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ORIGINAL ARTICLE

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On invariance properties of an extended energy balance

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Abstract Gradient plasticity theories are of utmost importance for accounting for size effects in metals, especially on the grain scale. Today, there are several methods used to derive the governing equations for the additional degrees of freedom in gradient plasticity theories. Here, the equivalence between an extended principle of virtual power and an extended energy balance is shown. The energy balance of a Boltzmann continuum is supplemented by contributions based on a scalar-valued degree of freedom. It is considered to be invariant with respect to a change of observer. This yields unambiguously the existence of a corresponding micro-stress vector, which is presumed from the outset in the context of an extended principle of virtual power. A thermodynamically consistent nonlocal evolution equation for the additional, scalar-valued degree of freedom is obtained by evaluation of the dissipation inequality in terms of the Clausius–Duhem inequality. Partitioning the nonlocal flow rule yields a partial differential equation, often referred to as micro-force balance. The approach presented is applied to derive a slip gradient crystal plasticity theory regarding single slip. Finally, the distribution of the plastic slip is exemplified with respect to a laminate material consisting of an elastic and an elastoplastic phase.

Keywords Extended energy balance · Green–Naghdi–Rivlin theorem · Additional field equations · Micro-stress · Conservation of micro-inertia

1 Introduction

Classical continuum mechanics considers a body as a set of undeformable material points. Each material point of such a Boltzmann continuum exhibits three degrees of freedom (DOFs), describing its displacement, cf. Hellinger [37, p. 606] and Eugster and dell'Isola [21–23]. The Boltzmann continuum, cf. Vardoulakis [71, p. 1], is also referred to as Cauchy continuum as described in Maugin [46, p. 3], cf. also [14,31,60]. Extended continuum models account for the underlying microstructure of the material by introducing additional, internal DOFs. One of the first suggested extended continuum models is the so-called Cosserat continuum [13]. It allows for the orientation of a material point. Thus, each material point is supplemented by three rotational DOFs in addition. Many authors addressed this topic in the mid of the twentieth century focusing on generalizations or extensions to the Cosserat continuum [17–19,30,33,47]. Especially the micromorphic continuum according

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to Eringen and Suhubi [20] can be considered as a direct generalization of the Cosserat continuum. It treats each material point as a micro-continuum. Consequently, a micro-deformation tensor associated with each material point is introduced in [20]. Conceptually similar is the consideration of a micro-medium as discussed by Mindlin [47]. Continua that account for couple stresses are discussed by Toupin [67,68]. Velocity gradients of higher order or multipolar displacements are introduced in the context of extended continua by Green and Rivlin [33,35]. An extensive overview of generalized continua is given in [8] and [50]. Further applications of extended continua are to be found in the context of liquid crystals [17,41], continuum theory of dislocations [27,28], nonlocal plasticity [53,54], nonlocal damage [29,55,57,58] and nonlocal diffusion [70].

