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ORIGINAL ARTICLE

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On viscous gradient fluids

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Abstract A framework for linear viscous gradient fluids as extensions of the Navier–Stokes law is derived. More precisely, the list of the independent variables is enlarged up to an Nth-order velocity gradient and, accordingly, also the list of the dependent variables up to dual hyperstress tensors of the same rank. It is shown that due to general invariance requirements, such models must be hemitropic functions. A complete representation for a second-order linear fluid and its balance equation are derived from a dissipation potential. In doing so, we follow Trostel (in: Trostel (ed) Beiträge zu den Ingenieurwissenschaften, Universitäts-Bibliothek der Technische Universität Berlin, Berlin, 96–134, 1985) and coworkers.

Keywords Gradient fluids · Turbulence · Navier-Stokes law · Gradient materials

1 Introduction

In his dissertation thesis published in 1867, Maurice Levy investigated velocity profiles in different channel flows measured by Darcy and Bazin. He saw that such *mouvements giratoires et oscillatoires* and *fort tumultueux* (turbulent flows) cannot be described neither by the linear Navier–Stokes law nor by nonlinear fluid laws like polynominals of the rate-of-deformation tensor. St.-Venant [15] reported on this thesis and suggested to include higher velocity gradients in the stress law. However, for a long time nobody seemed to have overtaken this challenging, although difficult task.

It was in the 1980s, when Trostel and coworkers started to work on this problem, obviously unaware of St.-Venant's proposal. The suggested viscous models with the inclusion of higher velocity gradients and the extension to corresponding hyperstresses were meant as a format to describe fully developed turbulence in incompressible fluids. The starting paper in this series was by Trostel [31], [1] presenting a rather general format of Nth order. Such extensions will be called *gradient fluids* in the sequel. It is necessary to emphasize that we do not mean gradient fluids in the sense of Korteweg [17] and his followers, where the gradient of the density (and not of the velocity) appears in the elastic part of the stress law. [2]

These gradient fluid models were applied to turbulence, spanning the whole range from a second-order material model, solution of the balance laws, material identification, and the discussion of appropriate boundary conditions. Shortly after this pioneering work, Silber [24] extended this format to include also third-order velocity gradients. However, the works of Trostel and his team were soon forgotten due to inadequate publishing policies.

[1] Trostel (1928–2016).
[2] See, e.g., Podio-Guidugli and Vianello [20].

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After these singular works on gradient fluids, hardly anybody seemed to be interested in such models, whereas gradient models for solids became very fashionable in elasticity, plasticity, fracture mechanics, etc.

It is therefore our intention here[3] to reproduce such models and bring them into a format which reflects the present knowledge on gradient materials.

2 Notations

In the present article, we do our best to use standard notations as far as possible, but we will be obliged to introduce new notations for higher-rank tensor operations.

Vectors are written as $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x} \in \mathscr{V}$, and dyadics or second-rank tensors as $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$, etc. For kth-rank tensors, we use $\overset{\langle k \rangle}{\boldsymbol{A}}$, $\overset{\langle k \rangle}{\boldsymbol{B}}$, $\overset{\langle k \rangle}{\boldsymbol{C}}$, etc. $\mathbf{I}$ is the second-rank identity, and $\overset{\langle 4 \rangle}{\boldsymbol{I}}$ the fourth-rank identity, etc. $\boldsymbol{\varepsilon}$ is the permutation triadic.

$\mathscr{O}rth$ is the orthogonal group within the dyadics. The tensor product is written as $\otimes$. For every contraction between tensors, we put one dot. More exactly, the ***P*-fold contraction o**f a K-fold tensor product $\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_K$ with an M-fold tensor product $\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_M$ for $K \geq P \leq M$ is the $(K + M - 2P)$-fold tensor product

$$\begin{aligned}&(\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_K) \cdot \ldots \cdot (\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_M) \\ &\quad = \mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_{K-P} \otimes \mathbf{x}_{P+1} \otimes \ldots \otimes \mathbf{x}_M (\mathbf{v}_{K-P+1} \cdot \mathbf{x}_1)(\mathbf{v}_{K-P+2} \cdot \mathbf{x}_2) \ldots (\mathbf{v}_K \cdot \mathbf{x}_P)\end{aligned} \tag{1}$$

wherein "$\cdot \ldots \cdot$" stands for P contraction dots. For better visibility, we will eventually arrange these contraction dots in groups with identical meaning, like $:$ for $\cdot\cdot$, $\therefore$ for $\cdot\cdot\cdot$, and $::$ for $\cdot\cdot\cdot\cdot$. These notions can be immediately and uniquely extended from tensor products to higher-rank tensors.

The transpose of a second-rank tensor $\mathbf{T}$ is $\mathbf{T}^{\mathrm{T}}$. The symmetric part of a dyadic is denoted as $sym\, \mathbf{T} = 1/2\, (\mathbf{T} + \mathbf{T}^{\mathrm{T}})$. Transpositions for higher-rank tensors are not unique. We use upper brackets for such transpositions, e.g., for a triadic $\overset{\langle 3 \rangle}{\boldsymbol{A}}{}^{[23]}$ which gives for the components with respect to an orthonormal vector basis $(\overset{\langle 3 \rangle}{\boldsymbol{A}}{}^{[23]})_{ijk} = (\overset{\langle 3 \rangle}{\boldsymbol{A}})_{ikj}$. If a tensor is symmetric with respect to this particular transposition, we call it **right subsymmetric**. By $sym^{[ij]}\, \overset{\langle 3 \rangle}{\boldsymbol{A}}$ we mean the symmetric part with respect to the indicated entries, e.g., $sym^{[23]} (\overset{\langle 3 \rangle}{\boldsymbol{A}}) = 1/2\, (\overset{\langle 3 \rangle}{\boldsymbol{A}} + \overset{\langle 3 \rangle}{\boldsymbol{A}}{}^{[23]})$.

We will also use the **Rayleigh product** that maps all basis vectors of a tensor simultaneously without changing its components. To be more precise, let $\mathbf{T}$ be a dyadic and $\overset{\langle k \rangle}{\boldsymbol{C}}$ a tensor of kth-rank ($k \geq 0$). Then the Rayleigh product between them is defined as

$$\mathbf{T} * \overset{\langle k \rangle}{\boldsymbol{C}} = \mathbf{T} * (C^{ik\ldots l}\, \mathbf{r}_i \otimes \mathbf{r}_k \otimes \ldots \otimes \mathbf{r}_l) := C^{ik\ldots l}\, (\mathbf{T} \cdot \mathbf{r}_i) \otimes (\mathbf{T} \cdot \mathbf{r}_k) \otimes \ldots \otimes (\mathbf{T} \cdot \mathbf{r}_l) \tag{2}$$

with respect to an arbitrary vector basis $\{\mathbf{r}_i\}$. For $k \equiv 0$ this is simply the identity on the reals.

3 Mechanical variables

Let $\boldsymbol{\chi}$ be the motion of the body and

$$\mathbf{L} = grad\, \mathbf{v} = grad\, \boldsymbol{\chi}^{\bullet} = \mathbf{D} + \mathbf{W} \tag{3}$$

the spatial velocity gradient with its symmetric part $\mathbf{D}$ and its skew one $\mathbf{W}$.

We denote the higher-order velocity gradients by

$$\overset{\langle i+1 \rangle}{\boldsymbol{V}} := grad^i\, \mathbf{v} \tag{4}$$

and collect them in the hypervector

$$\boldsymbol{V} = \left\{ \overset{\langle 2 \rangle}{\boldsymbol{V}}, \overset{\langle 3 \rangle}{\boldsymbol{V}}, \ldots, \overset{\langle N+1 \rangle}{\boldsymbol{V}} \right\} \tag{5}$$

[3] For more details see Bertram [11].

forming the **kinematical set** of our Nth-order gradient material. These tensors have all right subsymmetries like

$$\overset{\langle 3\rangle}{\boldsymbol{V}} = \overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[23]} \tag{6}$$

$$\overset{\langle 4\rangle}{\boldsymbol{V}} = \overset{\langle 4\rangle}{\boldsymbol{V}}{}^{[23]} = \overset{\langle 4\rangle}{\boldsymbol{V}}{}^{[24]} \tag{7}$$

etc. So only the first entry plays a particular role, while all the others are interchangeable.

The conjugate **dynamical set** consists of the stress and hyperstress tensors which we also collect in a hypervector

$$\boldsymbol{T} = \left\{ \overset{\langle 2\rangle}{\boldsymbol{T}}, \overset{\langle 3\rangle}{\boldsymbol{T}}, \dots, \overset{\langle N+1\rangle}{\boldsymbol{T}} \right\} \tag{8}$$

such that the stress power per unit mass for an Nth-order gradient material has the form

$$\pi = 1/\rho \left(\overset{\langle 2\rangle}{\boldsymbol{T}} : \overset{\langle 2\rangle}{\boldsymbol{V}} + \overset{\langle 3\rangle}{\boldsymbol{T}} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} + \overset{\langle 4\rangle}{\boldsymbol{T}} :: \overset{\langle 4\rangle}{\boldsymbol{V}} + \cdots + \overset{\langle N+1\rangle}{\boldsymbol{T}} \cdot \ldots \cdot \overset{\langle \mathrm{N+1}\rangle}{\boldsymbol{V}} \right) \tag{9}$$

where ρ is the density in the current placement, or in a more compact notation

$$\pi = \frac{1}{\rho} < \boldsymbol{T},\ \boldsymbol{V} > . \tag{10}$$

These hyperstresses are assumed to inherit the same subsymmetries from their kinematical duals, like

$$\overset{\langle 3\rangle}{\boldsymbol{T}} = \overset{\langle 3\rangle}{\boldsymbol{T}}{}^{[23]} \tag{11}$$

$$\overset{\langle 4\rangle}{\boldsymbol{T}} = \overset{\langle 4\rangle}{\boldsymbol{T}}{}^{[23]} = \overset{\langle 4\rangle}{\boldsymbol{T}}{}^{[24]} \tag{12}$$

etc. $\overset{\langle 2\rangle}{\boldsymbol{T}}$ is symmetric as a consequence of the moment of momentum balance.

