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ORIGINAL ARTICLE

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Some uniqueness results for thermoelastic materials with double porosity structure

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Abstract The main goal of the present paper was to obtain some new uniqueness results for the anisotropic thermoelastic bodies with double porosity structure. There are obtained some auxiliary results based on the Betti reciprocity relation that involve some thermoelastic processes.

Keywords Double porosity bodies · Thermoelasticity · Unicity · Betti type

1 Introduction

The applications of double porosity materials spread accross a wide range of domains, e.g., civil engineering, geotechniques [1,2], biomechanics [3]. First mathematical model of a thermoelastic material with double porosity structure was studied by Barenblatt [4,5]. Straughan [6] expressed the connection between the two porosities of a material referring to the pores of the body and the cracks of the skeleton.

Based on the results obtained by Biot [7] for the materials with single porosity and by Barenblatt et al. [4] for the bodies with double porosity, Wilson and Aifantis [8] have continued the research in the field of deformable bodies with double porosity.

The uniqueness, reciprocity and variational theorems for the basic equations that govern the elastic materials with voids were proved by Ieşan [9] and in [10] Ieşan included the thermal effect for such media. Based on the Nunziato–Cowin theory for materials with voids [11], Ieşan and Quintanilla [12] developed a nonlinear theory for thermoelastic bodies with double porosity.

The dynamical problems of the theory od elasticity, viscoelasticity, and thermoelasticity for bodies with double porosity were studied in many papers by Svanadze [13–19]. In [20], Svanadze obtained some theorems for the isotropic materials with double porosity structure. Straughan studied the stability and uniqueness for the materials with double porosity by introducing a novel functional in order to obtain Holder stability estimates [21]. Kansal [22] obtained uniqueness and reciprocity theorems for anisotropic materials with double porosity

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based on the Lord–Shuman model [23]. Mixed problem with initial and boundary conditions, in the context of thermoelasticity of dipolar bodies were recently studied in [24,25].

Uniqueness results for a mixed initial boundary problem for dipolar thermoelastic bodies were studied by Marin and Craciun [26] and for the microstretch thermoelastic materials were investigated in [27]. The uniqueness theorems for the classical solutions of the boundary value problems of steady vibrations in case of isotropic porous solids are proved in [28]. Existence and uniqueness of solutions for thermoelastic materials with two porosities in the case of one-dimensional theory was studied by Bazarra et al. [29].

The continuous dependence by means of some estimate regarding the gradient of deformations and the gradient of the function that describes the evolution of the voids is presented in [30]. In [31] was investigated an overdetermined problem for a general class of anisotropic equations on a cylindrical domain. Some estimates regarding the behavior of solutions for a mixed problem in the context of a semi-infinite cylinder composed of two sub-cylinders with an interface at the common boundary were proved in [32].

The structure of the present study is as follows. In Sect. 2, the mixed initial boundary value problem is defined in order to model a thermoelastic body with double porosity structure. In Sect. 3 are obtained the uniqueness theorems based on the Betti-type reciprocity relation that establishes a connection between two systems of external loadings and the solution of those loadings.

2 Basic equations

Let us consider a material with two porosities. In (1) are presented the equations that model this type of thermoelastic structure:

$$\begin{aligned} \rho \ddot{u}_i &= t_{ji,j} + \rho f_i, \\ K_1 \ddot{\varphi} &= \sigma_{j,j} + \mathfrak{p} + \rho G, \\ K_2 \ddot{\psi} &= \tau_{j,j} + \mathfrak{r} + \rho L. \end{aligned} \tag{1}$$

In the above equations, we noted the density of material by ρ and the displacement by u_i, the equilibrated inertia coefficients are K_1, K_2 and the volume fraction fields φ, ψ, that are considered in the reference configuration. The vectors of the equilibrated stress are σ_j, τ_j and respectively the stress tensors at the body's surface ∂B are t_{ij} . On the body act some forces that are further mentioned: $\mathfrak{p}$, $\mathfrak{r}$ are the intrinsic forces, f_i are the direct forces, G is the extrinsic forces that act on the pores, L is the extrinsic forces that act on the cracks.

Further we express the energy's equation:

$$\rho \mathcal{T}_0 \dot{\eta} = \mathcal{Q}_{j,j} + \varrho \mathfrak{h}. \tag{2}$$

In the reference configuration, the body has a constant absolute temperature T_0, the entropy is noted by η, and $\mathcal{Q}_j$, $\mathfrak{h}$ are the heat flux and the heat supply considered per unit mass and unit time, respectively.

The stress tensors of components t_{ij}, τ_i, σ_i, the intrinsic forces $\mathfrak{p}$, $\mathfrak{r}$, the specific entropy η, the heat conduction vector of components Q_i are defined through the constitutive equations in the linear theory for centrosymmetric materials that follows [33]:

$$\begin{aligned} t_{ij} &= C_{ijkl} u_{k,l} + B_{ij}\varphi + D_{ij}\psi - \beta_{ij}\theta, \\ \sigma_i &= \alpha_{ij}\varphi_{,j} + b_{ij}\psi_{,j}, \\ \tau_i &= b_{ij}\varphi_{,j} + \gamma_{ij}\psi_{,j}, \\ \mathfrak{p} &= -B_{ij}u_{i,j} - \alpha_1\varphi - \alpha_3\psi + \gamma_1\theta, \\ \mathfrak{r} &= -D_{ij}u_{i,j} - \alpha_3\varphi - \alpha_2\psi + \gamma_2\theta, \\ \rho\eta &= \beta_{ij}u_{i,j} + \gamma_1\varphi + \gamma_2\psi + a\theta, \\ \mathcal{Q}_i &= K_{ij}\theta_{,j}. \end{aligned} \tag{3}$$

In the above constitutive equations we have the terms: elasticity tensor is noted by $\mathcal{C}_{ijkl}$, the tensor of thermal dilatation is β_{ij}, the heat conductivity tensor is $\mathcal{K}_{ij}$ and θ is the temperature measured from the reference temperature $\mathcal{T}_0$. In the theory of the bodies with double porosity structure, there are some typical functions that in our constitutive equations are noted by: $\mathcal{B}_{ij}$, $\mathcal{D}_{ij}$, α_{ij}, b_{ij}, γ_{ij}, α_1, α_2, α_3, γ_1, γ_2, a.

Taking into account the motion's equations (1) and the energy's equation (2) and replacing them into the constitutive equations, we further obtain:

$$\begin{aligned} \rho\ddot{u}_i &= \left(C_{jikl}u_{k,l} + B_{ji}\varphi + D_{ji}\psi - \beta_{ji}\theta\right)_{,j} + \rho f_i, \\ \kappa_1\ddot{\varphi} &= \left(\alpha_{ij}\varphi_{,i} + b_{ij}\psi_{,i}\right)_{,j} - B_{ij}u_{i,j} - \alpha_1\varphi - \alpha_3\psi + \gamma_1\theta + \rho G, \\ \kappa_2\ddot{\psi} &= \left(b_{ij}\varphi_{,i} + \gamma_{ij}\psi_{,i}\right)_{,j} - D_{ij}u_{i,j} - \alpha_3\varphi - \alpha_2\psi + \gamma_2\theta + \rho L, \\ aT_0\dot{\theta} &= -T_0(\beta_{ij}\dot{u}_{i,j} + \gamma_1\dot{\varphi} + \gamma_2\dot{\psi}) + (K_{ij}\theta_{,j})_{,j} + \rho\mathfrak{h}. \end{aligned} \tag{4}$$

The constitutive coefficients are symmetric and they satisfy the next relations:

$$\begin{aligned} C_{ijkl} = C_{ijlk} = C_{klij}; \quad \alpha_{ij} = \alpha_{ji}; \quad \beta_{ij} = \beta_{ji}; \\ \gamma_{ij} = \gamma_{ji}; \quad B_{ij} = B_{ji}; \quad D_{ij} = D_{ji}; \quad K_{ij} = K_{ji}. \end{aligned} \tag{5}$$

The entropy inequality implies:

$$K_{ij}\theta_i\theta_j \geq 0. \tag{6}$$

We consider the not null initial conditions:

$$\begin{aligned} &u_i(x_0,0) = u_i^0(x_0),\ \dot{u}_i(x_0,0) = v_i^0(x_0),\ \varphi(x_0,0) = \varphi_0(x_0),\ \dot{\varphi}(x_0,0) = \tilde{\varphi}_0(x_0), \\ &\psi(x_0,0) = \psi_0(x_0),\ \dot{\psi}(x_0,0) = \tilde{\psi}_0(x_0),\ \eta(x_0,0) = \eta_0(x_0), \end{aligned} \tag{7}$$

where $x_0 \in \bar{B}$.

We consider that the surface ∂B has the subsets $\partial B_1, \partial B_2, \partial B_3, \partial B_4$ and their complements $\partial B_1^c, \partial B_2^c, \partial B_3^c, \partial B_4^c$. They are satisfying the following conditions:

$$\partial B_i \cup \partial B_i^c = \partial B, \quad \partial B_i \cap \partial B_i^c = \Phi, \text{ where } i = \overline{1,4}.$$

The boundary conditions are given by the relations that follows:

$$\begin{aligned} u_i = u_i^b \quad &\text{on} \quad \partial B_1 \times [0,\infty), \quad t_i = t_i^b \quad \text{on} \quad \partial B_1^c \times [0,\infty), \\ \varphi = \varphi^b \quad &\text{on} \quad \partial B_2 \times [0,\infty), \quad \alpha = \alpha^b \quad \text{on} \quad \partial B_2^c \times [0,\infty), \\ \psi = \psi^b \quad &\text{on} \quad \partial B_3 \times [0,\infty), \quad \beta = \beta^b \quad \text{on} \quad \partial B_3^c \times [0,\infty), \\ \theta = \theta^b \quad &\text{on} \quad \partial B_4 \times [0,\infty), \quad Q = Q^b \quad \text{on} \quad \partial B_4^c \times [0,\infty), \end{aligned} \tag{8}$$

where t_i represent the components of surface traction, α, β represent the components of surface couple and Q represents the heat flux, and they can be expressed as follows:

$$t_i = t_{ij}n_j, \quad \alpha = \sigma_i n_i, \quad \beta = \tau_i n_i, \quad Q = Q_i n_i. \tag{8$'$}$$

In (7) $u_i^0, v_i^0, \varphi_0, \tilde{\varphi}_0, \psi_0, \tilde{\psi}_0, \eta_0$ are set down functions and in (8) $u_i^b, t_i^b, \varphi^b, \alpha^b, \psi^b, \beta^b, \theta^b, Q^b$ are given functions.

The ordered array $(u_i, \varphi, \psi, \theta)$ is a solution for the mixed boundary value problem of a thermoelastic material with double porosity in the cylinder $\Omega_0 = B \times [0,\infty)$. This solution fulfills the differential equations in the system (4) for all $(x,\xi) \in \Omega_0$, the initial conditions (7) and the boundary conditions (8).

We consider the following assumptions to be valid:

(1) the constitutive coefficients, the density ρ, the coefficients of inertia κ_1, κ_2 and the prescribed functions in the initial conditions (7) are functions that are continuous on $\bar{B}$;
(2) the body forces f_i, the extrinsic forces G, L and the heat supply $\mathfrak{h}$ are functions that are continuous on the cylinder $\Omega_0 = B \times [0,\infty)$;
(3) $u_i^b, \varphi^b, \psi^b, \theta^b$ are functions that are continuous in their domains of definition;
(4) $t_i^b, \alpha^b, \beta^b, Q^b$ are functions that are continuous in time and piecewise regular in their domains of definition.

