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Uniqueness theorem in coupled strain gradient elasticity with mixed boundary conditions

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Abstract The equilibrium equations and the traction boundary conditions are evaluated on the basis of the condition of the stationarity of the Lagrangian for coupled strain gradient elasticity. The quadratic form of strain energy can be written as a function of the strain and the second gradient of displacement and contains a fourth-, a fifth- and a sixth-order stiffness tensor \mathbb{C}_4 , \mathbb{C}_5 and \mathbb{C}_6 , respectively. Assuming invariance under rigid body motions the balance of linear and angular momentum is obtained. The uniqueness theorem (Kirchhoff) for the mixed boundary value problem is proved for the case of the coupled linear strain gradient elasticity (novel). To this end, the total potential energy is altered to be presented as an uncoupled quadratic form of the strain and the modified second gradient of displacement vector. Such a transformation leads to a decoupling of the equation of the potential energy density. The uniqueness of the solution is proved in the standard manner by considering the difference between two solutions.

Keywords Coupled strain gradient elasticity · Coupling fifth-rank tensor · Uniqueness of solution

1 Introduction

Strain gradient elasticity, in which the strain energy density is a function of the strain and the second gradient of the displacement vector, is a natural extension of the classical theory of elasticity. Strain gradient elasticity is a particular case of higher-order gradient material theories and has a long history. Since the beginning of the last century, in order to avoid the shortcomings of the classical theory of elasticity, a variety of non-classical theories have been proposed. Cosserat and Cosserat [11] have created the polar media, in which the independent rotations and the associated coupled stresses were firstly introduced in the Eulerian equations of motion. After almost half century the more general theories have been originated in Toupin [50], Mindlin [36], Germain

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[25]. In these theories it was assumed that the potential energy density depends not only on the strain, but also on higher derivatives of the displacement vector. More recently, the generalized non-classical theories have been also applied to modeling of materials at the micro- and nano-scale [10,23] to describing of phenomena like dislocations [21], to analyzing of composites with a high difference of the material properties at a lower scale [2,20,42,45,51] to describing some phenomena in regions with stress concentrations [5], to accounting for boundary and surface energies [14,28] or to removing singularities caused by discontinues of boundary conditions (e.g., [6,24,46,49]). It has been shown in numerous papers (see, for example, [22,34,35,41]) that some restrictions of the classical theory of elasticity can be overcome with such gradient expansion.

The uniqueness of solution of equilibrium equations is a crux of any theory. Within the framework of the classical linear elasticity theory the first uniqueness theorem is given for isotropic materials in Kirchhoff [29], which has been completed by Cosserat and Cosserat [11] for boundary value problems with displacement boundary conditions. The generalization to anisotropic materials can be found in Knops and Payne [30] or Bertram and Glüge [9]. A uniqueness theorem for linearly elastic body with surface stresses is proved in Gurtin and Murdoch [26]. Sufficient condition for the uniqueness of the solution of the mixed boundary value problem is the definiteness of the stiffness tetrad \mathbb{C}_4 . Requiring additionally stability of the solution yields the condition that \mathbb{C}_4 needs to be positive definite. The necessity of this condition has been shown by Cosserat and Cosserat [11].

Modern existence and uniqueness theorems employ different versions of Korn's inequality [31,32], which is an adaption of Poincaré's inequality [43] to problems that involve a projection of the differential operator into the symmetric part. With its help, the requirements for existence and uniqueness of a solution to a variational form of the boundary value problem of classical linear elasticity are reduced to requirements according to the theorem of Babuška–Lax–Milgram [7,33,48].

Conditions sufficient for uniqueness solution are considered for uncoupled isotropic gradient elasticity in Mindlin and Eshel [38]. Usually, positive definiteness of the stored elastic energy is assumed, see Sect. 2, Eq. (1), which reduces to positive definiteness of the stiffness tensors \mathbb{C}_4 , and \mathbb{C}_6 , while $\mathbb{C}_5 = \mathbb{O}$ was presumed as well. These are sufficient conditions, but they are surely not necessary. The number of possibilities of boundary value problems in gradient elasticity is much larger compared to classical elasticity. Nevertheless, Mindlin's uniqueness proof is of reasonable generality: it holds for mixed boundary conditions, and no restrictions regarding the shape of the domain are necessary. Neglecting of the constitutive coupling tensor of fifth-rank \mathbb{C}_5 leads to requirement of positive definiteness of both first and second gradient constitutive stiffness tensors of fourth- and sixth-ranks \mathbb{C}_4 and \mathbb{C}_6 . Inequality constraints on first and on second gradient constitutive parameters in absence of coupling terms were also presented in Mindlin [36], dell'Isola et al. [12]. In Nazarenko et al. [39] results of Mindlin [36] and dell'Isola et al. [12] have been extended for the case of coupled strain gradient elasticity. In this regard, it has been introduced a block diagonalization of the composite stiffness in strain gradient elasticity. By such a formal transformation, which contains a fourth-, a fifth- and a sixth-order stiffness tensor \mathbb{C}_4 , \mathbb{C}_5 and \mathbb{C}_6 , necessary conditions for positive definiteness and convexity of the isotropic strain and strain gradient energy, accounting for the coupling stiffness \mathbb{C}_5 have been obtained. Closed-form relations for the compliance tensors in the frame of the linear theory of coupled gradient elasticity have been given for arbitrary material symmetry classes in Nazarenko et al. [40].

The uniqueness of the solution for an isotropic material featured by four constants (simplified Aifantis model, see, e.g., [3,47]) has been proved for the dynamic evolution problem in Polizzotto [44]. An existence and uniqueness theorem for weak solutions of the equilibrium problem for the simplified case of the linear isotropic strain gradient elasticity (dilatational strain gradient elasticity) is given in Eremeyev et al. [14] and was extended to nonlinear case in Eremeyev et al. [18] and to non-smooth domains with edges in Eremeyev and dell'Isola [15]. In particular, the presence of edges changes the regularity of solutions. In the case of degeneration of the elastic moduli, uniqueness of solutions is proved for a specific boundary conditions, see, e.g., Eremeyev et al. [13,17].