The number of independent variables of $\boldsymbol{V}$ and of $\boldsymbol{T}$ is

$$\text{for } N = 1 : \quad 3 \times [(1+2)] - 3 = 6 \tag{13}$$

$$\text{for } N = 2 : \quad 3 \times [(1+2) + (1+2+3)] - 3 = 6 + 18 = 24 \tag{14}$$

$$\text{for } N = 3 : \quad 3 \times [(1+2) + (1+2+3) + (1+2+3+4)] - 3 = 24 + 30 = 54 \tag{15}$$

$$\text{for } N \geq 1 : \quad 3/2 \times \sum_{j=1}^{N} [(1+j)^2 + (1+j)] - 3. \tag{16}$$

As it is well known since half a century,[4] the generalized Cauchy equations for an Nth-order gradient material are

$$div\, \overset{\langle 2\rangle}{\boldsymbol{T}} - div^2\, \overset{\langle 3\rangle}{\boldsymbol{T}} + div^3\, \overset{\langle 4\rangle}{\boldsymbol{T}} + \cdots - (-1)^N div^{\,N}\, \overset{\langle N+1\rangle}{\boldsymbol{T}} + \rho\mathbf{b} = \rho\mathbf{a} \tag{17}$$

$$\overset{\langle 2\rangle}{\boldsymbol{T}} = \overset{\langle 2\rangle}{\boldsymbol{T}}{}^{\mathrm{T}} \tag{18}$$

with the specific body force $\mathbf{b}$, the acceleration $\mathbf{a}$, and the mass density ρ. $\overset{\langle 2\rangle}{\boldsymbol{T}}$ is symmetric because of the balance of moment of momentum and, consequently, the first term in (9) can be substituted by $\overset{\langle 2\rangle}{\boldsymbol{T}} : \mathbf{D}$ and we can further on identify the first entry $\overset{\langle 2\rangle}{\boldsymbol{V}}$ in the kinematical set by $\mathbf{D}$.

Alternatively one could also use the form

$$< \underline{\boldsymbol{T}}, \boldsymbol{V} > = \overset{\langle 2\rangle}{\underline{\boldsymbol{T}}} : \mathbf{D} + \overset{\langle 3\rangle}{\underline{\boldsymbol{T}}} \therefore grad\, \mathbf{D} + \overset{\langle 4\rangle}{\underline{\boldsymbol{T}}} :: grad^2 \mathbf{D} + \cdots + \overset{\langle N+1\rangle}{\underline{\boldsymbol{T}}} \cdot \ldots \cdot grad^{N-1}\, \mathbf{D} \tag{19}$$

[4] See, e.g., Bertram and Forest [7] for its derivation.

for the stress power of a gradient material with slightly different stress tensors. But this format is mathematically equivalent to ours since the following identities hold

$$\overset{\langle 3 \rangle}{\boldsymbol{V}} = grad\ grad\ \ \mathbf{v} = grad\,\mathbf{D} + grad\,\mathbf{D}^{[23]} - grad\,\mathbf{D}^{[13]} \tag{20}$$

or inversely

$$grad\,\mathbf{D} = grad\ 1/2\,(grad\,\mathbf{v} + grad^{\mathrm{T}}\,\mathbf{v}) = 1/2\,(\overset{\langle 3 \rangle}{\boldsymbol{V}} + \overset{\langle 3 \rangle}{\boldsymbol{V}}{}^{[12]}). \tag{21}$$

So there is a one-to-one relation between $grad\,\mathbf{D}$ and $\overset{\langle 3 \rangle}{\boldsymbol{V}}$.

4 Viscous fluids

In the theory of gradient fluids, the kinematical set for the constitutive equations that determines the stresses is assumed to be

$$\boldsymbol{V} := \left\{ \mathbf{D}, \overset{\langle 3 \rangle}{\boldsymbol{V}}, \ldots, \overset{\langle \mathrm{N}+1 \rangle}{\boldsymbol{V}} \right\}. \tag{22}$$

This leads to the following definition.

Def. A **viscous fluid of order** $\boldsymbol{N}$ is constituted by *N-1* tensor functions of the kinematical set

$$\overset{\langle 2 \rangle}{\boldsymbol{T}} = f_1\left(\mathbf{D}, \overset{\langle 3 \rangle}{\boldsymbol{V}}, \ \ldots\ , \overset{\langle \mathrm{N}+1 \rangle}{\boldsymbol{V}}\right) \tag{23}$$

$$\overset{\langle 3 \rangle}{\boldsymbol{T}} = f_2\left(\mathbf{D}, \overset{\langle 3 \rangle}{\boldsymbol{V}}, \ \ldots\ , \overset{\langle \mathrm{N}+1 \rangle}{\boldsymbol{V}}\right)$$

$$\ldots \tag{24}$$

$$\overset{\langle N+1 \rangle}{\boldsymbol{T}} = f_N\left(\mathbf{D}, \overset{\langle 3 \rangle}{\boldsymbol{V}}, \ \ldots\ , \overset{\langle \mathrm{N}+1 \rangle}{\boldsymbol{V}}\right) \tag{25}$$

to determine the dynamical set, or in compact notation

$$\boldsymbol{T} = f(\boldsymbol{V}). \tag{26}$$

4.1 Invariance principles

The above stress functions (23)–(26) are constitutive equations and as such have to be submitted to the usual invariance postulates known as *Principles of Material Theory*. The first group of such principles is related to the invariance under Euclidean transformations. In the present context it is necessary to distinguish carefully between the passive and the active version of the Euclidean invariance postulates, see TRUESDELL/NOLL [36].

4.1.1 Principle of objectivity

Changes of observers play an important role since all these kinematical and dynamical variables depend on the observer (or the frame of reference). The spatial transformation induced by a change of observer is given by the **EUCLIDean transformation** which transforms the position vector $\mathbf{x}(P, t)$ of a particle or a material point P at a time t for one observer into that of some other observer indicated by an upper asterisk,

$$\mathbf{x}^*(P, t) = \mathbf{Q}(t) \cdot \mathbf{x}(P, t) + \mathbf{c}(t) \tag{27}$$

by a time-dependent vector $\mathbf{c}(t) \in \mathscr{V}$ and a time-dependent orthogonal tensor $\mathbf{Q}(t) \in \mathscr{O}rth$, both of which are determined (solely) by the two involved observers.

We will call a tensor quantity invariant if

$$\overset{\langle k\rangle}{\boldsymbol{C}}{}^* = \overset{\langle k\rangle}{\boldsymbol{C}} \tag{28}$$

holds under all changes of observer, and objective if

$$\overset{\langle k\rangle}{\boldsymbol{C}}{}^* = \mathbf{Q} * \overset{\langle k\rangle}{\boldsymbol{C}}\,. \tag{29}$$

For scalar quantities, invariance and objectivity coincide.

In this sense, all of our kinematical variables $\mathbf{D},\ \overset{\langle 3\rangle}{\boldsymbol{V}}, \ldots,\ \overset{\langle \mathrm{N+1}\rangle}{\boldsymbol{V}}$ are objective tensors.

After the *principle of objectivity*[5] (passive version) the hyperstresses must also be objective. This leads to the following relation between the stress function of two observers

$$\mathbf{Q} * f_i\left(\mathbf{D},\ \overset{\langle 3\rangle}{\boldsymbol{V}}, \ldots,\ \overset{\langle \mathrm{N+1}\rangle}{\boldsymbol{V}}\right) = f_i^*\left(\mathbf{Q} * \mathbf{D},\ \mathbf{Q} * \overset{\langle 3\rangle}{\boldsymbol{V}},\ \ldots, \mathbf{Q} * \overset{\langle \mathrm{N+1}\rangle}{\boldsymbol{V}}\right) \quad \text{for } i = 1, \ldots, N \tag{30}$$

or in a compact form

$$\mathbf{Q} * f(\boldsymbol{V}) = f^*(\mathbf{Q} * \boldsymbol{V}). \tag{31}$$

So if one observer has identified the stress function, it is also determined for all other observers by this transformation. This principle shall be carefully distinguished from the following one, since its physical content is different and so are the consequences for the constitutive laws.

4.1.2 Principle of invariance under superimposed rigid body modifications

This principle (active version) requires that

$$f_i(\mathbf{D},\ \overset{\langle 3\rangle}{\boldsymbol{V}}, \ldots,\ \overset{\langle \mathrm{N+1}\rangle}{\boldsymbol{V}}) = \mathbf{Q}^{\mathrm{T}} * f_i(\mathbf{Q} * \mathbf{D},\ \mathbf{Q} * \overset{\langle 3\rangle}{\boldsymbol{V}}, \ldots, \mathbf{Q} * \overset{\langle \mathrm{N+1}\rangle}{\boldsymbol{V}}) \text{ for } i = 1, \ldots, N \tag{32}$$

or in compact notation

$$\mathbf{Q} * f(\boldsymbol{V}) = f(\mathbf{Q} * \boldsymbol{V}) \tag{33}$$

holds for all values of the kinematical set $\mathbf{D},\ \overset{\langle 3\rangle}{\boldsymbol{V}}, \ldots,\ \overset{\langle \mathrm{N+1}\rangle}{\boldsymbol{V}}$ and all proper orthogonal tensors $\mathbf{Q}$. Functions with this property are called *hemitropic functions*. Note that on both sides of these equations the same constitutive equation appears. In contrast to the principle of objectivity, in the present principle only one observer is involved, observing two different motions. This is a strong restriction upon our constitutive equations, as will be shown below.