3 Main results

Theorem 1 *Let us consider the following functions:*

$$
\begin{aligned}
X(\xi) &= \int_B [\rho \dot{u}_i(\xi)\dot{u}_i(\xi) + K_1\dot{\varphi}(\xi)\dot{\varphi}(\xi) + K_2\dot{\psi}(\xi)\dot{\psi}(\xi)]\, \mathrm{d}V, \\
Y(\xi) &= \int_B \big[C_{ijkl}u_{k,l}(\xi)u_{i,j}(\xi) + 2B_{ij}u_{i,j}(\xi)\varphi(\xi) + 2D_{ij}u_{i,j}(\xi)\psi(\xi) \\
&\quad + \alpha_{ij}\varphi_{,i}(\xi)\varphi_{,j}(\xi) + 2b_{ij}\varphi_{,i}(\xi)\psi_{,j}(\xi) + \gamma_{ij}\psi_{,i}(\xi)\psi_{,j}(\xi) \\
&\quad + \alpha_1\varphi^2(\xi) + \alpha_2\psi^2(\xi) + 2\alpha_3\varphi(\xi)\psi(\xi) + a\theta^2(\xi)\big]\, \mathrm{d}V.
\end{aligned} \tag{9}
$$

The below identity is satisfied by the considered functions X and Y:

$$
\begin{aligned}
Y(\xi) - X(\xi) &= \int_B \big\{C_{ijkl}u_{i,j}(0)u_{k,l}(2\xi) + B_{ij}[u_{i,j}(0)\varphi(2\xi) + u_{i,j}(2\xi)\varphi(0)] \\
&\quad + D_{ij}[u_{i,j}(0)\psi(2\xi) + u_{i,j}(2\xi)\psi(0)] + \alpha_{ij}\varphi_{,i}(0)\varphi_{,j}(2\xi) \\
&\quad + b_{ij}[\varphi_{,i}(0)\psi_{,j}(2\xi) + \varphi_{,i}(2\xi)\psi_{,j}(0)] + \gamma_{ij}\psi_{,i}(0)\psi_{,j}(2\xi) \\
&\quad + \alpha_1\varphi(0)\varphi(2\xi) + \alpha_2\psi(0)\psi(2\xi) + \alpha_3[\varphi(0)\psi(2\xi) \\
&\quad + \varphi(2\xi)\psi(0)] + a\theta(0)\theta(2\xi)\big\}\, \mathrm{d}V \\
&\quad - \int_0^{\xi}\int_{\partial B} \big[t_i(\xi-\zeta)\dot{u}_j(\xi+\zeta) + \alpha(\xi-\zeta)\dot{\varphi}(\xi+\zeta) + \beta(\xi-\zeta)\dot{\psi}(\xi+\zeta) \\
&\quad - \frac{1}{\rho T_0}Q(\xi-\zeta)\theta(\xi+\zeta)\Big]\, \mathrm{d}A\, \mathrm{d}\xi \\
&\quad + \int_0^{\xi}\int_{\partial B} \big[t_i(\xi+\zeta)\dot{u}_j(\xi-\zeta) + \alpha(\xi+\zeta)\dot{\varphi}(\xi-\zeta) + \beta(\xi+\zeta)\dot{\psi}(\xi-\zeta) \\
&\quad - \frac{1}{\rho T_0}Q(\xi+\zeta)\theta(\xi-\zeta)\Big]\, \mathrm{d}A\, \mathrm{d}\xi \\
&\quad - \int_0^{\xi}\int_B \big[\rho f_i(\xi-\zeta)\dot{u}_j(\xi+\zeta) + \rho G(\xi-\zeta)\dot{\varphi}(\xi+\zeta) + \rho L(\xi-\zeta)\dot{\psi}(\xi+\zeta) \\
&\quad - \frac{1}{T_0}\mathfrak{h}(\xi-\zeta)\theta(\xi+\zeta)\Big]\, \mathrm{d}V\, \mathrm{d}\xi \\
&\quad + \int_0^{\xi}\int_B \big[\rho f_i(\xi+\zeta)\dot{u}_j(\xi-\zeta) + \rho G(\xi+\zeta)\dot{\varphi}(\xi-\zeta) + \rho L(\xi+\zeta)\dot{\psi}(\xi-\zeta) \\
&\quad - \frac{1}{T_0}\mathfrak{h}(\xi+\zeta)\theta(\xi-\zeta)\Big]\, \mathrm{d}V\, \mathrm{d}\xi,
\end{aligned} \tag{10}
$$

where $\xi \in [0, \infty)$.

Proof To be able to prove the theorem, we shall introduce the following function:

$$
\begin{aligned}
\mathfrak{W}(a, b) &= t_{ij}(a)\dot{u}_{i,j}(b) + \sigma_i(a)\dot{\varphi}_{,i}(b) + \tau_i(a)\dot{\psi}_{,i}(b) \\
&\quad + \theta(a)\dot{\eta}(b) - \mathfrak{p}(a)\dot{\varphi}(b) - \mathfrak{r}(a)\dot{\psi}(b).
\end{aligned} \tag{11}
$$

Replacing $a \leftrightarrow \xi - \zeta$ and $b \leftrightarrow \xi + \zeta$ and given the constitutive equations, is obtained:

$$\begin{aligned}\mathfrak{W}(\xi-\zeta,\xi+\zeta) = {} & \Big[t_{ij}(\xi-\zeta)\dot{u}_j(\xi+\zeta)+\sigma_i(\xi-\zeta)\dot{\varphi}(\xi+\zeta)\\ & +\tau_i(\xi-\zeta)\dot{\psi}(\xi+\zeta)+\frac{1}{\rho T_0}\theta(\xi-\zeta)Q_i(\xi+\zeta)\Big]_{,i}+\rho f_i(\xi-\zeta)\dot{u}_j(\xi+\zeta)\\ & +\rho G(\xi-\zeta)\dot{\varphi}(\xi+\zeta)+\rho L(\xi-\zeta)\dot{\psi}(\xi+\zeta)+\frac{1}{T_0}\theta(\xi-\zeta)\mathfrak{h}(\xi+\zeta)\\ & -\frac{K_{ij}}{\rho T_0}Q_{,i}(\xi-\zeta)Q_{,j}(\xi+\zeta)+\frac{\mathrm{d}}{\mathrm{d}\xi}[\rho\dot{u}_i(\xi-\zeta)\dot{u}_j(\xi+\zeta)]\\ & -\rho\dot{u}_i(\xi-\zeta)\ddot{u}_j(\xi+\zeta)+\frac{\mathrm{d}}{\mathrm{d}\xi}[K_1\dot{\varphi}(\xi-\zeta)\dot{\varphi}(\xi+\zeta)]\\ & -K_1\dot{\varphi}(\xi-\zeta)\ddot{\varphi}(\xi+\zeta)+\frac{\mathrm{d}}{\mathrm{d}\xi}[K_2\dot{\psi}(\xi-\zeta)\dot{\psi}(\xi+\zeta)]\\ & -K_2\dot{\psi}(\xi-\zeta)\ddot{\psi}(\xi+\zeta). \end{aligned} \tag{12}$$

Replacing $a \leftrightarrow \xi + \zeta$ and $b \leftrightarrow \xi - \zeta$ and given the constitutive equations, is obtained:

$$\begin{aligned}\mathfrak{W}(\xi+\zeta,\xi-\zeta) = {} & \Big[t_{ij}(\xi+\zeta)\dot{u}_j(\xi-\zeta)+\sigma_i(\xi+\zeta)\dot{\varphi}(\xi-\zeta)\\ & +\tau_i(\xi+\zeta)\dot{\psi}(\xi-\zeta)+\frac{1}{\rho T_0}\theta(\xi+\zeta)Q_i(\xi-\zeta)\Big]_{,i}+\rho f_i(\xi+\zeta)\dot{u}_j(\xi-\zeta)\\ & +\rho G(\xi+\zeta)\dot{\varphi}(\xi-\zeta)+\rho L(\xi+\zeta)\dot{\psi}(\xi-\zeta)+\frac{1}{T_0}\theta(\xi+\zeta)\mathfrak{h}(\xi-\zeta)\\ & -\frac{K_{ij}}{\rho T_0}Q_{,i}(\xi+\zeta)Q,j(\xi-\zeta)+\frac{\mathrm{d}}{\mathrm{d}\xi}[\rho\dot{u}_i(\xi+\zeta)\dot{u}_j(\xi-\zeta)]\\ & -\rho\dot{u}_i(\xi+\zeta)\ddot{u}_j(\xi-\zeta)+\frac{\mathrm{d}}{\mathrm{d}\xi}[K_1\dot{\varphi}(\xi+\zeta)\dot{\varphi}(\xi-\zeta)]\\ & -K_1\dot{\varphi}(\xi+\zeta)\ddot{\varphi}(\xi-\zeta)+\frac{\mathrm{d}}{\mathrm{d}\xi}[K_2\dot{\psi}(\xi+\zeta)\dot{\psi}(\xi-\zeta)]\\ & -K_2\dot{\psi}(\xi+\zeta)\ddot{\psi}(\xi-\zeta). \end{aligned} \tag{13}$$

From (12) and (13), we obtain:

$$\begin{aligned} & W(\xi-\zeta,\xi+\zeta)-W(\xi+\zeta,\xi-\zeta)\\ & = \Big[t_{ij}(\xi-\zeta)\dot{u}_j(\xi+\zeta)+\sigma_i(\xi-\zeta)\dot{\varphi}(\xi+\zeta)+\tau_i(\xi-\zeta)\dot{\psi}(\xi+\zeta)+\frac{1}{\rho T_0}\theta(\xi-\zeta)Q_i(\xi+\zeta)\\ & \quad -t_{ij}(\xi+\zeta)\dot{u}_j(\xi-\zeta)-\sigma_i(\xi+\zeta)\dot{\varphi}(\xi-\zeta)-\tau_i(\xi+\zeta)\dot{\psi}(\xi-\zeta)\\ & \quad -\frac{1}{\rho T_0}\theta(\xi+\zeta)Q_i(\xi-\zeta)\Big]_{,i}+\rho f_i(\xi-\zeta)\dot{u}_j(\xi+\zeta)-\rho f_i(\xi+\zeta)\dot{u}_j(\xi-\zeta)\\ & \quad +\rho G(\xi-\zeta)\dot{\varphi}(\xi+\zeta)-\rho G(\xi+\zeta)\dot{\varphi}(\xi-\zeta)+\rho L(\xi-\zeta)\dot{\psi}(\xi+\zeta)\\ & \quad -\rho L(\xi+\zeta)\dot{\psi}(\xi-\zeta)+\frac{1}{T_0}\theta(\xi-\zeta)\mathfrak{h}(\xi+\zeta)-\frac{1}{T_0}\theta(\xi+\zeta)\mathfrak{h}(\xi-\zeta)\\ & \quad -\rho\dot{u}_i(\xi-\zeta)\ddot{u}_j(\xi+\zeta)+\rho\dot{u}_i(\xi+\zeta)\ddot{u}_j(\xi-\zeta)\\ & \quad -K_1\dot{\varphi}(\xi-\zeta)\ddot{\varphi}(\xi+\zeta)+K_1\dot{\varphi}(\xi+\zeta)\ddot{\varphi}(\xi-\zeta)\\ & \quad -K_2\dot{\psi}(\xi-\zeta)\ddot{\psi}(\xi+\zeta)+K_2\dot{\psi}(\xi+\zeta)\ddot{\psi}(\xi-\zeta). \end{aligned} \tag{14}$$

By integration the identity (14) on B and using the divergence theorem we obtain:

$$\int_B \left(\mathfrak{W}(\xi-\zeta,\xi+\zeta)-\mathfrak{W}(\xi+\zeta,\xi-\zeta)\right)\mathrm{d}V$$

$$
\begin{aligned}
&= \int_{\partial B} \Big[t_i(\xi-\zeta)\dot{u}_j(\xi+\zeta) + \alpha(\xi-\zeta)\dot{\varphi}(\xi+\zeta) + \beta(\xi-\zeta)\dot{\psi}(\xi+\zeta) \\
&\quad - \frac{1}{\rho T_0} Q(\xi-\zeta)\theta(\xi+\zeta) \Big] \, \mathrm{d}A \\
&\quad - \int_{\partial B} \Big[t_i(\xi+\zeta)\dot{u}_j(\xi-\zeta) + \alpha(\xi+\zeta)\dot{\varphi}(\xi-\zeta) + \beta(\xi+\zeta)\dot{\psi}(\xi-\zeta) \\
&\quad - \frac{1}{\rho T_0} Q(\xi+\zeta)\theta(\xi-\zeta) \Big] \, \mathrm{d}A \\
&\quad + \int_B \Big[\rho f_i(\xi-\zeta)\dot{u}_j(\xi+\zeta) + \rho G(\xi-\zeta)\dot{\varphi}(\xi+\zeta) + \rho L(\xi-\zeta)\dot{\psi}(\xi+\zeta) \\
&\quad - \frac{1}{T_0} \mathfrak{h}(\xi-\zeta)\theta(\xi+\zeta) \Big] \, \mathrm{d}V \\
&\quad - \int_B \Big[\rho f_i(\xi+\zeta)\dot{u}_j(\xi-\zeta) + \rho G(\xi+\zeta)\dot{\varphi}(\xi-\zeta) + \rho L(\xi+\zeta)\dot{\psi}(\xi-\zeta) \\
&\quad - \frac{1}{T_0} \mathfrak{h}(\xi+\zeta)\theta(\xi-\zeta) \Big] \, \mathrm{d}V \\
&\quad + \int_B \frac{\mathrm{d}}{\mathrm{d}\xi} [\rho \dot{u}_i(\xi-\zeta)\dot{u}_i(\xi+\zeta) + K_1 \dot{\varphi}(\xi-\zeta)\dot{\varphi}(\xi+\zeta) \\
&\quad + K_2 \dot{\psi}(\xi-\zeta)\dot{\psi}(\xi+\zeta)] \, \mathrm{d}V.
\end{aligned}
\tag{15}
$$

