ORIGINAL ARTICLE

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A new approach to modeling of thermal and electrical conductivities by means of the Cosserat continuum

Abstract We develop a linear theory of the Cosserat continuum of a special type. This continuum possesses only rotational degrees of freedom. The constitutive equation for the moment stress tensor is the same as for the elastic continuum. The main feature of our model is that the differential equation relating the wryness tensor to the angular velocity vector contains a source term. Thanks to a special choice of the constitutive equation for the source term, we obtain a model of continuum that has some properties of a viscoelastic continuum. Considering such a continuum, we associate the main variables characterizing its stress–strain state with quantities characterizing electrodynamic and thermodynamic processes. Our new model describes all physical processes that were described by using our previous models, but at the same time, it gives us some important results that cannot be obtained in the framework of our previous models. In contrast to the previous model, which is based on Maxwell's model of a viscoelastic material, the new model allows us to arrive at Maxwell's equations for conductors without modifying constitutive equations. In addition, the new model describes the conversion of electrical energy into thermal energy due to Joule heat. Furthermore, the new model allows us to obtain the entropy balance equation.

1 Introduction

The idea of using mechanical models to describe physical processes was dominant in science until the late nineteenth century. Many authors proposed mechanical models of thermal, electrical, magnetic and electromagnetic processes. These models are known as models of the ether [1]. All mathematical models of the ether constructed in the nineteenth century are based on translational degrees of freedom. Such models continued to be seen in the twentieth and twentyfirst centuries (see, e.g., [2–17]). However, some scientists of the nineteenth century, e.g., Kelvin, Fitzgerald and Maxwell, came up with an idea of using models based on rotational degrees of freedom [1]. At that time, the level of development of continuum mechanics did not allow scientists

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to develop this idea. In the early twentieth century, the Cosserat brothers created a correct mathematical model of a continuum with rotational degrees of freedom. After that, it became possible to use such a continuum for modeling the ether.

The description of electromechanical and magnetomechanical effects using continuum models with rotational degrees of freedom was performed in works of other authors. We can refer, e.g., to one-component continuum models [18–25] and two-component continuum models [26–29]. We can also refer to [30–32], where analogues between curved beams and electrical circuits are used to design the multi-physics metamaterials. Zhilin was the first scientist of twentieth century who created models of physical processes based on the Cosserat continuum and who called these models the ether models. In 1996, Zhilin gave the lecture "Reality and mechanics" at XXIII Summer School "Nonlinear Oscillations in Mechanical Systems" (St. Petersburg, Russia). He showed that based on the Cosserat continuum possessing only rotation degrees of freedom one can obtain the well-known equations of quantum mechanics: the Schrödinger equation and the Klein–Gordon equation. In 2000, Zhilin gave the lecture "The main direction of the development of mechanics for XXI century" at XXVIII Summer School–Conference "Advanced Problems in Mechanics" (St. Petersburg, Russia). He presented the Cosserat continuum model also based only on rotation degrees of freedom. He showed that this model can be considered as a model of electromagnetic field in vacuum. Later, in 2005, Zhilin developed this model and created a nonlinear theory of electromagnetic field. The aforementioned lectures were published in [7]. The nonlinear theory of electromagnetic field was published in [33,34]. A discussion of the above Zhilin works can be found in [35]. The biography and scientific contributions of Zhilin are presented in [36–38].

There are many experiments interrelating thermal, electric and magnetic processes. We are convinced that it is important to describe these processes on the basis of the same approach and the same principles. We believe that this method allows us to find the ways of describing various experimentally discovered thermoelectric, thermomagnetic and thermoelectromagnetic effects. We mean *the Seebeck effect* (a phenomenon in which a temperature difference between two dissimilar electrical conductors or semiconductors produces a voltage difference between the two substances); *the Peltier effect* (the cooling of one junction and the heating of the other one when an electric current is maintained in a circuit consisting of two dissimilar conductors or semiconductors); *the Ettingshausen effect* (if there is a current in a conductor and a magnetic field normal to it, one observes a temperature gradient normal to the current and magnetic field); *the Nernst–Ettingshausen effect* (if a conductive sample undergoes the influence of a magnetic field and a temperature gradient normal to the magnetic field, one observes an electric field directed perpendicular to both the magnetic field and the temperature gradient).

Beginning in 2010, we have published a series of works [39–45] developing Zhilin's ideas applied to modeling thermodynamic processes and a series of works [46–53] developing Zhilin's ideas applied to modeling electromagnetic processes and mutual influence of thermodynamic and electromagnetic processes. Paper [52] is directly related to the subject of the present study, so we discuss the results of this paper in more detail. In this paper, we consider Maxwell's model of a viscoelastic material that differs from the classical one by that it is based on rotational degrees of freedom. We interpret equations describing the continuum as equations of thermodynamics and electrodynamics. We emphasize that this theory describes thermal and electromagnetic processes on the basis of the same approach and the same principles. If the thermal component is ignored, the obtained equations can be reduced to a three-dimensional analogue of Kirchhoff's laws for electric circuits. If the thermal component is taken into account, the obtained equations can be reduced to two three-dimensional telegraph equations: one for temperature and the other for the electric field vector. We believe that the telegraph equation describes electromagnetic processes in conducting media more accurately than the classical Maxwell's equations since it allows to model the static skin effect observed in many experiments. The telegraph equation for temperature may be useful in describing thermal and thermoelectric surface effects. At the same time, the theory proposed in [52] is not free from shortcomings. Firstly, this theory does not describe the conversion of electrical energy into thermal energy due to Joule heat. Secondly, in this theory, Maxwell's first equation contains the electric current density only if this equation is written in terms of the electric field vector and the magnetic field vector or in terms of the electric field vector and the magnetic induction vector. If this equation is written in terms of the electric induction vector and the magnetic field vector or in terms of the electric induction vector and the magnetic induction vector, the electric current density is absent in it. The study presented below aims to construct a theory without the aforementioned shortcomings.

The main ideas concerning the modification of the theory constructed in [52] are as follows. Firstly, we replace the constitutive equation for the viscoelastic material with the constitutive equation for the elastic material. Secondly, we modify the differential equation relating the wryness tensor to the angular velocity vector by adding a source term to this equation. Thirdly, we set the source term so to obtain a model of

continuum that possesses properties of a viscoelastic continuum and also to ensure the energy transfer from one degree of freedom to another. We note that modifying the differential equation for the wryness tensor we follow ideas of [54]. This paper considers the classical continuum based on translational degrees of freedom. The authors of this paper introduce the source term to the equation for the strain measure. In fact, paper [54] expands the concept of strains, taking them beyond purely geometrical characteristics. We emphasize that when adding a source term to the equation for the wryness tensor, we treat this equation as a balance equation. We believe that such an approach can bring continuum mechanics closer to nonequilibrium thermodynamics, where the balance equations are formulated for various quantities, not only for mass, momentum, angular momentum and energy (see [55,56]). Certainly, there are no strain balance equations with source terms in nonequilibrium thermodynamics. Such an equation was firstly introduced in [54].

We note that the model proposed in the present paper, as well as the model considered in [52], possesses the following properties. Firstly, there are two types of waves in this model: the bending (transverse) waves, which are associated with electromagnetic processes, and the torsional (longitudinal) waves, which are associated with thermodynamic processes. According to this model, at the interface between two media, an incident electromagnetic wave can generate both electromagnetic and thermal waves. An incident thermal wave can also generate both thermal and electromagnetic waves. Secondly, if the thermal component is ignored, the proposed model is described by the equations that can be reduced to three-dimensional analogues of Kirchhoff's laws. Thirdly, the proposed model includes three mutually orthogonal vectors: the electric field vector, the magnetic induction vector and the temperature gradient. It agrees with experimental facts discovered by Ettingshausen and Nernst (the Ettingshausen effect and the Nernst–Ettingshausen effect). Certainly, in order to achieve quantitative agreement with the experimental data we need to generalize the theory. But the proposed model includes all the quantities mentioned when describing the essence of the Ettingshausen and Nernst–Ettingshausen effects. This gives us a reason to believe that by generalizing our theory, we will be able to model various thermoelectric, thermomagnetic and thermoelectromagnetic phenomena.

2 Some known facts from thermo- and electrodynamics

2.1 Thermodynamics of isotropic non-deformable media

The energy balance equation for heat-conducting non-deformable media is formulated as

$$\rho \frac{\mathrm{d}U_T}{\mathrm{d}t} = -\nabla \cdot \mathbf{h} + q, \tag{1}$$

where ρ is the density of mass, U_T is the specific thermal energy, $\dfrac{\mathrm{d}}{\mathrm{d}t}$ is the total time derivative, $\mathbf{h}$ is the heat flux vector, q is the heat supply per unit volume per unit time.

Since there are different views on the physical meaning of time derivatives and some confusion in notation, we believe that it is not superfluous to explain what we mean by the total time derivative. In order to give a correct definition of the total time derivative [57,58], we need the concept of the frame of reference. Let us imagine in a point O three rigidly connected, mutually orthogonal pointers ("arrows"), $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The set $\{O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called a "frame." The body of reference is defined by a frame to which a set of points (in space) have been added, whereby a rigid body motion of all the points together with the frame is allowed. The position of the points is labeled relatively to the frame by establishing the reference coordinate system x_1, x_2, x_3 with origin O: $\mathbf{r}_* = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, where $-\infty < (x_1, x_2, x_3) < +\infty$. The frame and the reference coordinate system determine the reference body. They are "immutable." In order to describe motion, we must be able to measure not only distance but also time. Hence, we need a "clock." The reference body with a "clock" is called the "Frame of Reference." Let $f(x_1, x_2, x_3, t)$ be a function of the reference coordinates and of time. By the definition, the total time derivative of f is

$$\frac{\mathrm{d}f(x_1, x_2, x_3, t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{f(x_1, x_2, x_3, t + \Delta t) - f(x_1, x_2, x_3, t)}{\Delta t}, \tag{2}$$

under the condition that the reference coordinates x_1, x_2, x_3 are held constant and there is an increment in the function only because of the increment in time. We note that in addition to the reference coordinate system one is free to choose any mathematical coordinate system in which the equations are specified. However, the reference coordinate system is a distinctive one since it determines the frame of reference. Let

$g(x(x_1, x_2, x_3, t), y(x_1, x_2, x_3, t), z(x_1, x_2, x_3, t), t)$ be a composite function of several variables, namely x, y, z. Then the total time derivative of g is

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial g}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial g}{\partial t}. \tag{3}$$

In accordance with Eq. (3), the total time derivative is the partial derivative with the reference coordinates held constant.

Now, we return to thermodynamics. In the books on non-equilibrium thermodynamics, e.g., [55,56,59,60] and also in some books on continuum mechanics, e.g., [61] one can find the entropy balance equation

$$\rho\frac{\mathrm{d}\Theta_a}{\mathrm{d}t} = -\nabla \cdot \mathbf{h}_\Theta + q_\Theta, \tag{4}$$

where Θ_a is specific entropy, $\mathbf{h}_\Theta$ is the entropy flux vector, q_Θ is the entropy production per unit volume per unit time. In contrast to the energy balance equation (1), the entropy balance equation (4) is valid for arbitrary media, not only for non-deformable ones.

We note that absolute temperature and specific entropy can be defined in different ways. Two approaches are used in continuum mechanics. One of them is based on the Clausius–Duhem inequality, see, e.g., [62,63]. The other approach [6,7,33,64] is close to the non-equilibrium thermodynamics approach. According to this approach, in the case of heat-conducting non-deformable media, absolute temperature and specific entropy are introduced as conjugate quantities by means of the equality

$$\rho T_a \frac{\mathrm{d}\Theta_a}{\mathrm{d}t} = -\nabla \cdot \mathbf{h} + q, \tag{5}$$

where T_a is absolute temperature. In view of Eq. (5), the energy balance equation takes the form of the so-called reduced energy balance equation

$$\rho\frac{\mathrm{d}U_T}{\mathrm{d}t} = \rho T_a \frac{\mathrm{d}\Theta_a}{\mathrm{d}t}. \tag{6}$$

From Eq. (6) it follows that

$$U_T = U_T(\Theta_a) \quad \Rightarrow \quad T_a = \frac{\partial U_T}{\partial \Theta_a}. \tag{7}$$

The second formula in (7) is well known. It is generally accepted in classical thermodynamics, statistical physics, continuum mechanics and non-equilibrium thermodynamics.

We note that Eq. (5) can be reduced to the form

$$\rho\frac{\mathrm{d}\Theta_a}{\mathrm{d}t} = -\nabla \cdot \left(\frac{\mathbf{h}}{T_a}\right) - \frac{\mathbf{h} \cdot \nabla T_a}{T_a^2} + \frac{q}{T_a}. \tag{8}$$

Comparing Eq. (4) and Eq. (8), we infer that

$$\mathbf{h}_\Theta = \frac{\mathbf{h}}{T_a}, \qquad q_\Theta = \frac{q}{T_a} - \frac{\mathbf{h} \cdot \nabla T_a}{T_a^2}. \tag{9}$$

In the linear approximation relations (9) take the form

$$\mathbf{h}_\Theta = \frac{\mathbf{h}}{T_a^*}, \qquad q_\Theta = \frac{q}{T_a^*}, \tag{10}$$

where T_a^* is a reference value of T_a. We emphasize that Eqs. (9) and (10) are valid only if we deal with non-deformable media.

Now, we discuss two heat conduction equations: the classical one and hyperbolic-type one. In order to derive these equations from the energy balance equation (1), it is necessary to set the specific thermal energy U_T and the heat flux vector $\mathbf{h}$.

If we deal with a linear theory, we set the specific thermal energy as

$$U_T = U_T^* + T_a^* \Theta_a + \frac{1}{2} \frac{T_a^*}{c_v} \Theta_a^2,$$

(11)

where U_T^* is a reference value of the specific thermal energy, c_v is the specific heat at constant volume. Inserting Eq. (11) into Eq. (7) yields the linear relation between absolute temperature and specific entropy

$$T_a = T_a^* + \frac{T_a^*}{c_v} \Theta_a.$$

(12)

From Eqs. (11), (12), it follows that

$$U_T = U_T^* + \frac{1}{2} \frac{c_v}{T_a^*} \left(T_a^2 - (T_a^*)^2 \right).$$

(13)

In the linear approximation, Eq. (13) takes the form

$$U_T \approx U_T^* + c_v \left(T_a - T_a^* \right).$$

(14)

This formula can be used when deriving the linear heat conduction equations.