In contrast to hemitropic functions, isotropic functions are those for which (32) or (33) holds for all orthogonal tensors $\mathbf{Q}$, both proper and improper. The isotropic functions are included in the hemitropic ones. Our principle from above, however, gives no rise to include also improper tensors into this invariance requirement since the mirror image of a motion is not a motion in general.

4.1.3 Principle of material symmetry

Any constitutive equation must allow for the inherent symmetry properties of the material. In the present case we consider fluids, for which all unimodular tensors are symmetry transformations, by definition.

A symmetry transformation is a change of the reference placement under which the constitutive equations are invariant. In (26), neither the independent (kinematical) variables nor the dependent ones (spatial stresses) depend on the reference placement. Consequently, the symmetry group is maximal and all unimodular tensors are symmetry transformations. So our constitutive equation (26) describes in fact (isotropic) fluids. This has nothing to do with condition (32), which is often confused, see, e.g., Truesdell and Noll ([36], p. 476), Trostel ([31], p. 110), Altenbach ([5], p. 317).

[5] See Bertram and Svendsen [6], Bertram and Forest [7], and Bertram [10,11].

5 Quadratic dissipation potential

In the case of incompressible viscous fluids the stress power is completely dissipated. Often one assumes the existence of a **dissipation potential** as a scalar-valued function of the kinematical set

$$\delta(\overset{\langle 2\rangle}{\boldsymbol{V}}, \overset{\langle 3\rangle}{\boldsymbol{V}}, \dots, \overset{\langle N+1\rangle}{\boldsymbol{V}}) = \delta(\boldsymbol{V}) \tag{34}$$

so that the potential relations

$$\overset{\langle i\rangle}{\boldsymbol{T}} = \partial\delta/\partial \overset{\langle i\rangle}{\boldsymbol{V}} \qquad \text{for } i = 2, \dots, N+1 \tag{35}$$

hold.

If we particularize our concern to linear fluids, the dissipation potential must be a square form of the kinematical set

$$\begin{aligned}\delta = 1/2\, &\overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 4\rangle}{\boldsymbol{D}}_{22} : \overset{\langle 2\rangle}{\boldsymbol{V}} + \overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 5\rangle}{\boldsymbol{D}}_{23} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} + \dots + \overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 3+N\rangle}{\boldsymbol{D}}_{2\,N+1} \cdot \ldots \cdot \overset{\langle N+1\rangle}{\boldsymbol{V}} \\ &+ \frac{1}{2} \overset{\langle 3\rangle}{\boldsymbol{V}} \therefore \overset{\langle 6\rangle}{\boldsymbol{D}}_{33} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} + \overset{\langle 3\rangle}{\boldsymbol{V}} \therefore \overset{\langle 7\rangle}{\boldsymbol{D}}_{34} :: \overset{\langle 4\rangle}{\boldsymbol{V}} + \dots + \overset{\langle 3\rangle}{\boldsymbol{V}} \therefore \overset{\langle 4+N\rangle}{\boldsymbol{D}}_{3\,N+1} \cdot \ldots \cdot \overset{\langle N+1\rangle}{\boldsymbol{V}} \\ &\dots \\ &+ 1/2 \overset{\langle N+1\rangle}{\boldsymbol{V}} \cdot \ldots \cdot \overset{\langle 2(N+1)\rangle}{\boldsymbol{D}}_{N+1\,N+1} \cdot \ldots \cdot \overset{\langle N+1\rangle}{\boldsymbol{V}}\end{aligned} \tag{36}$$

such that the stress laws become after (23)–(25)

$$\overset{\langle 2\rangle}{\boldsymbol{T}} = f_1\left(\overset{\langle 2\rangle}{\boldsymbol{V}}, \overset{\langle 3\rangle}{\boldsymbol{V}}, \dots, \overset{\langle N+1\rangle}{\boldsymbol{V}}\right) = \overset{\langle 4\rangle}{\boldsymbol{D}}_{22} : \overset{\langle 2\rangle}{\boldsymbol{V}} + \overset{\langle 5\rangle}{\boldsymbol{D}}_{23} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} + \dots + \overset{\langle 3+N\rangle}{\boldsymbol{D}}_{2\,N+1} \cdot \ldots \cdot \overset{\langle N+1\rangle}{\boldsymbol{V}} \tag{37}$$

$$\overset{\langle 3\rangle}{\boldsymbol{T}} = f_2\left(\overset{\langle 2\rangle}{\boldsymbol{V}}, \overset{\langle 3\rangle}{\boldsymbol{V}}, \dots, \overset{\langle N+1\rangle}{\boldsymbol{V}}\right) = \overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 5\rangle}{\boldsymbol{D}}_{23} + \overset{\langle 6\rangle}{\boldsymbol{D}}_{33} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} + \overset{\langle 7\rangle}{\boldsymbol{D}}_{34} :: \overset{\langle 4\rangle}{\boldsymbol{V}} + \dots + \overset{\langle 4+N\rangle}{\boldsymbol{D}}_{3\,N+1} \cdot \ldots \cdot \overset{\langle N+1\rangle}{\boldsymbol{V}}$$

$$\dots \tag{38}$$

$$\overset{\langle N+1\rangle}{\boldsymbol{T}} = f_N\left(\overset{\langle 2\rangle}{\boldsymbol{V}}, \overset{\langle 3\rangle}{\boldsymbol{V}}, \dots, \overset{\langle N+1\rangle}{\boldsymbol{V}}\right) = \overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 3+N\rangle}{\boldsymbol{D}}_{2\,N+1} + \overset{\langle 3\rangle}{\boldsymbol{V}} \therefore \overset{\langle 4+N\rangle}{\boldsymbol{D}}_{3\,N+1} + \dots + \overset{\langle 2(N+1)\rangle}{\boldsymbol{D}}_{N+1\,N+1} \cdot \ldots \cdot \overset{\langle N+1\rangle}{\boldsymbol{V}} \tag{39}$$

with viscosity tensors $\overset{\langle i+j\rangle}{\boldsymbol{D}}_{i\,j}$, or in symbolic notation

$$\delta = 1/2\,\mathbb{D}[\boldsymbol{V}, \boldsymbol{V}] \tag{40}$$

represented by a hyper tensor $\mathbb{D}$ which gives the dynamical set

$$\boldsymbol{T} = \mathbb{D}[\boldsymbol{V}]. \tag{41}$$

The dissipation depends only on the symmetric part of $\mathbb{D}$ so that this hyper tensor can be chosen symmetric.

As a consequence of the Clausius–Duhem inequality, $\mathbb{D}$ must be positive semi-definite.

Because of invariance under rigid body modifications (8.6), all the viscosity tensors $\overset{\langle i+j\rangle}{\boldsymbol{D}}_{i\,j}$ must be hemitropic tensors, which is a strong restriction to be considered in detail in the sequel. Before doing so, we will give a complete list of the hemitropic tensors up to 6th rank needed for second-order fluids.

5.1 Hemitropic and isotropic tensors

Definition We call a tensor of kth-rank ($k \geq 1$) $\overset{\langle k\rangle}{\boldsymbol{C}}$ **isotropic** if

$$\overset{\langle k\rangle}{\boldsymbol{C}} = \mathbf{Q} * \overset{\langle k\rangle}{\boldsymbol{C}} \tag{42}$$

holds for all orthogonal tensors $\mathbf{Q}$. We call it **hemitropic** if (42) holds for all *proper* orthogonal tensors $\mathbf{Q}$.

Hemitropic tensors up to 8th-rank have been listed by Kearsly and Fong [16]. Those of 5th-rank can be already found in Cisotti [13] and Caldonazzo [12]. Racah [21] gives the number of independent hemitropic tensors of arbitrary rank. Weyl [37] shows that all even-rank hemitropic tensors can be composed by transpositions of the second-rank identity, while odd-rank ones need a permutation tensor in addition.

The following statements can be easily verified.

- The zero tensors of all ranks are both isotropic and hemitropic. Therefore we are only interested in non-trivial tensors.
- With each isotropic/hemitropic tensor also every scalar multiple of it is again isotropic/hemitropic. The same holds for linear combinations of isotropic/hemitropic tensors.
- Every isotropic tensor is also hemitropic.
- Every even-rank hemitropic tensor is also isotropic.
- Among the odd-rank tensors, there are only trivial isotropic tensors.

The following list contains all the hemitropic tensors which will be needed for the dissipation potential of a second-order gradient fluid.

1st-rank isotropic tensors

or vectors: only the zero vector is isotropic/hemitropic. Non-trivial isotropic/hemitropic tensors of this rank do not exist.

2nd-rank isotropic/ hemitropic tensors

are scalar multiples of the second-rank identity tensor $\mathbf{I}$.

3rd-rank hemitropic tensors

are scalar multiples of the epsilon or permutation tensor $\boldsymbol{\varepsilon}$. Non-trivial isotropic triadics do not exist.

4th-rank isotropic/ hemitropic tensors

are scalar multiples of

- $\mathbf{I} \otimes \mathbf{I}$
- the fourth-rank identity tensor $\overset{\langle 4 \rangle}{\boldsymbol{I}}$
- the transposer $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i$.