Going back to (12) we obtain the following identity:

$$
\begin{aligned}
&\mathfrak{W}(\xi-\zeta, \xi+\zeta) - \mathfrak{W}(\xi+\zeta, \xi-\zeta) \\
&= \frac{\mathrm{d}}{\mathrm{d}\xi} [C_{ijkl} u_{i,j}(\xi-\zeta) u_{k,l}(\xi+\zeta) + B_{ij} u_{i,j}(\xi-\zeta)\varphi(\xi+\zeta) \\
&\quad + B_{ij}\varphi(\xi-\zeta) u_{i,j}(\xi+\zeta) + D_{ij}\psi(\xi+\zeta) u_{i,j}(\xi-\zeta) + D_{ij} u_{i,j}(\xi+\zeta)\psi(\xi-\zeta) \\
&\quad + \alpha_{ij}\varphi_{,i}(\xi-\zeta)\varphi_{,j}(\xi+\zeta) + b_{ij}\varphi_{,i}(\xi-\zeta)\psi_{,j}(\xi+\zeta) + b_{ij}\psi_{,i}(\xi-\zeta)\varphi_{,j}(\xi+\zeta) \\
&\quad + \gamma_{ij}\psi_{,i}(\xi-\zeta)\psi_{,j}(\xi+\zeta) + a\theta(\xi-\zeta)\theta(\xi+\zeta) + \alpha_1\varphi(\xi-\zeta)\varphi(\xi+\zeta) \\
&\quad + \alpha_2\psi(\xi-\zeta)\psi(\xi+\zeta) + \alpha_3\varphi(\xi-\zeta)\psi(\xi+\zeta) + \alpha_3\psi(\xi-\zeta)\varphi(\xi+\zeta)].
\end{aligned}
\tag{16}
$$

We integrate (16) on B taking into account (15), we have:

$$
\begin{aligned}
&\int_B \frac{\mathrm{d}}{\mathrm{d}\xi} [C_{ijkl} u_{i,j}(\xi-\zeta) u_{k,l}(\xi+\zeta) + B_{ij} u_{i,j}(\xi-\zeta)\varphi(\xi+\zeta) \\
&\quad + B_{ij}\varphi(\xi-\zeta) u_{i,j}(\xi+\zeta) + D_{ij}\psi(\xi+\zeta) u_{i,j}(\xi-\zeta) \\
&\quad + D_{ij} u_{i,j}(\xi+\zeta)\psi(\xi-\zeta) + \alpha_{ij}\varphi_{,i}(\xi-\zeta)\varphi_{,j}(\xi+\zeta) \\
&\quad + b_{ij}\varphi_{,i}(\xi-\zeta)\psi_{,j}(\xi+\zeta) + b_{ij}\psi_{,i}(\xi-\zeta)\varphi_{,j}(\xi+\zeta) \\
&\quad + \gamma_{ij}\psi_{,i}(\xi-\zeta)\psi_{,j}(\xi+\zeta) + a\theta(\xi-\zeta)\theta(\xi+\zeta) \\
&\quad + \alpha_1\varphi(\xi-\zeta)\varphi(\xi+\zeta) + \alpha_2\psi(\xi-\zeta)\psi(\xi+\zeta) \\
&\quad + \alpha_3\varphi(\xi-\zeta)\psi(\xi+\zeta) + \alpha_3\psi(\xi-\zeta)\varphi(\xi+\zeta)] \, \mathrm{d}V \\
&= \int_{\partial B} \Big[t_i(\xi-\zeta)\dot{u}_j(\xi+\zeta) + \alpha(\xi-\zeta)\dot{\varphi}(\xi+\zeta) + \beta(\xi-\zeta)\dot{\psi}(\xi+\zeta) \\
&\quad - \frac{1}{\rho T_0} Q(\xi-\zeta)\theta(\xi+\zeta) \Big] \, \mathrm{d}A
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\partial B}\bigg[t_i(\xi+\zeta)\dot{u}_j(\xi-\zeta)+\alpha(\xi+\zeta)\dot{\varphi}(\xi-\zeta)+\beta(\xi+\zeta)\dot{\psi}(\xi-\zeta)\\
&-\frac{1}{\rho T_0}Q(\xi+\zeta)\theta(\xi-\zeta)\bigg]\,\mathrm{d}A\\
&+\int_{B}\bigg[\rho f_i(\xi-\zeta)\dot{u}_j(\xi+\zeta)+\rho G(\xi-\zeta)\dot{\varphi}(\xi+\zeta)+\rho L(\xi-\zeta)\dot{\psi}(\xi+\zeta)\\
&-\frac{1}{T_0}\mathfrak{h}(\xi-\zeta)\theta(\xi+\zeta)\bigg]\,\mathrm{d}V\\
&-\int_{B}\bigg[\rho f_i(\xi+\zeta)\dot{u}_j(\xi-\zeta)+\rho G(\xi+\zeta)\dot{\varphi}(\xi-\zeta)+\rho L(\xi+\zeta)\dot{\psi}(\xi-\zeta)\\
&-\frac{1}{T_0}\mathfrak{h}(\xi+\zeta)\theta(\xi-\zeta)\bigg]\,\mathrm{d}V\\
&+\int_{B}\frac{\mathrm{d}}{\mathrm{d}\xi}\,[\rho\dot{u}_i(\xi-\zeta)\dot{u}_i(\xi+\zeta)+K_1\dot{\varphi}(\xi-\zeta)\dot{\varphi}(\xi+\zeta)\\
&+K_2\dot{\psi}(\xi-\zeta)\dot{\psi}(\xi+\zeta)]\,\mathrm{d}V.
\end{aligned}
\tag{17}
$$

Further we integrate (17) on the range $[0,\xi]$ with respect to ζ and the theorem results is obtained. □

Theorem 2 *For* $\forall\xi\in[0,\infty)$ *the defined functions from* (9) *satisfy the following relations:*

$$
\begin{aligned}
Y(\xi)=&\frac{1}{2}[X(0)+Y(0)]\\
&+\frac{1}{2}\int_{B}[C_{ijkl}u_{i,j}(0)u_{k,l}(2\xi)+B_{ij}[u_{i,j}(0)\varphi(2\xi)+u_{i,j}(2\xi)\varphi(0)]\\
&+D_{ij}[u_{i,j}(0)\psi(2\xi)+u_{i,j}(2\xi)\psi(0)]+\alpha_{ij}\varphi_{,i}(0)\varphi_{,j}(2\xi)\\
&+b_{ij}[\varphi_{,i}(0)\psi_{,j}(2\xi)+\varphi_{,i}(2\xi)\psi_{,j}(0)]+\gamma_{ij}\psi_{,i}(0)\psi_{,j}(2\xi)\\
&+\alpha_1\varphi(0)\varphi(2\xi)+\alpha_2\psi(0)\psi(2\xi)+\alpha_3[\varphi(0)\psi(2\xi)+\varphi(2\xi)\psi(0)]\\
&+a\theta(0)\theta(2\xi)]\,\mathrm{d}V\\
&-\frac{1}{2}\int_0^{\xi}\int_{\partial B}\bigg[t_i(\xi-\zeta)\dot{u}_j(\xi+\zeta)+\alpha(\xi-\zeta)\dot{\varphi}(\xi+\zeta)+\beta(\xi-\zeta)\dot{\psi}(\xi+\zeta)\\
&-\frac{1}{\rho T_0}Q(\xi-\zeta)\theta(\xi+\zeta)\bigg]\,\mathrm{d}A\,\mathrm{d}\zeta\\
&+\frac{1}{2}\int_0^{\xi}\int_{\partial B}\bigg[t_i(\xi+\zeta)\dot{u}_j(\xi-\zeta)+\alpha(\xi+\zeta)\dot{\varphi}(\xi-\zeta)+\beta(\xi+\zeta)\dot{\psi}(\xi-\zeta)\\
&-\frac{1}{\rho T_0}Q(\xi+\zeta)\theta(\xi-\zeta)\bigg]\,\mathrm{d}A\,\mathrm{d}\zeta\\
&-\frac{1}{2}\int_0^{\xi}\int_{B}\bigg[\rho f_i(\xi-\zeta)\dot{u}_j(\xi+\zeta)+\rho G(\xi-\zeta)\dot{\varphi}(\xi+\zeta)+\rho L(\xi-\zeta)\dot{\psi}(\xi+\zeta)\\
&-\frac{1}{T_0}\mathfrak{h}(\xi-\zeta)\theta(\xi+\zeta)\bigg]\,\mathrm{d}V\,\mathrm{d}\zeta
\end{aligned}
$$

$$+\frac{1}{2}\int_0^{\xi}\int_B\Big[\rho f_i(\xi+\zeta)\dot{u}_j(\xi-\zeta)+\rho G(\xi+\zeta)\dot{\varphi}(\xi-\zeta)+\rho L(\xi+\zeta)\dot{\psi}(\xi-\zeta)$$
$$-\frac{1}{T_0}\mathfrak{h}(\xi+\zeta)\theta(\xi-\zeta)\Big]\,\mathrm{d}V\,\mathrm{d}\zeta$$
$$+\int_0^{\xi}\int_{\partial B}\Big[t_i(\xi)\dot{u}_j(\xi)+\alpha(\xi)\dot{\varphi}(\xi)+\beta(\xi)\dot{\psi}(\xi)-\frac{1}{\rho T_0}Q(\xi)\theta(\xi)\Big]\,\mathrm{d}A\,\mathrm{d}\zeta$$
$$+\int_0^{\xi}\int_B\Big[\rho f_i(\xi)\dot{u}_j(\xi)+\rho G(\xi)\dot{\varphi}(\xi)+\rho L(\xi)\dot{\psi}(\xi)-\frac{1}{T_0}\mathfrak{h}(\xi)\theta(\xi)\Big]\,\mathrm{d}V\,\mathrm{d}\zeta$$
$$-\int_0^{\xi}\int_B\frac{K_{ij}}{\rho T_0}\theta_{,i}(\xi)\theta_{,j}(\xi)\,\mathrm{d}V\,\mathrm{d}\zeta, \tag{18}$$

$$X(\xi)=\frac{1}{2}[X(0)+Y(0)]$$
$$-\frac{1}{2}\int_B[C_{ijkl}u_{i,j}(0)u_{k,l}(2\xi)+B_{ij}[u_{i,j}(0)\varphi(2\xi)+u_{i,j}(2\xi)\varphi(0)]$$
$$+D_{ij}[u_{i,j}(0)\psi(2\xi)+u_{i,j}(2\xi)\psi(0)]+\alpha_{ij}\varphi_{,i}(0)\varphi_{,j}(2\xi)$$
$$+b_{ij}[\varphi_{,i}(0)\psi_{,j}(2\xi)+\varphi_{,i}(2\xi)\psi_{,j}(0)]+\gamma_{ij}\psi,i(0)\psi,j(2\xi)$$
$$+\alpha_1\varphi(0)\varphi(2\xi)+\alpha_2\psi(0)\psi(2\xi)+\alpha_3[\varphi(0)\psi(2\xi)+\varphi(2\xi)\psi(0)]$$
$$+a\theta(0)\theta(2\xi)]\,\mathrm{d}V$$
$$+\frac{1}{2}\int_0^{\xi}\int_{\partial B}\Big[t_i(\xi-\zeta)\dot{u}_j(\xi+\zeta)+\alpha(\xi-\zeta)\dot{\varphi}(\xi+\zeta)+\beta(\xi-\zeta)\dot{\psi}(\xi+\zeta)$$
$$-\frac{1}{\rho T_0}Q(\xi-\zeta)\theta(\xi+\zeta)\Big]\,\mathrm{d}A\,\mathrm{d}\zeta$$
$$-\frac{1}{2}\int_0^{\xi}\int_{\partial B}\Big[t_i(\xi+\zeta)\dot{u}_j(\xi-\zeta)+\alpha(\xi+\zeta)\dot{\varphi}(\xi-\zeta)+\beta(\xi+\zeta)\dot{\psi}(\xi-\zeta)$$
$$-\frac{1}{\rho T_0}Q(\xi+\zeta)\theta(\xi-\zeta)\Big]\,\mathrm{d}A\,\mathrm{d}\zeta$$
$$+\frac{1}{2}\int_0^{\xi}\int_B\Big[\rho f_i(\xi-\zeta)\dot{u}_j(\xi+\zeta)+\rho G(\xi-\zeta)\dot{\varphi}(\xi+\zeta)+\rho L(\xi-\zeta)\dot{\psi}(\xi+\zeta)$$
$$-\frac{1}{T_0}\mathfrak{h}(\xi-\zeta)\theta(\xi+\zeta)\Big]\,\mathrm{d}V\,\mathrm{d}\zeta$$
$$-\frac{1}{2}\int_0^{\xi}\int_B\Big[\rho f_i(\xi+\zeta)\dot{u}_j(\xi-\zeta)+\rho G(\xi+\zeta)\dot{\varphi}(\xi-\zeta)+\rho L(\xi+\zeta)\dot{\psi}(\xi-\zeta)$$
$$-\frac{1}{T_0}\mathfrak{h}(\xi+\zeta)\theta(\xi-\zeta)\Big]\,\mathrm{d}V\,\mathrm{d}\zeta$$

$$+\int_0^{\xi}\int_{\partial B}\left[t_i(\xi)\dot{u}_j(\xi)+\alpha(\xi)\dot{\varphi}(\xi)+\beta(\xi)\dot{\psi}(\xi)-\frac{1}{\rho T_0}Q(\xi)\theta(\xi)\right]\mathrm{d}A\ \mathrm{d}\zeta$$
$$+\int_0^{\xi}\int_{B}\left[\rho f_i(\xi)\dot{u}_j(\xi)+\rho G(\xi)\dot{\varphi}(\xi)+\rho L(\xi)\dot{\psi}(\xi)-\frac{1}{T_0}\mathfrak{h}(\xi)\theta(\xi)\right]\mathrm{d}V\ \mathrm{d}\zeta$$
$$-\int_0^{\xi}\int_{B}\frac{K_{ij}}{\rho T_0}\theta_{,i}(\xi)\theta_{,j}(\xi)\ \mathrm{d}V\ \mathrm{d}\zeta, \tag{19}$$