Let us turn to the constitutive equations for the heat flux vector. Fourier's law is written as

$$\mathbf{h} = -\lambda \nabla T_a,$$

(15)

where λ is the thermal conductivity. The Maxwell–Cattaneo–Vernotte law has the form [65,66]

$$\tau_h \frac{d\mathbf{h}}{dt} + \mathbf{h} = -\lambda \nabla T_a,$$

(16)

where τ_h is the heat flux relaxation constant. According to the Maxwell–Cattaneo–Vernotte law, the heat flux does not appear or disappear instantly.

Substituting Eqs. (14), (15) in Eq. (1), we obtain the classical heat conduction equation

$$\Delta T_a - \frac{\rho c_v}{\lambda} \frac{dT_a}{dt} = -\frac{q}{\lambda}.$$

(17)

Inserting Eqs. (14), (16) into Eq. (1), we arrive at a hyperbolic-type heat conduction equation (the heat conduction equation containing the second time derivative of temperature)

$$\Delta T_a - \frac{\rho c_v}{\lambda} \frac{dT_a}{dt} - \frac{\tau_h \rho c_v}{\lambda} \frac{d^2 T_a}{dt^2} = -\frac{q}{\lambda} - \frac{\tau_h}{\lambda} \frac{dq}{dt}.$$

(18)

The classical heat conduction equation is in agreement with the second law of thermodynamics, whereas the hyperbolic-type heat conduction equation is in some contradiction with it. The problem of violation of the second law of thermodynamics is well known. There are many theoretical studies that express different opinions and propose different approaches to solve this problem (see, e.g., [67–76]).

Below, we compare the equations presented in this section with equations of our model in order to show that some equations describing our model can be interpreted as thermodynamic equations. We also use the equations presented in this section to express parameters of our model in terms of thermodynamic parameters.

2.2 Electrodynamics of isotropic conducting media

First of all, we formulate Maxwell's equations in the form most convenient for our purposes, namely

$$\nabla \times \mathcal{H} = \mathcal{J} + \frac{d\mathcal{D}}{dt}, \quad \nabla \times \mathcal{E} = -\frac{d\mathcal{B}}{dt}, \quad \nabla \cdot \mathcal{D} = \mathcal{Q}, \quad \nabla \cdot \mathcal{B} = 0. \tag{19}$$

Here $\mathcal{E}$ is the electric field vector, $\mathcal{H}$ is the magnetic field vector, $\mathcal{D}$ is the electric induction vector, $\mathcal{B}$ is the magnetic induction vector, $\mathcal{J}$ is the electric current density vector, $\mathcal{Q}$ is the electric charge density. The first equation in (19) is known as Maxwell's first equation, the second one is usually called Maxwell's second equation or the Maxwell–Faraday equation, the third and fourth equations are the Gauss laws for electric field and magnetic field, respectively. We note that, in most physics textbooks and monographs, Maxwell's equations are written in terms of partial time derivatives. However, there is a tendency in modern physics to rewrite Maxwell's equations using the total time derivatives instead of the partial time derivatives. For example, in the old editions of Pohl's book (see, e.g., [77]), all equations are written in terms of partial time derivatives, while in a new edition of this book (see [78]), the total time derivatives are used. Since we follow the concepts of continuum mechanics, we write Maxwell's equations using the total time derivatives (see [57,58]).

In the case of linear isotropic media, the constitutive equations take the form

$$\mathcal{D} = \varepsilon \varepsilon_0 \mathcal{E}, \quad \mathcal{B} = \mu \mu_0 \mathcal{H}, \tag{20}$$

where ε_0 and μ_0 are the permittivity and the permeability of vacuum, ε and μ are the relative permittivity and the relative permeability of a material. The charge conservation law

$$\frac{d\mathcal{Q}}{dt} + \nabla \cdot \mathcal{J} = 0 \tag{21}$$

follows from Eq. (19) and, in fact, is a condition for the solvability of Eq. (19).

Now, we turn to conducting media. In the case of isotropic conductors, Ohm's law

$$\mathcal{J} = \sigma \mathcal{E} \tag{22}$$

is usually used. The constant σ in Eq. (22) is called the electric conductivity. We note that Ohm's law application raises questions in a number of cases described in textbooks on physics, electrodynamics and electrical engineering (see, e.g., [79–81]). However, for metal conductors under normal conditions (at room temperature and atmospheric pressure), Ohm's law is fulfilled quite accurately. In view of Ohm's law (22), from Eqs. (19), (20) it follows that

$$\Delta \mathcal{B} - \sigma \mu \mu_0 \frac{d\mathcal{B}}{dt} - \varepsilon \varepsilon_0 \mu \mu_0 \frac{d^2\mathcal{B}}{dt^2} = 0, \quad \nabla \cdot \mathcal{B} = 0, \tag{23}$$

and

$$\Delta \mathcal{E} - \nabla(\nabla \cdot \mathcal{E}) - \sigma \mu \mu_0 \frac{d\mathcal{E}}{dt} - \varepsilon \varepsilon_0 \mu \mu_0 \frac{d^2\mathcal{E}}{dt^2} = 0, \quad \varepsilon \varepsilon_0 \frac{d(\nabla \cdot \mathcal{E})}{dt} + \sigma(\nabla \cdot \mathcal{E}) = 0. \tag{24}$$

Equations (23) and (24) are well known, see, e.g., [79–84]. If the charge density $\mathcal{Q}$ is assumed to be zero, Eq. (24) takes the form

$$\Delta \mathcal{E} - \sigma \mu \mu_0 \frac{d\mathcal{E}}{dt} - \varepsilon \varepsilon_0 \mu \mu_0 \frac{d^2\mathcal{E}}{dt^2} = 0, \quad \nabla \cdot \mathcal{E} = 0. \tag{25}$$

It is known that an alternating electric current is distributed within a conductor so that the current density is the largest near the surface and decreases with depth (see, e.g., [79]). This is called the skin effect. The skin depth is usually estimated by using the following equations:

$$\Delta \mathcal{E} - \sigma \mu \mu_0 \frac{d\mathcal{E}}{dt} = 0, \quad \nabla \cdot \mathcal{E} = 0, \tag{26}$$

which result from Eq. (25) if we neglect the term containing the second time derivative of $\mathcal{E}$ in the first equation. It is not difficult to show that we can use Eq. (26) instead of Eq. (25) if $f \ll \frac{\sigma}{\varepsilon \varepsilon_0}$, where f is a frequency. From Eq. (26) it follows that the skin depth of the conductor δ is determined as

$$\delta = \frac{1}{\sqrt{\sigma \mu \mu_0 \pi f}}. \tag{27}$$

Equation (27) is well known (see, e.g., [81, p. 537]).

2.3 A comparison of the equations describing the processes of thermal and electrical conductivities

As shown in the previous section, Maxwell's equations can be reduced to differential equation (25) analogous to the hyperbolic type heat conduction equation (18), and under the simplifying assumptions, Maxwell's equations can be reduced to differential equation (26) analogous to the classical heat conduction equation (17). It seems obvious that the differential equations describing the thermal conductivity process and the electrical conductivity process must be exactly the same. However, this is not quite true. Comparing the coefficients at the first time derivatives in these equations, we find one important difference. In heat conduction equations (17) and (18), the thermal conductivity λ is in the denominator of the coefficient at the first time derivative, whereas in equations of electrodynamics (25) and (26), the electrical conductivity σ is in the numerator of the coefficient at the first time derivative.

In 1853, Gustav Wiedemann and Rudolph Franz discovered that ratio λ/σ has approximately the same value for different metals at the same temperature. In 1872, Ludvig Lorenz has established the proportionality of λ/σ with absolute temperature:

$$\frac{\lambda}{\sigma} = LT_a, \tag{28}$$

where coefficient L is the Lorenz number. This coefficient is a constant, which is approximately the same for most metals. It is known that the Wiedemann–Franz law (28) is confirmed by experiments at high temperatures (above room temperature) and at very low temperatures (temperatures of a few kelvins), but it may not hold at intermediate temperatures. In other words, the Wiedemann–Franz law is fulfilled approximately.

In view of Eq. (28), the aforementioned difference between the heat conduction equations and the corresponding equations of electrodynamics looks very strange. Here, we refrain from further comments on this matter, since we discussed this issue in detail in [52].

2.4 Electrodynamics based on Kirchhoff's laws for electric circuits

An alternative approach based on Kirchhoff's laws for electric circuits is widely used for calculating of transmission lines (see books on electrical engineering, e.g., [81,85,86]). The comparison of the approach based on Kirchhoff's laws and the approach based on Maxwell's equations shows that they lead to different results. Let us discuss the situation in more detail. Kirchhoff's voltage law and Kirchhoff's current law have the form

$$\frac{\partial I}{\partial z} = -\left(\mathcal{G}V + \mathcal{C}\frac{\partial V}{\partial t} \right), \qquad \frac{\partial V}{\partial z} = -\left(\mathcal{R}I + \mathcal{L}\frac{\partial I}{\partial t} \right). \tag{29}$$

Here z is the spatial coordinate, I is the electric current, V is the electric voltage, $\mathcal{L}$ is the inductance, $\mathcal{C}$ is the capacitance, $\mathcal{G}$ is the shunt conductance, $\mathcal{R}$ is the series resistance. These parameters are specified per unit length. For all transmission lines, the following relations hold:

$$\mathcal{LC} = \varepsilon\varepsilon_0\mu\mu_0, \qquad \frac{\mathcal{G}}{\mathcal{C}} = \frac{\sigma_d}{\varepsilon\varepsilon_0}, \qquad \frac{\mathcal{R}}{\mathcal{L}} = \frac{1}{\sigma_c\mu\mu_0 l\delta}, \tag{30}$$

where σ_d and σ_c are two different electrical conductivities, l is a parameter, which depends on the geometrical properties of the transmission line, δ is the skin depth defined by Eq. (27).

We note that two quantities, I and $\mathcal{J}$, characterizing the electric current in electric circuits and in three-dimensional media, respectively, differ from each other by that: first, I is the electric current itself, whereas $\mathcal{J}$ is the electric current density; second, the current direction in three-dimensional media can be an arbitrary and therefore $\mathcal{J}$ is a vector, whereas the current direction in electric circuits is completely determined by the direction of the wires and therefore I is a scalar.

Eliminating I from system (29) and taking into account Eq. (30), we obtain the telegraph equation in V:

$$\frac{\partial^2 V}{\partial z^2} - \frac{\sigma_d}{\sigma_c\, l\delta}\, V - \left(\sigma_d\, \mu\mu_0 + \frac{\varepsilon\varepsilon_0}{\sigma_c\, l\delta} \right) \frac{\partial V}{\partial t} - \varepsilon\varepsilon_0\mu\mu_0\, \frac{\partial^2 V}{\partial t^2} = 0. \tag{31}$$

Eliminating V from system (29) and taking into account Eq. (30), we arrive at the telegraph equation in I:

$$\frac{\partial^2 I}{\partial z^2} - \frac{\sigma_d}{\sigma_c\, l\delta}\, I - \left(\sigma_d\,\mu\mu_0 + \frac{\varepsilon\varepsilon_0}{\sigma_c\, l\delta}\right)\frac{\partial I}{\partial t} - \varepsilon\varepsilon_0\mu\mu_0\,\frac{\partial^2 I}{\partial t^2} = 0. \tag{32}$$

It is easy to see that the last two equations have the same structure and the same coefficients. Thus, we have obtained the telegraph equation, which can be written in terms of voltage V as Eq. (31) or in terms of current I as Eq. (32).

2.5 A modified Maxwell's electrodynamics consistent with Kirchhoff's laws

Comparing Eqs. (31), (32) obtained on the basis of Kirchhoff's laws with Eqs. (23), (25) obtained in the framework of Maxwell's electrodynamics, we see that they differ from each other. In [52], we have proposed the linear theory of electromagnetism, which can be considered as a modified Maxwell's electrodynamics consistent with Kirchhoff's laws. In this section we provide some results obtained in [52]. Below, we use these results for determination of a number of parameters of the model proposed in the present paper. A comparison of the theory constructed in [52] and the theory constructed in the present paper can be found in Appendix.

In the framework of the model proposed in [52], we have obtained the telegraph equation for the magnetic induction vector

$$\Delta\mathcal{B} - \frac{\sigma_d}{\sigma_c(L\chi^2/a^2)}\,\mathcal{B} - \left(\sigma_d\,\mu\mu_0 + \frac{\varepsilon\varepsilon_0}{\sigma_c(L\chi^2/a^2)}\right)\frac{d\mathcal{B}}{dt} - \varepsilon\varepsilon_0\mu\mu_0\frac{d^2\mathcal{B}}{dt^2} = 0, \qquad \nabla\cdot\mathcal{B} = 0. \tag{33}$$

and the telegraph equation for the electric field vector

$$\Delta\mathcal{E} - \frac{\sigma_d}{\sigma_c(L\chi^2/a^2)}\,\mathcal{E} - \left(\sigma_d\,\mu\mu_0 + \frac{\varepsilon\varepsilon_0}{\sigma_c(L\chi^2/a^2)}\right)\frac{d\mathcal{E}}{dt} - \varepsilon\varepsilon_0\mu\mu_0\frac{d^2\mathcal{E}}{dt^2} = 0. \qquad \nabla\cdot\mathcal{E} = 0. \tag{34}$$

Here χ and a are the constants that do not depend on the properties of a specific material. We discuss the physical meaning of these constants in Sect. 4.1. Constant $L\chi^2/a^2$ in Eqs. (33) and (34) corresponds to constant $l\delta$ in Eqs. (31) and (32). Electrical conductivities σ_d and σ_c in Eqs. (33) and (34) correspond in some sense to the electrical conductivities σ_d and σ_c in Eqs. (31) and (32). Thus, in the framework of the model proposed in [52], electrical conductivity of a material is characterized by two different parameters, σ_d and σ_c. In the general case, both these electrical conductivities can differ from the electrical conductivity σ presented in physics handbooks.