We will later need such tetradics as linear mappings between *symmetric* second-rank tensors. In this particular case, the identity tetradic and the transposer give the same result

$$\overset{\langle 4 \rangle}{\boldsymbol{I}} : \mathbf{T} = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i : \mathbf{T} \tag{43}$$

for all symmetric dyadics $\mathbf{T}$. So only one of them will be needed.

It is also possible to implement the left and right subsymmetries already into the tetradic. But this is neither necessary nor essential. Therefore we will not consider this point anymore.

5th-rank hemitropic tensors

are scalar multiples of products between the second-rank identity and the permutation tensor after Weyl [37]. Ten of them have been listed by, e.g., Caldonazzo [12], Kearsley and Fong [16], and Silber [25]

$$\overset{\langle 5 \rangle}{\boldsymbol{H}}_1 = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_l \otimes \mathbf{e}_l \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{I} \otimes \mathbf{e}_j \otimes \mathbf{e}_k \tag{44}$$

$$\overset{\langle 5 \rangle}{\boldsymbol{H}}_2 = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_l \otimes \mathbf{e}_j \otimes \mathbf{e}_l \otimes \mathbf{e}_k = \overset{\langle 5 \rangle}{\boldsymbol{H}}_1{}^{[34]} \tag{45}$$

$$\overset{\langle 5 \rangle}{\boldsymbol{H}}_3 = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_l \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \overset{\langle 5 \rangle}{\boldsymbol{H}}_2{}^{[45]} \tag{46}$$

$$\overset{\langle 5 \rangle}{\boldsymbol{H}}_4 = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l \otimes \mathbf{e}_l \otimes \mathbf{e}_k = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{I} \otimes \mathbf{e}_k \tag{47}$$

$$\overset{\langle 5 \rangle}{\boldsymbol{H}}_5 = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \boldsymbol{\varepsilon} \cdot \overset{\langle 4 \rangle}{\boldsymbol{I}} \tag{48}$$

$$\overset{\langle 5 \rangle}{\boldsymbol{H}}_6 = \varepsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_l = \boldsymbol{\varepsilon} \otimes \mathbf{I} \tag{49}$$

$$\overset{\langle 5 \rangle}{\boldsymbol{H}}_7 = \varepsilon_{ijk} \mathbf{e}_l \otimes \mathbf{e}_l \otimes \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \mathbf{I} \otimes \boldsymbol{\varepsilon} \tag{50}$$

$$\overset{\langle 5 \rangle}{\boldsymbol{H}}_8 = \varepsilon_{ijk} \mathbf{e}_l \otimes \mathbf{e}_i \otimes \mathbf{e}_l \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \overset{\langle 4 \rangle}{\boldsymbol{I}} \cdot \boldsymbol{\varepsilon} \tag{51}$$

$$\overset{\langle 5\rangle}{\boldsymbol{H}}_9 = \varepsilon_{ijk}\mathbf{e}_l \otimes \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l \otimes \mathbf{e}_k = \overset{\langle 5\rangle}{\boldsymbol{H}}_2{}^{[12]} \tag{52}$$

$$\overset{\langle 5\rangle}{\boldsymbol{H}}_{10} = \varepsilon_{ijk}\mathbf{e}_l \otimes \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \overset{\langle 5\rangle}{\boldsymbol{H}}_9{}^{[45]}. \tag{53}$$

All of them can be mutually transformed into one another by transpositions.

Not all of these tensors are linearly independent. In fact, Caldonazzo [12] and Silber [25] give the following linear dependencies

$$\overset{\langle 5\rangle}{\boldsymbol{H}}_3 + \overset{\langle 5\rangle}{\boldsymbol{H}}_6 = \overset{\langle 5\rangle}{\boldsymbol{H}}_5 + \overset{\langle 5\rangle}{\boldsymbol{H}}_{10} \tag{54}$$

$$\overset{\langle 5\rangle}{\boldsymbol{H}}_2 + \overset{\langle 5\rangle}{\boldsymbol{H}}_6 = \overset{\langle 5\rangle}{\boldsymbol{H}}_4 + \overset{\langle 5\rangle}{\boldsymbol{H}}_9 \tag{55}$$

$$\overset{\langle 5\rangle}{\boldsymbol{H}}_1 + \overset{\langle 5\rangle}{\boldsymbol{H}}_5 = \overset{\langle 5\rangle}{\boldsymbol{H}}_4 + \overset{\langle 5\rangle}{\boldsymbol{H}}_8 \tag{56}$$

$$\overset{\langle 5\rangle}{\boldsymbol{H}}_2 + \overset{\langle 5\rangle}{\boldsymbol{H}}_7 = \overset{\langle 5\rangle}{\boldsymbol{H}}_1 + \overset{\langle 5\rangle}{\boldsymbol{H}}_3 \tag{57}$$

so that four hemitropic tensors can be purged from the list and only six remain.

In the sequel we will need such hemitropic pentadics as linear mappings between triadics and dyadics or in forms like

$$\overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 5\rangle}{\boldsymbol{H}} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} \tag{58}$$

with symmetric dyadics $\overset{\langle 2\rangle}{\boldsymbol{V}}$ and triadics $\overset{\langle 3\rangle}{\boldsymbol{V}}$ with right subsymmetry. Therefore we can demand symmetry in the first and in the last two entries. For this reason $\overset{\langle 5\rangle}{\boldsymbol{H}}_1, \overset{\langle 5\rangle}{\boldsymbol{H}}_4, \overset{\langle 5\rangle}{\boldsymbol{H}}_5, \overset{\langle 5\rangle}{\boldsymbol{H}}_6, \overset{\langle 5\rangle}{\boldsymbol{H}}_7, \overset{\langle 5\rangle}{\boldsymbol{H}}_8$ will not be needed.

Only scalar multiples of the following hemitropic pentadic

$$\begin{aligned} &\overset{\langle 5\rangle}{\boldsymbol{H}}_2 + \overset{\langle 5\rangle}{\boldsymbol{H}}_3 + \overset{\langle 5\rangle}{\boldsymbol{H}}_9 + \overset{\langle 5\rangle}{\boldsymbol{H}}_{10} \\ &\quad = \varepsilon_{ijk}\mathbf{e}_i \otimes \mathbf{e}_l \otimes \mathbf{e}_j \otimes \mathbf{e}_l \otimes \mathbf{e}_k + \varepsilon_{ijk}\mathbf{e}_i \otimes \mathbf{e}_l \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \\ &\qquad + \varepsilon_{ijk}\mathbf{e}_l \otimes \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l \otimes \mathbf{e}_k + \varepsilon_{ijk}\mathbf{e}_l \otimes \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \end{aligned} \tag{59}$$

show all the required symmetries. However, this gives the same results as any of them

$$\begin{aligned} &\overset{\langle 2\rangle}{\boldsymbol{V}} : {}^{1}/_{4} \left(\overset{\langle 5\rangle}{\boldsymbol{H}}_2 + \overset{\langle 5\rangle}{\boldsymbol{H}}_3 + \overset{\langle 5\rangle}{\boldsymbol{H}}_9 + \overset{\langle 5\rangle}{\boldsymbol{H}}_{10} \right) \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} \\ &= \overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 5\rangle}{\boldsymbol{H}}_2 \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} = \overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 5\rangle}{\boldsymbol{H}}_3 \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} = \overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 5\rangle}{\boldsymbol{H}}_9 \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} = \overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 5\rangle}{\boldsymbol{H}}_{10} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} \\ &= \overset{\langle 2\rangle}{\boldsymbol{V}} : \boldsymbol{\varepsilon} : \overset{\langle 3\rangle}{\boldsymbol{V}} = (\boldsymbol{\varepsilon} \cdot \overset{\langle 2\rangle}{\boldsymbol{V}}) \therefore \overset{\langle 3\rangle}{\boldsymbol{V}}. \end{aligned} \tag{60}$$

for all symmetric dyadics $\overset{\langle 2\rangle}{\boldsymbol{V}}$ and triadics $\overset{\langle 3\rangle}{\boldsymbol{V}}$ with right subsymmetry.

6th-rank isotropic/ hemitropic tensors

Kearsly and Fong [16] give the following complete list of 15 isotropic tensors

$$\overset{\langle 6\rangle}{\boldsymbol{H}}_1 = \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_m = \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \tag{61}$$

$$\overset{\langle 6\rangle}{\boldsymbol{H}}_2 = \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_k \otimes \mathbf{e}_m = \mathbf{I} \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_k \otimes \mathbf{e}_m \tag{62}$$

$$\overset{\langle 6\rangle}{\boldsymbol{H}}_3 = \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_m \otimes \mathbf{e}_k = \mathbf{I} \otimes \mathbf{e}_k \otimes \mathbf{I} \otimes \mathbf{e}_k \tag{63}$$

$$\overset{\langle 6\rangle}{\boldsymbol{H}}_4 = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_m = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{I} \tag{64}$$

$$\overset{\langle 6\rangle}{\boldsymbol{H}}_5 = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_k \otimes \mathbf{e}_m \tag{65}$$

$$\overset{\langle 6\rangle}{\boldsymbol{H}}_6 = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_m \otimes \mathbf{e}_k = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{I} \otimes \mathbf{e}_k \tag{66}$$

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_7 = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_m = \mathbf{e}_i \otimes \mathbf{I} \otimes \mathbf{e}_i \otimes \mathbf{I} \tag{67}$$

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_8 = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m = \overset{\langle 6 \rangle}{\boldsymbol{I}} \tag{68}$$

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_9 = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_k \tag{69}$$

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_{10} = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_i \otimes \mathbf{e}_m = \mathbf{e}_i \otimes \mathbf{I} \otimes \mathbf{e}_m \otimes \mathbf{e}_i \otimes \mathbf{e}_m \tag{70}$$

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_{11} = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_m \tag{71}$$

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_{12} = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_m \otimes \mathbf{e}_i \otimes \mathbf{e}_k = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{I} \otimes \mathbf{e}_i \otimes \mathbf{e}_k \tag{72}$$

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_{13} = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_m \otimes \mathbf{e}_i = \mathbf{e}_i \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{e}_i \tag{73}$$

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_{14} = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_i \tag{74}$$

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_{15} = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_m \otimes \mathbf{e}_k \otimes \mathbf{e}_i = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{I} \otimes \mathbf{e}_k \otimes \mathbf{e}_i \tag{75}$$

all of which are transpositions of the hexadic $\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}$.