Additional DOFs are associated with corresponding equations of motion, relating the kinematic of the DOFs to the underlying forces, cf. remark in Maugin [45]. Consequently, the total energy describing the system is supplemented by contributions related to the additionally introduced DOFs. The comparison of a mathematical pendulum with a double pendulum serves as an illustrative example with respect to discrete systems [40]. In the continuum mechanical context, several approaches exist to derive or motivate additional balance equations associated with additional DOFs. An overview is given in the review paper of Mariano [42] or others, e.g., [48,56]. In a nondissipative context, Hamilton's principle of least action is a suitable method for the derivation of associated field equations [37]. It can be considered as the predecessor of many other variational principles. An application to continua with a microstructure based on elastic micro-trusses is given by [61]. However, this is getting more involved for dissipative systems, cf. Planck [59, p. 81]. Closely related to variational principles is the principle of virtual power. Its classical formulation can be supplemented by additional work terms accounting for the virtual power of additional DOFs [24]. According to Mariano [42], a drawback of an extended principle of virtual power is that quantities, such as the stress and micro-stress tensor, are presumed. Another approach is to consider the invariance properties of an extended energy balance with respect to a superimposed rigid-body motion. Additionally, the 'tetrahedron' argument [15] is applied to prove the existence of, e.g., the stress and micro-stress tensor. This approach is often referred to as Green–Naghdi–Rivlin (GNR) theorem [43,44]. Its first application can be found in [34]. With this method, however, it is difficult to obtain additional field equations, as already noted by Planck [59]. In Germain [30, p. 574] it is stated that invariance considerations of an extended energy balance do not lead to the same field equations as obtained by an extended principle of virtual power. Maugin [44] confirms this issue, referring to the seven parameter invariance, that is commonly applied to the energy balance in this context. The number of field equations obtained by the energy balance is less compared to the number obtained by Hamilton's principle [59], or the principle of virtual power, Maugin [44, p. 63]. This topic is recaptured by Yavari and Marsden [74, p. 10]. They show that an extended energy balance only leads to modifications of the common balance equations if the ambient space is chosen Euclidean rather than a Riemannian manifold. However, it is not possible to obtain additional field equations. In Yavari and Marsden [74], the extensions to the energy balance, which is discussed in the context of the GNR-theorem, are due to additional vectorial DOFs. As discussed in Svendsen [65, footnote 2], it is regardless whether the invariance of the energy balance is considered with respect to a superimposed rigid-body motion or a change of observer. Both approaches yield the same balance equations.[1] In fact, some authors state the invariance of an extended energy balance with respect to a change of observer [9]. Moreover, the covariance of an extended energy balance with respect to a spatial and a microstructural diffeomorphism is considered in Yavari and Marsden [74]. This procedure is applied to a classical Boltzmann continuum in Marsden and Hughes [43, pp. 165-167]. However, they explicitly emphasize that the stress vector transforms objectively, irrespective of the underlying material behavior, if the considered spatial diffeomorphism describes a rigid deformation. Regarding a generic spatial diffeomorphism, an objective transformation of the stress vector is postulated only in the purely elastic case, cf. Marsden and Hughes [43, p. 163]. Consequently, the consideration of dissipative processes by means of this framework is quite involved. This limitation to purely elastic material behavior can be seen as the most critical point of this approach. Following Truesdell and Toupin [69, p. 529], a clear separation between balance equations and constitutive equations has to be drawn. This arises from the demand that balance equations should be of generic nature, valid for all materials. Hence, as stated in [69], constitutive laws cannot be obtained from balance equations. Contrarily, the approach of Marsden and Hughes [43] and Yavari and Marsden [74] yields a constitutive equation for the stress tensor, denoted as Doyle-Ericksen formula.

In this work, the invariance of an extended energy balance is considered with respect to a change of observer. The supplementary contributions are based on an additional, scalar-valued DOF related to the under-

[1] In the context of material theory, invariance considerations with respect to a change of observer or a superimposed rigid-body motion are denoted as PMO or PISM, respectively. In contrast to the PMO, the PISM is not always valid, cf. Krawietz [38, p. 161] and [65,66].

lying microstructure. A contribution of kinetic energy associated with the additional DOF is taken into account. In this context, it is not possible to obtain a corresponding conservation of micro-inertia. However, the existence of a vectorial micro-stress associated with a scalar-valued additional DOF can be shown unambiguously. This is of special interest, as a corresponding extended principle of virtual power does presume the unambiguous existence of a micro-stress vector from the outset. Exploitation of the dissipation inequality leads to a thermodynamically consistent, nonlocal evolution equation associated with the additional DOF. Partitioning the evolution equation yields the so-called micro-force balance. In this context, it is shown that the enhanced energy balance is equivalent to an extended principle of virtual power if the effects of micro-inertia and micro-body forces are neglected. Finally, a slip gradient crystal plasticity theory regarding small deformations is discussed as application of the approach presented. In this context, the additional scalar-valued DOF is considered as plastic slip within a slip system. For brevity, considerations are limited to single slip. An analytic solution for the plastic slip and the displacement field is discussed with respect to a laminate material that consists of an elastic and an elastoplastic phase.

Outline In Sect. 2, the balance equations of a classical continuum as well as the transformation used in the context of a change of observer are revisited. Moreover, an energy balance is proposed that contains generic extensions due to additional DOFs of arbitrary nature. The dissipation inequality in terms of the Clausius–Duhem inequality is briefly discussed. Regarding an additional scalar-valued DOF, the invariance of an extended energy balance with respect to a change of observer is discussed in Sect. 3. The relation to an extended principle of virtual power is shown. In Sect. 4, the application of the approach presented is illustrated regarding a slip gradient crystal plasticity theory. The manuscript is concluded by Sect. 5. A discussion concerning an additional vectorial DOF and the derivation of the corresponding conservation of micro-inertia is provided in "Appendix."

Notation A direct tensor notation is preferred throughout the manuscript. Vectors and second-order tensors are denoted by lowercase and uppercase bold letters, e.g., $\boldsymbol{a}$ and $\boldsymbol{A}$, respectively. A linear mapping of second-order tensors by a fourth-order tensor is written as $\boldsymbol{A} = \mathbb{C}[\boldsymbol{B}]$. The scalar product and the dyadic product are denoted, e.g., by $\boldsymbol{A} \cdot \boldsymbol{B}$ and $\boldsymbol{A} \otimes \boldsymbol{B}$, respectively. The composition of two second-order or two fourth-order tensors is formulated by $\boldsymbol{AB}$ and $\mathbb{AB}$.