The aim of the paper is to study the uniqueness of solution of the mixed boundary problem in the frame of the linear coupled strain gradient elasticity, which was not done in the previous papers (novelty). This paper has the following structure. Notations used in the paper are introduced in the next section. The equilibrium equation and the traction boundary conditions for the coupled strain gradient elasticity are re-derived on the basis of the principle of virtual power in Sect. 3. The difference between the equations presented here and in Mindlin and Eshel [38] is in index associations of the scalar products. This is the result of the presence of a coupled term in the strain and strain gradient energy Eq. (1) and of the symmetry of the stiffness tensor of fifth-rank. The potential energy density Eq. (1) of Sect. 2 is assumed to be a function of the strain and the second gradient of the displacement vector and contains a fourth-, a fifth- and a sixth-order stiffness tensors

\mathbb{C}_4 , \mathbb{C}_5 and \mathbb{C}_6 , respectively. In Sect. 4, the balances of the linear and the angular momentum are employed in the derivation of the equations of strain gradient elasticity. Starting with the balance of the linear and the angular momentum we recover the principle of virtual power. The uniqueness theorem (Kirchhoff) for mixed boundary value problem is extended to the coupled linear strain gradient elasticity in Sect. 5. Conclusions and discussion are presented in the last section.

2 Notation

Scalars, vectors, second- and higher-rank tensors are denoted by italic letters (like a or A), bold minuscules (like \mathbf{a}), bold majuscules (like \mathbf{A}), and blackboard bold majuscules (like \mathbb{A}), respectively. The basic mathematical operations are given, for example, in Altenbach [4], Eremeyev et al. [16].

The elastic energy density taking into account strain and strain gradient tensors can be written as

$$w = \frac{1}{2} \mathbf{E}_2 \cdot \mathbb{C}_4 \cdot \mathbf{E}_2 + \mathbf{E}_2 \cdot \mathbb{C}_5 \cdots \mathbb{E}_3 + \frac{1}{2} \mathbb{E}_3 \cdots \mathbb{C}_6 \cdots \mathbb{E}_3, \quad (1)$$

where \mathbb{C}_4 , \mathbb{C}_5 , \mathbb{C}_6 are the stiffness tensors of 4th, 5th and 6th order, respectively. The strain tensor and the second gradient of displacement vector are

$$\mathbf{E}_2 = \frac{1}{2} (\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u}) = \text{sym}(\mathbf{u} \otimes \nabla), \quad \mathbb{E}_3 = \mathbf{u} \otimes \nabla \otimes \nabla \quad (2)$$

functions of $\mathbf{u}(\mathbf{x})$, where $\mathbf{u}(\mathbf{x})$ is the displacement field, with the position vector of a material point \mathbf{x} . For the sake of simplicity, the independent variable \mathbf{x} is dropped. ∇ is the three-dimensional nabla operator defined as

$$\nabla \equiv \frac{\partial}{\partial x_i} \mathbf{e}_i \quad (i = 1, 2, 3),$$

where \mathbf{e}_i is an orthonormal base vector. It is implied a summation by the repeating indices. \otimes denotes the dyadic product. A gradient of the displacement field \mathbf{u} is defined by action of the nabla operator on \mathbf{u} :

$$\mathbf{u} \otimes \nabla = \frac{\partial u_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j = u_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j. \quad (3)$$

On a surface, the spatial nabla operator can be split into a normal and a tangential (surface) part,

$$\nabla = \nabla_n + \nabla_s, \quad (4)$$

with

$$\nabla_n \equiv \mathbf{n} \otimes \mathbf{n} \cdot \nabla = \nabla \cdot \mathbf{n} \otimes \mathbf{n} = \frac{\partial}{\partial x_j} n_j \mathbf{n}, \quad (5)$$

and

$$\nabla_s \equiv \nabla \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}). \quad (6)$$

Here \mathbf{I} is the three-dimensional unit dyadic.

The stresses and the double stresses are defined as

$$\mathbf{T}_2 = \frac{\partial w}{\partial \mathbf{E}_2} = \mathbb{C}_4 \cdot \mathbf{E}_2 + \mathbb{C}_5 \cdots \mathbb{E}_3, \quad (7)$$

$$\mathbb{T}_3 = \frac{\partial w}{\partial \mathbb{E}_3} = \mathbb{C}_5^T \cdot \mathbf{E}_2 + \mathbb{C}_6 \cdots \mathbb{E}_3. \quad (8)$$

The subscript indices correspond the order of tensors $\mathbf{E}_2, \mathbb{E}_3, \mathbb{C}_4, \mathbb{C}_5, \mathbb{C}_6$, which have the following symmetries (see, e.g., [27])

$$E_{ij} = E_{ji}, \quad (9)$$

$$E_{ijk} = E_{ikj}, \quad (10)$$

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \quad (11)$$

$$C_{ijklm} = C_{jiklm} = C_{ijkml}, \quad (12)$$

$$C_{ijklmn} = C_{lmnij} = C_{ikjlmn} = C_{ijklnm}. \quad (13)$$

The dots denote scalar contractions

$$\begin{aligned} & \mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_k \underbrace{\cdots}_{n \text{ dots}} \mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_l \\ &= (\mathbf{a}_{k-n} \cdot \mathbf{b}_1) \cdots (\mathbf{a}_k \cdot \mathbf{b}_n) \mathbf{a}_1 \otimes \mathbf{a}_{k-n-1} \otimes \mathbf{b}_{n+1} \otimes \cdots \otimes \mathbf{b}_l. \end{aligned} \quad (14)$$

The double and triple scalar contractions in Eqs. (1), (7), (8) act in accordance with the following rules

$$\underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \cdot \mathbf{e}_k \otimes \mathbf{e}_l}_{\text{double contraction}} = \delta_{ik} \delta_{jl}, \quad (15)$$

$$\underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \cdots \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n}_{\text{triple contraction}} = \delta_{il} \delta_{jm} \delta_{kn}, \quad (16)$$

where δ_{ij} is the Kronecker symbol.

