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ORIGINAL ARTICLE

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On the evolution of solutions of mixed problems in thermoelasticity of porous bodies with dipolar structure

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Abstract Our study deals with a thermoelastic body with pores. We have added a new independent variable, namely the time derivative of the voidage. Within the theory of such media, we analyze the spatial and temporal evolution of solutions. For the spatial behavior, we will prove certain estimations of the Saint-Venant type, in the situation the bodies are bounded. In the case the bodies are unbounded bodies, to describe the spatial evolution we consider certain estimations of the Phragmén–Lindelöf type.

Keywords Dipolar structure · Voids · Thermoelastic · Acceleration waves

1 Introduction

The considerations from our study can be used in applications regarding porous bodies, such as geological bodies, some granular solids and so on.

The granular theory of Goodman and Cowin from [1] is considered the first investigation on porous media. Here and, also, in the paper [2], the researchers introduced a supplementary degree of freedom to develop the mechanical evolution of solids with voids so that the interstices are voids and matrix is an elastic material. There are many applications of this theory, as such materials of geological type, like soil and rocks and, also, in artificially manufactured materials with pores, as such, ceramics media and pressed powders. In the theory of Cowin and Nunziato and also in the paper [3], the materials are non-conductors of heat. The basic concept of these theories is that for these materials the bulk density is stated as a product of two fields, the volume fraction field and the matrix material density (see also, [4–6]). After that, the theory was extended by Iesan in [4] to cover the materials with voids for which it is considered the thermal effect. However, the author does not consider the fact that the changes in the volume fraction have effect on the internal dissipation in the material.

The constitutive equations for porous elastic bodies with incompressible matrix material are derived in [7]. Chirita and Ciarletta proposed in [8] a time-weighted power function which we will use in the following. Ciarletta and Scarpetta in [9] give a variational characterization of Gurtin type for the incremental problem of thermoelasticity for porous dielectric materials. Some refinements to the behavior of solutions for different

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kind of bodies with microstructure and voids can be found in [10–21]. In our paper, we intend to generalize the theory of Cowin and Nunziato in order to cover the dipolar thermoelastic bodies with voids. To this aim, we consider the time derivative of the voidage as a new variable in the set of the constitutive variables. This is to take into account the inelastic effects.

2 Basic equations and conditions

We consider a regular domain D of the Euclidean space R^3 occupied at the moment $t = 0$ by a material which is dipolar elastic and have voids. The boundary of D is denoted by ∂D and is an enough regular surface to allow the application of the theorem of divergence. In the initial state of the body, the relation between the density of bulk, the density of matrix and the fraction of the matrix volume is given by:

$$\varrho_0 = \gamma_0 \nu_0,$$

where γ_0 and ν_0 are constants regarding the spatial variables. In order to describe the evolution of our dipolar body with pores, we will use the following independent variables:

- $v_m(x,t)$, $\phi_{mn}(x,t)$—the components of displacement and of dipolar displacement with regard to the initial configuration;
- ϑ—the variation of the temperature from T_0, i.e., $\vartheta(x,t) = T(x,t) - T_0$;
- φ—the variation in volume fraction, regarding the initial configuration, i.e., $\varphi(x,t) = \nu(x,t) - \nu_0$. With the help of the motion variables, we can define the tensors of strain, namely e_{mn}, ϵ_{mn}, γ_{mnr} by means of the following kinematic equations:

$$e_{mn} = \frac{1}{2}\left(v_{m,n} + v_{n,m}\right), \ \epsilon_{mn} = u_{n,m} - \phi_{mn}, \ \gamma_{mnr} = \phi_{mn,r}. \tag{1}$$

We will assume that the body has no flux rate, no intrinsic equilibrated mass forces and it has zero initial stress and dipolar stress. In all what follows, we will take into account only linear equations and conditions. Hence, we have to suppose that the internal energy is a quadratic application, with respect to its constitutive functions. Consequently, the energy principle helps us to state the internal energy in the following form:

$$\begin{aligned}\Psi = &\frac{1}{2}A_{ijmn}e_{ij}e_{mn} + G_{ijmn}e_{ij}\epsilon_{mn} + F_{ijmnr}e_{ij}\gamma_{mnr}\\ &+\frac{1}{2}B_{ijmn}\epsilon_{ij}\epsilon_{mn} + D_{ijmnr}\epsilon_{ij}\gamma_{mnr} + \frac{1}{2}C_{ijkmnr}\gamma_{ijk}\gamma_{mnr}\\ &+a_{ijk}e_{ij}\varphi_{,k} + b_{ijk}\epsilon_{ij}\varphi_{,k} + c_{ijkm}\gamma_{ijk}\varphi_{,m} - a_i\vartheta\varphi_{,i} - \frac{1}{2}c\vartheta^2\\ &-\alpha_{ij}e_{ij}\vartheta - \beta_{ij}\epsilon_{ij}\vartheta - \delta_{ijk}\gamma_{ijk}\vartheta + \frac{1}{2}d_{ij}\varphi_{,i}\varphi_{,j} + \frac{1}{2}\kappa_{ij}\vartheta_{,i}\vartheta_{,j}.\end{aligned} \tag{2}$$

We can use a procedure proposed in the paper Nunziato and Cowin [3] in order to obtain:

$$t_{mn} = \frac{\partial\Psi}{\partial e_{mn}}, \ \tau_{mn} = \frac{\partial\Psi}{\partial\epsilon_{mn}}, \ m_{ijk} = \frac{\partial\Psi}{\partial\gamma_{ijk}}, \ h_m = \frac{\partial\Psi}{\partial\varphi_{,m}}, \ S = -\frac{\partial\Psi}{\partial\vartheta}, \ q_m = \frac{\partial\Psi}{\partial\vartheta_{,m}}.$$

In this way, we obtain the connections between the tensors of deformation and the stress, namely the constitutive equations:

$$\begin{aligned} t_{ij} &= A_{ijmn}e_{mn} + G_{mnij}\epsilon_{mn} + F_{mnrij}\gamma_{mnr} + a_{ijk}\varphi_{,k} - \alpha_{ij}\vartheta,\\ \tau_{ij} &= G_{ijmn}e_{mn} + B_{ijmn}\epsilon_{mn} + D_{ijmnr}\gamma_{mnr} + b_{ijk}\varphi_{,k} - \beta_{ij}\vartheta,\\ m_{ijk} &= F_{ijkmn}e_{mn} + D_{mnijk}\epsilon_{mn} + C_{ijkmnr}\gamma_{mnr} + c_{ijkr}\varphi_{,r} - \delta_{ijk}\vartheta,\\ h_i &= a_{ijk}e_{jk} + b_{ijk}\epsilon_{jk} + c_{ijkr}\gamma_{jkr} + d_{ij}\varphi_{,j} - a_i\vartheta,\\ S &= \alpha_{ij}e_{ij} + \beta_{ij}\epsilon_{ij} + c_{ijk}\gamma_{ijk} + a_i\varphi_{,i} + c\theta,\\ q_i &= \kappa_{ij}\theta_{,j}.\end{aligned} \tag{3}$$

Based on the fact that the tensor of deformations e_{ij} is a symmetric one (see Eq. (1)$_1$), we obtain the following relations of symmetry:

$$A_{jkmn} = A_{kjmn} = A_{mnjk},\ G_{jkmn} = G_{kjmn},\ F_{jkmnr} = F_{kjmnr},$$
$$C_{jklmnr} = C_{mnrjkl},\ d_{mn} = d_{nm},\ \alpha_{mn} = \alpha_{nm},\ \kappa_{mn} = \kappa_{nm}. \tag{4}$$

With the help of the same suggestion from Nunziato and Cowin [2], we can deduce the next main balances (see also [4]):

- the equations of motion:

$$(t_{mn} + \tau_{mn})_{,n} + \varrho f_m = \varrho \ddot{v}_m,$$
$$m_{ijk,i} + \tau_{jk} + \varrho g_{jk} = I_{jm}\ddot{\phi}_{km}; \tag{5}$$

- **the equation of the equilibrated forces:

$$h_{m,m} + \varrho l = \varrho k \ddot{\varphi}; \tag{6}$$

- the balance of the energy:

$$\varrho T_0 \dot{\eta} = q_{m,m} + \varrho r. \tag{7}$$

In above equations remained unspecified the next notations: S— the mass entropy, k—the inertia of balancing, I_{mn}—the inertia, h_m—a vector of stress, q_m—the vector of flux of heat, f_m, g_m, l—body forces and r— heat supply. The entropy inequality implies

$$k_{mn}\vartheta_{,m}\vartheta_{,n} \geq 0. \tag{8}$$

The motion equations (5) are similar to the classical motion equations of motion and Eq. (6) is the same balance of energy as the classical case. A new equation is (5), which is for the balance of equilibrated force. A motivation for the presence of this equation can be made using a variational reason, as proposed in [2].