Proof We consider the relations (11) and (13) in which $a=\xi, b=\xi$ and if we take into consideration the constitutive equations (16), then the function $\mathfrak{W}(\xi,\xi)$ has the following structure:

$$\begin{aligned}\mathfrak{W}(\xi,\xi)=&\,C_{ijkl}u_{i,j}(\xi)\dot{u}_{k,l}(\xi)+B_{ij}[\dot{u}_{i,j}(\xi)\varphi(\xi)+u_{i,j}(\xi)\dot{\varphi}(\xi)]\\&+D_{ij}[\dot{u}_{i,j}(\xi)\psi(\xi)+u_{i,j}(\xi)\dot{\psi}(\xi)]+\alpha_{ij}\dot{\varphi}_{,i}(\xi)\varphi_{,j}(\xi)\\&+b_{ij}[\dot{\varphi}_{,i}(\xi)\psi_{,j}(\xi)+\dot{\psi}_{,i}(\xi)\varphi_{,j}(\xi)]+\gamma_{ij}\dot{\psi}_{,i}(\xi)\psi_{,j}(\xi)\\&+a\theta(\xi)\dot{\theta}(\xi)+\alpha_1\varphi(\xi)\dot{\varphi}(\xi)+\alpha_2\psi(\xi)\dot{\psi}(\xi)\\&+\alpha_3[\dot{\varphi}(\xi)\psi(\xi)+\dot{\psi}(\xi)\varphi(\xi)].\end{aligned} \tag{20}$$

We observe that if we derive (9) in relation to t, we have:

$$\begin{aligned}\dot{X}(\xi)=&\int_B[\rho\ddot{u}_i(\xi)\dot{u}_i(\xi)+\rho\dot{u}_i(\xi)\ddot{u}_i(\xi)+K_1\ddot{\varphi}(\xi)\dot{\varphi}(\xi)\\&+K_1\dot{\varphi}(\xi)\ddot{\varphi}(\xi)+K_2\ddot{\psi}(\xi)\dot{\psi}(\xi)+K_2\dot{\psi}(\xi)\ddot{\psi}(\xi)]\,\mathrm{d}V\\=&\,2\int_B[\rho\dot{u}_i(\xi)\ddot{u}_i(\xi)+K_1\dot{\varphi}(\xi)\ddot{\varphi}(\xi)+K_2\dot{\psi}(\xi)\ddot{\psi}(\xi)]\,\mathrm{d}V,\\\dot{Y}(\xi)=&\int_B[C_{ijkl}\dot{u}_{i,j}(\xi)u_{k,l}(\xi)+C_{ijkl}u_{i,j}(\xi)\dot{u}_{k,l}(\xi)+2B_{ij}[\dot{u}_{i,j}(\xi)\varphi(\xi)\\&+u_{i,j}(\xi)\dot{\varphi}(\xi)]+2D_{ij}[\dot{u}_{i,j}(\xi)\psi(\xi)+u_{i,j}(\xi)\dot{\psi}(\xi)]\\&+\alpha_{ij}[\dot{\varphi}_{,i}(\xi)\varphi_{,j}(\xi)+\varphi_{,i}(\xi)\dot{\varphi}_{,j}(\xi)]+2b_{ij}[\dot{\varphi}_{,i}(\xi)\psi_{,j}(\xi)\\&+\varphi_{,i}(\xi)\dot{\psi}_{,j}(\xi)]+\gamma_{ij}[\dot{\psi}_{,i}(\xi)\psi_{,j}(\xi)+\psi_{,i}(\xi)\dot{\psi}_{,j}(\xi)]\\&+2\alpha_1\varphi(\xi)\dot{\varphi}(\xi)+2\alpha_2\psi(\xi)\dot{\psi}(\xi)+2\alpha_3[\dot{\varphi}(\xi)\psi(\xi)+\varphi(\xi)\dot{\psi}(\xi)]+2a\theta(\xi)\dot{\theta}(\xi)]\,\mathrm{d}V\\=&\int_B 2\mathfrak{W}(\xi,\xi)\ \mathrm{d}V.\end{aligned}$$

Integrating on B and considering the divergence theorem, the following relation is deduced:

$$\begin{aligned}\dot{X}(\xi)+\dot{Y}(\xi)=&\,2\int_{\partial B}\left[t_i(\xi)u_j(\xi)+\alpha(\xi)\dot{\varphi}(\xi)+\beta(\xi)\dot{\psi}(\xi)\right.\\&\left.-\frac{1}{\rho T_0}Q(\xi)\theta(\xi)\right]\mathrm{d}A\\&+2\int_B\left[\rho f_i(\xi)\dot{u}_i(\xi)+\rho G(\xi)\dot{\varphi}(\xi)+\rho L(\xi)\dot{\psi}(\xi)\right.\\&\left.+\frac{1}{T_0}\theta(\xi)\mathfrak{h}(\xi)\right]\mathrm{d}V-2\int_B\frac{K_{ij}}{\rho T_0}\theta_{,i}(\xi)\theta_{,j}(\xi)\ \mathrm{d}V.\end{aligned} \tag{21}$$

After integration the relation (21) on $[0, \xi]$, we derive:

$$\begin{aligned}
&X(\xi) - X(0) + Y(\xi) - Y(0) \\
&= 2\int_0^{\xi}\int_{\partial B}\left[t_i(\xi)u_j(\xi) + \alpha(\xi)\dot{\varphi}(\xi) + \beta(\xi)\dot{\psi}(\xi) - \frac{1}{\rho T_0}Q(\xi)\theta(\xi)\right] \mathrm{d}A\, \mathrm{d}\zeta \\
&\quad + 2\int_0^{\xi}\int_{B}\left[\rho f_i(\xi)\dot{u}_i(\xi) + \rho G(\xi)\dot{\varphi}(\xi) + \rho L(\xi)\dot{\psi}(\xi) + \frac{1}{T_0}\theta(\xi)\mathfrak{h}(\xi)\right] \mathrm{d}V\, \mathrm{d}\zeta \\
&\quad - 2\int_0^{\xi}\int_{B}\frac{K_{ij}}{\rho T_0}\theta_{,i}(\xi)\theta_{,j}(\xi)\, \mathrm{d}V\, \mathrm{d}\zeta.
\end{aligned} \tag{22}$$

We will solve the system of equations (10) and (22) and we obtain $Y(\xi)$ and $X(\xi)$ which are the relations of Theorem 2. Thus, by summing, we obtain $Y(\xi)$ from formula (18) and by subtraction, we obtain $X(\xi)$ from formula (19). □

Theorems 1 and 2 allow us to obtain the following uniqueness theorem:

Theorem 3 *If the assumptions (1)–(5) are fulfilled then the mixed problem for thermoelastic bodies with double porosity (4) accompanied by the initial conditions (7) and the boundary conditions (8) admits a unique solution.*

Proof We will proof the theorem by contradiction. We consider that the proposed mixed problem admit two different solutions: $u_i^{(k)}, \varphi^{(k)}, \psi^{(k)}, \theta^{(k)}, t_{ij}^{(k)}, \sigma_i^{(k)}, \tau_i^{(k)}, \xi^{(k)}, \zeta^{(k)}, \eta^{(k)}, Q_i^{(k)}, k = 1, 2$. The difference between the two considered solutions will be noted by: $\hat{u}_i$, $\hat{\varphi}$, $\hat{\psi}$, $\hat{\theta}$, $\hat{t}_{ij}$, $\hat{\sigma}_i$, $\hat{\tau}_i$, $\hat{\xi}$, $\hat{\zeta}$, $\hat{\eta}$, $\hat{Q}_i$. Due to the linearity, the difference between the two solutions is also a solution of the problem corresponding to the null initial conditions and the null boundary conditions.

Considering the null boundary conditions, the expression of the function $X(\xi)$ from (19) becomes:

$$X(\xi) = -\int_0^{\xi}\int_B \frac{K_{ij}}{\rho T_0}\hat{\theta}_{,i}(\xi)\hat{\theta}_{,j}(\xi)\, \mathrm{d}V\, \mathrm{d}\zeta. \tag{23}$$

Considering the expression of $X(\xi)$ in (9) and from (23) we obtain

$$\begin{aligned}
&\int_B\left[\rho\hat{\dot{u}}_i(\xi)\hat{\dot{u}}_i(\xi) + K_1\hat{\dot{\varphi}}(\xi)\hat{\dot{\varphi}}(\xi) + K_2\hat{\dot{\psi}}(\xi)\hat{\dot{\psi}}(\xi)\right] \\
&\quad + \frac{1}{\rho T_0}\int_0^{\xi}\int_B K_{ij}\hat{\theta}_{,i}(\xi)\hat{\theta}_{,j}(\xi)\, \mathrm{d}V\, \mathrm{d}\zeta = 0,
\end{aligned} \tag{24}$$

where $\xi \in [0, \infty)$.

Due to the fact that $a > 0$, K_1, K_2 are positive definite tensors and K_{ij} is a positive semi-definite tensor, considering (24) we derive:

$$\hat{\dot{u}}_i(x, \xi) = 0, \quad \hat{\dot{\varphi}}(x, \xi) = 0, \quad \hat{\dot{\psi}}(x, \xi) = 0, \quad \forall (x, \xi) \in B \times [0, \infty), \tag{25}$$

respectively,

$$\int_0^{\xi}\int_B K_{ij}\hat{\theta}_{,i}(\xi)\hat{\theta}_{,j}(\xi)\, \mathrm{d}V\, \mathrm{d}\zeta = 0, \quad \forall \xi \in [0, \infty). \tag{26}$$

Taking into account the null initial conditions for the difference between the two solutions, and considering (25), we obtain:

$$\hat{u}_i(x, \xi) = 0, \quad \hat{\varphi}(x, \xi) = 0, \quad \hat{\psi}(x, \xi) = 0, \quad \forall (x, \xi) \in B \times [0, \infty). \tag{27}$$

If we now consider the null boundary conditions, then the expression of $Y(\xi)$ in (??) becomes:

$$Y(\xi) = -\int_0^\xi \int_B \frac{K_{ij}}{\rho T_0} \hat{\theta}_{,i}(\xi)\hat{\theta}_{,j}(\xi)\,\mathrm{d}V\,\mathrm{d}\zeta. \tag{28}$$

From the expression of $Y(\xi)$ in (9), taking into account (27) and (28), we obtain:

$$\int_B a\hat{\theta}^2(\xi)\,\mathrm{d}V = -\int_0^\xi \int_B \frac{K_{ij}}{\rho T_0} \hat{\theta}_{,i}(\xi)\hat{\theta}_{,j}(\xi)\,\mathrm{d}V\,\mathrm{d}\zeta.$$

Considering (26) we obtain:

$$\int_B a\hat{\theta}^2(\xi)\,\mathrm{d}V = 0. \tag{29}$$

Due to the fact that $a > 0$, the relation (29) leads to:

$$\hat{\theta}(x, \xi) = 0, \quad (x, \xi) \in B \times [0, \infty). \tag{30}$$

In conclusion, due to the fact that the difference of the two considered solutions is null, (27) and (30) lead to the fact that the considered problem has a unique solution. □

The basis of the following uniqueness theorem is given by the proof of the following Betti-type reciprocity relation that is in fact a relation between two systems of external loads and the solutions corresponding to these loads.

We now introduce the functions f and g defined on the cylinder $\Omega_0 = B \times [0, \infty)$, that are continuous in time and whose convolution product is defined by the relation:

$$(f * g)(x, \xi) = \int_0^\xi f(x, \xi - \zeta)g(\xi, \zeta)\,\mathrm{d}\zeta. \tag{31}$$

Integrating the energy equation (2) we obtain:

$$\rho T_0 \eta - \rho T_0 \eta_0 = \int_0^\xi Q_{j,j}(x, \zeta)\,\mathrm{d}\zeta + \rho \int_0^\xi \mathfrak{h}(x, \zeta)\,\mathrm{d}\zeta. \tag{32}$$

If in the convolution product (31) we consider that the function f is the identity function $f = 1$, then we have the following notation:

$$g_*(x, \xi) = \int_0^\xi 1 \cdot g(\xi, \zeta)\,\mathrm{d}\zeta = 1 * g(\xi, \zeta).$$

Using the notation above we have:

$$Q_{*j,j} = \int_0^\xi Q_{j,j}(x, \zeta)\,\mathrm{d}\zeta; \quad \mathfrak{h}_* = \int_0^\xi \mathfrak{h}(x, \zeta)\,\mathrm{d}\zeta.$$

Therefore, the energy equation (2) can also be written as follows:

$$\eta = \frac{1}{\rho T_0}(Q_{*j,j} + \omega), \tag{33}$$

where $\omega = \rho \mathfrak{h}_* + \rho T_0 \eta_0$.