2 Fundamentals

2.1 Energy balance

Energy balance of a classical continuum Regarding the current configuration of a material volume, the balance of total energy of a classical continuum is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}_t} \rho \left(e + \frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v} \right) \mathrm{d}v = \int_{\mathcal{V}_t} \rho \left(\boldsymbol{b} \cdot \boldsymbol{v} + r \right) \mathrm{d}v + \int_{\partial \mathcal{V}_t} \boldsymbol{t} \cdot \boldsymbol{v} + h \, \mathrm{d}a, \tag{1}$$

cf. Marsden and Hughes [43]. The volume of the continuum is referred to as $\mathcal{V}_t$ and its boundary as $\partial \mathcal{V}_t$. Equation (1) also holds true for a material volume that contains a singular surface. However, singular surfaces are not considered throughout the work at hand. Here, e denotes the mass specific internal energy and ρ the mass density. The spatial velocity field of the body is denoted by $\boldsymbol{v}$. Mechanical power is expended by the body and traction forces $\boldsymbol{b}$ and $\boldsymbol{t}$, respectively. The thermal contribution is due to the radiation r and the heat flux h.

Invariance consideration of the energy balance Given are two independent Euclidean vector spaces $\mathcal{W}$ and $\mathcal{W}'$. While Euclidean vector spaces are isomorph in general, $\mathcal{W}$ and $\mathcal{W}'$ are distinguished, here. Each vector space is associated with an observer. The relation between quantities described by the corresponding observer is given by the Euclidean transformation

$$\boldsymbol{x}'(t) = \boldsymbol{Q}(t)\boldsymbol{x}(t) + \boldsymbol{c}'(t). \tag{2}$$

In this context, $\boldsymbol{Q}(t)$ describes a time-dependent isometry between the two vector spaces. Consequently, $\boldsymbol{Q}$ is invertible and $\det(\boldsymbol{Q}) = 1$ holds true. The origins of both vector spaces are related to each other by the time-dependent vector $\boldsymbol{c}'(t)$. Regarding $\mathcal{W}$ and $\mathcal{W}'$, $\boldsymbol{x} \in \mathcal{W}$, $\boldsymbol{x}' \in \mathcal{W}'$, $\boldsymbol{c}' \in \mathcal{W}'$ and $\boldsymbol{Q} : \mathcal{W} \to \mathcal{W}'$ holds true. Thus, the isometry $\boldsymbol{Q}$ is given by $\boldsymbol{Q} = Q_{ij} \boldsymbol{e}'_i \otimes \boldsymbol{e}_j$ with $\boldsymbol{e}'_i \in \mathcal{W}'$ and $\boldsymbol{e}_j \in \mathcal{W}$, see , e.g., [39] for more details. Both observers consider the same physical process in their respective vector space. This motivates to assume the invariance of the energy balance with respect to a change of observer which yields the existence of the Cauchy stress tensor $\boldsymbol{\sigma}$ and the heat flux vector $\boldsymbol{q}$, reading

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}, \quad \text{and} \quad h = -\boldsymbol{q} \cdot \boldsymbol{n}, \tag{3}$$

cf. Šilhavý [62]. Moreover the balance of mass, linear and angular momentum, and the balance of internal energy are obtained

$$\begin{aligned} &\dot{\rho} + \rho \mathrm{div}\,(\boldsymbol{v}) = 0, \quad \rho\,(\boldsymbol{a} - \boldsymbol{b}) - \mathrm{div}\,(\boldsymbol{\sigma}) = \boldsymbol{0}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}, \\ &\rho\dot{e} - \rho r - \boldsymbol{\sigma} \cdot \boldsymbol{D} + \mathrm{div}\,(\boldsymbol{q}) = 0. \end{aligned} \tag{4}$$

The material time derivative is denoted by a $\dot{(\ )}$. The acceleration is abbreviated as $\boldsymbol{a} = \dot{\boldsymbol{v}}$, and the symmetric part of the velocity gradient is denoted as

$$\boldsymbol{D} = \mathrm{sym}(\mathrm{grad}\,(\boldsymbol{v})). \tag{5}$$

A continuum satisfying the balance equations (1) and (4) is referred to as classical Boltzmann continuum or Cauchy continuum in the literature.