For a dyadic \mathbf{D}_2 , the symmetric part is $\text{sym}(\mathbf{D}_2)$, the skew part is $\text{skw}(\mathbf{D}_2)$, and the axial vector of the skew part of \mathbf{D}_2 is noted as $\text{axi}(\mathbf{D}_2)$. $\text{axi}(\mathbf{D}_2)$ is defined by its action on arbitrary vector \mathbf{a} according to the rule

$$\text{skw}(\mathbf{D}_2) \cdot \mathbf{a} = \text{axi}(\mathbf{D}_2) \times \mathbf{a}. \quad (17)$$

Here \times denotes the vector product.

3 Variational formulation of the equilibrium equation in the linear strain gradient elasticity

The equilibrium equations and the corresponding natural boundary conditions can be derived on the basis of the Lagrangian variational principle (see, e.g., [1, 14]) applied to the coupled linear strain gradient continua.

3.1 Variation of the strain energy for coupled linear strain gradient continua

The total potential energy in a volume V with a variation of \mathbf{u} is

$$\begin{aligned} \delta \int_V w \, dV &= \int_V (\mathbb{T}_2 \cdot \cdot \delta \mathbf{E}_2 + \mathbb{T}_3 \cdot \cdot \cdot \delta \mathbb{E}_3) \, dV \\ &= \int_V [\mathbb{T}_2 \cdot \cdot (\delta \mathbf{u} \otimes \nabla) + \mathbb{T}_3 \cdot \cdot \cdot (\delta \mathbf{u} \otimes \nabla \otimes \nabla)] \, dV, \end{aligned} \quad (18)$$

where the stresses \mathbb{T}_2 and the double stresses \mathbb{T}_3 are determined in Eqs. (7) and (8).

Applying the chain rule and the divergence theorem, the right-hand of Eq. (18) is reduced to the sum of volume and surface integrals

$$\begin{aligned} \delta \int_V w \, dV &= - \int_V \delta \mathbf{u} \cdot (\mathbb{T}_2 - \mathbb{T}_3 \cdot \nabla) \cdot \nabla \, dV \\ &\quad + \int_S \delta \mathbf{u} \cdot (\mathbb{T}_2 - \mathbb{T}_3 \cdot \nabla) \cdot \mathbf{n} \, dS \\ &\quad + \int_S \delta \mathbf{u} \otimes \nabla \cdot \cdot \mathbb{T}_3 \cdot \mathbf{n} \, dS, \end{aligned} \quad (19)$$

where S is the boundary of V and \mathbf{n} is the unit outward normal to S .

It should be noted, that gradient of the displacement variation $\delta\mathbf{u} \otimes \nabla$ and variation of the displacement $\delta\mathbf{u}$ on the surface S are dependent. Indeed, knowing $\delta\mathbf{u}$ on S it is possible to determine the surface gradient of $\delta\mathbf{u}$. Following Toupin [50] and Mindlin [37] we decompose the gradient of $\delta\mathbf{u}$ into its normal and tangential parts in accordance with Eqs. (4)–(6)

$$\delta\mathbf{u} \otimes \nabla = \delta\mathbf{u} \otimes (\nabla_n + \nabla_s). \quad (20)$$

Using identity Eq. (20) the last term, in the integrand in Eq. (19), can be written down as a sum

$$\delta\mathbf{u} \otimes \nabla \cdot \cdot \mathbb{T}_3 \cdot \mathbf{n} = \delta\mathbf{u} \otimes \nabla_n \cdot \cdot \mathbb{T}_3 \cdot \mathbf{n} + \delta\mathbf{u} \otimes \nabla_s \cdot \cdot \mathbb{T}_3 \cdot \mathbf{n}. \quad (21)$$

After applying the chain rule, the last term in Eq.(21) containing the surface gradient can be presented in the following form:

$$\delta\mathbf{u} \otimes \nabla_s \cdot \cdot \mathbb{T}_3 \cdot \mathbf{n} = (\delta\mathbf{u} \cdot \mathbb{T}_3 \cdot \mathbf{n}) \cdot \nabla_s - \delta\mathbf{u} \cdot (\mathbb{T}_3 \cdot \mathbf{n}) \cdot \nabla_s. \quad (22)$$

We use the surface divergence theorem in the form (see, [37], p. 435)

$$\int_S \mathbf{v} \cdot \nabla_s dS = \int_S (\mathbf{n} \cdot \nabla_s) \mathbf{v} \cdot \mathbf{n} dS + \oint_C \mathbf{v} \cdot \mathbf{m} dC, \quad (23)$$

where C is the union of all edges of the domain V , \mathbf{t} is the unit tangent to the edge, and $\mathbf{m} = \mathbf{t} \times \mathbf{n}$ is the unit outward normal to C tangent to \mathbf{t} . Taking $\mathbf{v} = \delta\mathbf{u} \cdot \mathbb{T}_3 \cdot \mathbf{n}$ in (23) gives

$$\begin{aligned} \int_S (\delta\mathbf{u} \cdot \mathbb{T}_3 \cdot \mathbf{n}) \cdot \nabla_s dS &= \int_S \delta\mathbf{u} \cdot [\mathbb{T}_3 \cdot \cdot \mathbf{n} \otimes \mathbf{n}] (\mathbf{n} \cdot \nabla_s) dS \\ &+ \oint_C \delta\mathbf{u} \cdot [\mathbb{T}_3 \cdot \cdot \mathbf{n} \otimes \mathbf{m}] dC. \end{aligned} \quad (24)$$

Using identity (24) and Eq. (21) the last term, in Eq. (19) for the variation of the total potential energy can be written down in the following form:

$$\begin{aligned} &\int_S \delta\mathbf{u} \otimes \nabla \cdot \cdot \mathbb{T}_3 \cdot \mathbf{n} dS \\ &= \int_S \{\delta\mathbf{u} \otimes \nabla_n \cdot \cdot \mathbb{T}_3 \cdot \mathbf{n} + (\delta\mathbf{u} \cdot \mathbb{T}_3 \cdot \mathbf{n}) \cdot \nabla_s - \delta\mathbf{u} \cdot (\mathbb{T}_3 \cdot \mathbf{n}) \cdot \nabla_s\} dS \\ &= \int_S \delta\mathbf{u} \otimes \nabla_n \cdot \cdot \mathbb{T}_3 \cdot \mathbf{n} dS \\ &+ \int_S \delta\mathbf{u} \cdot \{\mathbb{T}_3 \cdot \cdot [(\mathbf{n} \cdot \nabla_s) \mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \nabla_s] - (\mathbb{T}_3 \cdot \nabla_s) \cdot \mathbf{n}\} dS \\ &+ \oint_C \delta\mathbf{u} \cdot [\mathbb{T}_3 \cdot \cdot \mathbf{n} \otimes \mathbf{m}] dC. \end{aligned} \quad (25)$$