Let us consider two systems of loads marked by $H^{(a)}$, where $a = 1, 2$:

$$H^{(a)} = \Big\{ f_i^{(a)}, G^{(a)}, L^{(a)}, r^{(a)}, u_i^{*(a)}, t_i^{*(a)}, \varphi^{*(a)}, \alpha^{*(a)}, \psi^{*(a)}, \beta^{*(a)}, \theta^{*(a)}, Q^{*(a)}, u_i^{0(a)}, v_i^{0(a)}, \varphi_0^{(a)}, \tilde{\varphi}_0^{(a)}, \psi_0^{(a)}, \tilde{\psi}_0^{(a)}, \eta_0^{(a)} \Big\}, \tag{34}$$

respectively, the set of corresponding solutions $S^{(a)}$, where $a = 1, 2$:

$$S^{(a)} = \Big\{ u_i^{(a)}, \varphi^{(a)}, \psi^{(a)}, \theta^{(a)}, t_{ij}^{(a)}, \sigma_i^{(a)}, \tau_i^{(a)}, Q_i^{(a)}, \mathfrak{r}^{(a)}, \mathfrak{p}^{(a)} \Big\}, \tag{35}$$

where $t_i^{(a)} = t_{ij}^{(a)} n_j, \quad \alpha^{(a)} = \sigma_i^{(a)} n_i, \quad \beta^{(a)} = \tau_i^{(a)} n_i, \quad Q^{(a)} = Q_i^{(a)} n_i$. Let us consider the function:

$$\mathcal{F}_{ab}(\xi, \zeta) = t_{ij}^{(a)}(\xi) u_{i,j}^{(b)}(\zeta) + \sigma_i^{(a)}(\xi) \varphi_{,i}^{(b)}(\zeta) + \tau_i^{(a)}(\xi) \psi_{,i}^{(b)}(\zeta) - \mathfrak{p}^{(a)}(\xi) \varphi^{(b)}(\zeta) - \mathfrak{r}^{(a)}(\xi) \psi^{(b)}(\zeta) - \eta^{(a)}(\xi) \theta^{(b)}(\zeta). \tag{36}$$

We introduce the constitutive equations (3) in (36) and we obtain:

$$\begin{aligned} \mathcal{F}_{ab}(\xi, \zeta) = {} & C_{ijkl} u_{k,l}^{(a)}(\xi) u_{i,j}^{(b)}(\zeta) + B_{ij} \Big[\varphi^{(a)}(\xi) u_{i,j}^{(b)}(\zeta) + u_{i,j}^{(a)}(\xi) \varphi^{(b)}(\zeta) \Big] \\ & + D_{ij} \Big[\psi^{(a)}(\xi) u_{i,j}^{(b)}(\zeta) + u_{i,j}^{(a)}(\xi) \psi^{(b)}(\zeta) \Big] - \beta_{ij} \Big[\theta^{(a)}(\xi) u_{i,j}^{(b)}(\zeta) + u_{i,j}^{(a)}(\xi) \theta^{(b)}(\zeta) \Big] \\ & + \alpha_{ij} \varphi_{,j}^{(a)}(\xi) \varphi_{,i}^{(b)}(\zeta) + b_{ij} \Big[\psi_{,j}^{(a)}(\xi) \varphi_{,i}^{(b)}(\zeta) + \varphi_{,j}^{(a)}(\zeta) \psi_{,i}^{(b)}(\zeta) \Big] \\ & + \gamma_{ij} \psi_{,j}^{(a)}(\xi) \psi_{,i}^{(b)}(\zeta) + \alpha_1 \varphi^{(a)}(\xi) \varphi^{(b)}(\zeta) + \alpha_2 \psi^{(a)}(\xi) \psi^{(b)}(\zeta) \\ & + \alpha_3 \Big[\varphi^{(a)}(\xi) \psi^{(b)}(\zeta) + \psi^{(a)}(\xi) \varphi^{(b)}(\zeta) \Big] - \gamma_1 \Big[\theta^{(a)}(\xi) \varphi^{(b)}(\zeta) + \varphi^{(a)}(\xi) \theta^{(b)}(\zeta) \Big] \\ & - \gamma_2 \Big[\theta^{(a)}(\xi) \psi^{(b)}(\zeta) + \psi^{(a)}(\xi) \theta^{(b)}(\zeta) \Big] - a \theta^{(a)}(\xi) \theta^{(b)}(\zeta). \end{aligned} \tag{37}$$

Taking into account the symmetry relations (5), we observe that the commutative takes place:

$$\mathcal{F}_{ab}(\xi, \zeta) = \mathcal{F}_{ba}(\zeta, \xi). \tag{38}$$

We observe that:

$$\begin{aligned} & \Big(t_{ij}^{(a)}(\xi) u_j^{(b)}(\zeta) + \sigma_i^{(a)}(\xi) \varphi^{(b)}(\zeta) + \tau_i^{(a)}(\xi) \psi^{(b)}(\zeta) \Big)_{,i} \\ & = t_{ij,i}^{(a)}(\xi) u_j^{(b)}(\zeta) + t_{ij}^{(a)}(\xi) u_{j,i}^{(b)}(\zeta) + \sigma_{i,i}^{(a)}(\xi) \varphi^{(b)}(\zeta) + \sigma_i^{(a)}(\xi) \varphi_{,i}^{(b)}(\zeta) \\ & \quad + \tau_{i,i}^{(a)}(\xi) \psi^{(b)}(\zeta) + \tau_i^{(a)}(\xi) \psi_{,i}^{(b)}(\zeta). \end{aligned}$$

Using the equations of motion (1) the above relation leads us to:

$$\begin{aligned} & t_{ij}^{(a)}(\xi) u_{j,i}^{(b)}(\zeta) + \sigma_i^{(a)}(\xi) \varphi_{,i}^{(b)}(\zeta) + \tau_i^{(a)}(\xi) \psi_{,i}^{(b)}(\zeta) \\ & = \Big(t_{ij}^{(a)}(\xi) u_j^{(b)}(\zeta) + \sigma_i^{(a)}(\xi) \varphi^{(b)}(\zeta) + \tau_i^{(a)}(\xi) \psi^{(b)}(\zeta) \Big)_{,i} - \Big(\rho \ddot{u}_i^{(a)}(\xi) \\ & \quad - \rho f_i^{(a)}(\xi) \Big) u_j^{(b)}(\zeta) - \Big(K_1 \ddot{\varphi}^{(a)}(\xi) - \mathfrak{p}^{(a)}(\xi) - \rho G^{(a)}(\xi) \Big) \varphi^{(b)}(\zeta) \\ & \quad - \Big(K_2 \ddot{\psi}^{(a)}(\xi) - \mathfrak{r}^{(a)}(\xi) - \rho L^{(a)}(\xi) \Big) \psi^{(b)}(\zeta). \end{aligned} \tag{39}$$

We use the energy equation written as in (33) and we have:

$$\eta^{(a)}(\xi) \theta^{(b)}(\zeta) = \frac{1}{\rho T_0} \Big(Q_{*i}^{(a)}(\xi) \theta^{(b)}(\zeta) \Big)_{,i} + \frac{1}{\rho T_0} \omega^{(a)}(\xi) \theta^{(b)}(\zeta) - \frac{1}{\rho T_0} Q_{*i}^{(a)}(\xi) \theta_{,i}^{(b)}(\zeta). \tag{40}$$

Substituting (39) and (40) in (36) we obtain:

$$\mathcal{F}_{ab}(\xi,\zeta)=\Big(t_{ij}^{(a)}(\xi)u_j^{(b)}(\zeta)+\sigma_i^{(a)}(\xi)\varphi^{(b)}(\zeta)+\tau_i^{(a)}(\xi)\psi^{(b)}(\zeta)$$
$$-\frac{1}{\rho T_0}Q_{*i}^{(a)}(\xi)\theta^{(b)}(\zeta)\Big)_{,i}-\Big(\rho\ddot{u}_i^{(a)}(\xi)-\rho f_i^{(a)}(\xi)\Big)u_i^{(b)}(\zeta)$$
$$-\Big(K_1\ddot{\varphi}^{(a)}(\xi)-\rho G^{(a)}(\xi)\Big)\varphi^{(b)}(\zeta)-\Big(K_2\ddot{\psi}^{(a)}(\xi)-\rho L^{(a)}(\xi)\Big)\psi^{(b)}(\zeta)$$
$$-\frac{1}{\rho T_0}\omega^{(a)}(\xi)\theta^{(b)}(\zeta)+\frac{1}{\rho T_0}Q_{*i}^{(a)}(\xi)\theta_{,i}^{(b)}(\zeta). \tag{41}$$

We integrate the previous relation on B and using the divergence theorem and take into account (8′). We have:

$$\int_B\Big[t_{ij}^{(a)}(\xi)u_i^{(b)}(\zeta)+\sigma_i^{(a)}(\xi)\varphi^{(b)}(\zeta)+\tau_i^{(a)}(\xi)\psi^{(b)}(\zeta)$$
$$-\frac{1}{\rho T_0}Q_{*i}^{(a)}(\xi)\theta^{(b)}(\zeta)\Big]_{,i}\,\mathrm{d}V$$
$$=\int_{\partial B}\Big[t_{ij}^{(a)}(\xi)u_i^{(b)}(\zeta)n_i+\sigma_i^{(a)}(\xi)\varphi^{(b)}(\zeta)n_i+\tau_i^{(a)}(\xi)\psi^{(b)}(\zeta)n_i$$
$$-\frac{1}{\rho T_0}Q_{*i}^{(a)}(\xi)\theta^{(b)}(\zeta)n_i\Big]\,\mathrm{d}A$$
$$=\int_{\partial B}\Big[t_i^{(a)}(\xi)u_i^{(b)}(\zeta)+\alpha^{(a)}(\xi)\varphi^{(b)}(\zeta)+\beta^{(a)}(\xi)\psi^{(b)}(\zeta)$$
$$-\frac{1}{\rho T_0}Q_*^{(a)}(\xi)\theta^{(b)}(\zeta)\Big]\,\mathrm{d}A. \tag{42}$$

Based on the previous relations, we can express the following result:

Lemma 1 *For the function:*

$$\mathcal{F}_{ab}(\xi,\zeta)=\int_B\Big[\rho f_i^{(a)}(\xi)u_i^{(b)}(\zeta)+\rho G^{(a)}(\xi)\varphi^{(b)}(\zeta)+\rho L^{(a)}(\xi)\psi^{(b)}(\zeta)$$
$$-\frac{1}{\rho T_0}\omega^{(a)}(\xi)\theta^{(b)}(\zeta)\Big]\,\mathrm{d}V$$
$$-\int_B\Big[\rho\ddot{u}_i^{(a)}(\xi)u_i^{(b)}(\zeta)+K_1\ddot{\varphi}^{(a)}(\xi)\varphi^{(b)}(\zeta)+K_2\ddot{\psi}^{(a)}(\xi)\psi^{(b)}(\zeta)\Big]\,\mathrm{d}V$$
$$+\frac{1}{\rho T_0}\int_B Q_{*i}^{(a)}(\xi)\theta_{,i}^{(b)}(\zeta)\,\mathrm{d}V$$
$$+\int_{\partial B}\Big[t_i^{(a)}(\xi)u_i^{(b)}(\zeta)+\alpha^{(a)}(\xi)\varphi^{(b)}(\zeta)+\beta^{(a)}(\xi)\psi^{(b)}(\zeta)$$
$$-\frac{1}{\rho T_0}Q_*^{(a)}(\xi)\theta^{(b)}(\zeta)\Big]\,\mathrm{d}A. \tag{43}$$

The following commutative relation takes place:

$$\mathcal{F}_{ab}(\xi,\zeta)=\mathcal{F}_{ba}(\zeta,\xi),\quad \forall\xi,\zeta\in[0,\infty),\quad a,b=1,2. \tag{44}$$

The result of this lemma will be used in the following reciprocity theorem:

Theorem 4 *Let us consider the systems of loads (*) and the solutions corresponding to these loads (**). Then the following relation of reciprocity occurs:*

$$\begin{aligned}
&\int_B \left[F_i^{(1)} * u_i^{(2)} + G_i^{(1)} * \varphi^{(2)} + L_i^{(1)} * \psi^{(2)} - \frac{1}{\rho T_0} \xi * \omega^{(1)} * \theta^{(2)} \right] \mathrm{d}V \\
&\quad + \int_{\partial B} \left[\xi * \left(t_i^{(1)} * u_i^{(2)} + \alpha^{(1)} * \varphi^{(2)} + \beta^{(1)} * \psi^{(2)} - \frac{1}{\rho T_0} Q^{(1)} * \theta^{(2)} \right) \right] \mathrm{d}A \\
&= \int_B \left[F_i^{(2)} * u_i^{(1)} + G_i^{(2)} * \varphi^{(1)} + L_i^{(2)} * \psi^{(1)} - \frac{1}{\rho T_0} \xi * \omega^{(2)} * \theta^{(1)} \right] \mathrm{d}V \\
&\quad + \int_{\partial B} \left[\xi * \left(t_i^{(2)} * u_i^{(1)} + \alpha^{(2)} * \varphi^{(1)} + \beta^{(2)} * \psi^{(1)} - \frac{1}{\rho T_0} Q^{(2)} * \theta^{(1)} \right) \right] \mathrm{d}A, \qquad (45)
\end{aligned}$$

where

$$\begin{aligned}
F_i^{(a)} &= \rho \xi * f_i^{(a)} + \rho u_i^{0(a)} + \rho \xi v_i^{0(a)}, \\
G_i^{(a)} &= \rho \xi * G^{(a)} + K_1 \varphi_0^{(a)} + K_1 \xi \tilde{\varphi}_0^{(a)}, \\
L_i^{(a)} &= \rho \xi * L^{(a)} + K_2 \psi_0^{(a)} + K_2 \xi \tilde{\psi}_0^{(a)}.
\end{aligned}$$