In the sequel we are interested in such hexadics as symmetric square forms of triadics like

$$\overset{\langle 3 \rangle}{\boldsymbol{V}} \therefore \overset{\langle 6 \rangle}{\boldsymbol{H}} \therefore \overset{\langle 3 \rangle}{\boldsymbol{V}} \tag{76}$$

with triadics $\overset{\langle 3 \rangle}{\boldsymbol{V}}$ which have the right subsymmetry. Accordingly, the hexadics can be symmetric in the second and third entries, as well as in the fourth and sixth entries, and also have the major symmetry. Under this assumption, only the following five hexadics are needed.

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_8 + \overset{\langle 6 \rangle}{\boldsymbol{H}}_9 = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m + \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_k \tag{77}$$

so that

$$\overset{\langle 3 \rangle}{\boldsymbol{V}} \therefore 1/2 \left(\overset{\langle 6 \rangle}{\boldsymbol{H}}_8 + \overset{\langle 6 \rangle}{\boldsymbol{H}}_9 \right) \therefore \overset{\langle 3 \rangle}{\boldsymbol{V}} = \overset{\langle 3 \rangle}{\boldsymbol{V}} \therefore \overset{\langle 6 \rangle}{\boldsymbol{I}} \therefore \overset{\langle 3 \rangle}{\boldsymbol{V}} = \overset{\langle 3 \rangle}{\boldsymbol{V}} \therefore \overset{\langle 3 \rangle}{\boldsymbol{V}} \tag{78}$$

Here $1/2 \left(\overset{\langle 6 \rangle}{\boldsymbol{H}}_8 + \overset{\langle 6 \rangle}{\boldsymbol{H}}_9 \right)$ does the same as the sixth-rank identity $\overset{\langle 6 \rangle}{\boldsymbol{I}}$.

$$\begin{aligned} &\overset{\langle 6 \rangle}{\boldsymbol{H}}_{11} + \overset{\langle 6 \rangle}{\boldsymbol{H}}_{14} + \overset{\langle 6 \rangle}{\boldsymbol{H}}_{12} + \overset{\langle 6 \rangle}{\boldsymbol{H}}_{15} \\ &\quad = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_m + \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_i \\ &\qquad + \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_m \otimes \mathbf{e}_i \otimes \mathbf{e}_k + \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_m \otimes \mathbf{e}_k \otimes \mathbf{e}_i \end{aligned} \tag{79}$$

so that

$$\begin{aligned} &\overset{\langle 3 \rangle}{\boldsymbol{V}} \therefore 1/4 \left(\overset{\langle 6 \rangle}{\boldsymbol{H}}_{11} + \overset{\langle 6 \rangle}{\boldsymbol{H}}_{14} + \overset{\langle 6 \rangle}{\boldsymbol{H}}_{12} + \overset{\langle 6 \rangle}{\boldsymbol{H}}_{15} \right) \therefore \overset{\langle 3 \rangle}{\boldsymbol{V}} = \overset{\langle 3 \rangle}{\boldsymbol{V}} \therefore 1/2 \left(\overset{\langle 6 \rangle}{\boldsymbol{I}}{}^{[12]} + \overset{\langle 6 \rangle}{\boldsymbol{I}}{}^{[13]} \right) \therefore \overset{\langle 3 \rangle}{\boldsymbol{V}} \\ &= \overset{\langle 3 \rangle}{\boldsymbol{V}} \therefore 1/2 \left(\overset{\langle 3 \rangle}{\boldsymbol{V}}{}^{[12]} + \overset{\langle 3 \rangle}{\boldsymbol{V}}{}^{[13]} \right) \end{aligned} \tag{80}$$

This hexadic does the same as the symmetric transposer $1/2 \left(\overset{\langle 6 \rangle}{\boldsymbol{I}}{}^{[12]} + \overset{\langle 6 \rangle}{\boldsymbol{I}}{}^{[13]} \right)$.

$$\overset{\langle 6 \rangle}{\boldsymbol{H}}_7 = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_m = \mathbf{e}_i \otimes \mathbf{I} \otimes \mathbf{e}_i \otimes \mathbf{I} \tag{81}$$

so that

$$\overset{\langle 3 \rangle}{\boldsymbol{V}} \therefore \overset{\langle 6 \rangle}{\boldsymbol{H}}_7 \therefore \overset{\langle 3 \rangle}{\boldsymbol{V}} = \left(\overset{\langle 3 \rangle}{\boldsymbol{V}} : \mathbf{I} \right) \cdot \left(\overset{\langle 3 \rangle}{\boldsymbol{V}} : \mathbf{I} \right) \tag{82}$$

$$\overset{\langle 6\rangle}{\boldsymbol{H}}_1+\overset{\langle 6\rangle}{\boldsymbol{H}}_4+\overset{\langle 6\rangle}{\boldsymbol{H}}_{13}+\overset{\langle 6\rangle}{\boldsymbol{H}}_{10}$$
$$=\mathbf{e}_i\otimes\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_k\otimes\mathbf{e}_m\otimes\mathbf{e}_m+\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_m\otimes\mathbf{e}_m$$
$$+\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_k\otimes\mathbf{e}_m\otimes\mathbf{e}_m\otimes\mathbf{e}_i+\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_k\otimes\mathbf{e}_m\otimes\mathbf{e}_i\otimes\mathbf{e}_m \tag{83}$$

so that

$$\overset{\langle 3\rangle}{\boldsymbol{V}}\therefore 1/4\left(\overset{\langle 6\rangle}{\boldsymbol{H}}_1+\overset{\langle 6\rangle}{\boldsymbol{H}}_4+\overset{\langle 6\rangle}{\boldsymbol{H}}_{13}+\overset{\langle 6\rangle}{\boldsymbol{H}}_{10}\right)\therefore\overset{\langle 3\rangle}{\boldsymbol{V}}=\overset{\langle 3\rangle}{\boldsymbol{V}}\therefore\mathbf{e}_i\otimes\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{e}_i\therefore\overset{\langle 3\rangle}{\boldsymbol{V}}$$
$$=(\overset{\langle 3\rangle}{\boldsymbol{V}}:\mathbf{I})\cdot(\mathbf{I}:\overset{\langle 3\rangle}{\boldsymbol{V}}) \tag{84}$$

$$\overset{\langle 6\rangle}{\boldsymbol{H}}_2+\overset{\langle 6\rangle}{\boldsymbol{H}}_3+\overset{\langle 6\rangle}{\boldsymbol{H}}_5+\overset{\langle 6\rangle}{\boldsymbol{H}}_6$$
$$=\mathbf{e}_i\otimes\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_m\otimes\mathbf{e}_k\otimes\mathbf{e}_m+\mathbf{e}_i\otimes\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_m\otimes\mathbf{e}_m\otimes\mathbf{e}_k$$
$$+\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_i\otimes\mathbf{e}_m\otimes\mathbf{e}_k\otimes\mathbf{e}_m+\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_i\otimes\mathbf{e}_m\otimes\mathbf{e}_m\otimes\mathbf{e}_k \tag{85}$$

so that

$$\overset{\langle 3\rangle}{\boldsymbol{V}}\therefore 1/4\left(\overset{\langle 6\rangle}{\boldsymbol{H}}_2+\overset{\langle 6\rangle}{\boldsymbol{H}}_3+\overset{\langle 6\rangle}{\boldsymbol{H}}_5+\overset{\langle 6\rangle}{\boldsymbol{H}}_6\right)\therefore\overset{\langle 3\rangle}{\boldsymbol{V}}=\overset{\langle 3\rangle}{\boldsymbol{V}}\therefore\mathbf{I}\otimes\mathbf{e}_i\otimes\mathbf{I}\otimes\mathbf{e}_i\otimes\overset{\langle 3\rangle}{\boldsymbol{V}}$$
$$=\left(\mathbf{I}:\overset{\langle 3\rangle}{\boldsymbol{V}}\right)\cdot\left(\mathbf{I}:\overset{\langle 3\rangle}{\boldsymbol{V}}\right) \tag{86}$$

6th-rank hemitropic tensors are also linear combinations of all transpositions of the hexadic $\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}$. However, due to the imposed subsymmetries they do not enter the square form (76) in a non-trivial way.

