{\omega}_i-\boldsymbol{\omega})\cdot\boldsymbol{Q}\times\bar{\boldsymbol{Q}}_i. \tag{5.18}$$

For the remaining two terms with translation velocities, we start by writing

$$-(\boldsymbol{v}\cdot\nabla\boldsymbol{Q})^T\cdot\boldsymbol{Q}_i-\boldsymbol{Q}^T\cdot\left(\boldsymbol{v}_i\cdot(\nabla\boldsymbol{Q}_i)\right)=-\boldsymbol{v}_i\cdot\nabla\left(\boldsymbol{Q}^T\cdot\boldsymbol{Q}_i\right)+(\boldsymbol{v}_i-\boldsymbol{v})\cdot\left(\nabla\boldsymbol{Q}^T\right)\cdot\boldsymbol{Q}_i, \tag{5.19}$$

where

$$\begin{aligned}(\boldsymbol{v}_i-\boldsymbol{v})\cdot\left(\nabla\boldsymbol{Q}^T\right)\cdot\boldsymbol{Q}_i&=\left[\boldsymbol{Q}_i^T\cdot((\boldsymbol{v}_i-\boldsymbol{v})\cdot\nabla\boldsymbol{Q})\right]^T=\left[\boldsymbol{Q}_i^T\cdot((\boldsymbol{v}_i-\boldsymbol{v})\cdot\boldsymbol{\Theta}\times\boldsymbol{Q})\right]^T\\&=\left[(\boldsymbol{v}_i-\boldsymbol{v})\cdot\boldsymbol{\Theta}\times\boldsymbol{Q}\right]^T\cdot\boldsymbol{Q}\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}_i=\left[\boldsymbol{Q}^T\cdot((\boldsymbol{v}_i-\boldsymbol{v})\cdot\boldsymbol{\Theta}\times\boldsymbol{I})\cdot\boldsymbol{Q}\right]^T\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}_i\end{aligned} \tag{5.20}$$

Bearing (5.15) in mind, we have

$$\begin{aligned}(\boldsymbol{v}_i-\boldsymbol{v})\cdot\left(\nabla\boldsymbol{Q}^T\right)\cdot\boldsymbol{Q}_i&=\left[\left(\boldsymbol{Q}^T\cdot((\boldsymbol{v}_i-\boldsymbol{v})\cdot\boldsymbol{\Theta})\right)\times\boldsymbol{I}\right]^T\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}_i\\&=-(\boldsymbol{v}_i-\boldsymbol{v})\cdot\boldsymbol{\Theta}\cdot\boldsymbol{Q}\times\boldsymbol{I}\cdot\boldsymbol{Q}^T\cdot\boldsymbol{Q}_i\end{aligned} \tag{5.21}$$

Substituting Eqs.(5.18), (5.19) and (5.21) into Eq. (5.11), we transform the latter into

$$\frac{d\bar{\boldsymbol{Q}}_i}{dt}+\boldsymbol{v}_i\cdot\nabla\bar{\boldsymbol{Q}}_i=(\boldsymbol{\omega}_i-\boldsymbol{\omega})\cdot\boldsymbol{Q}\times\bar{\boldsymbol{Q}}_i+(\boldsymbol{v}-\boldsymbol{v}_i)\cdot\boldsymbol{\Theta}\cdot\boldsymbol{Q}\times\bar{\boldsymbol{Q}}_i. \tag{5.22}$$

Appendix C: Rotation around a fixed axis

Every proper orthogonal tensor $\boldsymbol{Q}$ has the representation via the Euler' formula

$$\begin{aligned}&\boldsymbol{Q}(\phi\boldsymbol{m})=\cos\phi\boldsymbol{I}+(1-\cos\phi)\,\boldsymbol{mm}+\sin\phi\boldsymbol{m}\times\boldsymbol{I},\qquad \boldsymbol{m}=\text{const},\\&\boldsymbol{Q}(\phi\boldsymbol{m})\cdot\boldsymbol{m}=\boldsymbol{m}.\end{aligned} \tag{5.23}$$