Substituting this into Eq. (19) we obtain

$$\begin{aligned}
\delta \int_V w \, dV &= - \int_V \delta \mathbf{u} \cdot (\mathbf{T}_2 - \mathbb{T}_3 \cdot \nabla) \cdot \nabla \, dV \\
&+ \int_S \delta \mathbf{u} \cdot \{(\mathbf{T}_2 - \mathbb{T}_3 \cdot \nabla - \mathbb{T}_3 \cdot \nabla_s) \cdot \mathbf{n} \\
&+ \mathbb{T}_3 \cdot [(\mathbf{n} \cdot \nabla_s) \mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \nabla_s]\} \, dS \\
&+ \int_S \delta \mathbf{u} \otimes \nabla_n \cdot \mathbb{T}_3 \cdot \mathbf{n} \, dS \\
&+ \oint_C \delta \mathbf{u} \cdot [\mathbb{T}_3 \cdot \mathbf{n} \otimes \mathbf{m}] \, dC. \tag{26}
\end{aligned}$$

Remark 1 There is a qualitative difference between converting volume integrals into surface integrals and surface integrals into edge integrals. While the surface of the volume divides space into inside and outside of the body, this does not hold for edges on the surface. The surface is, so to say, unbounded, unlike the volume. The surface divergence theorem produces two integrals (Eq. 24), the first contains $(\mathbf{n} \cdot \nabla_s) dS$ and the second contains $\delta \mathbf{u} \cdot [\mathbb{T}_3 \cdot \mathbf{n} \otimes \mathbf{m}] dC$. Both can be zero, but not at the same time:

- for bodies bounded only by planar facets, $\mathbf{n} \cdot \nabla_s = 0$, hence the first integral vanishes, and
- for bodies without edges, the edge integral disappears because the surface is unbounded.

One might argue that the second case is the more general case, as one can regularize sharp corners by a tiny but smooth edge radius, and get the first case as the limit to a zero edge radius. Therefore, for simplicity the edge integrals can be summarized into the surface integrals, in the sense of a regularization, which allows to drop the edge integrals. In the remainder we keep the edge integrals for the sake of completeness, but it may be possible to neglect the edges and all the quantities that are defined on them if one is willing to regularize sharp corners, which simplifies the derivation considerably.

3.2 Variational equation of equilibrium and boundary conditions

The Lagrangian l can be written down in the following form:

$$l = a - \int_V w \, dV, \tag{27}$$

where a is a work of external forces and double forces and w is a strain and strain gradient energy density Eq. (1). Then the principle of the Lagrangian stationary (the principle of virtual power) is

$$\delta l = \delta a - \delta \int_V w \, dV.$$

The variation of the strain and strain gradient energy requires an admissible form of the work:

$$a = \int_V \mathbf{u} \cdot \mathbf{f} \, dV + \int_S (\mathbf{u} \cdot \mathbf{p} + \mathbf{u} \otimes \nabla_n \cdot \mathbf{R}) \, dS + \oint_C \mathbf{u} \cdot \mathbf{c} \, dC. \tag{28}$$

Here \mathbf{f} is a body force per unit volume, \mathbf{p} is a surface traction, \mathbf{R} is a surface normal double force on S (e.g., [14]), and \mathbf{c} is a line force on edge C .

Since the normal gradient of displacement is described as

$$\mathbf{u} \otimes \nabla_n = \frac{\partial u_i}{\partial x_j} n_j \mathbf{e}_i \otimes \mathbf{n} = (\mathbf{D}\mathbf{u}) \otimes \mathbf{n}, \tag{29}$$

or

$$\mathbf{D}\mathbf{u} = (\mathbf{u} \otimes \nabla) \cdot \mathbf{n} = \frac{\partial u_i}{\partial x_j} n_j \mathbf{e}_i, \quad (30)$$

the work of external forces can be written as

$$a = \int_V \mathbf{u} \cdot \mathbf{f} dV + \int_S (\mathbf{u} \cdot \mathbf{p} + \mathbf{D}\mathbf{u} \cdot \mathbf{r}_n) dS + \oint_C \mathbf{u} \cdot \mathbf{c} dC, \quad (31)$$

where \mathbf{r}_n is a double traction in normal direction on S (e.g., [8])

$$\mathbf{r}_n = \mathbf{R} \cdot \mathbf{n}. \quad (32)$$

For all admissible functions $\delta\mathbf{u}$ the variation of the potential energy has a form

$$\delta \int_V w dV = \int_V \delta\mathbf{u} \cdot \mathbf{f} dV + \int_S (\delta\mathbf{u} \cdot \mathbf{p} + \mathbf{D}\delta\mathbf{u} \cdot \mathbf{r}_n) dS + \oint_C \delta\mathbf{u} \cdot \mathbf{c} dC, \quad (33)$$

which leads with Eq. (26) to the equation of equilibrium:

$$(\mathbb{T}_2 - \mathbb{T}_3 \cdot \nabla) \cdot \nabla + \mathbf{f} = \mathbf{0}, \quad (34)$$

and to the dynamic boundary conditions for the surface tractions and the edge forces, which can be prescribed:

- the vector field of the tractions on the part of the surface of the body S_d

$$\mathbf{p}_{pr} = (\mathbb{T}_2 - \mathbb{T}_3 \cdot \nabla - \mathbb{T}_3 \cdot \nabla_s) \cdot \mathbf{n} + \mathbb{T}_3 \cdot \cdot [(\mathbf{n} \cdot \nabla_s) \mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \nabla_s], \quad (35)$$

- the double tractions in normal direction on the S_d

$$\mathbf{r}_{n pr} = \mathbb{T}_3 \cdot \cdot \mathbf{n} \otimes \mathbf{n}, \quad (36)$$

- the line forces on edge on the part of edge C_d

$$\mathbf{c}_{pr} = \mathbb{T}_3 \cdot \cdot \mathbf{n} \otimes \mathbf{m}. \quad (37)$$

Here, the subscript pr denotes *prescribed*.