In order to clarify the physical meaning of σ_d and σ_c, we draw attention to the modified formulation of Ohm's law and the modified formulation of the Wiedemann–Franz law, which have been suggested in [52]:

$$\mathcal{J} = \sigma_d\,\mathcal{E}, \qquad \frac{\lambda}{\sigma_c} = L T_a^*. \tag{35}$$

The first equation in (35) represents the modified form of Ohm's law, which differs from Ohm's law (22) by that coefficient σ is replaced by coefficient σ_d. The second equation in (35) represents the modified form of the Wiedemann–Franz law, which differs from of the Wiedemann–Franz law (28) by that coefficient σ is replaced by coefficient σ_c and the current temperature T_a is replaced by the reference temperature T_a^*. The replacement of T_a by T_a^* in the Wiedemann–Franz law is justified by that we consider the linear theory. We note that Eq. (35) can be used for the experimental determination of the electrical conductivities σ_d and σ_c. In the area of parameters, where both Ohm's law (22) and the Wiedemann–Franz law (28) are fulfilled exactly, the found values of σ_d and σ_c will coincide with each other and with the value of σ presented in physics handbooks. If either Ohm's law (22) or the Wiedemann–Franz law (28) does not hold exactly, then the found values of σ_d and σ_c will differ from each other.

According to Eqs. (33), (34), the well-known formula (27) for the skin depth δ takes the form

$$\delta = \frac{1}{\sqrt{\left(\sigma_d\,\mu\mu_0 + \dfrac{\varepsilon\varepsilon_0}{\sigma_c(L\chi^2/a^2)}\right)\pi f}}. \tag{36}$$

The skin depth δ can be found experimentally. If we know δ and use Eq. (35) to determine σ_d and σ_c, we can find constant $L\chi^2/a^2$ by means of Eq. (36). If it turns out that constant $L\chi^2/a^2$ is the same for various materials, this will be an argument in favor of Eqs. (33), (34).

In the static approximation, Eqs. (33), (34) take the form

$$\Delta \mathcal{B} - \frac{1}{\delta_*^2}\,\mathcal{B} = 0, \qquad \Delta \mathcal{E} - \frac{1}{\delta_*^2}\,\mathcal{E} = 0, \tag{37}$$

where δ_* is calculated by the formula

$$\delta_* = \sqrt{\frac{\sigma_c (L\chi^2/a^2)}{\sigma_d}}\,. \tag{38}$$

If δ_* is small enough, the solutions of equations (37) are functions of the boundary layer type. They decrease rapidly with distance from the boundary of the region and are almost equal to zero inside the region. In this case, we deal with the so-called static skin effect and δ_* is called the static skin depth. We emphasize that the static skin effect is not described by Maxwell's electrodynamics. At the same time, the static skin effect was discovered a long time ago [87]; the experimental studies have been performed at different times by many authors [88–91] and continue up to the present [92,93]. The generally accepted explanation of the static skin effect is based on the use of concepts of bulk conductivity and surface conductivity. Because of this, the experimental studies provide data on the bulk and surface conductivities instead of data on the skin depth δ_*. However, we believe that the experimental data on the static skin effect can also be interpreted in terms of the static skin depth δ_*. Equation (38), as well as Eq. (36), can be used for determination of constant $L\chi^2/a^2$.

We are convinced that telegraph equations (33), (34) can be useful for describing electromagnetic processes in conducting media. We give two arguments in favor of the latter statement. The first argument: telegraph equations (33), (34) are the three-dimensional analogues of the one-dimensional telegraph equations (31), (32), which are derived from two Kirchhoff's laws. The second argument: telegraph equations (33), (34) allow us to describe not only the well-known skin effect (see Eq. (36)), but also the static skin effect (see Eq. (38)).

3 A linear theory of the Cosserat continuum of a special type

3.1 Preliminary remarks

Below we present the basic equations of the Cosserat continuum of a special type, which we are going to use for modeling thermodynamic and electrodynamic processes. We start with the well-known linear theory of the isotropic elastic Cosserat continuum. More precisely, we give the equations describing a special case of the Cosserat continuum, which possesses only rotational degrees of freedom. Then we modify this theory by adding a source term to the equation relating the wryness tensor and the angular velocity vector. In fact, we replace the well-known geometrical definition of the wryness tensor by a new definition, which has a kinematic meaning rather than a geometrical one. Thus, we arrive at the equations for the Cosserat continuum, which is something more complex than just the elastic Cosserat continuum. After that, we formulate a number of simplifying assumptions regarding constitutive equations. Finally, we write down a summary of the equations, to which we are going to give the meaning of analogues of the basic equations of electrodynamics and thermodynamics.

3.2 The linear theory of the elastic Cosserat continuum

Now, we consider the isotropic Cosserat continuum possessing only rotational degrees of freedom. Let vector $\mathbf{r}$ identify the position of some point in space. We use the following notation: $\boldsymbol{\theta}(\mathbf{r}, t)$ is the rotation vector field; $\boldsymbol{\omega}(\mathbf{r}, t)$ is the angular velocity vector field; $\rho(\mathbf{r})$ is the mass density of the continuum at a given point in space; $\mathbf{J}(\mathbf{r})$ is the inertia tensor per unit mass. Following [94–104], we define tensor $\mathbf{J}(\mathbf{r})$ as $\mathbf{J}(\mathbf{r}) = \left(\sum_{k=1}^{N} \hat{\boldsymbol{J}}_k\right) \Big/ \left(\sum_{k=1}^{N} m_k\right)$, where $\hat{\boldsymbol{J}}_k$ are the inertia tensors of particles with masses m_k, which occupy a fixed elementary volume V located in space near the point identified by the position vector $\mathbf{r}$. Since we consider the isotropic continuum, the tensor $\mathbf{J}$ is proportional to the second rank identity tensor $\mathbf{E}$:

$$\mathbf{J}(\mathbf{r}) = J(\mathbf{r})\mathbf{E}, \tag{39}$$

where $J(\mathbf{r})$ is the moment of inertia per unit mass. In the linear approximation, the kinematic relation has the form

$$\boldsymbol{\omega} = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}. \tag{40}$$

The specific kinetic energy $\mathcal{T}$ and the specific angular momentum vector $\mathcal{K}$ are

$$\mathcal{T} = \frac{1}{2} J \, \boldsymbol{\omega} \cdot \boldsymbol{\omega}, \qquad \mathcal{K} = J \, \boldsymbol{\omega}. \tag{41}$$

Now, we introduce the moment stress vector $\mathbf{T}_n$ modeling the surrounding medium action on surface S of elementary volume V. By standard reasoning, we introduce the concept of the moment stress tensor $\mathbf{T}$ associated with the moment stress vector $\mathbf{T}_n$. This tensor is defined by the relation $\mathbf{T}_n = \mathbf{n} \cdot \mathbf{T}$, where $\mathbf{n}$ denotes the unit outer normal vector to surface S. Now we can write the angular momentum balance equation as

$$\nabla \cdot \mathbf{T} + \rho \mathbf{L} = \frac{\mathrm{d}(\rho J \boldsymbol{\omega})}{\mathrm{d}t}, \tag{42}$$

where $\mathbf{L}$ is the external moment per unit mass.

Under the assumption that the energy supply from external sources is absent, the energy balance equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho(\mathcal{T} + \mathcal{U}) \, \mathrm{d}V = \int_S \mathbf{T}_n \cdot \boldsymbol{\omega} \, \mathrm{d}S + \int_V \rho \mathbf{L} \cdot \boldsymbol{\omega} \, \mathrm{d}V, \tag{43}$$

where $\mathcal{U}$ is the specific internal energy. By standard reasoning, Eq. (43) can be reduced to the local form

$$\rho \frac{\mathrm{d}\mathcal{T}}{\mathrm{d}t} + \rho \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = \nabla \cdot (\mathbf{T} \cdot \boldsymbol{\omega}) + \rho \mathbf{L} \cdot \boldsymbol{\omega}. \tag{44}$$

Taking into account Eqs. (41), (42), we can eliminate kinetic energy and the power of external moment from Eq. (44). These transformations result in the equation

$$\rho \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = \mathbf{T}^T \cdot\cdot \nabla \boldsymbol{\omega}, \tag{45}$$

where, for arbitrary dyads, $(\mathbf{ab})^T = \mathbf{ba}$ and the double contraction is defined as $\mathbf{ab} \cdot\cdot \mathbf{cd} = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$. Let us introduce the notation

$$\boldsymbol{\Theta} = \nabla \boldsymbol{\theta}. \tag{46}$$

In view of the fact that in the spatial description $\frac{\mathrm{d}}{\mathrm{d}t} \nabla = \nabla \frac{\mathrm{d}}{\mathrm{d}t}$ (see [57,58]), from Eqs. (40), (46) it follows that

$$\frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}t} = \nabla \boldsymbol{\omega}. \tag{47}$$

Inserting Eq. (47) into Eq. (45), we obtain

$$\rho \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = \mathbf{T}^T \cdot\cdot \frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}t}, \tag{48}$$

where tensor $\boldsymbol{\Theta}$, defined by Eq. (46) or by Eq. (47), is the strain tensor. This strain tensor is associated with rotational degrees of freedom. It is called the wryness tensor. If the continuum is assumed to be elastic, i.e., $\mathcal{U} = \mathcal{U}(\boldsymbol{\Theta})$ and $\mathbf{T} = \mathbf{T}(\boldsymbol{\Theta})$ then from Eq. (48) it follows the Cauchy–Green relation

$$\mathbf{T} = \rho \frac{\partial \mathcal{U}}{\partial \boldsymbol{\Theta}}. \tag{49}$$

In the linear theory, the specific internal energy is specified as

$$\rho \mathcal{U} = \mathbf{T}_*^T \cdot\cdot \boldsymbol{\Theta} + \frac{1}{2} \boldsymbol{\Theta} \cdot\cdot \mathbf{C} \cdot\cdot \boldsymbol{\Theta}, \tag{50}$$

where $\mathbf{T}_*$ is the so-called initial moment stress tensor, $\mathbf{C}$ is the fourth rank stiffness tensor. Inserting Eq. (50) into Eq. (49), we arrive at the constitutive equation

$$\mathbf{T}^T = \mathbf{T}_*^T + \mathbf{C} \cdot\cdot \boldsymbol{\Theta}. \tag{51}$$

Thus, the linear theory of the considered continuum is given by Eqs. (42), (47), (51).

3.3 A modified linear theory of the elastic Cosserat continuum

The system of equations describing the deformation of solids consists of the equations of motion, the constitutive equations and the equations for the strain tensors. All material parameters are contained in the constitutive equations. What the equations of motion and the equations for the strain tensors have in common is that they are formulated in the same way for any materials. However, in contrast to the equations of motion, which can be written in both differential form and integral form, the equations for the strain tensors are always formulated in differential form. In [54], we have proposed a new approach to the definition of the strain tensor for the classical continuum, possessing only translational degrees of freedom. Now, we use the ideas of [54] in relation to the Cosserat continuum, which possesses only rotational degrees of freedom.

We start with Eq. (47) directly relating the angular velocity vector and the wryness tensor. We transform this equation to integral form. In order to do so, we rewrite Eq. (47) as

$$\frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}t} = -\nabla \cdot \mathbf{J}_{\Theta}, \tag{52}$$

where the third rank tensor $\mathbf{J}_{\Theta}$ has the form

$$\mathbf{J}_{\Theta} = -\mathbf{E}\,\boldsymbol{\omega}. \tag{53}$$

Integrating Eq. (52) over the fixed volume V and using the divergence theorem yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{\Theta}\,\mathrm{d}V = -\int_S \mathbf{n}\cdot\mathbf{J}_{\Theta}\,\mathrm{d}S. \tag{54}$$

It is evident that Eq. (54) has a balance equation structure and the tensor $\mathbf{J}_{\Theta}$ plays the role of the flux of the wryness tensor. We note that Eq. (54) is similar to that we have obtained in [54] for the stretch tensor, i.e., the strain tensor associated with translational degrees of freedom. In [54], we called the obtained equation the strain balance equation. Now, since we discuss two different strain tensors, in order to avoid confusion, we rename the equation obtained in [54] to the stretch tensor balance equation, and we call Eq. (54) the wryness tensor balance equation.

It is well known that, in the general case, the balance equations contain source terms. That is why, it seems logical to us to add a source term to the balance equations for the strain tensors. In [54], we have changed the concept of the stretch tensor by adding a source term to the stretch tensor balance equation. As a result, we have modified the basic equations of the classical continuum. Following ideas of [54], we add a source term $\boldsymbol{\Upsilon}_{\Theta}$ to Eq. (54). In this case, Eq. (54) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{\Theta}\,\mathrm{d}V = -\int_S \mathbf{n}\cdot\mathbf{J}_{\Theta}\,\mathrm{d}S + \int_V \boldsymbol{\Upsilon}_{\Theta}\,\mathrm{d}V. \tag{55}$$

In view of Eq. (53), the local form of Eq. (55) is written as

$$\frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}t} = \nabla\boldsymbol{\omega} + \boldsymbol{\Upsilon}_{\Theta}. \tag{56}$$

It is easy to see that Eq. (56) is a generalization of Eq. (47). We emphasize that Eqs. (47)–(56) are valid only in the case of the linear theory which, in addition, ignores translational degrees of freedom.

Now, we make some remarks regarding the physical meaning of the source terms in the stretch tensor balance equation and the wryness tensor balance equation. As stated in [54], the source terms can be used, firstly, for modeling chemical reactions which result in changes in mechanical states and mechanical properties of solids, and secondly, for describing phase transitions and structural changes that occur both with a change in mass and without a change in mass. They also provide additional opportunities to take into account the interrelation of thermal and mechanical processes.

We pay attention to an important circumstance. If we replace Eq. (47) by Eq. (56), then there are two ways to develop the theory: either we must reject kinematic relation (40) or we must reject geometric relation (46). A discussion of these two ways is beyond the scope of the present study. Here we choose the second way and reject Eq. (46). In this case, we should consider Eq. (56) as a definition of the wryness tensor $\boldsymbol{\Theta}$ and Eq. (40)

as a definition of the rotation vector $\boldsymbol{\theta}$. With this approach, the angular velocity vector $\boldsymbol{\omega}$ plays the role of the main variable.