5.2 Hemitropic square forms

For the dissipation potential (40) of our gradient fluids we need hemitropic square forms. In the sequel, we will particularize our concern to second-order fluids. According to the previous representations of hemitropic tensors, the **general hemitropic symmetric square form** of a dyadic and a triadic is in this case

$$\delta\left(\overset{\langle 2\rangle}{\boldsymbol{V}},\overset{\langle 3\rangle}{\boldsymbol{V}}\right)=1/2\,\overset{\langle 2\rangle}{\boldsymbol{V}}:\overset{\langle 4\rangle}{\boldsymbol{D}}_{22}:\overset{\langle 2\rangle}{\boldsymbol{V}}+\overset{\langle 2\rangle}{\boldsymbol{V}}:\overset{\langle 5\rangle}{\boldsymbol{D}}_{23}\therefore\overset{\langle 3\rangle}{\boldsymbol{V}}+1/2\,\overset{\langle 3\rangle}{\boldsymbol{V}}\therefore\overset{\langle 6\rangle}{\boldsymbol{D}}_{33}\therefore\overset{\langle 3\rangle}{\boldsymbol{V}} \tag{87}$$

with the three hemitropic tensors

$$\overset{\langle 4\rangle}{\boldsymbol{D}}_{22}\equiv\alpha_1\mathbf{I}\otimes\mathbf{I}+\alpha_2\overset{\langle 4\rangle}{\boldsymbol{I}} \tag{88}$$
$$\overset{\langle 5\rangle}{\boldsymbol{D}}_{23}\equiv\alpha_3/4\left(\overset{\langle 5\rangle}{\boldsymbol{H}}_2+\overset{\langle 5\rangle}{\boldsymbol{H}}_3+\overset{\langle 5\rangle}{\boldsymbol{H}}_9+\overset{\langle 5\rangle}{\boldsymbol{H}}_{10}\right) \tag{89}$$

or, e.g., $\overset{\langle 5\rangle}{\boldsymbol{D}}_{23}\equiv\alpha_3\overset{\langle 5\rangle}{\boldsymbol{H}}_2$

$$\overset{\langle 6\rangle}{\boldsymbol{D}}_{33}\equiv\alpha_4\overset{\langle 6\rangle}{\boldsymbol{I}}+\alpha_5/2\left(\overset{\langle 6\rangle}{\boldsymbol{I}}^{[12]}+\overset{\langle 6\rangle}{\boldsymbol{I}}^{[13]}\right)$$
$$+\alpha_6\mathbf{e}_i\otimes\mathbf{I}\otimes\mathbf{e}_i\otimes\mathbf{I}+\alpha_7\mathbf{e}_i\otimes\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{e}_i+\alpha_8\mathbf{I}\otimes\mathbf{e}_i\otimes\mathbf{I}\otimes\mathbf{e}_i \tag{90}$$

with eight scalar constants $\alpha_1,\ldots,\alpha_8$, which gives

$$2\delta\left(\overset{\langle 2\rangle}{\boldsymbol{V}},\overset{\langle 3\rangle}{\boldsymbol{V}}\right)=\alpha_1\,tr^2\,\overset{\langle 2\rangle}{\boldsymbol{V}}+\alpha_2\,\overset{\langle 2\rangle}{\boldsymbol{V}}:\overset{\langle 2\rangle}{\boldsymbol{V}}$$
$$+2\alpha_3\,\overset{\langle 2\rangle}{\boldsymbol{V}}:\left(\boldsymbol{\varepsilon}:\overset{\langle 3\rangle}{\boldsymbol{V}}\right)$$

$$+\alpha_4 \overset{\langle 3\rangle}{\boldsymbol{V}} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} + \alpha_5/2 \overset{\langle 3\rangle}{\boldsymbol{V}} \therefore \left(\overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[12]} + \overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[13]}\right)$$
$$+\alpha_6 \left(\overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I}\right) \cdot \left(\overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I}\right) + \alpha_7 \left(\mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{V}}\right) \cdot \left(\overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I}\right) + \alpha_8 \left(\mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{V}}\right) \cdot \left(\mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{V}}\right) \tag{91}$$

For $\alpha_3 \equiv 0$ it becomes the **general isotropic symmetric square form**.

The differential of this form is

$$d\delta\left(\overset{\langle 2\rangle}{\boldsymbol{V}}, \overset{\langle 3\rangle}{\boldsymbol{V}}, \overset{\langle 2\rangle}{\boldsymbol{dV}}, \overset{\langle 3\rangle}{\boldsymbol{dV}}\right) = \alpha_1 \left(\mathbf{I} : \overset{\langle 2\rangle}{\boldsymbol{V}}\right)\left(\mathbf{I} : \overset{\langle 2\rangle}{\boldsymbol{dV}}\right) + \alpha_2 \overset{\langle 2\rangle}{\boldsymbol{V}} : \overset{\langle 2\rangle}{\boldsymbol{dV}}$$
$$+\alpha_3 \overset{\langle 2\rangle}{\boldsymbol{dV}} : \left(\boldsymbol{\varepsilon} : \overset{\langle 3\rangle}{\boldsymbol{V}}\right) + \alpha_3 \overset{\langle 2\rangle}{\boldsymbol{V}} : \left(\boldsymbol{\varepsilon} : \overset{\langle 3\rangle}{\boldsymbol{dV}}\right)$$
$$+\alpha_4 \overset{\langle 3\rangle}{\boldsymbol{V}} \therefore \overset{\langle 3\rangle}{\boldsymbol{dV}} + \alpha_5/2 \left(\overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[12]} + \overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[13]}\right) \therefore \overset{\langle 3\rangle}{\boldsymbol{dV}} + \alpha_6 \left(\overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I}\right) \cdot \left(\overset{\langle 3\rangle}{\boldsymbol{dV}} : \mathbf{I}\right)$$
$$+\alpha_7/2 \left(\mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{dV}}\right) \cdot \left(\overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I}\right) + \alpha_7/2 \left(\mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{V}}\right) \cdot \left(\overset{\langle 3\rangle}{\boldsymbol{dV}} : \mathbf{I}\right) + \alpha_8 \left(\mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{V}}\right) \cdot \left(\mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{dV}}\right)$$
$$= \left[\alpha_1 \left(tr\, \overset{\langle 2\rangle}{\boldsymbol{V}}\right)\mathbf{I} + \alpha_2 \overset{\langle 4\rangle}{\boldsymbol{I}} : \overset{\langle 2\rangle}{\boldsymbol{V}} + \alpha_3\, \boldsymbol{\varepsilon} : \overset{\langle 3\rangle}{\boldsymbol{V}}\right] : \overset{\langle 2\rangle}{\boldsymbol{dV}}$$
$$+\left[\alpha_3\, \boldsymbol{\varepsilon} \cdot \overset{\langle 2\rangle}{\boldsymbol{V}} + \alpha_4 \overset{\langle 3\rangle}{\boldsymbol{V}} + \alpha_5/2 \left(\overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[12]} + \overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[13]}\right) + \alpha_6\, \mathbf{e}_i \otimes \mathbf{I} \therefore \mathbf{e}_i \otimes \mathbf{I} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}}\right.$$
$$\left. +\alpha_7/2 \left(\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} \therefore \overset{\langle 3\rangle}{\boldsymbol{V}} + \overset{\langle 3\rangle}{\boldsymbol{V}} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}\right) + \alpha_8\, \mathbf{I} \otimes \mathbf{e}_i \otimes \mathbf{I} \otimes \mathbf{e}_i \therefore \overset{\langle 3\rangle}{\boldsymbol{V}}\right] \therefore \overset{\langle 3\rangle}{\boldsymbol{dV}}. \tag{92}$$

The derivatives are

$$\delta_{\overset{\langle 2\rangle}{\boldsymbol{V}}} \delta\left(\overset{\langle 2\rangle}{\boldsymbol{V}}, \overset{\langle 3\rangle}{\boldsymbol{V}}\right) = \alpha_1 \left(tr\, \overset{\langle 2\rangle}{\boldsymbol{V}}\right)\mathbf{I} + \alpha_2 \overset{\langle 2\rangle}{\boldsymbol{V}} + \alpha_3\, \boldsymbol{\varepsilon} : \overset{\langle 3\rangle}{\boldsymbol{V}} \tag{93}$$

$$\delta_{\overset{\langle 3\rangle}{\boldsymbol{V}}} \delta\left(\overset{\langle 2\rangle}{\boldsymbol{V}}, \overset{\langle 3\rangle}{\boldsymbol{V}}\right) = \alpha_3\, \boldsymbol{\varepsilon} \cdot \overset{\langle 2\rangle}{\boldsymbol{V}} + \alpha_4 \overset{\langle 3\rangle}{\boldsymbol{V}} + \alpha_5/2 \left(\overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[12]} + \overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[13]}\right)$$
$$+\alpha_6 \overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I} \otimes \mathbf{I} + \alpha_7/2 \left(\mathbf{I} \otimes \overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I} + \mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{V}} \otimes \mathbf{I}\right) + \alpha_8 \mathbf{I} \otimes \mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{V}}. \tag{94}$$

These tensors can be further symmetrized.