The displacement or kinematic boundary conditions in terms of the displacement fields \mathbf{u} and its normal gradient $\mathbf{D}\mathbf{u}$ on the part of the surface of the body S_g ($S_d \cup S_g = S$), and the displacement \mathbf{u} on the part of the edge are apparent from Eq. (33)

$$\mathbf{u}_{pr} = \mathbf{u} \quad \text{on } S_g, \quad (38)$$

$$\mathbf{D}\mathbf{u}_{pr} = \mathbf{D}\mathbf{u} \quad \text{on } S_g, \quad (39)$$

$$\mathbf{u}_{pr} = \mathbf{u} \quad \text{on } C_g. \quad (40)$$

3.3 Equilibrium equation based on the balance of the virtual power

Let us consider the Lagrangian variational principle (principle of virtual power) Eq. (33) for rigid body motion

$$\delta\mathbf{u} = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{x}, \quad (41)$$

where \mathbf{u}_0 and $\boldsymbol{\omega}$ are two constant vectors. For such $\delta\mathbf{u}$ from Eq. (33) it is possible to obtain the equilibrium equation for a free solid body (e.g., [9, 14])

$$\begin{aligned} \delta a - \delta \int_V w dV = \mathbf{u}_0 \cdot \left\{ \int_V \mathbf{f} dV + \int_S \mathbf{p} dS + \oint_C \mathbf{c} dC \right\} \\ + \boldsymbol{\omega} \cdot \left\{ \int_V \mathbf{x} \times \mathbf{f} dV + \int_S \mathbf{x} \times \mathbf{p} dS + \mathbf{m} + \oint_C \mathbf{x} \times \mathbf{c} dC \right\} = 0, \end{aligned} \quad (42)$$

Thus we have two balance laws for the total force and the total torque in the form:

– the balance of linear momentum

$$\int_V \mathbf{f} dV + \int_S \mathbf{p} dS + \oint_C \mathbf{c} dC = \mathbf{0}, \quad (43)$$

– the balance of angular momentum

$$\int_V \mathbf{x} \times \mathbf{f} dV + \int_S \mathbf{x} \times \mathbf{p} dS + \mathbf{m} + \oint_C \mathbf{x} \times \mathbf{c} dC = \mathbf{0}, \quad (44)$$

where the tractions \mathbf{p} on S , and the line forces on edge \mathbf{c} on C are described as

$$\mathbf{p} = (\mathbf{T}_2 - \mathbb{T}_3 \cdot \nabla - \mathbb{T}_3 \cdot \nabla_s) \cdot \mathbf{n} + \mathbb{T}_3 \cdot [(\mathbf{n} \cdot \nabla_s) \mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \nabla_s], \quad (45)$$

$$\mathbf{c} = \mathbb{T}_3 \cdot \mathbf{n} \otimes \mathbf{m}, \quad (46)$$

and \mathbf{m} is the torque induced by double forces on the surface of the body by the rotational field $\boldsymbol{\omega} \times \mathbf{x}$

$$\boldsymbol{\omega} \cdot \mathbf{m} = \int_V (\boldsymbol{\omega} \times \mathbf{x}) \otimes \nabla_n \cdot \mathbf{R} dV, \quad (47)$$

with

$$\mathbf{R} = \mathbb{T}_3 \cdot \mathbf{n}. \quad (48)$$

For an arbitrary rotational field $\boldsymbol{\omega} \times \mathbf{x}$, where \mathbf{x} is position vector and $\boldsymbol{\omega}$ - any constant vector

$$(\boldsymbol{\omega} \times \mathbf{x}) \otimes \nabla_n = \boldsymbol{\omega} \times (\mathbf{x} \otimes \nabla_n) = \boldsymbol{\omega} \times \mathbf{n} \otimes \mathbf{n}, \quad (49)$$

is antisymmetric. Thus

$$\text{axi}_n[(\boldsymbol{\omega} \times \mathbf{x}) \otimes \nabla_n] = \text{axi}_n(\boldsymbol{\omega} \times \mathbf{n} \otimes \mathbf{n}) = \boldsymbol{\omega}, \quad (50)$$

and, consequently

$$\begin{aligned} (\boldsymbol{\omega} \times \mathbf{x}) \otimes \nabla_n \cdot \mathbf{R} &= (\boldsymbol{\omega} \times \mathbf{x}) \otimes \nabla_n \cdot \text{skw}(\mathbf{R}) \\ &= \text{axi}_n[(\boldsymbol{\omega} \times \mathbf{x}) \otimes \nabla_n] \cdot 2\text{axi}_n(\mathbf{R}) = \boldsymbol{\omega} \cdot 2\text{axi}_n(\mathbf{R}). \end{aligned} \quad (51)$$

Here the axial part of the vector $\text{axi}(\boldsymbol{\omega} \times \mathbf{n} \otimes \mathbf{n})$ is defined according to Eq. (17) and its normal part $\text{axi}_n(\boldsymbol{\omega} \times \mathbf{n} \otimes \mathbf{n})$ analogically as for normal gradient Eqs. (4)–(6).

With the above identity we achieve the balance of angular momentum in the form [8]

$$\int_V \mathbf{x} \times \mathbf{f} dV + \int_S \{\mathbf{x} \times \mathbf{p} + 2\text{axi}_n(\mathbf{R})\} dS + \oint_C \mathbf{x} \times \mathbf{c} dC = \mathbf{0}. \quad (52)$$

It is possible to obtain from the balance of linear and angular momentum the local balance of linear momentum or first Eulerian law of motion

$$(\mathbf{T}_2 - \mathbb{T}_3 \cdot \nabla) \cdot \nabla + \mathbf{f} = \mathbf{0}. \quad (53)$$

Since \mathbf{p} is determined from Eq. (45) and using the following integral transformation

$$\int_S \mathbf{x} \times (\mathbf{T}_2 \cdot \mathbf{n}) dS = \int_V (\mathbf{x} \times \mathbf{T}_2) \cdot \nabla dV = \int_V [\mathbf{x} \times (\mathbf{T}_2 \cdot \nabla) + 2\text{axi}(\mathbf{T}_2)] dV, \quad (54)$$

one can show that $\text{axi}(\mathbf{T}_2) = 0$ and as a consequence the symmetry of the stress tensor (second Eulerian law of motion)

$$\mathbf{T}_2 = \mathbf{T}_2^T \quad (55)$$

follows.