Energy balance with generic extensions To provide a basis for the subsequent sections, an energy balance is considered that is extended by generic additional contributions compared to Eq. (1). The considered extensions are introduced irrespective of the tensorial order of the additional DOFs. Thus, an extended energy balance can be formulated with respect to the current configuration as

$$\begin{aligned} \epsilon = &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}_t} \rho \left(e + \frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v} + \kappa \right) \mathrm{d}v - \int_{\mathcal{V}_t} \rho\,(\boldsymbol{b} \cdot \boldsymbol{v} + \beta + r)\,\mathrm{d}v \\ &- \int_{\partial\mathcal{V}_t} \boldsymbol{t} \cdot \boldsymbol{v} + s + h\,\mathrm{d}a = 0. \end{aligned} \tag{6}$$

Here, κ denotes an additional kinetic energy density, β and s an additional volume and surface specific mechanical power, respectively. Application of Reynold's transport theorem to Eq. (6) leads to

$$\begin{aligned} \epsilon = &\int_{\mathcal{V}_t} (\dot{\rho} + \rho \mathrm{div}\,(\boldsymbol{v})) \left(e + \frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v} + \kappa \right) + \rho\dot{e} + \rho\,(\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{v} + \rho\dot{\kappa} - \rho\beta - \rho r\,\mathrm{d}v \\ &- \int_{\partial\mathcal{V}_t} \boldsymbol{t} \cdot \boldsymbol{v} + s + h\,\mathrm{d}a = 0. \end{aligned} \tag{7}$$

The additional contributions κ, β and s differ for each considered approach. In most approaches, the extensions are related to additional kinematic DOFs of various kinds, cf., e.g., the micromorphic medium according to Eringen and Suhubi [20] and the director theory according to Green et al. [32]. However, it is also possible to consider more general DOFs that are not associated with kinematics, cf. the interstitial working according to [16], for instance. A treatment of generic scalar-valued DOFs is given by Svendsen [64]. Thermal contributions are often considered unaltered, as with a classical continuum. The idea of obtaining conservation laws by means of an invariance consideration of the total energy density is closely related to Noether's theorem [51].

2.2 Dissipation inequality

Entropy balance Regarding the current configuration of a material volume, the standard form of the entropy balance is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}_t} \rho\eta\,\mathrm{d}v = -\int_{\partial\mathcal{V}_t} \boldsymbol{\phi}^{\eta} \cdot \boldsymbol{n}\,\mathrm{d}a + \int_{\mathcal{V}_t} \rho\gamma + s^{\eta}\,\mathrm{d}v, \tag{8}$$

cf. Müller [49]. Here, η denotes the mass specific entropy considering the bulk material, $\boldsymbol{\phi}^{\eta}$ the entropy flux across the boundary $\partial\mathcal{V}_t$, γ the mass specific entropy production and s^{η} the entropy supply.

Clausius–Duhem inequality As common in classical thermodynamics [12], the entropy flux $\boldsymbol{\phi}^{\eta}$ is assumed to be given by the ratio of the heat flux $\boldsymbol{q}$ and the temperature θ, and the entropy supply is assumed to be $s^{\eta} = \rho r/\theta$.

The bulk dissipation is defined as $\delta := \gamma\theta$. Accounting for the previous assumptions, as well as Reynold's transport theorem and the divergence theorem, localization of Eq. (8) yields

$$\rho\delta = \rho\theta\dot{\eta} - \rho r + \theta \mathrm{div}\left(\frac{\boldsymbol{q}}{\theta}\right) \tag{9}$$

Moreover, the relation between the free energy ψ, the internal energy e and the entropy η is given as $\psi = e - \theta\eta$, resulting from the Legendre transformation [4]. The second law of thermodynamics states that the dissipation is always non-negative. This yields the dissipation inequality, based on Eq. (9), reading

$$\rho\delta = \rho\dot{e} - \rho\dot{\psi} - \rho\dot{\theta}\eta - \rho r + \mathrm{div}\,(\boldsymbol{q}) - \frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g} \geq 0, \tag{10}$$

with $\boldsymbol{g} = \mathrm{grad}\,(\theta)$. In this form, the dissipation inequality according to Eq. (10) is also often referred to as Clausius–Duhem inequality [11].