Proof In the continuity relation (25), we perform the change of variables $\xi = s$ and $\zeta = \xi - s$:

$$\mathcal{F}_{ab}(s, \xi - s) = \mathcal{F}_{ba}(\xi - s, s). \qquad (46)$$

We consider the symmetry relations (5) and we obtain:

$$\begin{aligned}
&\mathcal{F}_{ab}(s, \xi - s) \\
&= \int_B \Big[\rho f_i^{(a)}(s) u_i^{(b)}(\xi - s) + \rho G^{(a)}(s) \varphi^{(b)}(\xi - s) + \rho L^{(a)}(s) \psi^{(b)}(\xi - s) \\
&\quad - \frac{1}{\rho T_0} \omega^{(a)}(s) \theta^{(b)}(\xi - s) \Big] \mathrm{d}V \\
&\quad - \int_B \Big[\rho \ddot{u}_i^{(a)}(s) u_i^{(b)}(\xi - s) + K_1 \ddot{\varphi}^{(a)}(s) \varphi^{(b)}(\xi - s) + K_2 \ddot{\psi}^{(a)}(s) \psi^{(b)}(\xi - s) \Big] \mathrm{d}V \\
&\quad + \frac{1}{\rho T_0} \int_B Q_{*i}^{(a)}(s) \theta_{,i}^{(b)}(\xi - s) \, \mathrm{d}V \\
&\quad + \int_{\partial B} \Big[t_i^{(a)}(s) u_i^{(b)}(\xi - s) + \alpha^{(a)}(s) \varphi^{(b)}(\xi - s) + \beta^{(a)}(s) \psi^{(b)}(\xi - s) \\
&\quad - \frac{1}{\rho T_0} Q_*^{(a)}(s) \theta^{(b)}(\xi - s) \Big] \mathrm{d}A.
\end{aligned}$$

If we apply the integration on $[0, \xi]$, we notice that we have convolution products, and taking into account (14′) we obtain:

$$\begin{aligned}
\int_0^\xi \mathcal{F}_{ab}(s, \xi - s) \, \mathrm{d}s = \int_B \Big[& \rho f_i^{(a)} * u_i^{(b)} + \rho G^{(a)} * \varphi^{(b)} + \rho L^{(a)} * \psi^{(b)} \\
& - \frac{1}{\rho T_0} \omega^{(a)} * \theta^{(b)} \Big] \mathrm{d}V
\end{aligned}$$

$$
\begin{aligned}
&-\int_B \left[\rho \ddot{u}_i^{(a)} * u_i^{(b)} + K_1 \ddot{\varphi}^{(a)} * \varphi^{(b)} + K_2 \ddot{\psi}^{(a)} * \psi^{(b)}\right] \mathrm{d}V \\
&+\frac{1}{\rho T_0}\int_B \left(1 * Q_i^{(a)} * \theta_{,i}^{(b)}\right) \\
&+\int_{\partial B} \left[t_i^{(a)} * u_i^{(b)} + \alpha^{(a)} * \varphi^{(b)} + \beta^{(a)} * \psi^{(b)} - \frac{1}{\rho T_0} 1 * Q^{(a)} * \theta^{(b)}\right] \mathrm{d}A. \qquad (47)
\end{aligned}
$$

Taking into account (28), the relation (27) leads to:

$$
\begin{aligned}
&\int_B \left[\rho f_i^{(1)} * u_i^{(2)} + \rho G^{(1)} * \varphi^{(2)} + \rho L^{(1)} * \psi^{(2)}\right. \\
&\quad \left. -\frac{1}{\rho T_0}\omega^{(1)} * \theta^{(2)}\right] \mathrm{d}V \\
&\quad -\int_B \left[\rho \ddot{u}_i^{(1)} * u_i^{(2)} + K_1 \ddot{\varphi}^{(1)} * \varphi^{(2)} + K_2 \ddot{\psi}^{(1)} * \psi^{(2)}\right] \mathrm{d}V \\
&\quad +\frac{1}{\rho T_0}\int_B \left(1 * Q_i^{(1)} * \theta_{,i}^{(2)}\right) \mathrm{d}V \\
&\quad +\int_{\partial B} \left[t_i^{(1)} * u_i^{(2)} + \alpha^{(1)} * \varphi^{(2)} + \beta^{(1)} * \psi^{(2)} - \frac{1}{\rho T_0} 1 * Q^{(1)} * \theta^{(2)}\right] \mathrm{d}A \\
&= \int_B \left[\rho f_i^{(2)} * u_i^{(1)} + \rho G^{(2)} * \varphi^{(1)} + \rho L^{(2)} * \psi^{(1)}\right. \\
&\quad \left. -\frac{1}{\rho T_0}\omega^{(2)} * \theta^{(1)}\right] \mathrm{d}V \\
&\quad -\int_B \left[\rho \ddot{u}_i^{(2)} * u_i^{(1)} + K_1 \ddot{\varphi}^{(2)} * \varphi^{(1)} + K_2 \ddot{\psi}^{(2)} * \psi^{(1)}\right] \mathrm{d}V \\
&\quad +\frac{1}{\rho T_0}\int_B \left(1 * Q_i^{(2)} * \theta_{,i}^{(1)}\right) \mathrm{d}V \\
&\quad +\int_{\partial B} \left[t_i^{(2)} * u_i^{(1)} + \alpha^{(2)} * \varphi^{(1)} + \beta^{(2)} * \psi^{(1)} - \frac{1}{\rho T_0} 1 * Q^{(2)} * \theta^{(1)}\right] \mathrm{d}A. \qquad (48)
\end{aligned}
$$

We consider the function $g(\xi) = \xi$ and apply the product of convolution between $g(\xi)$ and the relation (29) which leads to obtaining the quantity:

$$
\begin{aligned}
M = &\int_B \left[\rho\xi * f_i^{(a)} * u_i^{(b)} + \rho\xi * G^{(a)} * \varphi^{(b)} + \rho\xi * L^{(a)} * \psi^{(b)}\right. \\
&\left. -\frac{1}{\rho T_0}\xi * \omega^{(a)} * \theta^{(b)}\right] \mathrm{d}V \\
&-\int_B \left[\rho\xi * \ddot{u}_i^{(a)} * u_i^{(b)} + K_1\xi * \ddot{\varphi}^{(a)} * \varphi^{(b)} + K_2\xi * \ddot{\psi}^{(a)} * \psi^{(b)}\right] \mathrm{d}V \\
&+\frac{1}{\rho T_0}\int_B \left(\xi * 1 * Q_i^{(a)} * \theta_{,i}^{(b)}\right) \mathrm{d}V
\end{aligned}
$$

$$+\int_{\partial B}\Big[\xi * t_i^{(a)} * u_i^{(b)} + \xi * \alpha^{(a)} * \varphi^{(b)} + \xi * \beta^{(a)} * \psi^{(b)} - \frac{1}{\rho T_0}\xi * 1 * Q^{(a)} * \theta^{(b)}\Big]\,\mathrm{d}A.$$

We have:

$$(f * g)(\xi) = \int_0^{\xi} f(\xi)g(\xi-\zeta)\,\mathrm{d}\zeta,$$

which leads to:

$$\xi * \ddot{g} = \int_0^{\xi} \zeta\ddot{g}(\xi-\zeta)\,\mathrm{d}\zeta = -\zeta\dot{g}(\xi-\zeta)|_0^{\xi} + \int_0^{\xi} g'(\xi-\zeta)\,\mathrm{d}\zeta - \xi\dot{g}(0) - g(\xi-\zeta)|_0^{\xi}$$
$$= -\xi\dot{g}(0) - g(0) + g(\xi).$$

Based on the above demonstration we conclude that:

$$\xi * \ddot{u}_i^{(a)} = u_i^{(a)} - u_i^{(a)}(0) - \xi\dot{u}_i^{(a)}(0) = u_i^{(a)} - u_i^{0(a)} - \xi v_i^{0(a)},$$
$$\xi * \ddot{\varphi}^{(a)} = \varphi^{(a)} - \varphi_0^{(a)} - \xi\tilde{\varphi}_0^{(a)},$$
$$\xi * \ddot{\psi}^{(a)} = \psi^{(a)} - \psi_0^{(a)} - \xi\tilde{\psi}_0^{(a)}.$$

Replacing in (29) the reciprocity relation (26) is obtained:

$$\int_B \Big[\rho\xi * f_i^{(1)} * u_i^{(2)} + \rho\xi * G^{(1)} * \varphi^{(2)} + \rho\xi * L^{(1)} * \psi^{(2)} - \frac{1}{\rho T_0}\xi * \omega^{(1)} * \theta^{(2)}$$
$$-\rho u_i^{(1)} * u_i^{(2)} + \rho u_i^{0(1)} * u_i^{(2)} + \rho\xi v_i^{0(1)} * u_i^{(2)}$$
$$-K_1\varphi^{(1)} * \varphi^{(2)} + K_1\varphi_0^{(1)} * \varphi^{(2)} + K_1\xi\tilde{\varphi}_0^{(1)} * \varphi^{(2)}$$
$$-K_2\psi^{(1)} * \psi^{(2)} + K_2\psi_0^{(1)} * \psi^{(2)} + K_2\xi\tilde{\psi}_0^{(1)} * \psi^{(2)} + \frac{1}{\rho T_0}\xi * Q_i^{(1)} * \theta_{,i}^{(2)}\Big]\,\mathrm{d}V$$
$$+\int_{\partial B}\xi * \Big[t_i^{(1)} * u_i^{(2)} + \alpha^{(1)} * \varphi^{(2)} + \beta^{(1)} * \psi^{(2)} - \frac{1}{\rho T_0}Q^{(1)} * \theta^{(2)}\Big]\,\mathrm{d}A$$
$$= \int_B \Big[\rho\xi * f_i^{(2)} * u_i^{(1)} + \rho\xi * G^{(2)} * \varphi^{(1)} + \rho\xi * L^{(2)} * \psi^{(1)} - \frac{1}{\rho T_0}\xi * \omega^{(2)} * \theta^{(1)}$$
$$-\rho u_i^{(2)} * u_i^{(1)} + \rho u_i^{0(2)} * u_i^{(1)} + \rho\xi v_i^{0(2)} * u_i^{(1)}$$
$$-K_1\varphi^{(2)} * \varphi^{(1)} + K_1\varphi_0^{(2)} * \varphi^{(1)} + K_1\xi\tilde{\varphi}_0^{(2)} * \varphi^{(1)}$$
$$-K_2\psi^{(2)} * \psi^{(1)} + K_2\psi_0^{(2)} * \psi^{(1)} + K_2\xi\tilde{\psi}_0^{(2)} * \psi^{(1)} + \frac{1}{\rho T_0}\xi * Q_i^{(2)} * \theta_{,i}^{(1)}\Big]\,\mathrm{d}V$$
$$+\int_{\partial B}\xi * \Big[t_i^{(2)} * u_i^{(1)} + \alpha^{(2)} * \varphi^{(1)} + \beta^{(2)} * \psi^{(1)} - \frac{1}{\rho T_0}Q^{(2)} * \theta^{(1)}\Big]\,\mathrm{d}A,$$

which is equivalent to:

$$\int_B \Big[\Big(\rho\xi * f_i^{(1)} + \rho u_i^{0(1)} + \rho\xi v_i^{0(1)}\Big) * u_i^{(2)} + \Big(\rho\xi * G^{(1)} + K_1\varphi_0^{(1)} + K_1\xi\tilde{\varphi}_0^{(1)}\Big) * \varphi^{(2)}$$

$$+\left(\rho\xi * L^{(1)} + K_2\psi_0^{(1)} + K_2\xi\tilde{\psi}_0^{(1)}\right) * \psi^{(2)} - \frac{1}{\rho T_0}\xi * \omega^{(1)} * \theta^{(2)}\Bigg] \,\mathrm{d}V$$
$$+\int_{\partial B} \xi * \left[t_i^{(1)} * u_i^{(2)} + \alpha^{(1)} * \varphi^{(2)} + \beta^{(1)} * \psi^{(2)} - \frac{1}{\rho T_0} Q^{(1)} * \theta^{(2)}\right] \mathrm{d}A$$
$$= \int_B \Bigg[\left(\rho\xi * f_i^{(2)} + \rho u_i^{0(2)} + \rho\xi v_i^{0(2)}\right) * u_i^{(1)} + \left(\rho\xi * G^{(2)} + K_1\varphi_0^{(2)} + K_1 t\tilde{\varphi}_0^{(2)}\right) * \varphi^{(1)}$$
$$+\left(\rho\xi * L^{(2)} + K_2\psi_0^{(2)} + K_2\xi\tilde{\psi}_0^{(2)}\right) * \psi^{(1)} - \frac{1}{\rho T_0}\xi * \omega^{(2)} * \theta^{(1)}\Bigg] \,\mathrm{d}V$$
$$+\int_{\partial B} \xi * \left[t_i^{(2)} * u_i^{(1)} + \alpha^{(2)} * \varphi^{(1)} + \beta^{(2)} * \psi^{(1)} - \frac{1}{\rho T_0} Q^{(2)} * \theta^{(1)}\right] \mathrm{d}A.$$