We are now able to reformulate the **principle of virtual power**.

Theorem 1 *A motion of the body is dynamically admissible if and only if the balance of virtual power in the form*

$$\begin{aligned} & \int_V \delta \mathbf{u} \cdot \mathbf{f} \, dV + \int_S (\delta \mathbf{u} \cdot \mathbf{p} + \delta \mathbf{u} \otimes \nabla_n \cdot \mathbf{R}) \, dS + \oint_C \delta \mathbf{u} \cdot \mathbf{c} \, dC \\ &= \int_V (\mathbf{T}_2 \cdot \text{sym}(\delta \mathbf{u} \otimes \nabla) + \mathbb{T}_3 \cdot \delta \mathbf{u} \otimes \nabla \otimes \nabla) \, dV, \end{aligned} \quad (56)$$

holds for all differentiable vector fields $\delta \mathbf{u}$.

Proof Let us assume that a motion of the body is dynamically admissible so that Eqs. (53)–(55) hold. Then the balance of linear momentum is fulfilled if

$$(\mathbf{T}_2 - \mathbb{T}_3 \cdot \nabla) \cdot \nabla + \mathbf{f} = \mathbf{0}, \quad (57)$$

holds in every point, and also

$$\delta \mathbf{u} \cdot \{(\mathbf{T}_2 - \mathbb{T}_3 \cdot \nabla) \cdot \nabla + \mathbf{f}\} = 0, \quad (58)$$

for any differentiable vector fields $\delta \mathbf{u}$. The integral over the body also vanishes

$$\int_V \delta \mathbf{u} \cdot \{(\mathbf{T}_2 - \mathbb{T}_3 \cdot \nabla) \cdot \nabla + \mathbf{f}\} \, dV = 0. \quad (59)$$

Accounting for Eqs. (18), (26) we obtain

$$\begin{aligned} & \int_V \delta \mathbf{u} \cdot \mathbf{f} \, dV + \int_S (\delta \mathbf{u} \cdot \mathbf{p} + \delta \mathbf{u} \otimes \nabla_n \cdot \mathbf{R}) \, dS + \oint_C \delta \mathbf{u} \cdot \mathbf{c} \, dC \\ &= \int_V (\mathbf{T}_2 \cdot \delta \mathbf{u} \otimes \nabla + \mathbb{T}_3 \cdot \delta \mathbf{u} \otimes \nabla \otimes \nabla) \, dV. \end{aligned} \quad (60)$$

In addition, the balance of moment of momentum Eq. (55) is fulfilled if

$$\mathbf{T}_2 \cdot \text{skw}(\delta \mathbf{u} \otimes \nabla) = 0, \quad (61)$$

or

$$\int_V \mathbf{T}_2 \cdot \text{skw}(\delta \mathbf{u} \otimes \nabla) \, dV = 0 \quad (62)$$

holds for any differentiable vector fields $\delta \mathbf{u}$. We subtract this from equation (60) and obtain Eq. (56). \square

4 Extension of Kirchhoff's uniqueness theorem to coupled strain gradient elasticity

In this subsection it is demonstrated that the solution of the mixed boundary value problem, if exists, is unique. We consider a **mixed boundary value problem**:

- the stress equation of equilibrium

$$(\mathbf{T}_2 - \mathbb{T}_3 \cdot \nabla) \cdot \nabla + \mathbf{f} = \mathbf{0}, \quad (63)$$

- prescribed are:

- the body force \mathbf{f} in the interior of the body V ,
- the vector field of the tractions \mathbf{p}_{pr} on the surface of the body S_d ,
- the displacement fields \mathbf{u} on the surface of the body S_g ,
- the double tractions in normal direction \mathbf{r}_n on the surface of the body S_d ,

- the normal gradient of the displacement fields $\mathbf{D}\mathbf{u}$ on the surface of the body S_g ,
 - the line forces on edge \mathbf{c}_{pr} on the edges C_d of the surface of the body S ,
 - the displacement fields \mathbf{u} on the edge C_g of the surface of the body S ,
- where the tractions \mathbf{p}_{pr} , the double tractions in normal direction \mathbf{r}_{npr} and the line forces on edge \mathbf{c}_{pr} are determined in Eqs. (35)–(37).

Both parts of the surface S_d and S_g and the edge C_d and C_g should be almost everywhere disjoint and should form the entire surface and its edges

$$S_d \cup S_g = S, \quad C_d \cup C_g = C. \quad (64)$$

Since all equations of the gradient elasticity theory are linear, we can use the **principle of superposition**:

Theorem 2 Consider two sets $\{\mathbf{u}_k, \mathbf{E}_{2k}, \mathbb{E}_{3k}, \mathbf{T}_{2k}, \mathbb{T}_{3k}\}$ for $k = 1, 2$ of solutions of two mixed boundary value problems for given body forces \mathbf{f}_k , surface tractions \mathbf{p}_k and \mathbf{r}_{nk} on the part of surface S_d and the line forces \mathbf{c}_k on the part of edge C_d and the displacement \mathbf{u} and the normal gradient of the displacement $\mathbf{D}\mathbf{u}$ on the complementary part of surface S_g . Then for all real numbers $\alpha_k \in \mathfrak{R}$, $k = 1, 2$, the same equations are fulfilled by $\{\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2, \alpha_1 \mathbf{E}_{21} + \alpha_2 \mathbf{E}_{22}, \alpha_1 \mathbb{E}_{31} + \alpha_2 \mathbb{E}_{32}, \alpha_1 \mathbf{T}_{21} + \alpha_2 \mathbf{T}_{22}, \alpha_1 \mathbb{T}_{31} + \alpha_2 \mathbb{T}_{32}\}$ for the body forces $\{\alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2\}$, the surface tractions $\{\alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2\}$ and $\{\alpha_1 \mathbf{r}_{n1} + \alpha_2 \mathbf{r}_{n2}\}$ on S_d , the line forces $\{\alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_2\}$ on C_d and the displacement $\{\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2\}$, the normal gradient of the displacement $\{\alpha_1 \mathbf{D}\mathbf{u}_1 + \alpha_2 \mathbf{D}\mathbf{u}_2\}$ on S_g .