Let us turn to the energy balance equation. The formulations of the energy balance given by Eqs. (43), (44) and (45) remain valid since these equations do not depend on the wryness tensor. Equation (48) should be modified as

$$\rho \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = \mathbf{T}^T \cdot\cdot \frac{\mathrm{d}\boldsymbol{\Theta}}{\mathrm{d}t} - \mathbf{T}^T \cdot\cdot \boldsymbol{\Upsilon}_\Theta. \tag{57}$$

It is evident that Eq. (57) coincides with Eq. (48) only if we assume that

$$\mathbf{T}^T \cdot\cdot \boldsymbol{\Upsilon}_\Theta = 0. \tag{58}$$

In this special case, the above reasoning regarding the constitutive equations remains valid. Namely, if the continuum is elastic, i.e., $\mathcal{U} = \mathcal{U}(\boldsymbol{\Theta})$ and $\mathbf{T} = \mathbf{T}(\boldsymbol{\Theta})$ then the Cauchy–Green relation takes the form of Eq. (49). If in addition, we assume the specific internal energy to be quadratic form (50), we arrive at the constitutive equation (51). Below we consider only the special case determined by Eq. (58).

3.4 Simplifying assumptions regarding the constitutive equations

Now, we are going to make a few simplifying assumptions about the Cosserat continuum presented above. It allows us to construct a model of the Cosserat continuum of a special type. In Sect. 4, we give a physical interpretation of this model by introducing thermodynamic and electrodynamic analogues of mechanical quantities.

Hypothesis 1 *The moment stress tensor* $\mathbf{T}$ *has the following structure:*

$$\mathbf{T} = T\mathbf{E} - \mathbf{M} \times \mathbf{E}, \tag{59}$$

where the scalar quantity T *characterizes the spherical part of tensor* $\mathbf{T}$ *and the vector quantity* $\mathbf{M}$ *characterizes the antisymmetric part of tensor* $\mathbf{T}$.

Assuming this structure of the moment stress tensor, we follow the ideas of our previous works (see [42,43,47,49,52]). The physical meaning of representation (59) will be clarified below.

Hypothesis 2 *The source term in the balance equation for the wryness tensor* $\boldsymbol{\Theta}$ *has the following structure:*

$$\boldsymbol{\Upsilon}_\Theta = \frac{1}{3} \Upsilon_\Theta \mathbf{E} - \frac{1}{2} \boldsymbol{\Upsilon}_\Psi \times \mathbf{E}, \tag{60}$$

where the scalar quantity Υ_Θ *characterizes the spherical part of tensor* $\boldsymbol{\Upsilon}_\Theta$ *and the vector quantity* $\boldsymbol{\Upsilon}_\Psi$ *characterizes the antisymmetric part of tensor* $\boldsymbol{\Upsilon}_\Theta$.

It is easy to see that we choose the structure of the source term in the wryness tensor balance equation (56) by the analogy with the structure of the moment stress tensor.

Hypothesis 3 *The spherical part of the source term* $\boldsymbol{\Upsilon}_\Theta$ *is related to its antisymmetric part as*

$$\Upsilon_\Theta = -\frac{\mathbf{M} \cdot \boldsymbol{\Upsilon}_\Psi}{T}. \tag{61}$$

Here the vector quantity $\boldsymbol{\Upsilon}_\Psi$ *should be determined by a constitutive equation and after that Eq. (61) can be used to calculate the scalar quantity* Υ_Θ.

We note that Eq. (61) has the same meaning as Eq. (58). Therefore, from the above hypotheses it follows that the energy balance equation (57) takes the form

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = T \frac{\mathrm{d}\Theta_\rho}{\mathrm{d}t} + \mathbf{M} \cdot \frac{\mathrm{d}\boldsymbol{\Psi}_\rho}{\mathrm{d}t}, \tag{62}$$

where Θ_ρ and $\boldsymbol{\Psi}_\rho$ are

$$\Theta_\rho = \frac{\Theta}{\rho}, \qquad \Theta = \operatorname{tr} \boldsymbol{\Theta}, \qquad \boldsymbol{\Psi}_\rho = \frac{\boldsymbol{\Psi}}{\rho}, \qquad \boldsymbol{\Psi} = \boldsymbol{\Theta}_\times. \tag{63}$$

Here $(\;)_\times$ denotes the vector invariant of a tensor that is defined for an arbitrary dyad as $(\mathbf{ab})_\times = \mathbf{a} \times \mathbf{b}$.

If we suppose that the specific internal energy depends only on Θ_ρ and $\boldsymbol{\Psi}_\rho$ and does not depend on their time derivatives, we can obtain the Cauchy–Green relations by standard reasoning. Then, setting the specific internal energy as a function of its arguments and inserting it into the Cauchy–Green relations, we obtain the constitutive equations for scalar T and vector $\mathbf{M}$ determining the moment stress tensor $\mathbf{T}$.

Hypothesis 4 *The specific internal energy has the following form:*

$$\mathcal{U} = T_* \Theta_\rho + \frac{1}{2} C_\Theta \Theta_\rho^2 + \frac{1}{2} C_\Psi \boldsymbol{\Psi}_\rho \cdot \boldsymbol{\Psi}_\rho, \tag{64}$$

where T_* is the reference value of T, constants C_Θ and C_Ψ are the stiffness parameters.

Inserting Eqs. (63), (64) into Eq. (62) yields the following constitutive equations:

$$T = T_* + C_\Theta \Theta_\rho, \qquad \mathbf{M} = C_\Psi \boldsymbol{\Psi}_\rho. \tag{65}$$

We note that, despite the fact that constitutive equations (65) have the form of elasticity relations, the considered continuum may possess more complex rheological properties. The reason is that differential equation (56) for the wryness tensor $\boldsymbol{\Theta}$ contains the source term $\boldsymbol{\Upsilon}_\Theta$. Using different constitutive equations for this source term, we can obtain continuum models with different rheological properties.

Hypothesis 5 *Vector $\boldsymbol{\Upsilon}_\Psi$ characterizing the antisymmetric part of the source term $\boldsymbol{\Upsilon}_\Theta$ is proportional to a vector $\mathbf{M}$ characterizing the antisymmetric part of the moment stress tensor:*

$$\boldsymbol{\Upsilon}_\Psi = -\kappa \mathbf{M}, \tag{66}$$

where parameter κ is assumed to be constant.

Adopting constitutive equation (66), we arrive at the model that possesses some properties of a viscoelastic material. This model is in some sense similar to Maxwell's model of a viscoelastic material, which was used in [52].

Hypothesis 6 *The external moment $\rho\mathbf{L}$ is the moment of linear viscous damping proportional to the proper angular momentum:*

$$\rho\mathbf{L} = -\beta \rho J \boldsymbol{\omega}. \tag{67}$$

Here β is the coefficient of damping. In a linear theory this quantity is assumed to be constant.

In fact, the external moment $\rho\mathbf{L}$ models the dissipation of energy of the continuum. This dissipation is caused by the interaction of the considered continuum with some other continuum, which is ignored in the model. The structure of moment $\rho\mathbf{L}$ is chosen in accordance with the results obtained by solving two model problems (see [40,41]).

3.5 The system of the basic equations

Here, we present the system of the basic equations that we are going to discuss in the next sections. This system can be divided into two independent systems. The first system includes as follows: five constitutive equations, the relation between the spherical part of the source term and its antisymmetric part, trace and vector invariant of the relation between the wryness tensor and the angular velocity, and also the angular momentum balance equation. The second system consists of the deviatoric part of the relation between the wryness tensor and the angular velocity.

Let us start with the first system. The constitutive equations are

$$T = T_* + C_\Theta \Theta_\rho, \qquad \mathbf{M} = \frac{C_\Psi}{\rho} \boldsymbol{\Psi}, \qquad \mathcal{K} = \rho J \boldsymbol{\omega}, \qquad \boldsymbol{\Upsilon}_\Psi = -\kappa \mathbf{M}, \qquad \rho\mathbf{L} = -\beta \rho J \boldsymbol{\omega}, \tag{68}$$

where $\mathcal{K}$ is the proper angular momentum per unit volume. The relation between the spherical part of the source term and its antisymmetric part is

$$\varUpsilon_\Theta = -\frac{\mathbf{M} \cdot \boldsymbol{\varUpsilon}_\Psi}{T} \tag{69}$$

The trace and the vector invariant of the relation between the wryness tensor and the angular velocity are

$$\rho \frac{\mathrm{d}\Theta_\rho}{\mathrm{d}t} = \nabla \cdot \boldsymbol{\omega} + \varUpsilon_\Theta, \qquad \nabla \times \boldsymbol{\omega} = -\boldsymbol{\varUpsilon}_\Psi + \frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}t}. \tag{70}$$

The angular momentum balance equation rewritten by using the proper angular momentum $\mathcal{K}$ and also scalar T and vector $\mathbf{M}$ determining the moment stress tensor takes the form

$$\nabla T - \nabla \times \mathbf{M} = -\rho \mathbf{L} + \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t}. \tag{71}$$

The second system of equations allows us to find deviator of the symmetric part of tensor $\boldsymbol{\Theta}$ when vector $\boldsymbol{\omega}$ is already known. This system of equations can be represented in tensor form as

$$\frac{\mathrm{d}(\mathrm{dev}\,\boldsymbol{\Theta}^s)}{\mathrm{d}t} = \mathrm{dev}\,(\nabla \boldsymbol{\omega})^s, \tag{72}$$

where the index s denotes the symmetric part of a tensor. It is easy to see that the first and the second systems are independent. Indeed, one can solve the first system and find vector $\boldsymbol{\omega}$. After that, one can insert vector $\boldsymbol{\omega}$ to the second system and find dev $\boldsymbol{\Theta}^s$. Below we discuss only the first system because, in the present paper, we do not give any physical interpretation of dev $\boldsymbol{\Theta}^s$. We note that, considering nonlinear theories, where the system of the basic equations cannot be divided into two independent systems, we suggest some physical interpretations of quantities that contain dev $\boldsymbol{\Theta}^s$ (see [50,51,53]).

3.6 Some consequences of the basic equations

Let us consider some consequences of Eqs. (68)–(71). First of all, we note that the potential part of the proper angular momentum $\mathcal{K}$ is related to the time derivative of the spherical part T of the moment stress tensor and the source term $\varUpsilon_\Theta$ in the strain balance equation. Indeed, from the first and third equations in (68) and the first equation in (70), it follows that

$$\nabla \cdot \mathcal{K} = \frac{\rho^2 J}{C_\Theta} \frac{\mathrm{d}T}{\mathrm{d}t} - \rho J\, \varUpsilon_\Theta. \tag{73}$$

Next, we turn to the system of the second equation in (70) and Eq. (71). In view of Eq. (68), these equations can be rewritten as

$$\nabla \times \mathcal{K} = \kappa \rho J\, \mathbf{M} + \frac{\rho^2 J}{C_\Psi} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}, \qquad \nabla T - \nabla \times \mathbf{M} = \beta \mathcal{K} + \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t}. \tag{74}$$

Taking the divergence of the second equation in (74) and eliminating $\nabla \cdot \mathcal{K}$ in view of Eq. (73), we arrive at the equation for the spherical part of the moment stress tensor:

$$\Delta T = \frac{\beta \rho^2 J}{C_\Theta} \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\rho^2 J}{C_\Theta} \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} - \beta \rho J\, \varUpsilon_\Theta - \rho J \frac{\mathrm{d}\varUpsilon_\Theta}{\mathrm{d}t}, \tag{75}$$

where $\varUpsilon_\Theta$ (the source term in the wryness tensor balance equation) plays the role of an external factor. From Eqs. (73), (75) it follows that

$$\Delta(\nabla \cdot \mathcal{K}) = \frac{\beta \rho^2 J}{C_\Theta} \frac{\mathrm{d}(\nabla \cdot \mathcal{K})}{\mathrm{d}t} + \frac{\rho^2 J}{C_\Theta} \frac{\mathrm{d}^2(\nabla \cdot \mathcal{K})}{\mathrm{d}t^2} - \rho J\, \Delta\varUpsilon_\Theta. \tag{76}$$

It is easy to see that Eqs. (75) and (76) differ from each other only by the terms containing the external factor Υ_Θ. Inserting the third equation in (68) into Eq. (76), we obtain the equation for the potential part of the angular velocity $\boldsymbol{\omega}$:

$$\Delta(\nabla \cdot \boldsymbol{\omega}) = \frac{\beta \rho^2 J}{C_\Theta} \frac{d(\nabla \cdot \boldsymbol{\omega})}{dt} + \frac{\rho^2 J}{C_\Theta} \frac{d^2(\nabla \cdot \boldsymbol{\omega})}{dt^2} - \Delta\Upsilon_\Theta. \tag{77}$$

Inserting the last equation in (68) into Eq. (77), we arrive at the equation for the potential part of the external moment $\rho\mathbf{L}$:

$$\Delta[\nabla \cdot (\rho\mathbf{L})] = \frac{\beta \rho^2 J}{C_\Theta} \frac{d[\nabla \cdot (\rho\mathbf{L})]}{dt} + \frac{\rho^2 J}{C_\Theta} \frac{d^2[\nabla \cdot (\rho\mathbf{L})]}{dt^2} + \beta\rho J \, \Delta\Upsilon_\Theta. \tag{78}$$

Equations (77) and (78) differ from each other and from Eq. (76) only by the coefficients at $\Delta\Upsilon_\Theta$.

Taking the divergence of the first equation in (74), we arrive at the following equation for the potential part of vector $\mathbf{M}$, characterizing the antisymmetric part of the moment stress tensor:

$$\frac{d(\nabla \cdot \mathbf{M})}{dt} + \frac{\kappa C_\Psi}{\rho} \nabla \cdot \mathbf{M} = 0. \tag{79}$$

Solving Eq. (79) for unknown $\nabla \cdot \mathbf{M}$, we have

$$\nabla \cdot \mathbf{M} = \nabla \cdot \mathbf{M}\big|_{t=0} \exp\left(-\frac{\kappa C_\Psi}{\rho} t\right). \tag{80}$$

From Eq. (80) it follows that if $\nabla \cdot \mathbf{M}\big|_{t=0} = 0$ then $\nabla \cdot \mathbf{M}\big|_{t=0}$ at any time point, and if $\nabla \cdot \mathbf{M}\big|_{t=0} \neq 0$ then $\nabla \cdot \mathbf{M}$ tends to zero as $t \to \infty$.