Suppose the coefficients in the constitutive relations (3) are functions of class $C^1(\bar{D})$. Moreover, we suppose that the functions a, ϱ and κ are strictly positive in the domain $\bar{D}$, that is

$$\varrho(x) \geq \varrho_0 > 0,\ \kappa(x) \geq \kappa_0 > 0,\ a(x) \geq a_0 > 0, \tag{9}$$

where ϱ_0, $\kappa_0(x)$ and a_0 are constants. The conductivity tensor k_{mn} is positive definite, is symmetric and satisfies the conditions:

$$k_m\vartheta_{,r}\vartheta_{,r} \leq k_{rs}\vartheta_{,r}\vartheta_{,s} \leq k_M\vartheta_{,r}\vartheta_{,r}, \tag{10}$$

where k_m and k_M represent the minimum value and maximum value of the conductivity tensor, respectively. Considering the constitutive relation (3)$_6$ and using the inequality of Schwartz, the double inequality (10) led to:

$$q_m q_m = \left(k_{mn}\vartheta_{,n}\right) q_i \leq \left(k_{rs}\vartheta_{,r}\vartheta_{,s}\right)^{1/2} \left(k_{mn}q_m q_n\right)^{1/2} \leq \left(k_{rs}\vartheta_{,r}\vartheta_{,s}\right)^{1/2} \left(k_M q_n q_n\right)^{1/2}, \tag{11}$$

such that we can conclude that

$$q_m q_m \leq k_M k_{mn}\vartheta_{,m}\vartheta_{,n}. \tag{12}$$

Assume that the function of free energy Ψ, expressed in (2), is a quadratic application for which we can find the constants $\mu_m > 0$ and $\mu_M > 0$ so that the following double inequality is satisfied:

$$\mu_m \left(e_{mn}e_{mn} + \epsilon_{mn}\epsilon_{mn} + \gamma_{mnr}\gamma_{mnr} + \varphi_{,m}\varphi_{,m}\right) \leq 2\mathcal{E}$$
$$\leq \mu_M \left(e_{mn}e_{mn} + \epsilon_{mn}\epsilon_{mn} + \gamma_{mnr}\gamma_{mnr} + \varphi_{,m}\varphi_{,m}\right). \tag{13}$$

In the following, we will use a linear space, with specific norm, as the set of all components of displacements. This will be denoted by $\mathcal{S}_{13}$ and is a thirteen-dimensional space containing the displacement fields $\mathbf{V}$, as follows:

$$\mathbf{V} = \{v_m,\ \phi_{mn},\ \varphi\}. \tag{14}$$

The space $\mathcal{S}_{13}$ can be equipped with the following inner product

$$\mathbf{V}\ .\ \mathbf{W} = v_m w_m + \phi_{mn}\psi_{mn} + \varphi\chi,\ \mathbf{V} = \{v_m,\ \phi_{mn},\ \varphi\},\ \mathbf{W} = \{w_m,\ \psi_{mn},\ \chi\}\ . \tag{15}$$

As usual, the norm for a vector field $\mathbf{W} = \{w_m,\ \psi_{mn},\ \chi\} \in \mathcal{S}_{13}$, induced by this inner product, is given by (see [22,23]):

$$|\mathbf{W}| = (\mathbf{W}\ .\ \mathbf{W})^{1/2} = \left(w_m w_m + \psi_{mn}\psi_{mn} + \chi^2\right)^{1/2}. \tag{16}$$

It is clear that the state of strain can be characterized with the help of the fields

$$E(\mathbf{V}) = \left\{e_{mn}(\mathbf{V}),\ \epsilon_{mn}(\mathbf{V}),\ \gamma_{mnr}(\mathbf{V}),\ \varphi_{,m}(\mathbf{V})\right\}, \tag{17}$$

where, according to (3), we have

$$e_{mn}(\mathbf{V}) = \frac{1}{2}\left(v_{m,n} + v_{n,m}\right),\ \epsilon_{mn}(\mathbf{V}) = v_{m,n} - \phi_{mn},\ \gamma_{mnr}(\mathbf{V}) = \phi_{nr,m}. \tag{18}$$

Let us introduce the vector space of the strains, that is, having components described in (17). We will denote by $\mathcal{E}$ the space of the strains and we will endow it with the next norm:

$$|E| = \sqrt{(E\ .\ E)} = \left(e_{mn}e_{mn} + \epsilon_{mn}\epsilon_{mn} + \gamma_{mnr}\gamma_{mnr} + \varphi_{,m}\varphi_{,m}\right)^{1/2}. \tag{19}$$

For any $E \in \mathcal{E}$, we consider the set $S(E)$ defined by:

$$S(E) = \left\{T_{ij}(E), \mathcal{T}_{ij}(E), M_{ijk}(E), H_i(E)\right\},$$

where we used the notations:

$$\begin{aligned}
T_{ij}(E) &= A_{ijmn}e_{mn} + G_{mnij}\epsilon_{mn} + F_{mnrij}\gamma_{mnr} + a_{ijk}\varphi_{,k},\\
\mathcal{T}_{ij}(E) &= G_{ijmn}e_{mn} + B_{ijmn}\epsilon_{mn} + D_{ijmnr}\gamma_{mnr} + b_{ijk}\varphi_{,k},\\
M_{ijk}(E) &= F_{ijkmn}e_{mn} + D_{mnijk}\epsilon_{mn} + C_{ijkmnr}\gamma_{mnr} + c_{ijkr}\varphi_{,r},\\
H_i(E) &= a_{ijk}e_{jk} + b_{ijk}\epsilon_{jk} + c_{ijkr}\gamma_{jkr} + d_{ij}\varphi_{,j}.
\end{aligned} \tag{20}$$

Considering the above definitions (17), (19), for any $\mathbf{S}(E) \in \mathcal{E}$ we introduce the following norm:

$$|S(E)| = \left\{\ T_{ij}(E)T_{ij}(E) + \mathcal{T}_{ij}(E)\mathcal{T}_{ij}(E) + M_{ijk}(E)M_{ijk}(E) + H_i(E)H_i(E)\right\}^{1/2}. \tag{21}$$

Taking into account (17) and (18), we can consider the bilinear application F defined by:

$$\begin{aligned}
F\left(E^{(1)}, E^{(2)}\right) = \frac{1}{2}\Big[\ & A_{ijmn}e^{(1)}_{ij}e^{(2)}_{mn} + G_{ijmn}\left(e^{(1)}_{ij}\epsilon^{(2)}_{mn} + e^{(2)}_{ij}\epsilon^{(1)}_{mn}\right)\\
&+F_{ijmnr}\left(e^{(1)}_{ij}\gamma^{(2)}_{mnr} + e^{(2)}_{ij}\gamma^{(1)}_{mnr}\right) + B_{ijmn}\epsilon^{(1)}_{ij}\epsilon^{(2)}_{mn}\\
&+D_{ijmnr}\left(\epsilon^{(1)}_{ij}\gamma^{(2)}_{mnr} + \epsilon^{(1)}_{ij}\gamma^{(1)}_{mnr}\right) + C_{ijkmnr}\gamma^{(1)}_{ijk}\gamma^{(2)}_{mnr}\\
&+a_{ijk}\left(\varphi^{(1)}_{,k}e^{(2)}_{ij} + \varphi^{(2)}_{,k}e^{(1)}_{ij}\right) + b_{ijk}\left(\varphi^{(1)}_{,k}\epsilon^{(2)}_{ij} + \varphi^{(2)}_{,k}\epsilon^{(1)}_{ij}\right)\\
&+ +c_{ijkm}\left(\varphi^{(1)}_{,m}\gamma^{(2)}_{ijk} + \varphi^{(2)}_{,m}\gamma^{(1)}_{ijk}\right) + d_{ij}\varphi^{(1)}_{,i}\varphi^{(2)}_{,j}\ \Big].
\end{aligned} \tag{22}$$

for every $E^{(\nu)} \in \mathcal{E}$, where

$$E^{(\nu)} = \left\{e^{(\nu)}_{ij}, \epsilon^{(\nu)}_{ij}, \gamma^{(\nu)}_{ijk}, \varphi^{(\nu)}_{,i}\right\},\ \nu = 1, 2.$$

Taking into account the symmetry relations (7), it is easy to deduce that

$$F\left(E^{(1)}, E^{(2)}\right) = F\left(E^{(2)}, E^{(1)}\right),\ \forall\ E^{(1)}, E^{(2)} \in \mathcal{E}. \tag{23}$$

Also, after simple calculations it is easy to find that

$$F\,(E,E)=\Psi\,(E)\,,\ \forall\,E\in\mathcal{E}, \tag{24}$$

where Ψ is the free energy function defined by (2). Based on the double inequality (13), with the help of the Schwarz's inequality, we can deduce:

$$F\left(E^{(1)},E^{(2)}\right)\le\left[\Psi\left(E^{(1)}\right)\right]^{1/2}\left[\Psi\left(E^{(2)}\right)\right]^{1/2},\ \forall\,E^{(1)},E^{(2)}\in\mathcal{E}. \tag{25}$$

By direct calculations, using the relations (20)-(22) we obtain the equality:

$$\begin{aligned}|S(E)|^2&=T_{ij}(E)T_{ij}(E)+\mathcal{T}_{ij}(E)\mathcal{T}_{ij}(E)+M_{ijk}(E)M_{ijk}(E)+H_i(E)H_i(E)\\&=A_{ijmn}T_{ij}e_{mn}+G_{ijmn}T_{ij}\epsilon_{mn}+F_{ijmnr}T_{ij}\gamma_{mnr}+a_{ijk}T_{ij}\varphi_{,k}\\&\quad+G_{mnij}\mathcal{T}_{ij}e_{mn}+B_{mnij}\mathcal{T}_{ij}\epsilon_{mn}+D_{ijmnr}\mathcal{T}_{ij}\gamma_{mnr}+b_{ijk}\mathcal{T}_{ij}\varphi_{,k}\\&\quad+F_{ijkmn}M_{ijk}e_{mn}+D_{ijmn}M_{ijk}\epsilon_{mn}+C_{ijmnr}M_{ijk}\gamma_{mnr}+c_{ijkm}M_{ijk}\varphi_{,m}\\&\quad+a_{mni}e_{mn}H_i+b_{mni}\epsilon_{mn}H_i+c_{mnri}\gamma_{mnr}H_i+d_{ij}\varphi_{,j}H_i=2F\,(E,S(E))\,.\end{aligned} \tag{26}$$