Now we can consider the **uniqueness of the solution** in the usual manner (see, e.g., [9]) - based on assumption of the positive definiteness of the strain and strain gradient energy. The **extension of the uniqueness theorem** is the next theorem.

Theorem 3 For linear elastic gradient material with positive definite potential energy density, two solutions \mathbf{u}_1 and \mathbf{u}_2 of the mixed boundary value problem differ only by an infinitesimal rigid body motion

$$\mathbf{u}_1(\mathbf{x}) = \mathbf{u}_2(\mathbf{x}) + \mathbf{u}_0 + \boldsymbol{\Omega} \cdot (\mathbf{x} - \mathbf{x}_0), \quad (65)$$

where \mathbf{u}_0 and \mathbf{x}_0 are two constant vectors and $\boldsymbol{\Omega}$ is a constant anti-symmetric tensor. If the displacements are prescribed for at least three not collinear points then the solution is unique

$$\mathbf{u}_1(\mathbf{x}) = \mathbf{u}_2(\mathbf{x}). \quad (66)$$

Proof The theorem assumptions and the principle of superposition lead to the following stating:

$$\delta \mathbf{u}(\mathbf{x}) = \mathbf{u}_1(\mathbf{x}) - \mathbf{u}_2(\mathbf{x}) \quad (67)$$

is a solution of the mixed boundary value problem with zero-boundary conditions: $\mathbf{f} = \mathbf{0}$ in V , $\mathbf{p} = \mathbf{0}$ and $\mathbf{r}_n = \mathbf{0}$ on S_d , $\mathbf{c} = \mathbf{0}$ on C_d , $\delta \mathbf{u} = \mathbf{0}$ and $\mathbf{D}\delta \mathbf{u} = \mathbf{0}$ on S_g . Following the principle of virtual displacements Eqs. (33) and (56) we have for the zero-boundary value problem

$$\begin{aligned} & \int_V \delta \mathbf{u} \cdot \mathbf{f} \, dV + \int_{S_d} (\delta \mathbf{u} \cdot \mathbf{p} + \mathbf{D}\delta \mathbf{u} \cdot \mathbf{r}_n) \, dS \\ & + \int_{S_g} (\delta \mathbf{u} \cdot \mathbf{p} + \mathbf{D}\delta \mathbf{u} \cdot \mathbf{r}_n) \, dS \\ & + \oint_{C_d} \delta \mathbf{u} \cdot \mathbf{c} \, dC + \oint_{C_g} \delta \mathbf{u} \cdot \mathbf{c} \, dC \\ & = \int_V (\mathbf{T}_2 \cdot \cdot \text{sym}(\delta \mathbf{u} \otimes \nabla) + \mathbb{T}_3 \cdot \cdot \delta \mathbf{u} \otimes \nabla \otimes \nabla) \, dV = 0, \end{aligned} \quad (68)$$

considering that at least one factor under each integral on the left-hand side of the above equation is zero in the integration domain. Here the constitutive relations are defined as

$$\begin{aligned} \mathbf{T}_2 &= \mathbb{C}_4 \cdot \cdot \text{sym}(\delta \mathbf{u} \otimes \nabla) + \mathbb{C}_5 \cdot \cdot \delta \mathbf{u} \otimes \nabla \otimes \nabla \\ &= \mathbb{C}_4 \cdot \cdot \delta \mathbf{E}_2 + \mathbb{C}_5 \cdot \cdot \delta \mathbb{E}_3, \end{aligned} \quad (69)$$

$$\begin{aligned}\mathbb{T}_3 &= \mathbb{C}_5^T \cdot \text{sym}(\delta \mathbf{u} \otimes \nabla) + \mathbb{C}_6 \cdot \cdot \delta \mathbf{u} \otimes \nabla \otimes \nabla \\ &= \mathbb{C}_5^T \cdot \cdot \delta \mathbb{E}_2 + \mathbb{C}_6 \cdot \cdot \cdot \delta \mathbb{E}_3,\end{aligned}\quad (70)$$

where

$$\delta \mathbb{E}_2 = \text{sym}(\delta \mathbf{u} \otimes \nabla), \quad \delta \mathbb{E}_3 = \delta \mathbf{u} \otimes \nabla \otimes \nabla. \quad (71)$$

With Eqs. (69) and (70) the integrand in equation (68) can be presented as

$$\begin{aligned}\mathbb{T}_2 \cdot \cdot \delta \mathbb{E}_2 + \mathbb{T}_3 \cdot \cdot \cdot \delta \mathbb{E}_3 \\ = \delta \mathbb{E}_2 \cdot \cdot \mathbb{C}_4 \cdot \cdot \delta \mathbb{E}_2 + 2\delta \mathbb{E}_2 \cdot \cdot \mathbb{C}_5 \cdot \cdot \cdot \delta \mathbb{E}_3 + \delta \mathbb{E}_3 \cdot \cdot \cdot \mathbb{C}_6 \cdot \cdot \cdot \delta \mathbb{E}_3.\end{aligned}\quad (72)$$

The presence on the right-hand side of the coupling term \mathbb{C}_5 complicates essential obtaining the inequality constraints needed for positive definiteness of w .