Let us return to the system of equations (74). Eliminating vector $\mathcal{K}$ from Eq. (74), we obtain

$$\Delta\mathbf{M} - \nabla(\nabla \cdot \mathbf{M}) = \beta\kappa\rho J \, \mathbf{M} + \left(\kappa\rho J + \frac{\beta\rho^2 J}{C_\Psi}\right) \frac{d\mathbf{M}}{dt} + \frac{\rho^2 J}{C_\Psi} \frac{d^2\mathbf{M}}{dt^2}. \tag{81}$$

Taking the divergence of Eq. (81) with regard to Eq. (79), we arrive at the identity. This means that Eq. (81), in fact, determines the vortex part of vector $\mathbf{M}$, whereas the potential part of vector $\mathbf{M}$ is determined by Eq. (79). Taking the curl of Eq. (81), we obtain the following form of the equation for the vortex part of vector $\mathbf{M}$:

$$\Delta(\nabla \times \mathbf{M}) = \beta\kappa\rho J \, (\nabla \times \mathbf{M}) + \left(\kappa\rho J + \frac{\beta\rho^2 J}{C_\Psi}\right) \frac{d(\nabla \times \mathbf{M})}{dt} + \frac{\rho^2 J}{C_\Psi} \frac{d^2(\nabla \times \mathbf{M})}{dt^2}. \tag{82}$$

The vectors $\boldsymbol{\Psi}$ and $\boldsymbol{\Upsilon}_\Psi$, characterizing the antisymmetric parts of the wryness tensor and the source term in the wryness tensor balance equation, respectively, are related to the vector $\mathbf{M}$ by linear algebraic equations, namely the second and the fourth equations in (68). Therefore, these vectors possess the same properties as the vector $\mathbf{M}$ and satisfy equations that coincide with Eqs. (79), (80), (81), (82).

Next, eliminating vector $\mathbf{M}$ from Eq. (74), we arrive at the following equation for the proper angular momentum $\mathcal{K}$:

$$\Delta\mathcal{K} - \nabla(\nabla \cdot \mathcal{K}) = \beta\kappa\rho J \, \mathcal{K} + \left(\kappa\rho J + \frac{\beta\rho^2 J}{C_\Psi}\right) \frac{d\mathcal{K}}{dt} + \frac{\rho^2 J}{C_\Psi} \frac{d^2\mathcal{K}}{dt^2} - \kappa\rho J \, \nabla T - \frac{\rho^2 J}{C_\Psi} \frac{d(\nabla T)}{dt}. \tag{83}$$

Taking the divergence of Eq. (83) yields

$$\left(\frac{d}{dt} + \frac{\kappa C_\Psi}{\rho}\right)\left[\frac{d(\nabla \cdot \mathcal{K})}{dt} + \beta(\nabla \cdot \mathcal{K}) - \Delta T\right] = 0. \tag{84}$$

Reducing Eq. (84) with regard to Eqs. (73), (75), we obtain the identity. Taking the curl of Eq. (83), we arrive at the equation for the vortex part of vector $\mathcal{K}$:

$$\Delta(\nabla \times \mathcal{K}) = \beta\kappa\rho J \, (\nabla \times \mathcal{K}) + \left(\kappa\rho J + \frac{\beta\rho^2 J}{C_\Psi}\right) \frac{d(\nabla \times \mathcal{K})}{dt} + \frac{\rho^2 J}{C_\Psi} \frac{d^2(\nabla \times \mathcal{K})}{dt^2}. \tag{85}$$

It is easy to see that Eq. (85) for the vortex part of the proper angular momentum $\mathcal{K}$ coincides with Eq. (82) for the vortex part of vector $\mathbf{M}$, characterizing the antisymmetric part of the moment stress tensor. The vortex parts of the angular velocity $\boldsymbol{\omega}$ and the external moment $\rho\mathbf{L}$ also satisfy the differential equations coinciding with Eqs. (82), (85).

4 A physical interpretation of the proposed theory

4.1 Mechanical analogues of thermodynamic and electromagnetic quantities

Following the ideas of [39–42], we interpret quantity T as a mechanical analogue of temperature and quantity Θ_ρ as a mechanical analogue of specific entropy. Thus, quantities T and Θ_ρ are related to absolute temperature T_a and specific entropy Θ_a as

$$T = aT_a, \qquad \Theta_\rho = \frac{1}{a}\,\Theta_a, \tag{86}$$

where a is the normalization factor.

Following the ideas of [46,47,50], we introduce the analogues of electromagnetic quantities as follows: the moment stress vector $\mathbf{M}$ is the analogue of the electric field vector $\mathcal{E}$; the volume density of proper angular momentum $\mathcal{K}$ is the analogue of the magnetic induction vector $\mathcal{B}$; the wryness vector $\boldsymbol{\Psi}$ (which is the vector invariant of the wryness tensor) is the analogue of the electric induction vector $\mathcal{D}$; the angular velocity vector $\boldsymbol{\omega}$ is the analogue of the magnetic field vector $\mathcal{H}$. Thus, we have the relations

$$\mathbf{M} = \chi\mathcal{E}, \qquad \mathcal{K} = \chi\mathcal{B}, \qquad \boldsymbol{\Psi} = \frac{1}{\chi}\,\mathcal{D}, \qquad \boldsymbol{\omega} = \frac{1}{\chi}\,\mathcal{H}, \tag{87}$$

where χ is the normalization factor.

We emphasize that the analogues given by Eqs. (86) and (87) coincide with those introduced in our previous papers. Now, we introduce the following additional analogues: vector $\boldsymbol{\Upsilon}_\psi$, characterizing the antisymmetric part of the source term, is the analogue of the electric current density vector $\mathcal{J}$; the external moment per unit volume $\rho\mathbf{L}$ is the analogue of *the electric voltage density vector* $\mathcal{V}$ (this term was first introduced in [50]); the angular velocity vector $\boldsymbol{\omega}$ is the analogue of the entropy flux vector $\mathbf{h}_\Theta$; scalar Υ_Θ, characterising the spherical part of the source term, is the analogue of the entropy production per unit volume per unit time q_Θ. Thus, we suppose the following relations:

$$\boldsymbol{\Upsilon}_\psi = -\frac{1}{\chi}\,\mathcal{J}, \qquad \rho\mathbf{L} = -\chi\mathcal{V}, \qquad \boldsymbol{\omega} = -\frac{1}{a}\,\mathbf{h}_\Theta, \qquad \Upsilon_\Theta = \frac{1}{a}\,q_\Theta. \tag{88}$$

The physical meaning of the quantities in the first, third and fourth equations in (88) is clear. The physical meaning of vector $\mathcal{V}$ in the second equation in (88) needs to be explained. The appearance of vector $\boldsymbol{\omega}$ simultaneously in the third equation in (88) and in the fourth equation in (87) also needs to be explained. All necessary explanations will be given below. The treatment of the source term Υ_Θ as the entropy production requires Υ_Θ to be non-negative. This condition is satisfied due to the hypotheses given by Eqs. (61), (66).

4.2 Physical interpretation of equations and parameters of the model

We start with the equations that do not contain material constants. First of all, we consider Eq. (70), which represents the trace and the vector invariant of the relation between the wryness tensor and the angular velocity vector. In view of analogues (86), (87), (88), we can rewrite Eq. (70) as

$$\rho\frac{\mathrm{d}\Theta_a}{\mathrm{d}t} = -\nabla\cdot\mathbf{h}_\Theta + q_\Theta, \qquad \nabla\times\mathcal{H} = \mathcal{J} + \frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t}. \tag{89}$$

The first equation in (89) coincides with the entropy balance equation (4). The second one is Maxwell's first equation (see Eq. (19)). We note that we have obtained the first equation in (89) using the analogue for the angular velocity vector $\boldsymbol{\omega}$ given by the third equation in (88), and we have obtained the second equation in (89) using the analogue for the angular velocity vector $\boldsymbol{\omega}$ given by the last equation in (87). Comparing the third equation in (88) and the last equation in (87), we infer that

$$\mathbf{h}_\Theta = -\frac{a}{\chi}\,\mathcal{H}. \tag{90}$$

At first glance, relation (90) may seem strange. However, as seen from Eq. (89), the entropy balance equation contains only the potential part of the entropy flux vector $\mathbf{h}_\Theta$, whereas Maxwell's first equation contains only the vortex part of the magnetic field vector $\mathcal{H}$. That is why we do not find any contradiction in formula (90).

Next, we consider the angular momentum balance equation (71). Taking into account analogues (86), (87), (88), we can reduce Eq. (71) to the form

$$\frac{a}{\chi}\,\nabla T_a - \nabla \times \mathcal{E} = \mathcal{V} + \frac{d\mathcal{B}}{dt}. \tag{91}$$

Comparing Eq. (91) with Maxwell's equations (19), we see that Eq. (91) contains two terms, $\nabla \times \mathcal{E}$ and $\frac{d\mathcal{B}}{dt}$, that constitute the Maxwell–Faraday equation, and also two additional terms: *the electric voltage density vector* $\mathcal{V}$ and the term proportional to the temperature gradient $\frac{a}{\chi}\,\nabla T_a$. The presence of the terms constituting the Maxwell–Faraday equation gives us reason to treat Eq. (91) as the modified Maxwell–Faraday equation or the generalized Maxwell–Faraday equation. The physical meaning of *the electric voltage density vector* $\mathcal{V}$ and its role in the modified Maxwell's equations is discussed in Sect. 4.3 and Sect. 4.8. Here we only note that it is precisely due to the presence of vector $\mathcal{V}$ in Eq. (91) the modified Maxwell's equations turn out to be in agreement with Kirchhoff's laws for electric circuits. The mutual influence of thermodynamic and electrodynamic processes caused by the presence of the term $\frac{a}{\chi}\,\nabla T_a$ in Eq. (91) is discussed in Sect. 4.4 and Sect. 4.6.

Now, we consider the constitutive equations (68). In view of the analogues (86), (87), (88), these constitutive equations can be rewritten in terms of thermodynamic and electrodynamic quantities as

$$T_a = T_a^* + \frac{C_\Theta}{a^2}\Theta_a, \qquad \mathcal{E} = \frac{C_\Psi}{\rho\chi^2}\,\mathcal{D}, \qquad \mathcal{B} = \frac{\rho J}{\chi^2}\,\mathcal{H}, \qquad \mathcal{J} = \kappa\chi^2\mathcal{E}, \qquad \mathcal{V} = \frac{\beta\rho J}{\chi^2}\,\mathcal{H}. \tag{92}$$

The first equation in (92) relates absolute temperature to specific entropy; the rest of the constitutive equations in (92) have the electrodynamic meaning. Comparing the first four equations in (92) with the corresponding equations written in terms of the physical parameters, namely with the constitutive equations (12), (20) and Ohm's law in the form of the first equation in (35), we can identify four parameters of the mechanical model as

$$C_\Theta = \frac{a^2 T_a^*}{c_v}, \qquad \frac{C_\Psi}{\rho} = \frac{\chi^2}{\varepsilon\varepsilon_0}, \qquad \rho J = \chi^2\mu\mu_0, \qquad \kappa = \frac{\sigma_d}{\chi^2}. \tag{93}$$

Next, taking into account Eq. (90) and also the third and the last equations in (92), we can rewrite Eq. (91) in terms of the entropy flux vector $\mathbf{h}_\Theta$ as

$$\nabla T_a - \frac{a}{\chi}\,\nabla \times \mathcal{E} = -\frac{\beta\rho J}{a^2}\,\mathbf{h}_\Theta - \frac{\rho J}{a^2}\,\frac{d\mathbf{h}_\Theta}{dt}. \tag{94}$$

In view of the first equation in (10) relating the entropy flux to the heat flux, in the linear approximation, Eq. (94) can be rewritten in terms of the heat flux vector $\mathbf{h}$ as

$$\beta^{-1}\frac{d\mathbf{h}}{dt} + \mathbf{h} = -\frac{a^2 T_a^*}{\beta\rho J}\,\nabla T_a + \frac{\chi a T_a^*}{\beta\rho J}\,\nabla \times \mathcal{E}. \tag{95}$$

If we neglect the last term on the right-hand side of Eq. (95), this equation takes the form of the Maxwell–Cattaneo–Vernotte law (16) relating the heat flux vector to the temperature gradient. As is known, the heat conduction equations contain the divergence of the heat flux vector. This means that the last term on the right-hand side of Eq. (95) does not effect on the heat conduction equations. However, this is true only for isotropic materials. In the case of an anisotropic material, the scalar coefficient at $\nabla \times \mathcal{E}$ must be replaced by the corresponding tensor coefficient. This will result in that the divergence of the last term on the right-hand side of Eq. (95) will be non-zero. Thus, in the case of anisotropic materials, the curl of the electric field vector can effect on the heat conduction equations.

Let us compare Eq. (95) with the Maxwell–Cattaneo–Vernotte law (16) and take into account the Wiedemann–Franz law in the form of the second equation in (35). As a result, we have three expressions for parameter β:

$$\beta = \frac{1}{\tau_h}, \qquad \beta = \frac{a^2}{\chi^2}\,\frac{T_a^*}{\lambda\mu\mu_0}, \qquad \beta = \frac{1}{\sigma_c\,\mu\mu_0\,(L\chi^2/a^2)}. \tag{96}$$

The first relation in (96) is obtained by comparing the coefficients at the first time derivatives of the heat flux vector; the second relation is obtained by comparing the coefficients at the temperature gradients; the third relation is obtained in view of the Wiedemann–Franz law.

We draw attention to important consequences of formulas (96):

$$\frac{a}{\chi} = \sqrt{\frac{\lambda \mu \mu_0}{\tau_h T_a^*}}, \qquad \frac{a}{\chi} = \sqrt{\frac{L \sigma_c \, \mu \mu_0}{\tau_h}}. \tag{97}$$

The ratio on the left-hand side of both equations in (97) is a fundamental constant, i.e., the constant which cannot depend on any parameters of a material. Therefore, the ratios on the right-hand side of these equations must also be identical for all materials. Unfortunately, at present it is difficult to check whether these ratios are really independent of the parameters of a specific material. The point is that the experimental values of the heat flux relaxation time τ_h given by different authors sometimes differ by several orders of magnitude. For example, according to [105,106], the experimentally determined τ_h of metals is of the order of tens of nanoseconds, whereas according to [107–109] it is of the order of 0.01 nanoseconds. Apparently, the values of τ_h depend significantly on the measurement technique. For example, [110] shows that, depending on the type of scattering (phonon–phonon and electron–phonon), τ_h of gold is 12 and 1.52 picoseconds, respectively. Such experimental data do not allow to estimate fundamental constant a/χ correctly.