3 Additional scalar-valued DOF

3.1 Extended energy balance

Deformation and microstructure function The energy balance Eq. (7) is given with respect to a current configuration referred to as $\mathcal{S}$. The current configuration is occupied by a body that is exposed to arbitrary, external loads. An arbitrary reference configuration is denoted as $\mathcal{B}$. Within the scope of this manuscript, $\mathcal{S}$ and $\mathcal{B}$ are considered to be embedded in the Euclidean space, i.e., an Euclidean ambient space is considered [74]. A material point, identified by its position vector $\boldsymbol{X}$, is mapped from $\mathcal{B}$ to $\mathcal{S}$ by the deformation function $\boldsymbol{\varphi}_t$. Regarding the current configuration, a material point is identified by its position vector $\boldsymbol{x}$. The spatial velocity field $\boldsymbol{v}$ is obtained by means of the time derivative of the deformation mapping $\boldsymbol{\varphi}_t$. It is calculated by

$$\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{V}(\boldsymbol{X},t)\circ{\boldsymbol{\varphi}_t}^{-1} \quad , \quad \boldsymbol{V}(\boldsymbol{X},t) = \left.\frac{\partial\boldsymbol{\varphi}_t(\boldsymbol{X})}{\partial t}\right|_{\boldsymbol{X}=\mathrm{const.}} \quad , \quad \boldsymbol{x} = \boldsymbol{\varphi}_t(\boldsymbol{X}). \tag{11}$$

Here, the composition between two maps, f and g, is denoted as $f \circ g$. The additionally considered DOF is referred to as p. It is given in terms of the microstructure function $\tilde{\varphi}_t$. The spatial velocity $\tilde{v}$ of p is calculated by means of the time derivative of $\tilde{\varphi}_t$, i.e.,

$$\tilde{v}(\boldsymbol{x},t) = \tilde{V}(\boldsymbol{X},t)\circ{\boldsymbol{\varphi}_t}^{-1}, \quad \tilde{V}(\boldsymbol{X},t) = \left.\frac{\partial\tilde{\varphi}_t(\boldsymbol{X})}{\partial t}\right|_{\boldsymbol{X}=\mathrm{const.}} \quad , \quad p = \tilde{\varphi}_t(\boldsymbol{X}). \tag{12}$$

Energy balance Regarding Eq. (7), the considered extensions to the energy balance, based on the additional DOF, are given by

$$\kappa = \frac{1}{2}\tilde{A}\tilde{v}^2, \quad \beta = \tilde{b}\tilde{v}\,, \quad s = \tilde{t}\tilde{v}\,. \tag{13}$$

Here, $\tilde{b}$ denotes a generalized micro-body force and $\tilde{t}$ a generalized micro-traction. The micro-inertia is referred to as $\tilde{A}$. Thus, the extended energy balance Eq. (7) can be written in the form

$$\begin{aligned} \epsilon = \int_{\mathcal{V}_t} (\dot{\rho} + \rho\mathrm{div}\,(\boldsymbol{v}))\left(e + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + \frac{1}{2}\tilde{A}\tilde{v}^2\right) + \rho\dot{e} + \rho\,(\boldsymbol{a}-\boldsymbol{b})\cdot\boldsymbol{v} \\ + \rho\left(\tilde{A}\tilde{a} - \tilde{b}\right)\tilde{v} + \frac{1}{2}\dot{\tilde{A}}\tilde{v}^2 - \rho r\,\mathrm{dv} - \int_{\partial\mathcal{V}_t} \boldsymbol{t}\cdot\boldsymbol{v} + \tilde{t}\tilde{v} + h\,\mathrm{da} = 0. \end{aligned} \tag{14}$$

As discussed in Sect. 2.1, it is assumed that the energy balance is invariant with respect to a change of observer, described by Eq. (2). The additionally considered DOF is scalar-valued and, thus, not affected by a change of observer. The energy balance described by the observer of the vector space $\mathcal{W}$ is denoted by ϵ. Consequently, the observer of the vector space $\mathcal{W}'$ refers to the energy balance as ϵ'. To obtain the balance laws, the difference

$$\Delta\epsilon_{t_0} = \left(\epsilon - \epsilon'\right)\big|_{t=t_0} \tag{15}$$

is introduced and evaluated. The considered calculations are closely related to the discussion given in Marsden and Hughes [43, pp. 145-149]. Instead of the Euclidean transformation according to Eq. (2), it is also possible to consider a pure translational transformation first and a pure rotational one subsequently.