Dotting both sides of equation

$$\frac{\delta\boldsymbol{Q}}{\delta t}=\boldsymbol{\omega}(t)\times\boldsymbol{Q} \tag{5.24}$$

by $\boldsymbol{m}$ from the right and taking into account that

$$\frac{\delta\boldsymbol{Q}}{\delta t}\cdot\boldsymbol{m}=\frac{\delta}{\delta t}(\boldsymbol{Q}\cdot\boldsymbol{m})=\frac{\delta\boldsymbol{m}}{\delta t}=\boldsymbol{0}, \tag{5.25}$$

we have

$$\boldsymbol{\omega}(t)\times\boldsymbol{m}=\boldsymbol{0}. \tag{5.26}$$

It follows

$$\boldsymbol{\omega}(t)=\omega(t)\,\boldsymbol{m}, \tag{5.27}$$

and

$$\frac{\delta\boldsymbol{Q}}{\delta t}=\omega(t)\,\boldsymbol{m}\times\boldsymbol{Q}. \tag{5.28}$$

Let us take a trace of both sides of Eq. (5.28)

$$\begin{aligned}\operatorname{tr}\left(\frac{\delta\boldsymbol{Q}}{\delta t}\right)&=\frac{\delta}{\delta t}\operatorname{tr}\boldsymbol{Q}=\frac{\delta}{\delta t}(1+2\cos\phi)=-2\sin\phi\,\frac{\delta\phi}{\delta t}\\\operatorname{tr}(\omega(t)\boldsymbol{m}\times\boldsymbol{Q})&=\omega(t)\operatorname{tr}(\cos\phi\boldsymbol{m}\times\boldsymbol{I}+\sin\phi\boldsymbol{m}\times\boldsymbol{I}\times\boldsymbol{m})\\&=\omega(t)\sin\phi\operatorname{tr}(\boldsymbol{mm}-\boldsymbol{m}\cdot\boldsymbol{m}\boldsymbol{I})=-2\omega(t)\sin\phi.\end{aligned} \tag{5.29}$$

Therefore,

$$\boldsymbol{\omega}(t)=\frac{\delta\phi}{\delta t}\boldsymbol{m}, \tag{5.30}$$

Finally, let us show that for $\boldsymbol{m} = \text{const}$ the wryness tensor in the form $\boldsymbol{\Theta} = (\nabla\phi)\,\boldsymbol{m}$ satisfies the equation

$$\nabla\left(\boldsymbol{Q}(\phi\boldsymbol{m})\right) = \boldsymbol{\Theta} \times \boldsymbol{Q}(\phi\boldsymbol{m}). \tag{5.31}$$

Indeed from Eq. (5.23) follows

$$\begin{aligned} \nabla\left(\boldsymbol{Q}(\phi\boldsymbol{m})\right) &= (\nabla\phi)\left(\sin\phi\,\left(\boldsymbol{m}\boldsymbol{m} - \boldsymbol{I}\right) + \cos\phi\,\boldsymbol{m}\times\boldsymbol{I}\right) \\ \boldsymbol{\Theta}\times\boldsymbol{Q}(\phi\boldsymbol{m}) &= (\nabla\phi)\left(\cos\phi\,\boldsymbol{m}\times\boldsymbol{I} + \sin\phi\,\boldsymbol{m}\times(\boldsymbol{m}\times\boldsymbol{I})\right) \\ &= (\nabla\phi)\left(\cos\phi\,\boldsymbol{m}\times\boldsymbol{I} + \sin\phi\,\left(\boldsymbol{m}\,(\boldsymbol{m}\cdot\boldsymbol{I}) - (\boldsymbol{m}\cdot\boldsymbol{m})\,\boldsymbol{I}\right)\right). \end{aligned} \tag{5.32}$$

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