By using Eqs. (13), (19), (25) and (26), it results:

$$|S(E)|^2\le 2\mu_M\Psi(E). \tag{27}$$

Taking into account norm (21) and inequality (27), we get

$$T_{ij}(E)T_{ij}(E)+\mathcal{T}_{ij}(E)\mathcal{T}_{ij}(E)+M_{ijk}(E)M_{ijk}(E)+H_i(E)H_i(E)\le 2\mu_M\Psi(E),\ \forall\,E\in\mathcal{E}. \tag{28}$$

Given two real numbers a and b, we have:

$$(a+b)\,(a+b)\le(1+\varepsilon)a^2+\left(1+\frac{1}{\varepsilon}\right)b^2, \tag{29}$$

for any arbitrary positive number ε. With the help of relations (3), (20) and inequality (29), inequality (28) gives us:

$$\begin{aligned}t_{ij}t_{ij}+\tau_{ij}\tau_{ij}+m_{ijk}m_{ijk}+h_ih_i&=\left(T_{ij}-\alpha_{ij}\vartheta\right)\left(T_{ij}-\alpha_{ij}\vartheta\right)\\&+\left(\mathcal{T}_{ij}-\beta_{ij}\vartheta\right)\left(\mathcal{T}_{ij}-\beta_{ij}\vartheta\right)+\left(M_{ijk}-\delta_{ijk}\vartheta\right)\left(M_{ijk}-\delta_{ijk}\vartheta\right)\\&+\left(H_i-\gamma_i\vartheta\right)\left(H_i-\gamma_i\vartheta\right)\le(1+\varepsilon)T_{ij}T_{ij}+\left(1+\frac{1}{\varepsilon}\right)\alpha_{ij}\alpha_{ij}\vartheta^2\\&+(1+\varepsilon)\mathcal{T}_{ij}\mathcal{T}_{ij}+\left(1+\frac{1}{\varepsilon}\right)\beta_{ij}\beta_{ij}\vartheta^2+(1+\varepsilon)M_{ijk}M_{ijk}\\&+\left(1+\frac{1}{\varepsilon}\right)\delta_{ijk}\delta_{ijk}\vartheta^2+(1+\varepsilon)H_iH_i+\left(1+\frac{1}{\varepsilon}\right)a_ia_i\vartheta^2\\&\le(1+\varepsilon)2\mu_M\Psi(E)+\left(1+\frac{1}{\varepsilon}\right)M^2\vartheta^2,\ \forall\,\varepsilon>0,\end{aligned} \tag{30}$$

where we have used the notation

$$M^2=\max_{\bar{D}}\left(\alpha_{ij}\alpha_{ij}+\beta_{ij}\beta_{ij}+\delta_{ijk}\delta_{ijk}+a_ia_i\right). \tag{31}$$

In order to complete the basic mixed problem in the context of theory of thermoelastic media with pores and dipolar structure, we need to give some of the boundary relations and initial values. Moreover, we need to add some of the initial values. So, initial values have the form:

$$\begin{aligned}&v_m(0,x)=v_m^0(x),\ \dot{v}_m(0,x)=v_m^1(x),\ x\in\bar{D},\\&\phi_{mn}(0,x)=\phi_{mn}^0(x),\ \dot{\phi}_{mn}(0,x)=\phi_{mn}^1(x),\ x\in\bar{D},\\&\vartheta(0,x)=\vartheta^0(x),\ \varphi(0,x)=\varphi^0(x),\ \dot{\varphi}(0,x)=\varphi^1(x),\ x\in\bar{D}.\end{aligned} \tag{32}$$

We prescribe the boundary relations in the following form:

$$
\begin{aligned}
v_m &= \bar{v}_m \ on\ \partial D_1 \times [0,\infty),\ \ t_m \equiv (t_{km} + \tau_{km})\, n_k = \bar{t}_m \ on\ \partial D_1^c \times [0,\infty),\\
\phi_{mn} &= \bar{\phi}_{mn} \ on\ \partial D_2 \times [0,\infty),\ \ m_{ij} \equiv m_{ijk} n_k = \bar{m}_{ij} \ on\ \partial D_2^c \times [0,\infty),\\
\varphi &= \bar{\varphi} \ on\ \partial D_3 \times [0,\infty),\ \ h \equiv h_k n_k = \bar{h} \ on\ \partial D_3^c \times [0,\infty),\\
\vartheta &= \bar{\vartheta} \ on\ \partial D_4 \times [0,\infty),\ \ q \equiv q_k n_k = \bar{q} \ on\ \partial D_4^c \times [0,\infty),
\end{aligned} \tag{33}
$$

where $\partial D_1, \partial D_2, \partial D_3$ and ∂D_4 with respective complements $\partial D_1^c, \partial D_2^c, \partial D_3^c$ and ∂D_4^c are subsets of ∂D, n_k are the elements of the normal oriented to the exterior of ∂D. Also $v_m^0,\ v_m^1,\ \phi_{mn}^0,\ \phi_{mn}^1,\ \vartheta^0,\ \varphi^0,\ \varphi^1,\ \bar{v}_m,\ \bar{t}_m,\ \bar{\phi}_{mn},\ \bar{m}_{ij},\ \bar{\varphi},\ \bar{\vartheta},\ \bar{q}$ and $\bar{h}$ are prescribed continuous functions in their domains. Introducing Eq. (3) into Eqs. (5), (6) and (7), we obtain the following system of equations

$$
\begin{aligned}
\varrho \ddot{v}_i &= \big[\left(A_{ijmn} + G_{mnij}\right) e_{mn} + \left(G_{mnij} + B_{mnij}\right) \epsilon_{mn}\\
&\quad + \left(F_{mnrij} + D_{ijmnr}\right) \gamma_{mnr} + \left(a_{ijk} + b_{ijk}\right) \varphi_{,k} - \left(\alpha_{ij} + \beta_{ij}\right) \vartheta\big]_{,j} + \varrho f_i,\\
I_{km}\ddot{\phi}_{lm} &= \left(F_{jklmn} e_{mn} + D_{mnjkl} \epsilon_{mn} + C_{kljmnr} \gamma_{mnr} + c_{jklm} \varphi_{,m} - \delta_{kjl} \vartheta\right)_{,j}\\
&\quad + G_{klmn} e_{mn} + B_{klmn} \epsilon_{mn} + D_{klmnr} \gamma_{mnr} + b_{klm} \varphi_{,m} - \beta_{kl} \vartheta + \varrho g_{kl},\\
\varrho\kappa \ddot{\varphi} &= \left(a_{mni} e_{mn} + b_{mni} \epsilon_{mn} + c_{mnri} \gamma_{mnr} + d_{ij} \varphi_{,j} - a_i \vartheta\right)_{,i} + \varrho l,\\
a\dot{\vartheta} &= \frac{1}{\varrho T_0}(k_{ij}\vartheta_{,j})_{,i} + \frac{1}{T_0} r - \alpha_{ij}\dot{e}_{ij} - \beta_{ij}\dot{\epsilon}_{ij} - \delta_{ijk}\dot{\gamma}_{ijk} - a_i\dot{\phi}_i.
\end{aligned} \tag{34}
$$

We denote by $\mathcal{P}$ the mixed problem composed of system of Eq. (37), the initial data (35) and the boundary relations (36). An ordered array $(v_m, \phi_{mn}, \varphi, \vartheta)$ is a solution of the mixed problem $\mathcal{P}$ in the thermoelasticity theory of dipolar porous media, if it satisfies the system of Eq. (34) for all $(x,t) \in \Omega_0 = D \times [0,\infty)$, the boundary relations (33) and the initial data (32).

3 Preliminary auxiliary estimates

The integral identities that we demonstrate in this section are helpful in obtaining the behavior of any solution of the mixed problem $\mathcal{P}$.

Theorem 1 *Consider a solution $(v_m, \phi_{mn}, \varphi,\ \vartheta)$ of the mixed problem $\mathcal{P}$. Then, the following law of conservation for the energy is satisfied:*

$$
\begin{aligned}
&\int_D e^{-\lambda t}\left\{\frac{1}{2}\left[\varrho \dot{v}_m(t)\dot{v}_m(t) + I_{mn}\dot{\phi}_{mr}(t)\dot{\phi}_{nr}(t) + \varrho\kappa\dot{\varphi}^2(t)\right] + \Psi(E(t)) + \frac{1}{2}a\vartheta^2(t)\right\} dV\\
&\quad + \int_0^t \int_D e^{-\lambda s}\frac{\lambda}{2}\left[\varrho \dot{v}_m(s)\dot{v}_m(s) + I_{mn}\dot{\phi}_{mr}(s)\dot{\phi}_{nr}(s) + \varrho\kappa\dot{\varphi}^2(s)\right] dV d\tau\\
&\quad + \int_0^t \int_D e^{-\lambda s}\left[\frac{\lambda}{2}a\vartheta^2(s) + \frac{1}{T_0}k_{mn}\vartheta_{,m}(s)\vartheta_{,n}(s) + \lambda\Psi(E(s))\right] dV d\tau\\
&= \int_D \left\{\frac{1}{2}\left[\varrho \dot{v}_m(0)\dot{v}_m(0) + I_{mn}\dot{\phi}_{mr}(0)\dot{\phi}_{nr}(0) + \varrho\kappa\dot{\varphi}^2(0)\right] + \Psi(E(0)) + \frac{1}{2}a\vartheta^2(0)\right\} dV\\
&\quad + \int_0^t \int_D e^{-\lambda s}\varrho\left[\dot{v}_m(s) f_m(s) + \dot{\phi}_{mn}(s) g_{mn}(s) + \dot{\varphi}(s) l(s) + \frac{1}{T_0}\vartheta(s) r(s)\right] dV d\tau\\
&\quad + \int_0^t \int_{\partial D} e^{-\lambda s}\left[t_m(s)\dot{v}_m(s) + m_{kl}(s)\dot{\phi}_{kl}(s) + h(s)\dot{\varphi}(s) + \frac{1}{T_0}q(s)\vartheta(s)\right] dA d\tau,
\end{aligned} \tag{35}
$$

for $\lambda > 0$ a known parameter, the sizes t_i, m_i, h and q introduced in (33) *and for $t \geq 0$.*