6 Second-order linear gradient fluids

If we apply the representation (91) to our kinematical variables, the complete ansatz for the dissipation potential as a hemitropic square form is

$$\begin{aligned}
&\delta(\mathbf{D}, grad\ grad\ \mathbf{v}) \\
&\quad = \alpha_1/2\, tr^2\, \mathbf{D} + \alpha_2/2\, \mathbf{D} : \mathbf{D} \\
&\qquad + \alpha_3\, \mathbf{D} : (\boldsymbol{\varepsilon} : grad\ grad\ \mathbf{v}) \\
&\qquad + \alpha_4/2\, grad\ grad\ \mathbf{v} \therefore grad\ grad\ \mathbf{v} \\
&\qquad + \alpha_5/4\, grad\ grad\ \mathbf{v} \therefore (grad\ grad\ \mathbf{v}^{[12]} + grad\ grad\ \mathbf{v}^{[13]}) \\
&\qquad + \alpha_6/2\, grad\ grad\ \mathbf{v} : \mathbf{I}) \cdot (grad\ grad\ \mathbf{v} : \mathbf{I}) \\
&\qquad + \alpha_7/2\, (\mathbf{I} : grad\ grad\ \mathbf{v}) \cdot (grad\ grad\ \mathbf{v} : \mathbf{I}) \\
&\qquad + \alpha_8/2 (\mathbf{I} : grad\ grad\ \mathbf{v}) \cdot (\mathbf{I} : grad\ grad\ \mathbf{v})
\end{aligned} \tag{95}$$

with eight scalar constants $\alpha_1, \ldots, \alpha_8$. So the coupling between the first and the second velocity gradient results only from the hemitropic tensor $\overset{\langle 5\rangle}{\boldsymbol{D}}_{23}$. This representation leads to the stress functions

$$\mathbf{T} = \alpha_1 \left(tr \overset{\langle 2\rangle}{\boldsymbol{V}} \right) \mathbf{I} + \alpha_2 \overset{\langle 2\rangle}{\boldsymbol{V}} + \alpha_3/2 \left(\boldsymbol{\varepsilon} \colon \overset{\langle 3\rangle}{\boldsymbol{V}} + \boldsymbol{\varepsilon} \colon \overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[23]} \right) \tag{96}$$

$$\begin{aligned} \overset{\langle 3\rangle}{\boldsymbol{T}} &= \alpha_3 \, \boldsymbol{\varepsilon} \cdot \overset{\langle 2\rangle}{\boldsymbol{V}} + \alpha_4 \overset{\langle 3\rangle}{\boldsymbol{V}} + \alpha_5/2 (\overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[12]} + \overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[13]}) \\ &\quad + \alpha_6 \overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I} \otimes \mathbf{I} + \alpha_7/2 \left(1/2 \, \mathbf{I} \otimes \overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I} + 1/2 \, \mathbf{I} \otimes \overset{\langle 3\rangle}{\boldsymbol{V}} : \mathbf{I}^{[23]} + \mathbf{I} : \overset{\langle 3\rangle}{\boldsymbol{V}} \otimes \mathbf{I} \right) \\ &\quad + \alpha_8/2 \left(\mathbf{I} \otimes \mathbf{I} \colon \overset{\langle 3\rangle}{\boldsymbol{V}} + \mathbf{I} \otimes \mathbf{I} \colon \overset{\langle 3\rangle}{\boldsymbol{V}}{}^{[23]} \right). \end{aligned} \tag{97}$$

6.1 Second-order linear incompressible gradient fluids

If we assume **incompressibility**, we have the constraint equation[6] in an Eulerian form which is more appropriate for fluids

$$div \;\; \mathbf{v} = \mathbf{I} : \mathbf{D} = 0. \tag{98}$$

If this constraint holds not only in one point but also in its neighborhood, then we can also state the second-order constraint

$$grad \; div \, \mathbf{v} \cdot \mathbf{r} = \mathbf{I} \otimes \mathbf{r} \therefore grad \; grad \, \mathbf{v} = 0 \tag{99}$$

with an arbitrary vector field $\mathbf{r}$. Then the second-order reaction stress is a hydrostatic pressure

$$\mathbf{T}_R = -p \, \mathbf{I} \tag{100}$$

and the third-order reaction stress is

$$\overset{\langle 3\rangle}{\boldsymbol{T}}_R = sym^{[23]} (\mathbf{I} \otimes \mathbf{r}). \tag{101}$$

Due to the incompressibility, the terms with α_1, α_7, and α_8 vanish. So the remainders are after some renaming

$$\begin{aligned} &\delta(\mathbf{D}, grad \; grad \, \mathbf{v}) \\ &\quad = b_1/2 \, \mathbf{D} : \; \mathbf{D} \; + b_2 \mathbf{D} \colon (\boldsymbol{\varepsilon} \colon grad \; grad \, \mathbf{v}) \\ &\qquad + b_3/2 \; grad \; grad \;\; \mathbf{v} \therefore \; grad \; grad \, \mathbf{v} \\ &\qquad + b_4/4 \; grad \; grad \;\; \mathbf{v} \therefore \left(grad \; grad \, \mathbf{v}^{[12]} + grad \; grad \;\; \mathbf{v}^{[13]} \right) \\ &\qquad + b_5/2 \Delta \mathbf{v} \cdot \Delta \mathbf{v}. \end{aligned} \tag{102}$$

As an immediate consequence of the dissipation postulate, the constants b_1, b_3, and b_5 must be nonnegative. The restrictions upon the other constants are less obvious.

The resulting extra stresses are now

$$\begin{aligned} \mathbf{T}_E &= b_1 \mathbf{D} + b_2 sym(\boldsymbol{\varepsilon} \colon grad \; grad \;\; \mathbf{v}) \\ &= b_1 \mathbf{D} - b_2/2 (grad \; curl \, \mathbf{v} + grad^{\mathrm{T}} \; curl \;\; \mathbf{v}) \end{aligned} \tag{103}$$

and

$$\begin{aligned} \overset{\langle 3\rangle}{\boldsymbol{T}}_E &= sym^{[23]} \Big[b_2 \boldsymbol{\varepsilon} \cdot \mathbf{D} + b_3 \overset{\langle 6\rangle}{\boldsymbol{I}} \therefore grad \; grad \, \mathbf{v} + b_4 \overset{\langle 6\rangle}{\boldsymbol{I}}{}^{[13]} \therefore grad \; grad \, \mathbf{v} \\ &\qquad + b_5 \mathbf{e}_i \otimes \mathbf{I} \otimes \mathbf{e}_i \otimes \mathbf{I} \therefore grad \; grad \, \mathbf{v} \Big] \\ &= b_2 sym^{[23]} [\boldsymbol{\varepsilon} \cdot \mathbf{D}] + b_3 grad \; grad \;\; \mathbf{v} \end{aligned}$$

6 For non-classical internal constraints see Bertram and Glüge [9] and Bertram [11].

$$+ b_4/2 \left(grad\ grad\ \mathbf{v}^{[12]} + grad\ grad\ \mathbf{v}^{[13]} \right) + b_5 \Delta \mathbf{v} \otimes \mathbf{I}. \tag{104}$$

This is the ***general form of an incompressible linear viscous second-order fluid***. The following tensor appears in the local balance of linear momentum (17)

$$\begin{aligned}
&\mathbf{T}_E + \mathbf{T}_R - div \overset{\langle 3 \rangle}{\boldsymbol{T}}_E - div \overset{\langle 3 \rangle}{\boldsymbol{T}}_R \\
&\quad = b_1 \mathbf{D} - b_2 sym(grad\ curl\ \mathbf{v}) - p\,\mathbf{I} \\
&\qquad + b_2/4\ (\mathbf{I} \times \Delta \mathbf{v} - grad\ curl\ \mathbf{v}) \\
&\qquad - (b_3 + b_5) grad\ \ \Delta \mathbf{v} - b_4/2\ grad^{\mathrm{T}} \Delta \mathbf{v} - \frac{1}{2}(div\, \mathbf{r}\, \mathbf{I} + grad^{\mathrm{T}} \mathbf{r}).
\end{aligned} \tag{105}$$

The local balance of linear momentum (1.142) is therefore

$$\begin{aligned}
\rho(\mathbf{a} - \mathbf{b}) = div \left[\mathbf{T}_E + \mathbf{T}_R - div \overset{\langle 3 \rangle}{\boldsymbol{T}}_E - div \overset{\langle 3 \rangle}{\boldsymbol{T}}_R \right] \\
= b_1/2\ \ \Delta \mathbf{v} - b_2\ curl\ \Delta \mathbf{v} - (b_3 + b_5)\ \Delta\Delta \mathbf{v} \\
- grad\ p - grad\ div\ \mathbf{r}
\end{aligned} \tag{106}$$

forming a vectorial partial differential equation of fourth-order in the velocities with two fields p and $\mathbf{r}$ for which no constitutive equation is given.

6.2 Boundary conditions

Boundary conditions play an important role in such theories since the classical Dirichlet and Neumann conditions are not sufficient for gradient models. Clearly, boundary value problems can only be solved if sufficient boundary conditions are prescribed. The higher the order of the model, the more boundary conditions are needed. More precisely, for each increment of the order, one additional vectorial boundary value on all boundary points is needed.

A fundamental problem of experimental fluid mechanics is the determination of the boundary conditions by measurements. The profile of the velocities becomes feral and undeterminable near the walls. There are indications that the classical assumption of bonding of the fluid with the walls does not hold in the turbulent case anymore. Instead, one has to account for slip phenomena, which are, however, hardly measurable. This point has been discussed in more detail by Trostel [32].

In Bertram [10] it is shown that for second-order gradient materials the power of the tensions and couple stresses on a smooth boundary (without edges and corners) is

$$\mathbf{t}_2 \cdot \mathbf{v} + \left(\overset{\langle 3 \rangle}{\boldsymbol{T}} \cdot \mathbf{n} \right) : grad_n\ \mathbf{v} \tag{107}$$

with

$$\mathbf{t}_2 := \left(\overset{\langle 2 \rangle}{\boldsymbol{T}} - div_n \overset{\langle 3 \rangle}{\boldsymbol{T}} - 2 div_t \overset{\langle 3 \rangle}{\boldsymbol{T}} \right) \cdot \mathbf{n} + \overset{\langle 3 \rangle}{\boldsymbol{T}} : (div_t \mathbf{n}\, \mathbf{n} \otimes \mathbf{n} - grad_t\ \mathbf{n}). \tag{108}$$

For a third-gradient material they become even more complicated.