4.3 The physical meaning of the electrical conductivities σ_d and σ_c

In order to clarify the physical meaning of parameters σ_d and σ_c, we rewrite the second equation in (89) (Maxwell's first equation) and Eq. (91) (the Maxwell–Faraday equation) taking into account the constitutive equations (20). In doing so, for simplicity sake, we ignore the thermodynamic term in Eq. (91). As a result, we have

$$\nabla \times \mathcal{H} = \mathcal{J} + \varepsilon \varepsilon_0 \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}, \qquad -\nabla \times \mathcal{E} = \mathcal{V} + \mu \mu_0 \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}. \tag{98}$$

In view of the expressions for the parameters given by Eqs. (93), (96) the last two constitutive equations in (92) take the form

$$\mathcal{J} = \sigma_d \, \mathcal{E}, \qquad \mathcal{V} = \frac{1}{\sigma_c \, (L\chi^2/a^2)} \, \mathcal{H}. \tag{99}$$

Thus, we have two constitutive equations: the first one relates the electric current density vector $\mathcal{J}$ to the electric field vector $\mathcal{E}$ and contains the electrical conductivity σ_d; the second one relates *the electric voltage density vector* $\mathcal{V}$ to the magnetic field vector $\mathcal{H}$ and contains the electrical conductivity σ_c. The first equation in (99) is, in fact, Ohm's law for the electric current density. The second equation can be treated as an analogue of Ohm's law for the electric voltage density. We emphasize that *the electric voltage density vector* $\mathcal{V}$ plays the same role in the modified Maxwell–Faraday equation (91) as the electric current density vector $\mathcal{J}$ plays in Maxwell's first equation (89).

Inserting Eq. (99) into Eq. (98) yields

$$\nabla \times \mathcal{H} = \sigma_d \, \mathcal{E} + \varepsilon \varepsilon_0 \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}, \qquad -\nabla \times \mathcal{E} = \frac{1}{\sigma_c \, (L\chi^2/a^2)} \, \mathcal{H} + \mu \mu_0 \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}. \tag{100}$$

We draw attention to that two modified Maxwell's equations given by Eq. (100) have the same structure as Kirchhoff's laws (29) for electrical circuits. It is evident that the first terms on the right-hand sides of both equations in (100) provide energy dissipation. The difference is that the vector $\sigma_d \, \mathcal{E}$ provides the electric field energy dissipation, whereas the vector $\dfrac{1}{\sigma_c \, (L\chi^2/a^2)} \, \mathcal{H}$ provides the magnetic field energy dissipation.

From the mechanical point of view, the electric field energy dissipation is due to the internal friction, whereas the magnetic field energy dissipation is due to the external friction. Thus, we can infer that the electrical conductivity σ_d is associated with the internal friction, which occurs due to the source term $\mathbf{\Upsilon}_\Psi$ in the wryness tensor balance equation, and the electrical conductivity σ_c is associated with the external friction, which is provided by the external moment $\rho \mathbf{L}$ in the angular momentum balance equation. We refer to [52], where we discuss two types of friction in more detail. Here, we only note that the energy dissipation vanishes when $\sigma_d = 0$ and $\sigma_c \to \infty$.

4.4 The hyperbolic-type heat conduction equation with Joule heat

Let us return to the generalized Maxwell–Cattaneo–Vernotte law (95) for the heat flux vector $\mathbf{h}$. In view of the constitutive equations (93), (96), this equation takes the form

$$\tau_h \frac{d\mathbf{h}}{dt} + \mathbf{h} = -\lambda \left(\nabla T_a - \frac{\chi}{a} \nabla \times \mathcal{E} \right). \tag{101}$$

We note that the right-hand side of Eq. (101) contains the potential term depending on the temperature gradient, ∇T_a, and the vortex term depending on the curl of the electric field vector, $\nabla \times \mathcal{E}$. Hence, according to Eq. (101), the potential part of the heat flux vector $\mathbf{h}$ is completely determined by ∇T_a, whereas the vortex part of vector $\mathbf{h}$ is completely determined by $\nabla \times \mathcal{E}$. The energy balance equation for heat-conducting media (1) depends only on the potential part of the heat flux vector $\mathbf{h}$. Therefore, if we use Eq. (101) instead of the previously established Maxwell–Cattaneo–Vernotte law (16) to derive the heat conduction equation by the standard method presented in Sect. 2.1, we arrive at the well-known hyperbolic-type heat conduction equation (18).

In addition, within the framework of our theory, we can obtain the hyperbolic-type heat conduction equation (18) from Eq. (75) for the spherical part of the moment stress tensor. Indeed, taking into account the analogues, given by the first equation in (86) and the last equation in (88), and also equations for parameters (93), (96), we can rewrite Eq. (75) as

$$\Delta T_a - \frac{\rho c_v}{\lambda} \frac{dT_a}{dt} - \frac{\tau_h \rho c_v}{\lambda} \frac{d^2 T_a}{dt^2} = -\frac{T_a^*}{\lambda} q_\Theta - \frac{\tau_h T_a^*}{\lambda} \frac{dq_\Theta}{dt}. \tag{102}$$

In view of the analogues, given by Eqs. (86), (87), (88), we rewrite Eq. (61) for the source term as follows:

$$q_\Theta = \frac{\mathcal{E} \cdot \mathcal{J}}{T_a} \quad \Rightarrow \quad q_\Theta \approx \frac{\mathcal{E} \cdot \mathcal{J}}{T_a^*} \quad \Rightarrow \quad q \approx \mathcal{E} \cdot \mathcal{J}. \tag{103}$$

Taking into account the constitutive equation (99), we infer that the entropy production q_Θ is positive:

$$q_\Theta = \sigma_d \, \mathcal{E} \cdot \mathcal{E} \geq 0. \tag{104}$$

Next, inserting the second equation in (103) into Eq. (102), we arrive at the equation

$$\Delta T_a - \frac{\rho c_v}{\lambda} \frac{dT_a}{dt} - \frac{\tau_h \rho c_v}{\lambda} \frac{d^2 T_a}{dt^2} = -\frac{1}{\lambda} \mathcal{E} \cdot \mathcal{J} - \frac{\tau_h}{\lambda} \frac{d(\mathcal{E} \cdot \mathcal{J})}{dt}. \tag{105}$$

It is easy to see that Eq. (105) coincides with the hyperbolic-type heat conduction equation (18) where the heat production per unit volume per unit time q equals to Joule heat $\mathcal{E} \cdot \mathcal{J}$.

4.5 The telegraph equations in electrodynamics

Now, we show that, in the framework of our theory, we can obtain telegraph equations for the electric field vector $\mathcal{E}$ and the magnetic induction vector $\mathcal{B}$.

In view of expressions for parameters (93), (96) and the analogue between the moment stress vector $\mathbf{M}$ and the electric field vector $\mathcal{E}$ given by the first equation in (87), we can rewrite Eq. (81) for the moment stress tensor as

$$\Delta \mathcal{E} - \nabla(\nabla \cdot \mathcal{E}) - \frac{\sigma_d}{\sigma_c(L\chi^2/a^2)} \mathcal{E} - \left(\sigma_d \mu\mu_0 + \frac{\varepsilon\varepsilon_0}{\sigma_c(L\chi^2/a^2)} \right) \frac{d\mathcal{E}}{dt} - \varepsilon\varepsilon_0\mu\mu_0 \frac{d^2\mathcal{E}}{dt^2} = 0, \tag{106}$$

and Eq. (82) for the curl of the moment stress tensor as

$$\Delta(\nabla \times \mathcal{E}) - \frac{\sigma_d}{\sigma_c(L\chi^2/a^2)} (\nabla \times \mathcal{E}) - \left(\sigma_d \mu\mu_0 + \frac{\varepsilon\varepsilon_0}{\sigma_c(L\chi^2/a^2)} \right) \frac{d(\nabla \times \mathcal{E})}{dt}$$
$$- \varepsilon\varepsilon_0\mu\mu_0 \frac{d^2(\nabla \times \mathcal{E})}{dt^2} = 0. \tag{107}$$

Thus, we have obtained the telegraph equation (107) for the vortex part of the electric field vector $\mathcal{E}$ and Eq. (106) for vector $\mathcal{E}$ itself. The last equation differs from the telegraph one only by the presence of term $\nabla(\nabla \cdot \mathcal{E})$. Hence, if $\nabla \cdot \mathcal{E} = 0$, Eq. (106) turns into the telegraph equation.

Taking into account expressions for parameters (93), (96), the analogue between spherical part of the moment stress tensor T and absolute temperature T_a given by the first equation in (91), and also analogue between the angular momentum density vector $\mathcal{K}$ and the magnetic induction vector $\mathcal{B}$ given by second equation in (87), we rewrite Eq. (83) for the angular momentum density vector as

$$\Delta\mathcal{B} - \nabla(\nabla \cdot \mathcal{B}) - \frac{\sigma_d}{\sigma_c(L\chi^2/a^2)}\,\mathcal{B} - \left(\sigma_d\,\mu\mu_0 + \frac{\varepsilon\varepsilon_0}{\sigma_c(L\chi^2/a^2)}\right)\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}t} - \varepsilon\varepsilon_0\mu\mu_0\,\frac{\mathrm{d}^2\mathcal{B}}{\mathrm{d}t^2}$$
$$= \frac{a}{\chi}\left(\sigma_d\,\mu\mu_0\,\nabla T_a + \varepsilon\varepsilon_0\mu\mu_0\,\frac{\mathrm{d}(\nabla T_a)}{\mathrm{d}t}\right), \qquad (108)$$

and Eq. (85) for the curl of the angular momentum density vector as

$$\Delta(\nabla \times \mathcal{B}) - \frac{\sigma_d}{\sigma_c(L\chi^2/a^2)}\,(\nabla \times \mathcal{B}) - \left(\sigma_d\,\mu\mu_0 + \frac{\varepsilon\varepsilon_0}{\sigma_c(L\chi^2/a^2)}\right)\frac{\mathrm{d}(\nabla \times \mathcal{B})}{\mathrm{d}t}$$
$$-\varepsilon\varepsilon_0\mu\mu_0\,\frac{\mathrm{d}^2(\nabla \times \mathcal{B})}{\mathrm{d}t^2} = 0. \qquad (109)$$

It is easy to see that Eq. (109) for the vortex part of the magnetic induction vector $\mathcal{B}$ is the same as Eq. (107) describing the behavior of the vortex part of the electric field vector $\mathcal{E}$. Equation (108) for vector $\mathcal{B}$ differs from the telegraph equation by the presence of term $\nabla(\nabla \cdot \mathcal{B})$, and it differs from Eq. (106) for vector $\mathcal{E}$ by the presence of the terms containing absolute temperature. If thermal effects are ignored and $\nabla \cdot \mathcal{B}$ is assumed to be equal to zero, Eq. (108) turns into the telegraph equation.

Taking into account the constitutive equations (20) and (99), we infer that the electric induction vector $\mathcal{D}$ and the electric current density vector $\mathcal{J}$ satisfy Eqs. (106) and (107), and also the magnetic field vector $\mathcal{H}$ and *the electric voltage density vector* $\mathcal{V}$ satisfy Eqs. (108) and (109).

Thus, in the framework of our theory, we have obtained telegraph equations for the vortex parts of all vectors characterizing the state of electromagnetic field. This means that, in contrast to Maxwell's electrodynamics, our theory leads to the equations that are consistent with electrodynamics based on Kirchhoff's laws. We emphasize that only the behavior of the vortex parts of vectors $\mathcal{E}$, $\mathcal{D}$, $\mathcal{J}$, $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{V}$ is in agreement with Kirchhoff's laws. The behavior of the potential parts of these vectors is described by completely different equations. As shown below, the potential parts of vectors $\mathcal{E}$, $\mathcal{D}$ and $\mathcal{J}$ satisfy the diffusion-type equations, whereas the potential parts of vectors $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{V}$ satisfy some differential equations relating them to thermodynamic quantities.

4.6 Equations for the potential parts of vectors $\mathcal{E}$ and $\mathcal{B}$

Let us rewrite Eq. (79) for the moment stress vector $\mathbf{M}$ taking into account the analogue between this vector and the electric field vector $\mathcal{E}$ given by the first equation in (87). In view of expressions for parameters (93), we arrive at the equation for the potential part of the electric field vector:

$$\frac{\mathrm{d}(\nabla \cdot \mathcal{E})}{\mathrm{d}t} + \frac{\sigma_d}{\varepsilon\varepsilon_0}\,\nabla \cdot \mathcal{E} = 0. \qquad (110)$$

We emphasize that the coefficient in Eq. (110) depends on parameter σ_d, not on parameter σ as in classical electrodynamics. This is because we use modified Ohm's law that contains parameter σ_d (see the first equation in (99)). Taking into account the constitutive equations (20), (99), we can infer that the electric induction vector $\mathcal{D}$ and the electric current density vector $\mathcal{J}$ satisfy the differential equations that coincide with Eq. (110).

Now we consider some equations relating the divergence of the magnetic induction vector $\nabla \cdot \mathcal{B}$ to thermodynamic quantities.

Firstly, in view of analogues (86), (87) and expressions for parameters (93), (95), from Eq. (84) for the angular momentum density vector $\mathcal{K}$ it follows that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\sigma_d}{\varepsilon\varepsilon_0}\right)\left[\frac{\mathrm{d}(\nabla \cdot \mathcal{B})}{\mathrm{d}t} + \frac{\nabla \cdot \mathcal{B}}{\sigma_c\,\mu\mu_0\,(L\chi^2/a^2)} - \frac{a}{\chi}\,\Delta T_a\right] = 0. \qquad (111)$$

If temperature T_a is assumed to be known function, Eq. (111) can be considered as the ordinary differential equation for $\nabla \cdot \mathcal{B}$.