Proof By direct calculations, based on Eq. (37), the constitutive relations (3), the kinematic conditions (1) and the relations of symmetry (7), it results:

$$\begin{aligned}
&\frac{d}{d\tau}\left\{\frac{1}{2}\left[\varrho\dot{v}_m(s)\dot{v}_m(s)+I_{mn}\dot{\phi}_{mr}(s)\dot{\phi}_{nr}(s)+\varrho\kappa\dot{\varphi}^2(s)\right]+\Psi(E(s))+\frac{1}{2}a\vartheta^2(s)\right\}\\
&\quad+\frac{1}{T_0}k_{mn}\vartheta_{,m}(s)\vartheta_{,n}(s)\\
&\quad=\varrho\left[\dot{v}_m(s)f_m(s)+\dot{\phi}_{mn}(s)g_{mn}(s)+\dot{\varphi}(s)l(s)+\frac{1}{T_0}\vartheta(s)r(s)\right]\\
&\quad+\left[t_{mj}(s)\dot{v}_m(s)+m_{ijk}(s)\dot{\phi}_{ik}(s)+h_j(s)\dot{\varphi}(s)+\frac{1}{T_0}q_j(s)\vartheta(s)\right]_{,j}
\end{aligned} \tag{36}$$

In (36), we multiply by $e^{-\lambda s}$, after that the obtained equality is integrated over $D\times[0,t]$. But the boundary ∂D has the degree of regularity that allows the application of the theorem of divergence and, based on this, we obtain the proposed identity (35) and so the proof of Theorem 1 is ended. □

Theorem 2 *Let $(v_m, \phi_{mn}, \varphi, \vartheta)$ be an arbitrary solution of $\mathcal{P}$. Then, the following equality is satisfied:*

$$\begin{aligned}
&2\int_D\left[\varrho v_m(t)\dot{v}_m(t)+I_{mn}\phi_{mr}(t)\dot{\phi}_{nr}(t)+\varrho\kappa\varphi(t)\dot{\varphi}(t)\right.\\
&\quad\left.+\frac{1}{T_0}k_{mn}\left(\int_0^t\vartheta_{,m}(s)d\tau\right)\left(\int_0^t\vartheta_{,n}(s)d\tau\right)\right]dV\\
&=2\int_0^t\int_D\left[\varrho\dot{v}_m(s)\dot{v}_m(s)+I_{mn}\dot{\phi}_{mr}(s)\dot{\phi}_{nr}(s)+\varrho\kappa\dot{\varphi}^2(s)-2\Psi(E(s))-a\vartheta^2(s)\right]dVd\tau\\
&\quad+2\int_0^t\int_D\varrho\eta(0)\vartheta(s)dVd\tau+2\int_D\left[\varrho v_m(0)\dot{v}_m(0)+I_{mn}\phi_{mr}(0)\dot{\phi}_{nr}(0)+\varrho\kappa\varphi(0)\dot{\varphi}(0)\right]dV\\
&\quad+2\int_0^t\int_D\varrho\left[f_m(s)v_m(s)+g_{mn}(s)\phi_{mn}(s)+l(s)\varphi(s)+\frac{1}{T_0}\vartheta(s)\int_0^s r(z)dz\right]dVd\tau\\
&\quad+2\int_0^t\int_D\varrho\eta(0)\vartheta(s)dVd\tau+2\int_D\left[\varrho v_m(0)\dot{v}_m(0)+I_{mn}\phi_{mr}(0)\dot{\phi}_{nr}(0)+\varrho\kappa\varphi(0)\dot{\varphi}(0)\right]dV\\
&\quad+2\int_0^t\int_{\partial D}\left[t_m(s)v_m(s)+m_{kl}(s)\phi_{kl}(s)+h(s)\varphi(s)+\frac{1}{T_0}\vartheta(s)\int_0^s q(z)dz\right]dAd\tau.
\end{aligned} \tag{37}$$

Proof If we use the equations of motion $(5)_1$ and take into account the kinematic equations (1), it results:

$$\frac{d}{d\tau}\left[\varrho v_m(s)\dot{v}_m(s)\right]=\varrho\dot{v}_m(s)\dot{v}_m(s)+[t_{mn}(s)v_m(s)]_{,n}-t_{mn}(s)v_{m,n}(s)+\varrho v_m(s)f_m(s). \tag{38}$$

By considering of motion equations $(5)_2$ and, again, kinematic equations (1), we obtain:

$$\begin{aligned}
&\frac{d}{d\tau}\left[I_{mn}\phi_{mr}(s)\dot{\phi}_{nr}(s)\right]=I_{mn}\dot{\phi}_{mr}(s)\dot{\phi}_{nr}(s)+\left[m_{klj}(s)\phi_{kl}(s)\right]_{,j}\\
&\quad-m_{klj}(s)\varphi_{kl,j}(s)+\varrho\phi_{mn}(s)g_{mn}(s).
\end{aligned} \tag{39}$$

By adding equalities (38) and (39), we find the relation:

$$\begin{aligned}
&\frac{d}{d\tau}\left[\varrho v_m(s)\dot{v}_m(s)+I_{mn}\phi_{mr}(s)\dot{\phi}_{nr}(s)\right]=\varrho\dot{v}_m(s)\dot{v}_m(s)\\
&\quad+I_{mn}\dot{\phi}_{mr}(s)\dot{\phi}_{nr}(s)+[t_{mn}(s)v_m(s)+m_{kln}(s)\phi_{kl}(s)]_{,n}\\
&\quad-t_{mn}(s)e_{mn}(s)-\tau_{mn}(s)\epsilon_{mn}(s)-m_{klj}(s)\gamma_{klj}(s).
\end{aligned} \tag{40}$$

Based on the constitutive Eqs. (3)$_1$ and (3)$_2$, we have:

$$\begin{aligned} &t_{ij}(s)e_{ij}(s) = A_{ijmn}e_{ij}(s)e_{mn}(s) + G_{ijmn}e_{ij}(s)\epsilon_{mn}(s) + F_{mnrij}e_{ij}(s)\gamma_{mnr}(s) \\ &+2a_{ijk}\phi_{,k}(s)e_{ij}(s) - \left[a_{ijk}\phi_{,k}(s)e_{ij}(s) + \alpha_{ij}\vartheta(s)e_{ij}(s)\right], \\ &\tau_{ij}(s)\epsilon_{ij}(s) = G_{ijmn}\epsilon_{ij}(s)e_{mn}(s) + B_{ijmn}\epsilon_{ij}(s)\epsilon_{mn}(s) + D_{ijmnr}\epsilon_{ij}(s)\gamma_{mnr}(s) \\ &+2b_{ijk}\phi_{,k}(s)\epsilon_{ij}(s) - \left[b_{ijk}\phi_{,k}(s)\epsilon_{ij}(s) + \beta_{ij}\vartheta(s)\epsilon_{ij}(s)\right]. \end{aligned} \tag{41}$$

On the other hand, using the constitutive relation (3)$_3$, we deduce:

$$\begin{aligned} &m_{ijk}(s)\gamma_{ijk}(s) = F_{ijmnr}e_{ij}(s)\gamma_{mnr}(s) + D_{mnrij}\epsilon_{ij}(s)\gamma_{mnr}(s) \\ &+C_{ijkmnk}\gamma_{ijk}(s)\gamma_{mnr}(s) + 2c_{ijkl}\phi_{,l}(s)\gamma_{ijk}(s) - \left[c_{ijkl}\phi_{,l}(s)\gamma_{ijk}(s) + \delta_{ijk}\vartheta(s)\gamma_{ijk}(s)\right]. \end{aligned} \tag{42}$$

By adding relations (41) and (42) together, we obtain

$$\begin{aligned} &t_{ij}(s)e_{ij}(s) + \tau_{ij}(s)\epsilon_{ij}(s) + m_{ijk}(s)\gamma_{ijk}(s) \\ &= +A_{ijmn}e_{ij}(s)e_{mn}(s) + 2G_{mnij}e_{ij}(s)\epsilon_{mn}(s) + 2F_{mnrij}e_{ij}(s)\gamma_{mnr}(s) \\ &+B_{ijmn}\epsilon_{ij}(s)\epsilon_{mn}(s) + 2D_{ijmnr}\epsilon_{ij}(s)\gamma_{mnr}(s) + C_{ijkmnr}\gamma_{ijk}(s)\gamma_{mnr}(s) \\ &+2a_{ijk}\phi_{,k}(s)e_{ij}(s) + 2b_{ijk}\phi_{,k}(s)\epsilon_{ij}(s) + 2c_{ijkl}\phi_{,l}(s)\gamma_{ijk}(s) \\ &-\left[a_{ijk}\phi_{,k}(s)e_{ij}(s) + \alpha_{ij}\vartheta(s)e_{ij}(s)\right] \\ &-\left[b_{ijk}\phi_{,k}(s)\epsilon_{ij}(s) + \beta_{ij}\vartheta(s)\epsilon_{ij}(s)\right] \\ &-\left[c_{ijkl}\phi_{,l}(s)\gamma_{ijk}(s) + \delta_{ijk}\vartheta(s)\gamma_{ijk}(s)\right]. \end{aligned} \tag{43}$$