□

We will further consider $a = b = 1$. In relation (25) we will make the variable changes $\xi \to \xi + \zeta$, $\zeta \to \xi - \zeta$, which leads to:

$$F_{11}(\xi + \zeta, \xi - \zeta) = F_{11}(\xi - \zeta, \xi + \zeta). \tag{49}$$

We integrate the left member of the function from (30) on the interval $[0, \xi]$:

$$\int_0^\xi F_{11}(\xi + \zeta, \xi - \zeta)\,\mathrm{d}\zeta$$
$$= \int_0^\xi \int_B \Bigg[\rho f_i(\xi+\zeta)u_i(\xi-\zeta) + \rho G(\xi+\zeta)\varphi(\xi-\zeta) + \rho L(\xi+\zeta)\psi(\xi-\zeta)$$
$$- \frac{1}{\rho T_0}\omega(\xi+\zeta)\theta(\xi-\zeta)\Bigg] \,\mathrm{d}V\,\mathrm{d}\zeta$$
$$- \int_0^\xi \int_B \left[\rho\ddot{u}_i(\xi+\zeta)u_i(\xi-\zeta) + K_1\ddot{\varphi}(\xi+\zeta)\varphi(\xi-\zeta) + K_2\ddot{\psi}(\xi+\zeta)\psi(\xi-\zeta)\right] \mathrm{d}V\,\mathrm{d}\zeta$$
$$+ \frac{1}{\rho T_0}\int_0^\xi \int_B Q_{*i}(\xi+\zeta)\theta_{,j}(\xi-\zeta)\,\mathrm{d}V\,\mathrm{d}\zeta$$
$$+ \int_0^\xi \int_{\partial B} [t_i(\xi+\zeta)u_i(\xi-\zeta) + \alpha(\xi+\zeta)\varphi(\xi-\zeta) + \beta(\xi+\zeta)\psi(\xi-\zeta)$$
$$- \frac{1}{\rho T_0}Q_*(\xi+\zeta)\theta(\xi-\zeta)\Bigg] \,\mathrm{d}A\,\mathrm{d}\zeta. \tag{50}$$

We perform integration by parts:

$$\int_0^\xi \ddot{u}_i(\xi+\zeta)u_i(\xi-\zeta)\mathrm{d}\zeta$$
$$= u_i(\xi-\zeta)\dot{u}_i(\xi+\zeta)\big|_0^\xi + \int_0^\xi \dot{u}_i(\xi+\zeta)\dot{u}_i(\xi-\zeta)\mathrm{d}\zeta$$

$$= u_i(0)\dot{u}_i(2\xi) - u_i(\xi)\dot{u}_i(\xi) + \int_0^{\xi} \dot{u}_i(\xi+\zeta)\dot{u}_i(\xi-\zeta)\mathrm{d}\zeta.$$

Analogously:

$$\int_0^{\xi} \ddot{\varphi}(\xi+\zeta)\varphi(\xi-\zeta)\mathrm{d}\zeta = \varphi(0)\dot{\varphi}(2\xi) - \varphi(\xi)\dot{\varphi}(\xi) + \int_0^{\xi} \dot{\varphi}(\xi+\zeta)\dot{\varphi}(\xi-\zeta)\mathrm{d}\zeta,$$

$$\int_0^{\xi} \ddot{\psi}(\xi+\zeta)\psi(\xi-\zeta)\mathrm{d}\zeta = \psi(0)\dot{\psi}(2\xi) - \psi(\xi)\dot{\psi}(\xi) + \int_0^{\xi} \dot{\psi}(\xi+\zeta)\dot{\psi}(\xi-\zeta)\mathrm{d}\zeta.$$

Consequently it is obtained:

$$\begin{aligned}
&\int_0^{\xi} F_{11}(\xi+\zeta, \xi-\zeta)\,\mathrm{d}\zeta \\
&\quad = \int_0^{\xi}\int_B \big[\rho f_i(\xi+\zeta)u_i(\xi-\zeta) + \rho G(\xi+\zeta)\varphi(\xi-\zeta) + \rho L(\xi+\zeta)\psi(\xi-\zeta) \\
&\quad -\frac{1}{\rho T_0}\omega(\xi+\zeta)\theta(\xi-\zeta)\Big]\,\mathrm{d}V\,\mathrm{d}\zeta \\
&\quad + \int_0^{\xi}\int_{\partial B} \big[t_i(\xi+\zeta)u_i(\xi-\zeta) + \alpha(\xi+\zeta)\varphi(\xi-\zeta) + \beta(\xi+\zeta)\psi(\xi-\zeta) \\
&\quad -\frac{1}{\rho T_0}Q_*(\xi+\zeta)\theta(\xi-\zeta)\Big]\,\mathrm{d}A\,\mathrm{d}\zeta \\
&\quad - \frac{1}{\rho T_0}\int_0^{\xi}\int_B K_{ij}\theta_{*,i}(\xi+\zeta)\theta_{,j}(\xi-\zeta)\,\mathrm{d}V\,\mathrm{d}\zeta \\
&\quad - \rho\int_B \left(u_i(0)\dot{u}_i(2t) - u_i(\xi)\dot{u}_i(\xi) + \int_0^{\xi} \dot{u}_i(\xi+\zeta)\dot{u}_i(\xi-\zeta)\mathrm{d}\zeta\right)\,\mathrm{d}V \\
&\quad - K_1\int_B \left(\varphi(0)\dot{\varphi}(2t) - \varphi(\xi)\dot{\varphi}(\xi) + \int_0^{\xi} \dot{\varphi}(\xi+\zeta)\dot{\varphi}(\xi-\zeta)\mathrm{d}\zeta\right)\,\mathrm{d}V \\
&\quad - K_2\int_B \left(\psi(0)\dot{\psi}(2t) - \psi(\xi)\dot{\psi}(\xi) + \int_0^{\xi} \dot{\psi}(\xi+\zeta)\dot{\psi}(\xi-\zeta)\mathrm{d}\zeta\right)\,\mathrm{d}V.
\end{aligned} \tag{51}$$

We proceed analogously for the right member from (30) integrating the function on the interval $[0, \xi]$:

$$\begin{aligned}
&\int_0^{\xi} F_{11}(\xi-\zeta, \xi+\zeta)\,\mathrm{d}\zeta \\
&\quad = \int_0^{\xi}\int_B \big[\rho f_i(\xi-\zeta)u_i(\xi+\zeta) + \rho G(\xi-\zeta)\varphi(\xi+\zeta) + \rho L(\xi-\zeta)\psi(\xi+\zeta)
\end{aligned}$$

$$-\frac{1}{\rho T_0}\omega(\xi-\zeta)\theta(\xi+\zeta)\Big]\,\mathrm{d}V\,\mathrm{d}\zeta$$
$$-\int_0^\xi\int_B\left[\rho\ddot{u}_i(\xi-\zeta)u_i(\xi+\zeta)+K_1\ddot{\varphi}(\xi-\zeta)\varphi(\xi+\zeta)+K_2\ddot{\psi}(\xi-\zeta)\psi(\xi+\zeta)\right]\mathrm{d}V\,\mathrm{d}\zeta$$
$$+\frac{1}{\rho T_0}\int_0^\xi\int_B Q_{*i}(\xi-\zeta)\theta_{,j}(\xi+\zeta)\,\mathrm{d}V\,\mathrm{d}\zeta$$
$$+\int_0^\xi\int_{\partial B}[t_i(\xi-\zeta)u_i(\xi+\zeta)+\alpha(\xi-\zeta)\varphi(\xi+\zeta)+\beta(\xi-\zeta)\psi(\xi+\zeta)$$
$$-\frac{1}{\rho T_0}Q_*(\xi-\zeta)\theta(\xi+\zeta)\Big]\,\mathrm{d}A\,\mathrm{d}\zeta. \tag{52}$$

We perform integration by parts:

$$\int_0^\xi \ddot{u}_i(\xi-\zeta)u_i(\xi+\zeta)\mathrm{d}\zeta$$
$$=-u_i(\xi+\zeta)\dot{u}_i(\xi-\zeta)|_0^\xi+\int_0^\xi \dot{u}_i(\xi+\zeta)\dot{u}_i(\xi-\zeta)\mathrm{d}\zeta$$
$$=-u_i(2\xi)\dot{u}_i(0)+u_i(\xi)\dot{u}_i(\xi)+\int_0^\xi \dot{u}_i(\xi+\zeta)\dot{u}_i(\xi-\zeta)\mathrm{d}\zeta.$$

Analogiously:

$$\int_0^\xi \ddot{\varphi}(\xi-\zeta)\varphi(\xi+\zeta)\mathrm{d}\zeta=-\varphi(2\xi)\dot{\varphi}(0)+\varphi(\xi)\dot{\varphi}(\xi)+\int_0^\xi \dot{\varphi}(\xi+\zeta)\dot{\varphi}(\xi-\zeta)\mathrm{d}\zeta,$$
$$\int_0^\xi \ddot{\psi}(\xi-\zeta)\psi(\xi+\zeta)\mathrm{d}\zeta=-\psi(2\xi)\dot{\psi}(0)+\psi(\xi)\dot{\psi}(\xi)+\int_0^\xi \dot{\psi}(\xi+\zeta)\dot{\psi}(\xi-\zeta)\mathrm{d}\zeta.$$

Thus (33) will be written:

$$\int_0^\xi F_{11}(\xi-\zeta,\xi+\zeta)\,\mathrm{d}\zeta$$
$$=\int_0^\xi\int_B[\rho f_i(\xi-\zeta)u_i(\xi+\zeta)+\rho G(\xi-\zeta)\varphi(\xi+\zeta)+\rho L(\xi-\zeta)\psi(\xi+\zeta)$$
$$-\frac{1}{\rho T_0}\omega(\xi-\zeta)\theta(\xi+\zeta)\Big]\,\mathrm{d}V\,\mathrm{d}\zeta$$
$$-\rho\int_B\left(-u_i(2\xi)\dot{u}_i(0)+u_i(\xi)\dot{u}_i(\xi)+\int_0^\xi \dot{u}_i(\xi+\zeta)\dot{u}_i(\xi-\zeta)\mathrm{d}\zeta\right)\mathrm{d}V$$

$$
\begin{aligned}
&- K_1 \int_B \left(-\varphi(2\xi)\dot{\varphi}(0) + \varphi(\xi)\dot{\varphi}(\xi) + \int_0^\xi \dot{\varphi}(\xi+\zeta)\dot{\varphi}(\xi-\zeta)\mathrm{d}\zeta \right) \mathrm{d}V \\
&- K_2 \int_B \left(-\psi(2\xi)\dot{\psi}(0) + \psi(\xi)\dot{\psi}(\xi) + \int_0^\xi \dot{\psi}(\xi+\zeta)\dot{\psi}(\xi-\zeta)\mathrm{d}\zeta \right) \mathrm{d}V \\
&- \frac{1}{\rho T_0} \int_0^\xi \int_B K_{ij}\theta_{*,i}(\xi-\zeta)\theta_{,j}(\xi+\zeta)\, \mathrm{d}V\, d\zeta \\
&+ \int_0^\xi \int_{\partial B} [t_i(\xi-\zeta)u_i(\xi+\zeta) + \alpha(\xi-\zeta)\varphi(\xi+\zeta) + \beta(\xi-\zeta)\psi(\xi+\zeta) \\
&- \frac{1}{\rho T_0} Q_*(\xi-\zeta)\theta(\xi+\zeta) \Big]\, \mathrm{d}A\, \mathrm{d}\zeta.
\end{aligned} \tag{53}
$$