3.2 Translational transformation

Applied transformations A pure translational transformation implies that both observers exhibit the same orientation, i.e., $\boldsymbol{Q} = \boldsymbol{e}_i' \otimes \boldsymbol{e}_i$. Consequently, the Euclidean transformation according to Eq. (2) is simplified. The relations between $\boldsymbol{x}$ and $\boldsymbol{x}'$, $\boldsymbol{v}$ and $\boldsymbol{v}'$, as well as $\boldsymbol{a}$ and $\boldsymbol{a}'$ are given by

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{c}', \quad \boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{w}, \quad \boldsymbol{a}' = \boldsymbol{a} + \dot{\boldsymbol{w}} \tag{16}$$

with $\boldsymbol{w} = \dot{\boldsymbol{c}}'$. All scalar quantities, including the contributions associated with the additional DOF, are invariant under the considered transformations. Moreover, the surface traction $\boldsymbol{t}$ remains unchanged under the translational transformation. Regarding the body force $\boldsymbol{b}'$, an additional contribution associated with the acceleration $\ddot{\boldsymbol{c}}'$ of the relative translation has to be taken into account. This contribution is referred to as fictitious body force, cf. Marsden and Hughes [43, p. 147]. Consequently, the transformation

$$\boldsymbol{a}' - \boldsymbol{b}' = \boldsymbol{a} - \boldsymbol{b} \tag{17}$$

holds true. Alternatively, the requirement of $\ddot{\boldsymbol{c}}' = \boldsymbol{0}$ also ensures the validity of Eq. (17), cf. Yavari and Marsden [74, p. 9] and Marsden and Hughes [43, p. 146]. As a consequence of the transformations according to Eq. (16), the kinetic contribution to the energy $\boldsymbol{v} \cdot \boldsymbol{v}/2$ is not invariant under the translational transformation. It contains additional terms that are linear and quadratic in $\dot{\boldsymbol{c}}'$. The arbitrariness of $\boldsymbol{c}'$ and, thus, $\dot{\boldsymbol{c}}'$ is used to derive the balance of mass, subsequently. This is an essential aspect for further invariance considerations of the extended energy balance.

Existence of the Cauchy stress tensor Evaluation of Eq. (15) yields

$$\Delta\epsilon_{t_0} = \int_{\mathcal{V}_t} (\dot{\rho} + \rho \operatorname{div}(\boldsymbol{v})) \left(\boldsymbol{v} \cdot \boldsymbol{w} + \frac{1}{2} \boldsymbol{w} \cdot \boldsymbol{w} \right) + \rho\, (\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{w} \,\mathrm{d}v - \int_{\partial\mathcal{V}_t} \boldsymbol{t} \cdot \boldsymbol{w} \,\mathrm{d}a = 0. \tag{18}$$

Application of Eq. (18) to an infinitesimal tetrahedron yields the existence of the Cauchy stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} \boldsymbol{n} = \boldsymbol{t}, \tag{19}$$

cf. Bertram [5, p. 138].

Conservation of mass Accounting for Eq. (19), Eq. (18) can be formulated as

$$\begin{aligned} \Delta\epsilon_{t_0} = \int_{\mathcal{V}_t} (\dot{\rho} + \rho \operatorname{div}(\boldsymbol{v})) \left(\boldsymbol{v} \cdot \boldsymbol{w} + \frac{1}{2} \boldsymbol{w} \cdot \boldsymbol{w} \right) \\ + (\rho\, (\boldsymbol{a} - \boldsymbol{b}) - \operatorname{div}(\boldsymbol{\sigma})) \cdot \boldsymbol{w} - \boldsymbol{\sigma} \cdot \operatorname{grad}(\boldsymbol{w}) \,\mathrm{d}v = 0. \end{aligned} \tag{20}$$

The vector field $\boldsymbol{w}$ is arbitrary and given by $\boldsymbol{w} = \dot{\boldsymbol{c}}'$. Thus, $\boldsymbol{w} = \lambda \boldsymbol{u}$ is considered, with the constant unit vector $\boldsymbol{u}$ and $\lambda \neq \lambda(\boldsymbol{x})$, cf. Marsden and Hughes [43, p. 148]. This leads to

$$\begin{aligned} \Delta\epsilon_{t_0} = \int_{\mathcal{V}_t} (\dot{\rho} + \rho \operatorname{div}(\boldsymbol{v})) \left(\lambda \boldsymbol{v} \cdot \boldsymbol{u} + \frac{1}{2} \lambda^2 \boldsymbol{u} \cdot \boldsymbol{u} \right) \\ + \lambda\, (\rho\, (\boldsymbol{a} - \boldsymbol{b}) - \operatorname{div}(\boldsymbol{\sigma})) \cdot \boldsymbol{u} \,\mathrm{d}v = 0. \end{aligned} \tag{21}$$

Differentiating Eq. (21) twice with respect to λ yields

$$\frac{d^2 \Delta\epsilon_{t_0}}{d\lambda^2} = \int_{\mathcal{V}_t} (\dot{\rho} + \rho \operatorname{div}(\boldsymbol{v}))\, (\boldsymbol{u} \cdot \boldsymbol{u}) \,\mathrm{d}v = 0. \tag{22}$$