Using formulas (3)$_3$–(3)$_5$ and (1), we can write the previous parentheses as follows:

$$\begin{aligned} &\left[a_{ijk}e_{ij}(s) + b_{ijk}\epsilon_{ij}(s) + c_{ijlk}\gamma_{ijl}(s)\right]\phi_{,k}(s) \\ &+\left[\alpha_{ij}e_{ij}(s) + \beta_{ij}\epsilon_{ij}(s) + \delta_{ijk}\gamma_{ijk}(s)\right]\vartheta(s) \\ &= -d_{ij}\phi_{,i}(s)\phi_{,j}(s) - a\vartheta^2(s) + \varrho\eta(s)\vartheta(s). \end{aligned} \tag{44}$$

By integrating the equation of energy (7), we obtain the following equality:

$$\varrho\eta(s) - \varrho\eta(0) = \frac{1}{T_0}\int_0^s q_{m,m}(z)dz + \frac{\varrho}{T_0}\int_0^s r(z)dz. \tag{45}$$

With the help of relations (6) and (45), we get:

$$\begin{aligned} &h_{m,m}(s)\varphi(s) - \varrho\eta(s)\vartheta(s) = \left[\varrho\kappa\ddot{\varphi}(s) - \varrho L(s)\right]\varphi(s) - \varrho\eta(0)\vartheta(s) \\ &-\frac{\varrho}{T_0}\int_0^s r(z)dz - \left[\frac{1}{T_0}\vartheta(s)\int_0^s q_m(z)dz\right]_{,m} + \frac{1}{T_0}\vartheta_{,m}(s)\int_0^s q_m(z)dz. \end{aligned} \tag{46}$$

Based on the constitutive relation (3)$_6$, equality (46) can be rewritten as follows:

$$\begin{aligned} &h_{m,m}(s)\varphi(s) - \varrho\eta(s)\vartheta(s) = -\varrho\kappa\dot{\varphi}^2(s) - \varrho\eta(0)\vartheta(s) \\ &+\frac{d}{d\tau}\left[\varrho\kappa\varphi(s)\dot{\varphi}(s) + \frac{1}{2T_0}k_{mn}\left(\int_0^s \vartheta_{,m}(z)dz\right)\left(\int_0^s \vartheta_{,n}(z)dz\right)\right] \\ &-\varrho\left[L(s)\varphi(s) + \frac{1}{T_0}\vartheta(s)\int_0^s r(z)dz\right] - \left[\frac{1}{T_0}\vartheta(s)\int_0^s q_m(z)dz\right]_{,m}. \end{aligned} \tag{47}$$

If we enter the results from (43), (44) and (47) into the identity (40), we are led to the equality:

$$\begin{aligned}
&\frac{d}{d\tau}\left[2\varrho v_m(s)\dot{v}_m(s)+2I_{mn}\phi_{mr}(s)\dot{\phi}_{nr}(s)+2\varrho\kappa\varphi(s)\dot{\varphi}(s)\right.\\
&\quad\left.+\frac{1}{T_0}k_{mn}\left(\int_0^s\vartheta_{,m}(z)dz\right)\left(\int_0^s\vartheta_{,n}(z)dz\right)\right]\\
&=2\varrho\dot{v}_m(s)\dot{v}_m(s)+2I_{mn}\dot{\phi}_{mr}(s)\dot{\phi}_{nr}(s)+2\varrho\kappa\dot{\varphi}^2(s)-2\left[2\Psi(E(s))+a\vartheta^2(s)\right]\\
&\quad+2\varrho\left[f_m(s)v_m(s)+g_{mn}(s)\phi_{mn}(s)+l(s)\varphi(s)+\frac{1}{T_0}\vartheta(s)\int_0^s r(z)dz\right]\\
&\quad+2\left[t_{nm}(s)v_m(s)+m_{mnj}(s)\phi_{mj}(s)+h_n(s)\varphi(s)+\frac{1}{T_0}\vartheta(s)\int_0^s q_n(z)dz\right]_{,n}\\
&\quad+2\varrho\eta(0)\vartheta(s).
\end{aligned}\tag{48}$$

Finally, by integrating identity (51) on $D\times[0,t]$ and, after that, using the theorem of divergence we obtain equality (40) and so we end the proof of Theorem 2. □

Theorem 3 *Consider $(v_m,\phi_{mn},\varphi,\vartheta)$ a solution of $\mathcal{P}$. The following equality is satisfied:*

$$\begin{aligned}
&2\int_D\left[\varrho v_m(t)\dot{v}_m(t)+I_{mn}\phi_{mr}(t)\dot{\phi}_{nr}(t)+\varrho\kappa\varphi(t)\dot{\varphi}(t)\right.\\
&\quad\left.+\frac{1}{T_0}k_{mn}\left(\int_0^t\vartheta_{,m}(s)d\tau\right)\left(\int_0^t\vartheta_{,n}(s)d\tau\right)\right]dV\\
&=\int_D\left\{\varrho\left[v_m(0)\dot{v}_m(2t)+\dot{v}_m(0)v_m(2t)\right]+I_{mn}\left[\phi_{mr}(0)\dot{\phi}_{nr}(2t)+\dot{\phi}_{mr}(0)\phi_{nr}(2t)\right]\right\}dV\\
&\quad+\int_D\varrho\kappa\left[\varphi(0)\dot{\varphi}(2t)+\dot{\varphi}(0)\varphi(2t)\right]dV+\int_0^t\int_D\varrho\eta(0)\left[\vartheta(t-\tau)-\vartheta(t+\tau)\right]dVd\tau\\
&\quad+\int_0^t\int_D\varrho\left[v_m(t+\tau)f_m(t-\tau)-v_m(t-\tau)f_m(t+\tau)\right]dVd\tau\\
&\quad+\int_0^t\int_D I_{mn}\left[\phi_{mr}(t+\tau)g_{nr}(t-\tau)-\phi_{mr}(t-\tau)g_{nr}(t+\tau)\right]dVd\tau\\
&\quad+\int_0^t\int_D\left[\varphi(t+\tau)l(t-\tau)-\varphi(t-\tau)l(t+\tau)\right]dVd\tau\\
&\quad+\int_0^t\int_D\frac{1}{T_0}\left[\vartheta(t-\tau)\int_0^{t+s}r(z)dz-\vartheta(t+\tau)\int_0^{t-s}r(z)dz\right]dVd\tau\\
&\quad+\int_0^t\int_{\partial D}\left[v_m(t+\tau)t_m(t-\tau)-v_m(t-\tau)t_m(t+\tau)\right]dAd\tau\\
&\quad+\int_0^t\int_{\partial D}\left[\phi_{mn}(t+\tau)g_{mn}(t-\tau)-\phi_{mn}(t-\tau)g_{mn}(t+\tau)\right]dAd\tau\\
&\quad+\int_0^t\int_{\partial D}\left[\varphi(t+\tau)h(t-\tau)-\varphi(t-\tau)h(t+\tau)\right]dAd\tau\\
&\quad+\int_0^t\int_{\partial D}\frac{1}{T_0}\left[\vartheta(t-\tau)\int_0^{t+s}q(z)dz-\vartheta(t+\tau)\int_0^{t-s}q(z)dz\right]dAd\tau.
\end{aligned}\tag{49}$$

Proof Using simple calculations, we deduce:

$$\begin{aligned}
&-\frac{d}{d\tau}\left\{\varrho\left[v_m(t+\tau)\dot{v}_m(t-\tau)+\dot{v}_m(t+\tau)v_m(t-\tau)\right]\right\}\\
&=\varrho\left[v_m(t+\tau)\ddot{v}_m(t-\tau)-v_m(t-\tau)\ddot{v}_m(t+\tau)\right],\ s\in[0,t],\ t\in[0,\infty).
\end{aligned}\tag{50}$$

With the help of the motion equations $(5)_1$, can be rewrite term from the right side of (50) in the following form:

$$\begin{aligned}&\varrho\left[v_m(t+\tau)\ddot{v}_m(t-\tau)-v_m(t-\tau)\ddot{v}_m(t+\tau)\right]\\&\quad=\varrho\left[v_m(t+\tau)f_m(t-\tau)-v_m(t-\tau)f_m(t+\tau)\right]\\&\qquad+\left[v_m(t+\tau)t_{nm}(t-\tau)-v_m(t-\tau)t_{nm}(t+\tau)\right]_{,n}\\&\qquad+\left[v_{m,n}(t-\tau)t_{nm}(t+\tau)-v_{m,n}(t+\tau)t_{nm}(t-\tau)\right].\end{aligned}\tag{51}$$