Based on the relations (32) and (34) the equality (30) becomes:

$$
\begin{aligned}
&\int_0^\xi \int_B [\rho f_i(\xi+\zeta)u_i(\xi-\zeta) + \rho G(\xi+\zeta)\varphi(\xi-\zeta) + \rho L(\xi+\zeta)\psi(\xi-\zeta) \\
&\quad - \frac{1}{\rho T_0}\omega(\xi+\zeta)\theta(\xi-\zeta) \Big]\, \mathrm{d}V\, \mathrm{d}\zeta \\
&\quad + \int_0^\xi \int_{\partial B} [t_i(\xi+\zeta)u_i(\xi-\zeta) + \alpha(\xi+\zeta)\varphi(\xi-\zeta) + \beta(\xi+\zeta)\psi(\xi-\zeta) \\
&\quad - \frac{1}{\rho T_0} Q_*(\xi+\zeta)\theta(\xi-\zeta) \Big]\, \mathrm{d}A\, \mathrm{d}\zeta \\
&\quad - \frac{1}{\rho T_0} \int_0^\xi \int_B K_{ij}\theta_{*,i}(\xi+\zeta)\theta_{,j}(\xi-\zeta)\, \mathrm{d}V\, \mathrm{d}\zeta \\
&\quad - \rho \int_B u_i(0)\dot{u}_i(2\xi)\, \mathrm{d}V + \rho \int_B u_i(\xi)\dot{u}_i(\xi)\, \mathrm{d}V - \rho \int_B \left(\int_0^\xi \dot{u}_i(\xi+\zeta)\dot{u}_i(\xi-\zeta)\mathrm{d}\zeta \right) \mathrm{d}V \\
&\quad - K_1 \int_B \varphi(0)\dot{\varphi}(2\xi)\, \mathrm{d}V + K_1 \int_B \varphi(\xi)\dot{\varphi}(\xi)\, \mathrm{d}V - K_1 \int_B \left(\int_0^\xi \dot{\varphi}(\xi+\zeta)\dot{\varphi}(\xi-\zeta)\mathrm{d}\zeta \right) \mathrm{d}V \\
&\quad - K_2 \int_B \psi(0)\dot{\psi}(2\xi)\, \mathrm{d}V + K_2 \int_B \psi(\xi)\dot{\psi}(\xi)\, \mathrm{d}V - K_2 \int_B \left(\int_0^\xi \dot{\psi}(\xi+\zeta)\dot{\psi}(\xi-\zeta)\mathrm{d}\zeta \right) \mathrm{d}V \\
&= \int_0^\xi \int_B \Big[\rho f_i(\xi-\zeta)u_i(\xi+\zeta) + \rho G(\xi-\zeta)\varphi(\xi+\zeta) + \rho L(\xi-\zeta)\psi(\xi+\zeta) \\
&\quad - \frac{1}{\rho T_0}\omega(\xi-\zeta)\theta(\xi+\zeta) \Big]\, \mathrm{d}V\, \mathrm{d}\zeta
\end{aligned}
$$

$$+\int_0^{\xi}\int_{\partial B}\left[t_i(\xi-\zeta)u_i(\xi+\zeta)+\alpha(\xi-\zeta)\varphi(\xi+\zeta)+\beta(\xi-\zeta)\psi(\xi+\zeta)\right.$$

$$\left.-\frac{1}{\rho T_0}Q_*(\xi-\zeta)\theta(\xi+\zeta)\right]\,\mathrm{d}A\,\mathrm{d}\zeta$$

$$-\frac{1}{\rho T_0}\int_0^{\xi}\int_B K_{ij}\theta_{*,i}(\xi-\zeta)\theta_{,j}(\xi+\zeta)\,\mathrm{d}V\,\mathrm{d}\zeta$$

$$+\rho\int_B u_i(2\xi)\dot{u}_i(0)\,\mathrm{d}V-\rho\int_B u_i(\xi)\dot{u}_i(\xi)\,\mathrm{d}V-\rho\int_B\left(\int_0^{\xi}\dot{u}_i(\xi+\zeta)\dot{u}_i(\xi-\zeta)\mathrm{d}\zeta\right)\,\mathrm{d}V$$

$$+K_1\int_B\varphi(2\xi)\dot{\varphi}(0)\,\mathrm{d}V-K_1\int_B\varphi(\xi)\dot{\varphi}(\xi)\,\mathrm{d}V-K_1\int_B\left(\int_0^{\xi}\dot{\varphi}(\xi+\zeta)\dot{\varphi}(\xi-\zeta)\mathrm{d}\zeta\right)\,\mathrm{d}V$$

$$+K_2\int_B\psi(2\xi)\dot{\psi}(0)\,\mathrm{d}V-K_2\int_B\psi(\xi)\dot{\psi}(\xi)\,\mathrm{d}V-K_2\int_B\left(\int_0^{\xi}\dot{\psi}(\xi+\zeta)\dot{\psi}(\xi-\zeta)\mathrm{d}\zeta\right)\,\mathrm{d}V,$$

$$2\rho\int_B u_i(\xi)\dot{u}_i(\xi)\,\mathrm{d}V+2K_1\int_B\varphi(\xi)\dot{\varphi}(\xi)\,\mathrm{d}V+2K_2\int_B\psi(\xi)\dot{\psi}(\xi)\,\mathrm{d}V$$

$$-\rho\int_B\left(u_i(2\xi)\dot{u}_i(0)+\dot{u}_i(2\xi)u_i(0)\right)\,\mathrm{d}V$$

$$-K_1\int_B\left(\varphi(2\xi)\dot{\varphi}(0)+\dot{\varphi}(2\xi)\varphi(0)\right)\,\mathrm{d}V$$

$$-K_2\int_B\left(\psi(2\xi)\dot{\psi}(0)+\dot{\psi}(2\xi)\psi(0)\right)\,\mathrm{d}V$$

$$=\int_0^{\xi}\int_B\left[\rho f_i(\xi-\zeta)u_i(\xi+\zeta)+\alpha(\xi-\zeta)\varphi(\xi+\zeta)+\beta(\xi-\zeta)\psi(\xi+\zeta)\right.$$

$$\left.-\frac{1}{\rho T_0}\omega(\xi-\zeta)\theta(\xi+\zeta)\right]\,\mathrm{d}V\,\mathrm{d}\zeta$$

$$-\int_0^{\xi}\int_B\left[\rho f_i(\xi+\zeta)u_i(\xi-\zeta)+\alpha(\xi+\zeta)\varphi(\xi-\zeta)+\beta(\xi+\zeta)\psi(\xi-\zeta)\right.$$

$$\left.-\frac{1}{\rho T_0}\omega(\xi+\zeta)\theta(\xi-\zeta)\right]\,\mathrm{d}V\,\mathrm{d}\zeta$$

$$+\int_0^{\xi}\int_{\partial B}\left[t_i(\xi-\zeta)u_i(\xi+\zeta)+\alpha(\xi-\zeta)\varphi(\xi+\zeta)+\beta(\xi-\zeta)\psi(\xi+\zeta)\right.$$

$$\left.-\frac{1}{\rho T_0}Q_*(\xi-\zeta)\theta(\xi+\zeta)\right]\,\mathrm{d}A\,\mathrm{d}\zeta$$

$$-\int_0^{\xi}\int_{\partial B}\left[t_i(\xi+\zeta)u_i(\xi-\zeta)+\alpha(\xi+\zeta)\varphi(\xi-\zeta)+\beta(\xi+\zeta)\psi(\xi-\zeta)\right.$$

$$-\frac{1}{\rho T_0} Q_*(\xi+\zeta)\theta(\xi-\zeta)\Big] \, \mathrm{d}A \, \mathrm{d}\zeta$$

$$+\frac{1}{\rho T_0}\int_0^{\xi}\int_B K_{ij}\Big[\theta_{*,i}(\xi+\zeta)\theta_{,j}(\xi-\zeta)-\theta_{*,i}(\xi-\zeta)\theta_{,j}(\xi+\zeta)\Big] \, \mathrm{d}V \, \mathrm{d}\zeta. \tag{54}$$

We notice that:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}[\theta_{*,i}(\xi+\zeta)\theta_{*,j}(\xi-\zeta)] = \theta_{,i}(\xi+\zeta)\theta_{*,j}(\xi-\zeta)-\theta_{*,i}(\xi+\zeta)\theta_{,j}(\xi-\zeta),$$

therefore the last term in (35) can be written:

$$\frac{1}{\rho T_0}\int_0^{\xi}\int_B K_{ij}[\theta_{*,i}(\xi+\zeta)\theta_{,j}(\xi-\zeta)-\theta_{*,i}(\xi-\zeta)\theta_{,j}(\xi+\zeta)] \, \mathrm{d}V \, \mathrm{d}\zeta$$

$$=-\frac{1}{\rho T_0}\int_0^{\xi}\int_B K_{ij}[\theta_{,j}(\xi+\zeta)\theta_{*,i}(\xi-\zeta)-\theta_{*,i}(\xi+\zeta)\theta_{,j}(\xi-\zeta)] \, \mathrm{d}V \, d\zeta$$

$$=-\frac{1}{\rho T_0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\int_0^{\xi}\int_B K_{ij}\theta_{*,i}(\xi+\zeta)\theta_{*,j}(\xi-\zeta) \, \mathrm{d}V \, \mathrm{d}\zeta\right].$$

In conclusion, we can state the following lemma which represents a useful auxiliary result in deducing a new result of uniqueness for thermoelastic media with double porosity structure.

Lemma 2 *Based on the above, the following differential relationship occurs:*

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\int_B (\rho u_i(\xi)u_i(\xi)+K_1\varphi(\xi)\varphi(\xi)+K_2\psi(\xi)\psi(\xi)) \, \mathrm{d}V\right.$$

$$\left.+\frac{1}{\rho T_0}\int_0^{\xi}\int_B K_{ij}\theta_{*,i}(\xi)\theta_{*,j}(\xi) \, \mathrm{d}V \, \mathrm{d}\zeta\right]$$

$$-\int_B [\rho(u_i(2\xi)\dot{u}_i(0)+\dot{u}_i(2\xi)u_i(0))+K_1(\varphi(2\xi)\dot{\varphi}(0)$$

$$+\dot{\varphi}(2\xi)\varphi(0))+K_2(\psi(2\xi)\dot{\psi}(0)+\dot{\psi}(2\xi)\psi(0))] \, \mathrm{d}V$$

$$=\int_0^{\xi}\int_B\Big[\rho f_i(\xi-\zeta)u_i(\xi+\zeta)+\alpha(\xi-\zeta)\varphi(\xi+\zeta)+\beta(\xi-\zeta)\psi(\xi+\zeta)$$

$$-\frac{1}{\rho T_0}\omega(\xi-\zeta)\theta(\xi+\zeta)\Big] \, \mathrm{d}V \, \mathrm{d}\zeta$$

$$-\int_0^{\xi}\int_B\Big[\rho f_i(\xi+\zeta)u_i(\xi-\zeta)+\alpha(\xi+\zeta)\varphi(\xi-\zeta)+\beta(\xi+\zeta)\psi(\xi-\zeta)$$

$$-\frac{1}{\rho T_0}\omega(\xi+\zeta)\theta(\xi-\zeta)\Big] \, \mathrm{d}V \, \mathrm{d}\zeta$$

$$+\int_0^{\xi}\int_{\partial B}\Big[t_i(\xi-\zeta)u_i(\xi+\zeta)+\alpha(\xi-\zeta)\varphi(\xi+\zeta)+\beta(\xi-\zeta)\psi(\xi+\zeta)$$

$$-\frac{1}{\rho T_0}Q_*(\xi-\zeta)\theta(\xi+\zeta)\Big]\,\mathrm{d}A\,\mathrm{d}\zeta$$
$$-\int_0^{\xi}\int_{\partial B}\Big[t_i(\xi+\zeta)u_i(\xi-\zeta)+\alpha(\xi+\zeta)\varphi(\xi-\zeta)+\beta(\xi+\zeta)\psi(\xi-\zeta)$$
$$-\frac{1}{\rho T_0}Q_*(\xi+\zeta)\theta(\xi-\zeta)\Big]\,\mathrm{d}A\,\mathrm{d}\zeta. \qquad (55)$$

Theorem 5 *If the thermal conductivity tensor K_{ij} is positive definite and the assumptions* (1)–(5) *are satisfied, then the mixed problem consisting in* (4)*,* (7)*,* (8) *admits a unique solution.*

Proof We will assume by contradiction that our mixed problem consisting in (4), (7), (8) admits two distinct solutions: $u_i^{(k)}, \varphi^{(k)}, \psi^{(k)}, \theta^{(k)}, t_{ij}^{(k)}, \sigma_i^{(k)}, \tau_i^{(k)}, \xi^{(k)}, \zeta^{(k)}, \eta^{(k)}, Q_i^{(k)}, k = 1, 2$. The difference between the two solutions will be marked as follows: $\hat{u}_i, \hat{\varphi}, \hat{\psi}, \hat{\theta}, \hat{t}_{ij}, \hat{\sigma}_i, \hat{\tau}_i, \hat{\xi}, \hat{\zeta}, \hat{\eta}, \hat{Q}_i$. Due to the linearity, the difference between the two solutions is also a solution of the problem corresponding to the null initial conditions and the null boundary conditions.

Given that the boundary conditions are null then relation (36) leads us to:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Bigg[\int_B\Big(\rho\hat{u}_i(\xi)\hat{u}_i(\xi)+K_1\hat{\varphi}(\xi)\hat{\varphi}(\xi)+K_2\hat{\psi}(\xi)\hat{\psi}(\xi)\Big)\,\mathrm{d}V$$
$$+\frac{1}{\rho T_0}\int_0^{\xi}\int_B K_{ij}\hat{\theta}_{*,i}(\xi)\hat{\theta}_{*,j}(\xi)\,\mathrm{d}V\,\mathrm{d}\zeta\Bigg]=0. \qquad (56)$$

From (56), we deduce:

$$\hat{u}_i(x,\xi)=0,\quad \hat{\varphi}(x,\xi)=0,\quad \hat{\psi}(x,\xi)=0,\quad \hat{\theta}_{*,i}(x,\xi)=0,\quad \forall(x,\xi)\in B\times[0,\infty).$$

If $\hat{\theta}_{*,i}(x,\xi)=0$, then $\hat{Q}_{*i}(x,\xi)=0, \forall(x,\xi)\in B\times[0,\infty)$.

We consider (2) and we have $\dot{\hat{\eta}}(x,\xi)=0$. Due to the null initial conditions, we have $\hat{\eta}(x,\xi)=0$ and the constitutive equations lead to the fact that $\hat{\theta}(x,\xi)=0, \forall(x,\xi)\in B\times[0,\infty)$. □

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interests.

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