Since $\boldsymbol{u}$ is a constant unit vector, i.e., $\boldsymbol{u} \cdot \boldsymbol{u} = 1$, conservation of mass is obtained in its local form, reading

$$\dot{\rho} + \rho \operatorname{div}(\boldsymbol{v}) = 0. \tag{23}$$

Balance of linear momentum Accounting for conservation of mass, cf. Eq. (23), Eq. (21) can be formulated as

$$\Delta\epsilon_{t_0} = \int_{\mathcal{V}_t} \lambda\, (\rho\, (\boldsymbol{a} - \boldsymbol{b}) - \operatorname{div}(\boldsymbol{\sigma})) \cdot \boldsymbol{u} \,\mathrm{d}v = 0. \tag{24}$$

Since λ and $\boldsymbol{u}$ are arbitrary, localization of Eq. (24) yields the local form of the balance of linear momentum as

$$\rho\, (\boldsymbol{a} - \boldsymbol{b}) - \operatorname{div}(\boldsymbol{\sigma}) = 0. \tag{25}$$

3.3 Rotational transformation

Applied transformations Considering a pure rotational transformation implies that both observers share the same origin, i.e., $\boldsymbol{c}' = \boldsymbol{0}$. Consequently, the relation between $\boldsymbol{x}$ and $\boldsymbol{x}'$, respectively, and between $\boldsymbol{v}$ and $\boldsymbol{v}'$ is given by

$$\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x}, \quad \boldsymbol{v}' = \boldsymbol{Q}\boldsymbol{v} + \boldsymbol{w}, \tag{26}$$

with $\boldsymbol{w} = \dot{\boldsymbol{Q}}\boldsymbol{x}$. The traction force is assumed to transform objectively, reading

$$\boldsymbol{t}' = \boldsymbol{Q}\boldsymbol{t}. \tag{27}$$

Contrarily, the acceleration $\boldsymbol{a}$ does not transform objectively, which is a consequence of the transformation law for $\boldsymbol{v}'$, cf. Eq. (26). Additional contributions associated with the centripetal and Coriolis forces are added. Taking into account these fictitious forces, the difference between body force and acceleration is assumed to transform objectively

$$\left(\boldsymbol{a}' - \boldsymbol{b}'\right) = \boldsymbol{Q}\left(\boldsymbol{a} - \boldsymbol{b}\right), \tag{28}$$

cf. Marsden and Hughes [43]. It is assumed that $\boldsymbol{Q}(t_0) = \boldsymbol{I}$ holds true. According to the assumptions of the previous section, all scalar quantities are considered invariant with respect to the applied transformations. This implies that p, $\tilde{t}$, $\tilde{b}$ and $\tilde{v}$ are invariant with respect to the rotational transformations.

Conservation of micro-inertia Evaluation of Eq. (15) leads to a form that does not contain any microstructural quantities associated with the additional DOFs, reading

$$\Delta\epsilon_{t_0} = \int_{\mathcal{V}_t} -\boldsymbol{\sigma} \cdot \operatorname{grad}\left(\boldsymbol{w}\right) \, \mathrm{d}v = 0. \tag{29}$$

Thus, the existence of a micro-inertia conservation cannot be proved based on invariance considerations regarding a change of observer. The same holds true if invariance with respect to a superimposed rigid-body motion is considered. This is different from the consideration of an additional vectorial DOF, cf. Eq. (74).

Balance of angular momentum Substituting $\boldsymbol{w}$ according to Eq. (26) into Eq. (29) yields

$$\Delta\epsilon_{t_0} = \int_{\mathcal{V}_t} -\boldsymbol{\sigma} \cdot \dot{\boldsymbol{Q}} \, \mathrm{d}v = 0. \tag{30}$$

Since $\dot{\boldsymbol{Q}}(t_0) \in Skw$, the localization of Eq. (30) yields the standard balance of angular momentum, as given in Eq. (4), reading

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}. \tag{31}$$

Existence of micro-stress vector Accounting for the results in Eqs. (19), (23) and (25), Eq. (14) reads

$$\int_{\mathcal{V}_t} \rho\dot{e} + \rho\left(\tilde{A}\tilde{a} - \tilde{b}\right)\tilde{v} + \frac{1}{2}\dot{\tilde{A}}\tilde{v}^2 - \rho r - \boldsymbol{\sigma} \cdot \operatorname{grad}\left(\boldsymbol{v}\right) \, \mathrm{d}v - \int_{\partial\mathcal{V}_t} \tilde{t}\tilde{v} + h \, \mathrm{d}a = 0. \tag{32}$$