Hence, based on Eq. (51), equality (50) becomes:

$$\begin{aligned}&-\frac{d}{d\tau}\left\{\varrho\left[v_m(t+\tau)\dot{v}_m(t-\tau)+\dot{v}_m(t+\tau)v_m(t-\tau)\right]\right\}\\&\quad=\varrho\left[v_m(t+\tau)f_m(t-\tau)-v_m(t-\tau)f_m(t+\tau)\right]\\&\qquad+\left[v_m(t+\tau)t_{nm}(t-\tau)-v_m(t-\tau)t_{nm}(t+\tau)\right]_{,n}\\&\qquad+\left[v_{m,n}(t-\tau)t_{nm}(t+\tau)-v_{m,n}(t+\tau)t_{nm}(t-\tau)\right].\end{aligned}\tag{52}$$

Clearly, we have

$$\begin{aligned}&-\frac{d}{d\tau}\left\{I_{mn}\left[\phi_{mr}(t+\tau)\dot{\phi}_{nr}(t-\tau)+\dot{\phi}_{mr}(t+\tau)\phi_{nr}(t-\tau)\right]\right\}\\&\quad=I_{mn}\left[\phi_{mr}(t+\tau)\ddot{\phi}_{nr}(t-\tau)-\phi_{mr}(t-\tau)\ddot{\phi}_{nr}(t+\tau)\right],\ s\in[0,t],\ \ t\in[0,\infty).\end{aligned}\tag{53}$$

Based on Eq. $(5)_2$, the last term in (53) receives the following form:

$$\begin{aligned}&I_{mn}\left[\phi_{mr}(t+\tau)\ddot{\phi}_{nr}(t-\tau)-\phi_{mr}(t-\tau)\ddot{\phi}_{nr}(t+\tau)\right]\\&\quad=\varrho\left[\phi_{mn}(t+\tau)g_{mn}(t-\tau)-\phi_{mn}(t-\tau)g_{mn}(t+\tau)\right]\\&\qquad+\left[\phi_{kl}(t+\tau)m_{klj}(t-\tau)-\phi_{kl}(t-\tau)m_{klj}(t+\tau)\right]_{,j}\\&\qquad+\left[\phi_{kl,j}(t-\tau)m_{klj}(t+\tau)-\phi_{kl,j}(t+\tau)m_{klj}(t-\tau)\right].\end{aligned}\tag{54}$$

Considering identity (54), equality (53) can be rewritten as follows:

$$\begin{aligned}&-\frac{d}{d\tau}\left\{I_{mn}\left[\phi_{mr}(t+\tau)\dot{\phi}_{nr}(t-\tau)+\dot{\phi}_{mr}(t+\tau)\phi_{nr}(t-\tau)\right]\right\}\\&\quad=\varrho\left[\phi_{mn}(t+\tau)g_{mn}(t-\tau)-\phi_{mn}(t-\tau)g_{mn}(t+\tau)\right]\\&\qquad+\left[\phi_{kl}(t+\tau)m_{klj}(t-\tau)-\phi_{kl}(t-\tau)m_{klj}(t+\tau)\right]_{,j}\\&\qquad+\left[\phi_{kl,j}(t-\tau)m_{klj}(t+\tau)-\phi_{kl,j}(t+\tau)m_{klj}(t-\tau)\right].\end{aligned}\tag{55}$$

By adding the relations (55) and (52) and using the kinematic equations (1), we find the following equality:

$$\begin{aligned}&-\frac{d}{d\tau}\left\{\varrho\left[v_m(t+\tau)\dot{v}_m(t-\tau)+\dot{v}_m(t+\tau)v_m(t-\tau)\right]\right\}\\&\quad-\frac{d}{d\tau}\left\{I_{mn}\left[\phi_{mr}(t+\tau)\dot{\phi}_{nr}(t-\tau)+\dot{\phi}_{mr}(t+\tau)\phi_{nr}(t-\tau)\right]\right\}\\&\quad=\varrho\left[v_m(t+\tau)f_m(t-\tau)-v_m(t-\tau)f_m(t+\tau)\right]\\&\qquad+\varrho\left[\phi_{mn}(t+\tau)g_{mn}(t-\tau)-\phi_{mn}(t-\tau)g_{mn}(t+\tau)\right]\\&\qquad+\left[v_m(t+\tau)t_{mn}(t-\tau)-v_m(t-\tau)t_{mn}(t+\tau)\right]_{,n}\\&\qquad+\left[\phi_{kl}(t+\tau)m_{klj}(t-\tau)-\phi_{kl}(t-\tau)m_{klj}(t+\tau)\right]_{,j}\\&\qquad+\left[t_{mn}(t+\tau)e_{mn}(t-\tau)-t_{mn}(t-\tau)e_{mn}(t+\tau)\right]\\&\qquad+\left[\tau_{mn}(t+\tau)\epsilon_{mn}(t-\tau)-\tau_{mn}(t-\tau)\epsilon_{mn}(t+\tau)\right]\\&\qquad+\left[m_{klj}(t+\tau)\gamma_{klj}(t-\tau)-m_{klj}(t-\tau)\gamma_{klj}(t+\tau)\right].\end{aligned}\tag{56}$$

We now intend to obtain another expression for the last two terms from identity (56). With the help of the constitutive relations $(3)_1$-$(3)_5$, we get:

$$
\begin{aligned}
&[t_{mn}(t+\tau)e_{mn}(t-\tau)-t_{mn}(t-\tau)e_{mn}(t+\tau)] \\
&\quad [t_{mn}(t+\tau)\epsilon_{mn}(t-\tau)-t_{mn}(t-\tau)\epsilon_{mn}(t+\tau)] \\
&\quad +\left[m_{klj}(t+\tau)\gamma_{klj}(t-\tau)-m_{klj}(t-\tau)\gamma_{klj}(t+\tau)\right] \\
&\quad =\left[h_m(t-\tau)\varphi_{,m}(t+\tau)-h_m(t+\tau)\varphi_{,m}(t-\tau)\right] \\
&\quad +\varrho\left[\vartheta(t-\tau)\eta(t+\tau)-\vartheta(t+\tau)\eta(t-\tau)\right].
\end{aligned} \tag{57}
$$

Considering Eq. (6) for the balance of the equilibrated forces and the kinematic equations (1), we deduce:

$$
\begin{aligned}
&h_m(t-\tau)\varphi_m(t+\tau)-h_m(t+\tau)\varphi_m(t-\tau) \\
&\quad =[h_m(t-\tau)\varphi(t+\tau)-h_m(t+\tau)\varphi(t-\tau)]_{,m} \\
&\quad +\varrho\left[\varphi(t+\tau)l(t-\tau)-\varphi(t-\tau)l(t+\tau)\right] \\
&\quad +\varrho\kappa\left[\varphi(t-\tau)\ddot{\varphi}(t+\tau)-\varphi(t+\tau)\ddot{\varphi}(t-\tau)\right].
\end{aligned} \tag{58}
$$

Now we use the relation (7) in order to obtain:

$$
\begin{aligned}
&\varrho\left[\vartheta(t-\tau)\eta(t+\tau)-\vartheta(t+\tau)\eta(t-\tau)\right]=\varrho\eta(0)\left[\vartheta(t-\tau)-\vartheta(t+\tau)\right] \\
&\quad +\frac{\varrho}{T_0}\left[\vartheta(t-\tau)\int_0^{t+\tau}r(z)dz-\vartheta(t+\tau)\int_0^{t-\tau}r(z)dz\right] \\
&\quad +\frac{1}{T_0}\left[\vartheta(t-\tau)\int_0^{t+\tau}q_m(z)dz-\vartheta(t+\tau)\int_0^{t-\tau}q_m(z)dz\right]_{,m} \\
&\quad +\frac{1}{T_0}k_{mn}\left[\vartheta_{,m}(t+\tau)\int_0^{t-s}\vartheta_{,m}(z)dz-\vartheta_{,m}(t-\tau)\int_0^{t+s}\vartheta_{,m}(z)dz\right].
\end{aligned} \tag{59}
$$

We now substitute the results from identities (59) and (58) into (57) and the identity that results is substituted in (56). In this way, we deduce:

$$
\begin{aligned}
&-\frac{d}{d\tau}\Big\{\varrho\left[v_m(t+\tau)\dot{v}_m(t-\tau)+\dot{v}_m(t+\tau)v_m(t-\tau)\right]\Big\} \\
&\quad -\frac{d}{d\tau}\Big\{I_{mn}\left[\phi_{mr}(t+\tau)\dot{\phi}_{nr}(t-\tau)+\dot{\phi}_{mr}(t+\tau)\phi_{nr}(t-\tau)\right]\Big\} \\
&\quad -\frac{d}{d\tau}\Big\{\varrho\kappa\left[\varphi(t-\tau)\dot{\varphi}(t+\tau)+\varphi(t+\tau)\dot{\varphi}(t-\tau)\right]\Big\} \\
&\quad -\frac{d}{d\tau}\left[\frac{1}{T_0}k_{mn}\left(\int_0^{t+\tau}\vartheta_{,m}(z)dz\right)\left(\int_0^{t-\tau}\vartheta_{,n}(z)dz\right)\right] \\
&\quad =\varrho\left[v_m(t+\tau)f_m(t-\tau)-v_m(t-\tau)f_m(t+\tau)\right] \\
&\quad +\varrho\left[\phi_{mn}(t+\tau)g_{mn}(t-\tau)-\phi_{mn}(t-\tau)g_{mn}(t+\tau)\right] \\
&\quad +\varrho\left[\varphi(t+\tau)l(t-\tau)-\varphi(t-\tau)l(t+\tau)\right] \\
&\quad +\frac{\varrho}{T_0}\left[\vartheta(t-\tau)\int_0^{t+\tau}r(z)dz-\vartheta(t+\tau)\int_0^{t-\tau}r(z)dz\right] \\
&\quad +\varrho\eta(0)\left[\vartheta(t-\tau)-\vartheta(t+\tau)\right] \\
&\quad +\left[v_m(t+\tau)t_{mn}(t-\tau)-v_m(t-\tau)t_{mn}(t+\tau)\right]_{,n} \\
&\quad +\left[\phi_{kl}(t+\tau)m_{klj}(t-\tau)-\phi_{kl}(t-\tau)m_{klj}(t+\tau)\right]_{,j} \\
&\quad +\left[h_m(t-\tau)\varphi(t+\tau)-h_m(t+\tau)\varphi(t-\tau)\right]_{,m} \\
&\quad +\frac{1}{T_0}\left[\vartheta(t-\tau)\int_0^{t+\tau}q_m(z)dz-\vartheta(t+\tau)\int_0^{t-\tau}q_m(z)dz\right]_{,m}.
\end{aligned} \tag{60}
$$

Finally, identity (60) is integrated over $D\times[0,t]$ so that with the help of the theorem of divergence, we find the equality (49) and so we end the proof of Theorem 3. □

4 Evolution of solutions

We need some auxiliary results in order to obtain the basic results of the present section, regarding the evolution of solutions of $\mathcal{P}$, as it is defined in Sect. 2.