Secondly, taking into account analogues (86), (87), (88) and expressions for parameters (93), we can reduce Eq. (73), relating the angular momentum density vector $\mathcal{K}$ to the spherical parts of the moment stress tensor and the source term in the wryness tensor balance equation, to the following expression for $\nabla \cdot \mathcal{B}$:

$$\nabla \cdot \mathcal{B} = \mu\mu_0 \frac{\chi}{a}\left(\frac{\rho c_v}{T_a^*}\frac{\mathrm{d}T_a}{\mathrm{d}t} - q_\Theta\right).\tag{112}$$

Thirdly, eliminating temperature T_a from Eqs. (111), (112), we arrive at the equation containing only the entropy production q_Θ as an external factor:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\sigma_d}{\varepsilon\varepsilon_0}\right)\left[\Delta(\nabla\cdot\mathcal{B}) - \frac{\rho c_v}{\lambda}\frac{\mathrm{d}(\nabla\cdot\mathcal{B})}{\mathrm{d}t} - \frac{\tau_h\rho c_v}{\lambda}\frac{\mathrm{d}^2(\nabla\cdot\mathcal{B})}{\mathrm{d}t^2} - -\frac{\chi\mu\mu_0}{a}\Delta q_\Theta\right] = 0.\tag{113}$$

We note that, in view of the constitutive equations (20), (99), we can obtain equations analogues to Eqs. (111), (112), (113) for the magnetic field vector $\mathcal{H}$ and *the electric voltage density vector* $\mathcal{V}$.

As seen from Eqs. (111), (112), (113), in the general case $\nabla \cdot \mathcal{B} \neq 0$. As a rule, the authors of theories that generalize Maxwell's equations believe that the nonzero right-hand side in equation for $\nabla \cdot \mathcal{B}$ should be associated with the presence of the so-called magnetic monopoles (see, e.g., [111–113]). In the proposed theory, we do not introduce any magnetic monopoles. Indeed, the terms in brackets on the right-hand side of Eq. (112) are thermodynamic quantities, not magnetic ones. Thus, Eq. (112) does not give a relationship between $\nabla \cdot \mathcal{B}$ and magnetic monopoles. It describes the mutual influence of $\nabla \cdot \mathcal{B}$ and the thermal field.

As seen from Eq. (112), we can obtain the well-known equation $\nabla \cdot \mathcal{B} = 0$ if we suppose that

$$q_\Theta = \frac{\rho c_v}{T_a^*}\frac{\mathrm{d}T_a}{\mathrm{d}t}.\tag{114}$$

In this case, the heat conduction equation (102) and Eq. (103) relating the entropy production q_Θ to Joule heat take the form

$$\Delta T_a = 0, \qquad \frac{\mathrm{d}T_a}{\mathrm{d}t} \approx \frac{\mathcal{E}\cdot\mathcal{J}}{\rho c_v}.\tag{115}$$

Certainly, if we deal with Joule heat $\mathcal{J}\cdot\mathcal{E}$, which does not depend on space coordinates or linearly depends on them, two equations in (115) are in agreement with each other. If $\mathcal{E}\cdot\mathcal{J}$ is an arbitrary function of space coordinates, the aforementioned equations contradict each other. The physical meaning of Eq. (115) is that Joule heat is completely spent on changing the temperature at a given point in space, and at the same time, there is no change in temperature due to the heat flux.

4.7 The energy balance equation for thermo-electromagnetic field

We start with the energy balance equation in the form of (44). In view of analogues (86), (87), (88), we can rewrite this equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathcal{B}\cdot\mathcal{H} + \frac{1}{2}\mathcal{D}\cdot\mathcal{E} + \frac{1}{2}\rho(T_a + T_a^*)\Theta_a\right) = -\nabla\cdot\mathcal{S} - \nabla\cdot\mathbf{h} - \mathcal{V}\cdot\mathcal{H}.\tag{116}$$

Equation (116) is the balance of total energy. The first term on the left-hand side of this equation is the magnetic field energy. It corresponds to the kinetic energy of the mechanical model. The second term is the electric field energy. The third term is the thermal energy. The second and the third terms correspond to the internal energy of the mechanical model. The right-hand side of Eq. (116) contains the following quantities: the Poynting vector $\mathcal{S} = \mathcal{E}\times\mathcal{H}$, which is the electromagnetic energy flux vector; the heat flux vector, which can be expressed in terms of temperature and the entropy flux vector as $\mathbf{h} = T_a\mathbf{h}_\Theta$; and an additional term $\mathcal{V}\cdot\mathcal{H}$, which corresponds to the power of external moment $\rho\mathbf{L}\cdot\boldsymbol{\omega}$ in the mechanical model. The last term determines the energy dissipation due to the interaction with the free ether. We refer to [40,41,43], where we illustrate the mechanism of this interaction between the ponderable matter and the free ether by solving several model problems. We draw attention to the fact that term $\mathcal{V}\cdot\mathcal{H}$ characterizes the irrecoverable energy loss. It

has nothing to do with the conversion of electrical energy into heat. Equation (116) does not contain Joule heat because, in the model under consideration, Joule heat is associated with the power of internal interactions.

In order to obtain the energy balance equation containing Joule heat, we eliminate the thermal energy from Eq. (116) by using the entropy balance equation (4) and the constitutive equation (12). As a result, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\,\mathcal{B}\cdot\mathcal{H}+\frac{1}{2}\,\mathcal{D}\cdot\mathcal{E}\right)=-\nabla\cdot\mathcal{S}-\mathcal{J}\cdot\mathcal{E}-\mathbf{h}_{\Theta}\cdot\nabla T_a-\mathcal{V}\cdot\mathcal{H}. \tag{117}$$

The left-hand side of Eq. (117) is the same as in the classical electrodynamics. It is important to note that the right-hand side of this equation contains two terms that are contained in the classical equation: the divergence of the Poynting vector $\mathcal{S}$ and Joule heat $\mathcal{J}\cdot\mathcal{E}$. But it is also important to draw attention to two additional terms on the right-hand side of Eq. (117). Firstly, Eq. (117), as well as Eq. (116), contains term $\mathcal{V}\cdot\mathcal{H}$ characterizing the irrecoverable energy loss. Secondly, Eq. (117) contains term $\mathbf{h}_{\Theta}\cdot\nabla T_a$ characterizing the change of the electric field energy and the magnetic field energy due to thermal processes.

Thus, both energy balance equations that are obtained within the framework of our theory differ from the classical one. This is not surprising, since using the proposed mechanical model we arrive at the more complex equations compared to Maxwell's equations. Since these equations describe not only electrodynamic, but also thermodynamic processes, it is quite natural that this is reflected in the formulations of the energy balance equation.

4.8 The role of the electric voltage density $\mathcal{V}$ in the proposed model

The appearance of the electric voltage density $\mathcal{V}$ in the Maxwell–Faraday equation brings up a question: whether we should find some reasons for neglecting this quantity or it makes sense to keep it in the Maxwell–Faraday equation and try to find some physical arguments substantiating this decision. First of all, we refer to our previous work [50] where we consider this issue from various points of view. In particular, in [50], we discuss the physical meaning of the electric voltage density $\mathcal{V}$ in view of the experimentally established laws of Ampère and Faraday. In [50], we also discuss some theoretical reasoning that allowed Maxwell to pass from the original formulations of the Ampère and Faraday laws to the system of equations that are now known as Maxwell's equations. After that, we infer that it would be logical to write Maxwell's equations in the symmetric form, keeping vector $\mathcal{V}$ in the Maxwell–Faraday equation. A question arises: Why did Maxwell do it differently? Perhaps, the reason is that the presence of the electric voltage density $\mathcal{V}$ in the Maxwell–Faraday equation contradicts the Gauss law for the magnetic field, according to which $\nabla\cdot\mathcal{B}=0$. However, using the proposed mechanical model and the mechanical analogues of thermodynamic and electromagnetic quantities, given in the previous section, we do not face any difficulties arising from the violation of the Gauss law $\nabla\cdot\mathcal{B}=0$.

In our model, the angular momentum balance equation plays a dual role. On the one hand, it can be treated as the modified Maxwell–Faraday equation, which contains the temperature gradient, see Eq. (91). On the other hand, it can be treated as the generalized Maxwell–Cattaneo–Vernotte law, which contains the curl of the electric field vector (see Eq. (101)). We emphasize that there is no contradiction in these two interpretations of the angular momentum balance equation. The point is that only the vortex parts of the magnetic induction vector $\mathcal{B}$, of the magnetic field vector $\mathcal{H}$ and of the electric voltage density vector $\mathcal{V}$ have the electrodynamic meaning, whereas their potential parts have the thermodynamic meaning. Similar to this, only the potential parts of the heat flux vector $\mathbf{h}$ and of the entropy flux vector $\mathbf{h}_{\Theta}$ have the thermodynamic meaning, whereas their vortex parts have the electrodynamic meaning. That is why, taking the divergence of Eq. (91), we can obtain the heat conduction equation instead of some equations having the electrodynamic meaning. Thus, the appearance of the electric voltage density $\mathcal{V}$ in our model does not lead to the difficulties, which arise when adding such quantity to the classical Maxwell–Faraday equation. Furthermore, in order to obtain the equations describing electrodynamic and thermodynamic processes within the framework of our mechanical model, we are forced to take into account the electric voltage density $\mathcal{V}$. Indeed, without this quantity, we can obtain neither the hyperbolic-type heat conduction equation, nor the classical heat conduction equation. In addition, without this quantity, we cannot obtain modified Maxwell's equations (100), which are three-dimensional analogues of Kirchhoff's laws (29) for electrical circuits. Finally, without this quantity, we cannot obtain the telegraph equations for the electric field vector and the magnetic induction vector. The last statement applies both to the model considered above and to the model based on the viscoelastic Cosserat continuum, which is described in [52].

5 Propagation of thermal waves in matter and vacuum

To avoid misunderstandings, we start by clarifying the terminology. First, the thermal waves that we are discussing here have nothing to do with what is called thermal radiation in physics. Thermal radiation is the radiation of electromagnetic waves that occurs due to the fact that the radiating body is heated. It is well known that electromagnetic waves are transverse. In our model, there are two types of waves: transverse ones, which are treated to be electromagnetic, and longitudinal ones, which are treated as thermal. Thus, we consider thermal waves to be something different from electromagnetic radiation. Second, our concept of temperature has nothing to do with the concept adopted in statistical thermodynamics where temperature is associated with the so-called thermal motion, i.e., the chaotic motion of atoms and molecules. According to the concept of statistical thermodynamics, the thermal wave propagation occurs due to the change in space and time of thermal motion of ponderable matter. In our model, the thermal waves propagation occurs due to the motion of the ether, and hence, our model allows the propagation of thermal waves both in the ether associated with ponderable matter and in the free ether (in vacuum). Thus, the thermal waves inherent to our model are different from the thermal waves that are considered in statistical thermodynamics. Now we abstract from the models and try to define what the thermal waves are from the point of view of experimental physics, i.e., from the point of view of being able to detect them directly. We call thermal waves the change in temperature (the temperature measured by a thermometer!) that occurs in space and time so that it can be described by some wave equation. According to our definition, the temperature change may be accompanied by a change in some quantities of non-thermodynamic nature or maybe not. In other words, the propagation of thermal waves can be an independent process, not related to the propagation of waves of a different nature.

In the nineteenth century, some scientists developed luminiferous ether models containing not only transverse waves, but also longitudinal waves. In particular, Cauchy proposed several such models, and he believed that the existence of longitudinal waves would be verified experimentally. Cauchy hypothesized that the longitudinal waves are of thermal nature (see [1, p. 134]), where the reference to the original work of Cauchy is given. William Thomson (Lord Kelvin) thought that an explanation of all electromagnetic phenomena should be sought in the existence of inertia or pressure of a certain substance, the motion of which is associated with thermal effects. In his opinion, the resultant angular momentum of thermal motion of this substance could be a measure of the magnetic moment (see [1, p. 245]), where several references to Thomson's original works are given. We draw attention to that also our theory contains some equations relating the magnetic induction vector and characteristics of thermal processes (see Eqs. (111), (112), (113)).

What are the thermal waves from a physical point of view, and do they exist in nature? Thermal waves are well known in modern thermodynamics (see, e.g., [56,65,114]). However, the views of most modern scientists on the physical nature of thermal waves differ significantly from Cauchy's and Thomson's ideas. In accordance with contemporary views, thermal waves propagate only in matter. Thomson describes the substance the motion of which is associated with thermal effects, in a very vague way [1, p. 245]: "Whether this matter is or is not electricity, whether it is a continuous fluid interpermeating the space between molecular nuclei, or is itself molecularly grouped; or whether all matter is continuous, and molecular heterogeneousness consists in finite vortical or other relative motion of contiguous parts of a body; it is impossible to decide, and, perhaps, in vain to speculate, in the present state of science." According to Cauchy's views, thermal waves can propagate both in the ether coupled with matter and in the free ether, i.e., in vacuum [1, p. 134].

In the late twentieth and early twentyfirst centuries, Winterberg published a number of works devoted to Maxwell's equations, quantum mechanics and cosmology (see [115–123]). In these papers, Winterberg develops some models of the universe, which he call the Planck aether. In particular, Winterberg assumed that there exist thermal waves in the universe, i.e., in vacuum. He considers these waves to be longitudinal and gives an estimate of their propagation velocity. We quote from [117, p. 1222]: "As is known from the theory of superfluidity, the existence of a normal component leads to a longitudinal wave mode called second sound. Compared with the propagation velocity of the first sound, the second sound has a propagation velocity smaller by the factor of $l/\sqrt{3}$. *In the Planck aether the first sound propagates with the velocity of light. The propagation velocity of the second sound therefore is $c_s = c/\sqrt{3} = 173,000$ km/s. The second sound consists of propagating temperature oscillations, which behave like scalar photons in a medium with a refractive index equal to $l/\sqrt{3}$. During the cosmic expansion, the second sound must therefore have cooled down by the same proportion as the temperature of the cosmic microwave background radiation. Because the second sound consists of temperature or entropy waves, it should smoothed out any temperature inhomogeneities during the initial phase of the cosmic expansion. It therefore can account for very smooth initial conditions, mimicking the inflationary scenario of the standard model."*

Our theory contains the thermal waves, which are similar to Cauchy's thermal waves and, perhaps, similar to Winterberg's thermal waves. The only difference between the thermal waves in our theory and Cauchy's thermal waves is the following. The Cauchy models are based on translational degrees of freedom, and the thermal waves in these models are tension–compression waves. Our model is based on rotational degrees of freedom, and the thermal waves in our model are torsional waves. Torsional waves and tension–compression waves have in common that they are longitudinal waves. As noted above, Winterberg's thermal waves are also longitudinal. To briefly explain the essence of the Winterberg's model, we quote from [117, p. 1217]: "The Planck aether is superfluid, and it can without expenditure of energy form a tangle of quantized vortex filaments permitting the transmission of two types of waves: One associated with a symmetric displacement of the vortex lattice leading to gravitational waves, and one associated with an antisymmetric displacement leading to electromagnetic waves. Dirac spinors are explained in this model as excitons made up from positive and negative energy resonances of the vortex lattice."