Trostel calls the boundary conditions *isoenergetic* if (107) vanishes in all boundary points, in contrast to *real* boundary conditions otherwise. Two special cases of isoenergetic conditions are

- the fully fixed boundary where both $\mathbf{v}$ and $grad_n \mathbf{v}$ vanish,
- and the fully free boundary where both $\mathbf{t}_2$ and $\overset{\langle 3 \rangle}{\boldsymbol{T}} : \mathbf{n} \otimes \mathbf{n}$ vanish

in all boundary points. Of course, these are not the only choices for isoenergetic boundary conditions.

Trostel's suggestion for *real* boundary conditions consists of a linear viscous ansatz between the surface friction tensions and couple stresses with the tangential relative velocity between the fluid and the wall $\mathbf{v}_{\mathrm{rel}}$ and its normal derivative $\partial_n \mathbf{v}_{\mathrm{rel}}$

$$\mathbf{t}_2 = \lambda_{11} \mathbf{v}_{\mathrm{rel}} + \lambda_{12} \partial_n \mathbf{v}_{\mathrm{rel}} \tag{109}$$

$$\overset{\langle 3 \rangle}{\boldsymbol{T}} : \mathbf{n} \otimes \mathbf{n} = \lambda_{21} \mathbf{v}_{\text{rel}} + \lambda_{22} \partial_n \mathbf{v}_{\text{rel}} \tag{110}$$

with **porosity coefficients** λ_{ij} which must be chosen in a way that the dissipation of these tensions is positive definite.[7]

7 Works of the Berlin school

The first work on gradient viscosity was probably that by Trostel [31].[8] It starts with a rather general framework for Nth-order generalizations of the Navier–Stokes law and later particularizes these representations for a second-order fluid similar to the previous section.

However, some differences between Trostel's approach and the present one shall be mentioned.

(i) Trostel considers only the isotropic parts, but not the hemitropic one. He believes that the isotropy of the stress laws for such fluids was a material property. In our format, however, the hemitropy of these laws results from the *principle of invariance under superimposed rigid body modifications* and has nothing to do with material symmetry properties. For us, all fluids are isotropic by definition.

(ii) In our theory, the second-rank stress tensor is symmetric due to the local balance of moment of momentum (1.93). In contrast, Trostel assumes symmetry for the entire tensor

$$\overset{\langle 2 \rangle}{\boldsymbol{T}}_E + \overset{\langle 2 \rangle}{\boldsymbol{T}}_R - div\, \overset{\langle 3 \rangle}{\boldsymbol{T}}_E - div\, \overset{\langle 3 \rangle}{\boldsymbol{T}}_R \,. \tag{111}$$

(iii) Trostel considers the internal constraint incompressibility without third-rank reaction stresses. As a consequence, his stress law contains a power neutral additional term which does not appear in our stress law for $\overset{\langle 3 \rangle}{\boldsymbol{T}}_E$.

In the sequel these differences are present in almost all papers of this school.

A more systematic and much more detailed description of gradient fluids can be found in Silber [24]. Here the extension of the Navier–Stokes theory is given up to third order, which naturally includes the second-order case. Silber starts from a dissipation potential as a positive-definite square form, which is automatically symmetric. This form is assumed to be isotropic (not hemitropic), so that the representations of isotropic tensors by, e.g., Caldonazzo [12] can be used. By such an isotropic dissipation potential, Silber derives the stress laws for the second-, third-, and fourth-rank hyperstresses. By implementing them into the local balance of linear momentum (1.142), he obtains sixth-order partial differential equations in the velocities.

The intention of these works is the description of steady fully developed turbulent flows. In Reichardt [22,23], experimental measurements of the velocity profile in a Couette flow (Fig. 1) and a channel flow for this case are presented. In both cases, these profiles deviate from the results of a (linear or nonlinear) first-order (simple) viscous law. In particular, the first-order theory would give a linear profile for the Couette flow and a parabola for the channel flow, while Reichardt's measurements render other profiles.

If the balance equation for the linear momentum is specified for these two cases, the only non-trivial equation is a sixth-order linear ordinary differential equation in the transverse direction of the channel. This can be solved analytically.

For the determination of a unique solution, one needs boundary conditions. The problem to determine these conditions by measurement in experiments have already been mentioned before. This general problem has been discussed in almost all of the works of the Berlin School.

Due to the poverty of reliable boundary conditions, Silber has determined the material constants by an optimization algorithm to fit the simulated velocity profile to the measured ones within the flow field, neglecting the boundary zone at the walls. By this procedure, Silber can reproduce the measured profiles with an acceptable accuracy (Figs. 2, 3). He demonstrates that the third-order theory gives better results than the second-order theory, although both are proven to be much better than the classical Navier–Stokes solutions with their principal deficits as mentioned before.

In Silber [25], the second-order fluid is compared with a micro-polar fluid.[9] Here also the hemitropic part (89) of the gradient fluid is included which leads to a coupling of the second- and third-order terms in the stress law.

7 See Silber et al. [28].
8 See also Trostel [35].
9 See Alexandru [1].

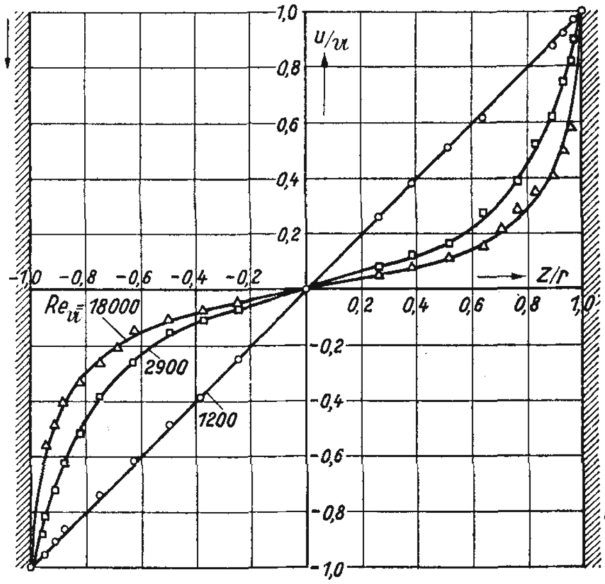

Fig. 1 Velocity profile in the Couette flow by Reichardt [23]

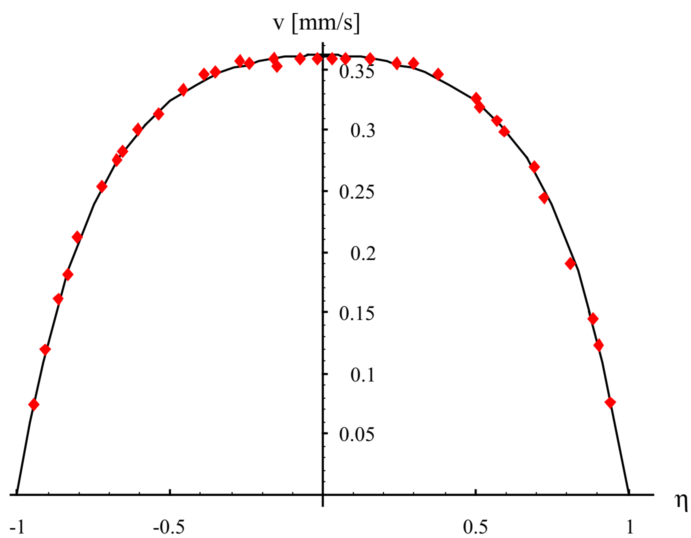

Fig. 2 Velocity profile in a Poiseuille flow from Silber et al. [28]. dots: measurement, curve: simulation

Of course, one might generally question whether the macroscopic behavior of a turbulent flow allows for the *principle of invariance under superimposed rigid body modifications*, see, e.g., Lumley [19], Speziale [30], Dafalias [14].

Another application of gradient fluids is the flow of blood which is a mixture of plasma and cells. For such mixtures we also find severe deviations from the classical results of Navier–Stokes fluids. In Silber [26][10] the model of an incompressible isotropic second-order fluid is used to solve the boundary value problem for a Poiseuille flow in a pipe with kinematical boundary conditions, and for an annulus flow between two concentric rotating cylinders with mixed boundary conditions. The analytical solutions reproduce the measured flow profiles in a satisfying precision.[11]

The boundary value problem of a flow in the annulus has been further investigated by Silber and Alizadeh [27] with the same gradient fluid ansatz. The boundary conditions are here dissipative and allow for a slip

[10] See also Alizadeh et al. [3].

[11] See also Silber et al. [28].

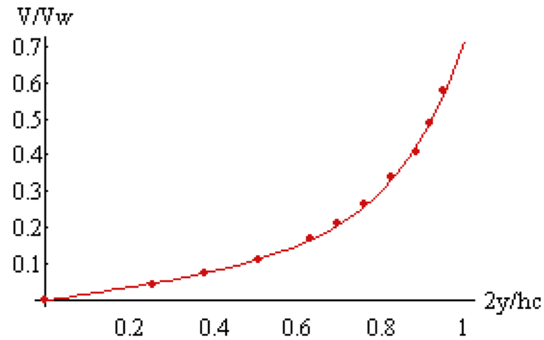

Fig. 3 Measured (dots) and calculated velocity profiles of a fully developed turbulent Couette flow of water from Silber et al. [28]

between fluid and wall. Besides the steady flow also a harmonic oscillation between the two cylinders is considered.

Porosity boundary conditions have been also applied in Trostel [32], Silber et al. [28], Alizadeh [2], Silber et al. [29], and Alizadeh et al. [4].

These early works have not been adequately published at the time and seem to be almost forgotten in the sequel, although they are highly innovative and interesting. With them, Trostel and his group opened a completely new insight into material modeling of fluidity. Naturally, many interesting questions came up and widely open the doors for further investigations.

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