We assume that the dipolar porous body occupies, at the initial time $t = 0$, a regular domain D of three-dimensional space R^3. The border of D is noted by ∂D and it must allow the application of the theorem of divergence. Fixing $T > 0$, we can define a new space Ω_T, consists of any $x \in \bar{D}$, in the following situations:

1. If $x \in D$, then

$$\begin{aligned} &v_m^0(x) \neq 0 \;\text{ or }\; v_m^1(x) \neq 0 \;\text{ or }\; \phi_{mn}^0(x) \neq 0 \;\text{ or }\; \phi_{mn}^1(x) \neq 0 \;\text{ or} \\ &\varphi^0(x) \neq 0 \;\text{ or }\; \varphi^1(x) \neq 0 \;\text{ or }\; \vartheta^0(x) \neq 0 \;\text{ or }\; \eta^0(x) \neq 0 \;\text{ or} \quad (61)\\ &f_m(t,x) \neq 0 \text{ or } g_{mn}(t,x) \neq 0 \text{ or } l(t,x) \neq 0 \text{ or } r(t,x) \neq 0,\;\; t \in [0,T]. \quad (62) \end{aligned}$$

2. If $x \in \partial D$, then

$$\begin{aligned} &\bar{v}_m(t,x) \neq 0 \;\text{ or }\; \bar{t}_m(t,x) \neq 0 \;\text{ or }\; \bar{\phi}_{kl}(t,x) \neq 0 \;\text{ or }\; \bar{m}_{kl}(t,x) \neq 0 \;\text{ or} \\ &\bar{\varphi}(t,x) \neq 0 \;\text{ or }\; \bar{h}(t,x) \neq 0 \;\text{ or }\; \bar{\vartheta}(t,x) \neq 0 \;\text{ or }\; \bar{\eta}(t,x) \neq 0,\;\; t \in [0,T]. \end{aligned} \quad (63)$$

From the above situations, we can observe that the space Ω_T is, in fact, the support for the boundary and initial conditions and, also, for the body charges for the problem $\mathcal{P}$, considered on $[0, T]$. For $R \geq 0$, we define the set Ω_R, defined by

$$\Omega_R = \left\{ \bar{x} \in \bar{D} \;:\; \Omega_R^* \cap \bar{S}(x,t) \neq 0 \right\}. \quad (64)$$

Here, we denoted by $\bar{S}(x,t)$ the ball with center at x and having radius r. We have also noted with Ω_T^* the smallest regular surface of ∂D that includes Ω_T.

In what follows, we will use two new notations. So, we will denote by B_R a subset of D such that $B_R = D \setminus D_r$ and for $R_1 > R_2$ we set $B(R_1, R_2) = B_{R_2} \setminus B_{R_1}$. Another notation is S_R and it is a subset of ∂D_R, included inside of D and having the normal oriented to the exterior of D_R. Let us consider a solution $(v_m, \phi_{mn}, \varphi, \vartheta)$ of the problem $\mathcal{P}$ and associate it with the following time-weighted surface power function:

$$I(R,t) = -\int_0^t \int_{S_R} e^{-\lambda s} \left[t_m(s)\dot{v}_m(s) + m_{kl}(s)\dot{\phi}_{kl}(s) + h(s)\dot{\varphi}(s) + \frac{1}{T_0} q(s)\vartheta(s) \right] dA d\tau. \quad (65)$$

This function is well defined for any $t \in [0, T]$ and any $R \geq 0$. In (65) $\lambda > 0$ is a given parameter. Also, the functions $t_M(s)$, $m_{kl}(s)$, $h(s)$ and $q(s)$ are introduced in Eq. (33). In the following, we will use the integral of function I, denoted by J, defined by:

$$J(R,t) = \int_0^R I(R,s) d\tau, \;\; R \geq 0, t \in [0,T]. \quad (66)$$

In the following theorem, we will formulate and prove some properties of the time-weighted surface power function I, defined in (65).

Theorem 4 *Consider the time-weighted function $I(R,t)$, corresponding to a solution $(v_m,\ \phi_{mn},\ \varphi,\ \vartheta)$ of problem $\mathcal{P}$. For every $t \in [0, T]$ and $R \geq 0$, the function $I(R,t)$ has the following properties:*
(i). If $0 \leq R_2 \leq R_1$, then

$$\begin{aligned} &I(R_1,t) - I(R_2,t) \\ &= \int_{B(R_1,R_2)} e^{-\lambda t} \left\{ \frac{1}{2}\left[\varrho \dot{v}_m(t)\dot{v}_m(t) + I_{mn}\dot{\phi}_{mr}(t)\dot{\phi}_{nr}(t) + \varrho\kappa\dot{\varphi}^2(t) \right] + \frac{1}{2} a\vartheta^2(t) + \Psi(E(t)) \right\} dV \\ &\quad - \int_0^t \int_{B(R_1,R_2)} e^{-\lambda s} \left\{ \frac{1}{2}\left[\varrho \dot{v}_m(s)\dot{v}_m(s) + I_{mn}\dot{\phi}_{mr}(s)\dot{\phi}_{nr}(s) + \varrho\kappa\dot{\varphi}^2(s) \right] \right\} dV d\tau \\ &\quad + - \int_0^t \int_{B(R_1,R_2)} e^{-\lambda s} \left\{ \lambda\Psi(E(s)) + \frac{\lambda}{2} a\vartheta^2(s) + \frac{1}{T_0} k_{mn}\vartheta_{,m}(s)\vartheta_{,m}(s) \right\} dV d\tau. \end{aligned} \quad (67)$$

(ii). $I(R,t)$ is a continuous differentiable function, with regard to the variable R. By direct derivation, we obtain:

$$\frac{\partial I}{\partial R}=\int_{S_R} e^{-\lambda t}\left\{\frac{1}{2}\left[\varrho \dot{v}_m(t)\dot{v}_m(t)+I_{mn}\dot{\phi}_{mr}(t)\dot{\phi}_{nr}(t)+\varrho\kappa\dot{\varphi}^2(t)\right]+\Psi(E)+\frac{1}{2}a\vartheta^2(t)\right\}dA$$
$$-\int_0^t\int_{S_R} e^{-\lambda s}\left\{\frac{1}{2}\left[\varrho \dot{v}_m(s)\dot{v}_m(s)+I_{mn}\dot{\phi}_{mr}(s)\dot{\phi}_{nr}(s)+\varrho\kappa\dot{\varphi}^2(s)\right]\right\}dAd\tau$$
$$+-\int_0^t\int_{S_R} e^{-\lambda s}\left\{\lambda\Psi(E(s))+\frac{\lambda}{2}a\vartheta^2(s)+\frac{1}{T_0}k_{mn}\vartheta_{,m}(s)\vartheta_{,n}(s)\right\}dAd\tau. \tag{68}$$

(iii). $I(R,t)$ is a non-increasing function with respect to R.
(iv). For each $R\geq 0$, $I(R,t)$ is a solution of differential inequality, of first order, of the form (see also [24]):

$$\frac{\partial I}{\partial R}(R,t)+\frac{\lambda}{c}\left|\,I(R,t)\right|\leq 0, \tag{69}$$

in which we have used the notation:

$$c=\sqrt{\frac{(1+\varepsilon_0)\,\mu_M}{\varrho_0}}. \tag{70}$$

Also, $\varepsilon_0>0$ is a solution root of the following second-order equation:

$$x^2+x\left(1-\frac{M^2}{a_0\mu_M}-\frac{\lambda\varrho_0 k_M}{2a_0T_0\mu_M}\right)-\frac{M^2}{a_0\mu_M}=0. \tag{71}$$

(v). Function $I(R,t)$ is positive.