We believe that the thermal waves propagating in vacuum, as well as the electromagnetic waves propagating in vacuum, are described by the wave equation without any damping terms:

$$\Delta T_a = \frac{1}{c_r^2} \frac{\mathrm{d}^2 T_a}{\mathrm{d}t^2}, \qquad c_r^2 = \frac{a^2}{\chi^2} \frac{T_a^*}{\rho c_v \mu \mu_0}. \tag{118}$$

There are only two differences between the thermal and electromagnetic waves in vacuum: the difference in the wave propagation velocities and that the electromagnetic waves are transverse whereas the thermal waves are supposed to be longitudinal. At present, the thermal waves in vacuum are hypothetical. These waves have never been observed. However, if such thermal waves actually exist, then they can be discovered experimentally, and it makes sense to think about how the experiment can be performed.

It is well known that electromagnetic waves decay rapidly in conductive media, propagate almost without attenuation in dielectrics and propagate in vacuum without attenuation at all. But what do we know about thermal waves? We know that thermal waves exist, but they decay very quickly in all the substances that have been used in research. Are there materials in which thermal waves propagate as well as electromagnetic waves propagate in dielectrics? Maybe there are no such materials, or maybe they exist, but we do not know about them. It is possible that such materials do not exist in nature, but they can be created artificially. And what can we say about thermal waves in a vacuum? Are they not there or have we not been able to find them yet? If thermal waves exist in a vacuum, what are the ways to detect them? These questions are yet to be answered. Below we discuss the ideas of two experiments through which the thermal waves in vacuum could be detected.

The first method that can allow us to experimentally study the thermal waves is based on the properties of wave behavior passing through an interface between two media. It is well known that at the interface between two optically isotropic media, an incident electromagnetic wave should generate no more than one refracted wave. At the same time, there are experimental data on birefringence in optically isotropic solids (see, e.g., [124–128]). We quoit from [125, pp. 158, 159]: "It is generally considered, both from its morphology and etching figures, that diamond belongs to the holohedral cubic system and that it should not show birefringence unless strained. Yet it has boon common knowledge since 1815, when Brewster described double refraction in some diamonds, that many diamonds are, in fact, birefringent. It has generally been assumed that these were strained either during growth or later. There have been conflicting views as to how widespread is this diamond birefringence. The most significant contribution to this subject is that of G. Friedel, who described a polariscope investigation of 200 diamonds and of 150 thin cleavage sheets. He reported that all the crystals showed some degree of birefringence and that so also did many of the cleavages. ... I have acquired 350 carats of small-mesh South Mrican mine tailings and a search through this has yielded 3000 clear transparent well-formed microdiamonds. ... Surprisingly enough, without exception, all 3000 exhibited birefringence. Despite the small thickness, in most, the double refraction can be fairly large, and frequently several bands appeared under the polarizing microscope. Observations were deliberately made with a light source of only moderate intensity, so that any near approach to extinction would not be over-illuminated. The occurrence of birefringence in all these small, clear, well-formed crystals was so unexpected that arrangements were made with the central diamond distributing organization (Industrial Distributors) to examine a considerable number of selected gem-quality diamonds for birefringence. ... All 2000 diamonds were examined, singly, with a simple pocket polariscope consisting of two crossed 'Polaroid' disks and *a* × 10 *eyepiece, the combination fitting over a small hand torch. Without exception, all these gem-quality selected crystals showed some degree of birefringence.*" There are various explanations for birefringence in optically isotropic solids (see [124–128]). We suppose that one of the possible explanations can be based on the assumption of the existence of thermal waves. In the linear approximation, the thermal and electromagnetic waves propagate independently of each

other. But they can influence each other when passing through an interface between two media. In accordance with the laws of wave propagation, when passing through the interface, an electromagnetic wave can generate not only the electromagnetic waves, but also the thermal waves, and a thermal wave can generate not only the thermal waves, but also the electromagnetic waves. Let us suppose that we have a heated body radiating the thermal and electromagnetic waves. These waves propagate in a vacuum, or perhaps in air. When these waves reach a surface of some optically isotropic transparent solid, each of them can generate two refracted waves, one thermal wave and one electromagnetic wave. The thermal waves must decay very quickly in the solid. The electromagnetic waves must propagate in the solid since the solid is transparent. It is well known that the presence or absence of a refracted wave is determined by the parameters of the two media and the angle between the incident wave and the interface. Thus, when two waves, the thermal one and the electromagnetic one, reach a surface of an optically isotropic solid, we can observe two, one, or no refracted electromagnetic waves. When an electromagnetic wave reaches a surface of an optically isotropic solid, we can observe only one refracted electromagnetic wave or no one. Hence, if we observe two refracted electromagnetic waves in an optically isotropic solid, we can infer that there are two incident waves: one electromagnetic wave and one thermal wave. If we accept this explanation, we will be able to determine the propagation velocity of thermal waves in vacuum using the experimental data on birefringence in optically isotropic media.

The second method to detect the thermal waves can be based on using the Dewar flask. The original Dewar flask was a double-walled glass flask with air evacuated from the space between them. To reduce heat loss through radiation, both inner surfaces of the bulb were covered with a reflective layer. Let us suppose that we have two Dewar's flasks. The first Dewar flask must shield both electromagnetic and thermal waves. The second Dewar flask must shield only electromagnetic waves. Let us put two identical bodies inside these two Dewar flasks and place these two Dewar flasks near a strongly heated body that radiates thermal and electromagnetic waves. Let us take care that the heat is transferred only by radiation. To do this, we can put the entire system in a vacuum. If the thermal waves exist then, under such conditions, it should be observed the constancy of the temperature of the body placed in the first Dewar flask, shielding both electromagnetic and thermal waves, and an increase in the temperature of the body placed in the second Dewar flask, shielding only electromagnetic waves. The question arises: is it possible to perform this experiment? This question is closely related to another question: are there materials that shield electromagnetic waves and transmit thermal waves? Now it is difficult to answer this question. But as seen from the hyperbolic-type heat conduction equation (18), such experiments require materials with large values of the heat flux relaxation time τ_h.

6 Conclusion and outlook

In the present paper, we propose the linear theory of thermo-electrodynamics. Our theory generalizes Maxwell's equations in two ways. First, in our theory, the modified Maxwell–Faraday equation contains the electric voltage density $\mathcal{V}$. Thanks to this, the generalized Maxwell's equations are in agreement with Kirchhoff's laws for electrical circuits and with the experimental data concerning the static skin effect. Second, in our theory, the modified Maxwell–Faraday equation contains the temperature gradient, the constitutive equation for the heat flux vector contains the curl of the electric field vector, and the potential part of the magnetic induction vector is related in some way to the thermodynamic quantities. This opens up new possibilities for describing thermoelectric, thermomagnetic, and thermoelectromagnetic effects. Certainly, the thermo-electrodynamics based on the proposed mechanical model does not allow one to describe quantitatively such effects as, e.g., the Ettingshausen effect or the Nernst–Ettingshausen effect. However, this mechanical model includes a number of the constitutive equations that can be changed without changing the general concept of the model. We mean the constitutive equations for the spherical and antisymmetric parts of the moment stress tensor $\mathbf{T}$, the constitutive equation for the external moment $\rho\mathbf{L}$, and the constitutive equation for the antisymmetric part of the source term $\boldsymbol{\Upsilon}_{\psi}$ in the wryness tensor balance equation. In order to describe the nonlinear thermoelectromagnetic effects one can try to replace the linear constitutive equations of the proposed theory by some nonlinear constitutive equations. In this case, one will obtain a geometrically linear, but physically nonlinear theory. Theories of this kind are well known and quite correct from a mechanical point of view. We are convinced that the development of such theories is a very promising direction in thermo-electrodynamics.

A comparison with the model based on the viscoelastic Cosserat continuum

Here, we compare the model considered in the present paper with the model proposed in [52]. There are only two differences between these models. Firstly, in contrast to the present paper, in [52] we use the classical relation between the wryness tensor $\boldsymbol{\Theta}$ and the angular velocity vector $\boldsymbol{\omega}$, namely relation(47), which does not contain the source term. Secondly, in [52] we use more complicated constitutive equations. The constitutive equation for the spherical part of the moment stress tensor **T** is

$$\frac{1}{C_\Theta} \frac{dT}{dt} + \frac{1}{\eta_\Theta} (T - T_*) = \frac{d\Theta_\rho}{dt}, \tag{119}$$

where C_Θ is the stiffness parameter, η_Θ is the viscosity parameter, T_* is the reference value of T. The constitutive equation for vector **M** characterizing the antisymmetric part of the moment stress tensor **T** has the form

$$\frac{1}{C_\Psi} \frac{d\mathbf{M}}{dt} + \frac{1}{\eta_\Psi} \mathbf{M} = \frac{d\boldsymbol{\Psi}_\rho}{dt}, \tag{120}$$

where C_Ψ and η_Ψ are the stiffness and viscosity parameters, respectively. According to constitutive equations (119), (120), the rheology of the continuum is similar to the classical Maxwell's model of a viscoelastic material.

The model proposed in [52] allows us to obtain two three-dimensional telegraph equations: one for temperature and the other for the electric field vector. The telegraph equation for the electric field vector coincides with Eq. (107). If $\eta_\Theta \to \infty$, the telegraph equation for temperature reduces to the hyperbolic-type heat conduction equation, which coincides with Eq. (102) under the condition that quantity q_Θ on the right-hand side of Eq. (102) is neglected. At the same time, the model proposed in [52] has a number of disadvantages. Firstly, this model leads to the relation between the electric induction vector $\mathcal{D}$ and the electric field vector $\mathcal{E}$ different from the classical one. Secondly, it leads to Maxwell's first equation without the electric current density vector

Table 1 A comparison of two models based on the Cosserat continuum

The model of viscoelastic Cosserat continuum	The model based on the strain balance equation
The constitutive equation relating $\mathcal{D}$ to $\mathcal{E}$: $\frac{d\mathcal{D}}{dt} = \varepsilon\varepsilon_0 \frac{d\mathcal{E}}{dt} + \sigma_d \mathcal{E}$	$\mathcal{D} = \varepsilon\varepsilon_0 \mathcal{E}$
The other constitutive equations: $\mathcal{B} = \mu\mu_0 \mathcal{H}, \quad \mathcal{J} = \sigma_d \mathcal{E}, \quad \mathcal{V} = \frac{1}{\sigma_c (L\chi^2/a^2)} \mathcal{H}$	
Maxwell's first equation in terms of vectors $\mathcal{H}$ and $\mathcal{D}$: $\nabla \times \mathcal{H} = \frac{d\mathcal{D}}{dt}$	$\nabla \times \mathcal{H} = \mathcal{J} + \frac{d\mathcal{D}}{dt}$
Maxwell's first equation in terms of vectors $\mathcal{B}$ and $\mathcal{E}$: $\nabla \times \mathcal{B} = \mu\mu_0 \mathcal{J} + \varepsilon\varepsilon_0\mu\mu_0 \frac{d\mathcal{E}}{dt}$	
The generalized Maxwell–Faraday equation: $\frac{a}{\chi} \nabla T_a - \nabla \times \mathcal{E} = \mathcal{V} + \frac{d\mathcal{B}}{dt}$	
The telegraph equation for vector $\mathcal{E}$: $\Delta\mathcal{E} = \frac{\sigma_d}{\sigma_c(L\chi^2/a^2)} \mathcal{E} + \left(\sigma_d \mu\mu_0 + \frac{\varepsilon\varepsilon_0}{\sigma_c(L\chi^2/a^2)}\right) \frac{d\mathcal{E}}{dt} + \varepsilon\varepsilon_0\mu\mu_0 \frac{d^2\mathcal{E}}{dt^2}$	
The hyperbolic-type heat conduction equation: $\Delta T_a - \frac{\rho c_v}{\lambda} \frac{dT_a}{dt} - \frac{\tau_h \rho c_v}{\lambda} \frac{d^2 T_a}{dt^2} = 0$	$\Delta T_a - \frac{\rho c_v}{\lambda} \frac{dT_a}{dt} - \frac{\tau_h \rho c_v}{\lambda} \frac{d^2 T_a}{dt^2} = -\frac{1}{\lambda} \mathcal{E} \cdot \mathcal{J} - \frac{\tau_h}{\lambda} \frac{d(\mathcal{E} \cdot \mathcal{J})}{dt}$
The entropy balance equation: —	$\rho \frac{d\Theta_a}{dt} = -\nabla \cdot \mathbf{h}_\Theta + q_\Theta$
The relation between vector $\mathcal{B}$ and thermodynamic quantities: $\nabla \cdot \mathcal{B} = \frac{\chi}{a} \frac{\mu\mu_0 \rho c_v}{T_a^*} \frac{dT_a}{dt}$	$\nabla \cdot \mathcal{B} = \mu\mu_0 \frac{\chi}{a} \left(\frac{\rho c_v}{T_a^*} \frac{dT_a}{dt} - q_\Theta\right)$

$\mathcal{J}$, if this equation is written in terms of the magnetic field vector $\mathcal{H}$ and the electric induction vector $\mathcal{D}$. Thirdly, this model does not describe the conversion of electrical energy into thermal energy due to Joule heat. The new model is free from all these shortcomings. Furthermore, it allows to obtain the entropy balance equation. In addition, within the framework of the new model, we arrive at the relation between magnetic induction vector $\mathcal{B}$ and thermodynamic quantities (112), which seems more adequate than the corresponding relation obtained in [52]. A comparison of the equations of electrodynamics and thermodynamics, obtained within the framework of two models, can be found in Table 1.

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