In order to separate the contributions of the strain and of the strain gradient, this equation can be transformed by using a formal modification of the second gradient of displacement and of the stiffness tensor of fourth-rank as shown in Nazarenko et al. [39]:

$$\mathbb{T}_2 \cdot \cdot \delta \mathbb{E}_2 + \mathbb{T}_3 \cdot \cdot \cdot \delta \mathbb{E}_3 = \delta \mathbb{E}_2 \cdot \cdot \mathbb{C}_4^m \cdot \cdot \delta \mathbb{E}_2 + \delta \mathbb{E}_3^m \cdot \cdot \cdot \mathbb{C}_6 \cdot \cdot \cdot \delta \mathbb{E}_3^m. \quad (73)$$

The modified second gradient of displacement and the modified stiffness tensor are marked by superscript m , and are specified as

$$\delta \mathbb{E}_3^m = \delta \mathbb{E}_3 + \delta \mathbb{E}_2 \cdot \cdot \mathbb{C}_5 \cdot \cdot \cdot \mathbb{C}_6^{-1} \quad (74)$$

and

$$\mathbb{C}_4^m = \mathbb{C}_4 - \mathbb{C}_5 \cdot \cdot \cdot \mathbb{C}_6^{-1} \cdot \cdot \cdot \mathbb{C}_5^T. \quad (75)$$

Because of the assumed positive definiteness of the quadratic form of the potential energy density,

$$\delta \mathbb{E}_2 \cdot \cdot \mathbb{C}_4^m \cdot \cdot \delta \mathbb{E}_2 + \delta \mathbb{E}_3^m \cdot \cdot \cdot \mathbb{C}_6 \cdot \cdot \cdot \delta \mathbb{E}_3^m \quad (76)$$

must be nonnegative everywhere. Then integrating the last expression gives

$$\begin{aligned}\int_V \delta \mathbb{E}_2 \cdot \cdot \mathbb{C}_4^m \cdot \cdot \delta \mathbb{E}_2 + \delta \mathbb{E}_3^m \cdot \cdot \cdot \mathbb{C}_6 \cdot \cdot \cdot \delta \mathbb{E}_3^m dV \\ \geq \int_V \delta \mathbb{E}_2 \cdot \cdot \mathbb{C}_4^m \cdot \cdot \delta \mathbb{E}_2 dV \geq 0,\end{aligned}\quad (77)$$

and after Korn's inequality (see, e.g., [19,30])

$$k \int_V |\delta \mathbb{E}_2|^2 dV \geq \int_V |\delta \mathbf{u} \otimes \nabla|^2 dV \quad (78)$$

can be zero, if and only if the integrand $\delta \mathbf{u} \otimes \nabla$ is zero everywhere. Here the magnitude of a second-rank tensor \mathbf{M} is defined as

$$|\mathbf{M}| = \sqrt{\mathbf{M} : \mathbf{M}}, \quad (79)$$

and k is a constant depending only on the region of integration V .

Therefore, the displacement $\delta \mathbf{u}$ can be strain-free only, and with Eq. (67)

$$\delta \mathbf{u}(\mathbf{x}) = \mathbf{u}_1(\mathbf{x}) - \mathbf{u}_2(\mathbf{x}) = \mathbf{u}_0 + \boldsymbol{\Omega} \cdot (\mathbf{x} - \mathbf{x}_0). \quad (80)$$

It is the proof of the first part of the theorem. The second part of the theorem proof is rather evident, indeed, if S_g has at least three not collinear points, for which the displacements are prescribed, then $\delta \mathbf{u}$ must vanish in these points, and that $\delta \mathbf{u}$ will be zero field everywhere

$$\mathbf{u}_1(\mathbf{x}) = \mathbf{u}_2(\mathbf{x}). \quad (81)$$

Remark 2 We would like to point out that Eq. (72) can also be modified in order to present the variation of the strain energy density as block matrices by introducing the variation of the modified strain and the modified stiffness tensor of sixth-rank (see [40])

$$\mathbf{T}_2 \cdot \delta \mathbf{E}_2 + \mathbf{T}_3 \cdots \delta \mathbf{E}_3 = \delta \mathbf{E}_2^m \cdot \mathbf{C}_4 \cdot \delta \mathbf{E}_2^m + \delta \mathbf{E}_3 \cdots \mathbf{C}_6^m \cdots \delta \mathbf{E}_3, \quad (82)$$

with

$$\delta \mathbf{E}_2^m = \delta \mathbf{E}_2 + \delta \mathbf{E}_3 \cdots \mathbf{C}_5^T \cdot \mathbf{C}_4^{-1} \quad (83)$$

and

$$\mathbf{C}_6^m = \mathbf{C}_6 - \mathbf{C}_5^T \cdot \mathbf{C}_4^{-1} \cdot \mathbf{C}_5. \quad (84)$$

Such a transformation is mathematically equivalent to Eq. (73) and leads to the same proving scheme. Thus, the both modifications can be used for proving and can be considered as an additional validation of the presented proof.

5 Conclusions

The equilibrium equation and the corresponding natural boundary conditions are re-derived on the basis of the Lagrange variational principle modified for the coupled linear strain gradient continua. The potential energy density Eq. (1) is assumed to be a function of the strain and the second gradient of displacement and contains a fourth-, a fifth- and a sixth-order stiffness tensor \mathbf{C}_4 , \mathbf{C}_5 and \mathbf{C}_6 .

Assuming rigid body motion the balance laws for the total forces and total torque are obtained from the principle of virtual power. These balance laws are employed in the derivation of the equilibrium equations of the coupled strain gradient elasticity. It is shown that the balance of angular momentum leads to the symmetry of the stress tensor \mathbf{T}_2 as a consequence. Then starting with the balance of the linear and the angular momentum the principle of the virtual power is recovered.

Kirchhoff's uniqueness theorem for the mixed boundary value problem is proven for the equilibrium problem for the coupled linear strain gradient elasticity. The existence of the coupling term \mathbf{C}_5 makes the problem more complicated. By a transformation which may be considered as a block diagonalization, the equation of the potential energy density Eq. (1) was modified to present w as an uncoupled quadratic form of the strain and the modified second gradient of displacement (see details in [39,40]). Then, one may apply Mindlin's uniqueness proof for isotropic materials. Here we proved the uniqueness of the solution in the usual way by contradiction by considering two solutions and their difference, similar to Kirchhoff's proof in classical elasticity. The proof requires positive definiteness of the potential energy density. The positive definiteness conditions for constitutive parameters are presented in Nazarenko et al. [39] in a tensorial form without restricting the symmetry class, but the inequalities are therein given for the 8 elasticity parameters in hemitropy (invariance under the action of the special orthogonal group).

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