Proof For $R_1\geq R_2\geq 0$, we will insert $B(R_1,R_2)$ instead of D in Theorem 1. Taking into account the definitions of $B(R_1,R_2)$ and $I(R,t)$, by using identity (35) we obtain the statement i). If we use the hypotheses (9) and (10), taking into account the equality (67) we obtain the affirmation ii). Also, part iii). follows from (67) with the aid of inequalities (13). Let us prove the assertion iv). Applying the inequality of Schwarz and the mean inequality from (65), we get:

$$\left|\,I(R,t)\right|\leq\int_0^t\int_{S_R} e^{-\lambda s}\left\{\frac{\varepsilon_1}{2\varrho_0}\left[t_{mn}(s)t_{mn}(s)+\tau_{mn}(s)\tau_{mn}(s)+m_{mnr}(s)m_{mnr}(s)+\right.\right.$$
$$\left.+h_m(s)h_m(s)\right]+\frac{1}{2\varepsilon_1}\left[\varrho\dot{v}_m(s)\dot{v}_m(s)+I_{mn}\dot{\phi}_{mr}(s)\dot{\phi}_{nr}(s)+\varrho\kappa\dot{\varphi}^2\right]$$
$$\left.+\frac{\varepsilon_2}{2T_0a_0}q_m(s)q_m(s)+\frac{1}{2T_0\varepsilon_2}a\vartheta^2(s)\right\}dAd\tau$$
$$\leq\int_0^t\int_{S_R} e^{-\lambda s}\left\{\frac{1}{\lambda\varepsilon_1}.\frac{\lambda}{2}\left[\varrho\dot{v}_m(s)\dot{v}_m(s)+I_{mn}\dot{\phi}_{mr}(s)\dot{\varphi}_{nr}(s)+\varrho\kappa\dot{\varphi}^2(s)\right]+\right.$$
$$+\frac{\varepsilon_1(1+\varepsilon)\mu_M}{\lambda\varrho_0}.\lambda\Psi(E(s))+\left[\frac{\varepsilon_1M^2}{\lambda a_0\varrho_0}\left(\varepsilon+\frac{1}{\varepsilon}\right)+\frac{1}{\lambda T_0\varepsilon_2}\right].\frac{\lambda}{2}a\vartheta^2(s)$$
$$\left.+\frac{\varepsilon_2k_M}{2a_0}.\frac{1}{T_0}k_{mn}\vartheta_{,m}(s)\vartheta_{,n}(s)\right\}dAd\tau,\ \ R\geq 0,\ 0\leq t\leq T,\ \forall\,\varepsilon_1,\ \varepsilon_2>0. \tag{72}$$

Now, the integral from the right-side hand of (72), we equate energy coefficients:

$$\frac{1}{\lambda\varepsilon_1}=\frac{\varepsilon_1(1+\varepsilon)\mu_M}{\lambda\varrho_0}=\frac{\varepsilon_1M^2}{\lambda a_0\varrho_0}\left(\varepsilon+\frac{1}{\varepsilon}\right)+\frac{1}{\lambda T_0\varepsilon_2}=\frac{\varepsilon_2k_M}{2a_0} \tag{73}$$

In view of (73) we set

$$\varepsilon_1=\frac{1}{c},\ \ \varepsilon_2=\frac{2a_0c}{\lambda k_M}, \tag{74}$$

where c has the expression (70). So, taking into account relations (68) and (72) we obtain the relation (69). To prove the result v) it is sufficient to use the definitions of the set Ω_T and of the power function $I(R,t)$ and, also, the assertion iii). With this, the proof of Theorem 4 is complete. □

Corollary *The function $J(R,t)$ defined in (69) satisfies a first-order differential inequality of the form*

$$\left| J(R,t)\right| + t\gamma(t)\frac{\partial J}{\partial r}(R,t) \leq 0,\ 0 \leq t \leq T,\ R \geq 0, \tag{75}$$

where we have used the notation:

$$\gamma(t) = \sqrt{\frac{(1+\delta_0(t))\,\mu_M}{\varrho_0}}, \tag{76}$$

in which $\delta_0(t)$ is a solution of the following second-order equation:

$$\xi^2 + \xi\left(1 - \frac{M^2}{a_0\mu_M} - \frac{\varrho_0 k_M}{2ta_0T_0\mu_M}\right) - \frac{M^2}{a_0\mu_M} = 0. \tag{77}$$

Proof It is no difficult to prove the inequality:

$$\int_0^t \int_0^s f^2(\xi)d\xi d\tau \leq t\int_0^s f^2(\xi)dz\xi. \tag{78}$$

Using the same procedure as in proof of point iv), in Theorem 4, and taking into account inequality (78), we obtain the inequality (75). □

Now we can prove the result on the spatial evolution of any solution of the problem $\mathcal{P}$ if the domain D is bounded. As such, behaviors will be appreciated by using the functions $J(t,R)$ and $I(t,R)$.

Theorem 5 *Consider a bounded domain D and the time-weighted function $I(R,t)$, corresponding to a solution $(v_m,\ \phi_{mn},\ \varphi,\ \vartheta)$ of the mixed problem $\mathcal{P}$. We assume that the body charges, the boundary relations and initial values have as support the set Ω_T, included in the interval $[0,T]$. For each $t\in[0,T]$, any solution of $\mathcal{P}$ decays, regarding to the measures $I(t,R)$ and $J(t,R)$, namely*

$$I(t,R) \leq I(t,0)e^{-\lambda R/c},\ 0 \leq R \leq D_d, \tag{79}$$

$$J(t,R) \leq J(t,0)e^{-R/(t\gamma(t))},\ \ 0 \leq R \leq D_d, \tag{80}$$

where the diameter D_d is for the domain $D\setminus\Omega_T^*$.

Proof In view of the fact that $I(R,t)$ is a positive function and taking into account the expression of the function $J(t,R)$, we can rewrite both differential inequalities that are fulfilled by the functions $I(t,R)$ and $J(t,R)$ in the following form:

$$\frac{\partial}{\partial R}\left[e^{\lambda R/c}I(t,R)\right] \leq 0,\ 0 \leq R \leq D_d, \tag{81}$$

$$\frac{\partial}{\partial R}\left[e^{R/(t\gamma(t))}J(t,R)\right] \leq 0,\ 0 \leq R \leq D_d. \tag{82}$$

If we integrate inequality (81) with respect to variable R, we obtain the estimation (79), and by integrating inequality (82) with regard to R, then we deduce the estimation (80). So, the proof of Theorem 5 is completed□.

We now propose to evaluate the spatial evolution of solution of the mixed problem $\mathcal{P}$ in the situation the dipolar thermoelastic body with pores occupies a domain which is unbounded. In order to achieve this, we will use certain estimations of the Phragmén–Lindelöf type.

Theorem 6 *Let us consider a domain D, which is unbounded, and the time-weighted function $I(r,t)$, corresponding to a solution $(v_m,\ \phi_{mn},\ \varphi,\ \vartheta)$ of the mixed problem $\mathcal{P}$, defined on D. We assume that the body charges and the boundary and initial values have as support the set Ω_T, included in $[0,T]$. For each fixed $t\in[0,T]$, the corresponding solution of the mixed problem $\mathcal{P}$ spatially decays, with respect to functions $J(t,R)$ and $I(t,R)$, in accordance to one of the next cases:*

1. If $I(t,R)\geq 0$ for all $R\geq 0$, then

$$I(t,R) \leq I(t,0)e^{-\lambda R/c},\quad R \geq 0, \tag{83}$$

$$J(t,R) \leq J(t,0)e^{-R/(t\gamma(t))},\quad R \geq 0. \tag{84}$$

2. Suppose $\exists R_1 \geq 0$ *so that* $I(t, R_1) < 0$*. Then, from Theorem 4, point iii), we get* $I(t, R) \leq I(t, R_1) < 0$ *and* $J(t, R) < 0$*, for all* $R \geq R_1$*. In addition, the following estimates hold:*

$$-I(t, R) \geq -I(t, R_1)e^{\lambda(R-R_1)/c}, \quad R \geq R_1, \tag{85}$$

$$-J(t, R) \geq -J(t, R_1)e^{(R-R_1)/c}, \quad 0 \leq R \geq R_1. \tag{86}$$

Proof Taking into account the fact $I(t, R)$ is a non-increasing function with respect to r, according to Theorem 4, part (iii), we obtain:

$$I(t, R) \geq 0, \quad \text{for any} \quad R \geq 0.$$

Then, we are led to the conclusion that the differential inequality (69), fulfilled by the function $I(t, R)$, can be stated in the form (81). So, we obtained the estimation (83). Similarly, inequality (75), fulfilled by the function $J(t, R)$, can be stated as in (82). So, we obtained the estimation (84). If we assume that there exists $R_1 \geq 0$ so that $I(t, R) \leq 0$, then from Theorem 4, part iii) we obtain:

$$I(t, R) < I(t, R_1) \leq 0,$$

for any $R \geq R_1$. Under these conditions, the differential inequality (69) becomes:

$$\frac{\partial}{\partial R}\left[e^{-\lambda R/c} I(t, R)\right] \leq 0, \;\; R \leq R_1, \tag{87}$$

and hence, by integration with respect to R, we obtain (85). Also, since $I(t, R) \leq 0$ we obtain $J(t, R) \leq 0$, considering the expression (66) of the function $J(t, R)$. Because of this, inequality (75) becomes:

$$\frac{\partial}{\partial R}\left[e^{-\lambda R/(t\gamma(t))} J(t, R)\right] \leq 0, \;\; R \leq R_1, \tag{88}$$

and hence, by integration with respect to R, we obtain (86). This ends the proof of Theorem 6. □

5 Conclusions

Let us make an analysis of the previous estimates demonstrated in our study. So, we deduced the estimates (79), (83) and (85), which are conveniently for certain short moments of time, while the estimations (80), (84) and (86) are conveniently for certain long values of the time variable. This is why we have coupled the demonstrations of the previous estimates, like this: (79) is coupled with (80), (83) is coupled with (84), and (85) is coupled with (86). With these couplings, we can get a comprehensive description for the spatial evolution of any solution of the mixed problem $\mathcal{P}$.

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