Application of Eq. (32) to an infinitesimal tetrahedron yields

$$\int_{\partial\mathcal{V}_t} \tilde{t}\tilde{v} + h \, \mathrm{d}a = 0. \tag{33}$$

The integrand of the surface integral consists of the contribution due to the micro-traction $\tilde{t}$ and the heat flux h. The existence of a flux term $\boldsymbol{k}(\boldsymbol{x}, t)$ can be proven for which $\boldsymbol{k}(\boldsymbol{x}, t) \cdot \boldsymbol{n} = \tilde{t}(\boldsymbol{x}, t, \boldsymbol{n})\tilde{v}(\boldsymbol{x}, t) + h(\boldsymbol{x}, t, \boldsymbol{n})$ holds, cf. Marsden and Hughes [43, p. 127]. Both, $\tilde{t}$ and h do not depend on $\tilde{v}$. Thus, the only possible choice for $\boldsymbol{k}(\boldsymbol{x}, t)$ that provides the integrand of Eq. (33) is given by

$$\boldsymbol{k}(\boldsymbol{x}, t) = \boldsymbol{\xi}\tilde{v} + \boldsymbol{h}, \quad \text{with } \tilde{t} = \boldsymbol{\xi} \cdot \boldsymbol{n} \quad \text{and } h = \boldsymbol{h} \cdot \boldsymbol{n}. \tag{34}$$

Consequently, the existence of the micro-stress vector $\boldsymbol{\xi}$ is shown unambiguously. This is different from the treatment of extended continua by an extended principle of virtual power. In this context, the existence of both the Cauchy stress tensor and the micro-stress vector is presumed from the outset, cf. the remark on this topic in the review paper of Mariano [42, p. 14].

Existence of heat flux vector Regarding Eq. (34), the heat flux vector $\boldsymbol{q}$ is introduced such that

$$\boldsymbol{q} = -\boldsymbol{h}, \quad \text{with } \boldsymbol{q} \cdot \boldsymbol{n} = -h, \tag{35}$$

cf. Marsden and Hughes [43, p. 148].

Balance of internal energy Considering the results in Eqs. (19), (23), (25), (31) and (34), localization of Eq. (14) yields the local form of the balance of internal energy, reading

$$\rho \dot{e} + \rho \left(\tilde{A}\tilde{a} - \tilde{b} \right) \tilde{v} + \frac{1}{2}\dot{\tilde{A}}\tilde{v}^2 - \rho r - \boldsymbol{\sigma} \cdot \boldsymbol{D} - \boldsymbol{\xi} \cdot \operatorname{grad}(\tilde{v}) - \operatorname{div}(\boldsymbol{\xi})\, \tilde{v} + \operatorname{div}(\boldsymbol{q}) = 0. \tag{36}$$

Simplifying assumptions Additionally considered DOFs are commonly used to describe the evolution of the underlying microstructure. Nonlocal damage [29], nonlocal diffusion [70] and nonlocal plasticity [72] are prominent examples for the application of extended continua. In this context, effects due to micro-inertia and micro-body forces are usually neglected, i.e., $\tilde{A} = 0$ and $\tilde{b} = 0$. Only micro-traction forces are considered. Thus, the balance of internal energy Eq. (36) reads

$$\rho \dot{e} - \rho r - \boldsymbol{\sigma} \cdot \boldsymbol{D} - \boldsymbol{\xi} \cdot \operatorname{grad}(\tilde{v}) - \operatorname{div}(\boldsymbol{\xi})\, \tilde{v} + \operatorname{div}(\boldsymbol{q}) = 0. \tag{37}$$

3.4 Nonlocal evolution equation for an additional DOF

Exploitation of the Clausius–Duhem inequality To discuss the evolution equation for the additional DOF, the simplified balance of internal energy according to Eq. (37) is considered, subsequently. Moreover, a small strain framework is considered for brevity, i.e., $\boldsymbol{D} = \dot{\boldsymbol{\varepsilon}}$ holds. Consequently, the Clausius–Duhem inequality according to Eq. (10) is given by

$$\rho\delta = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\xi} \cdot \nabla \dot{p} + \operatorname{div}(\boldsymbol{\xi})\, \dot{p} - \rho\dot{\psi} - \rho\dot{\theta}\eta - \frac{1}{\theta}\boldsymbol{q} \cdot \boldsymbol{g} \geq 0, \tag{38}$$

where $\dot{p} = \tilde{v}$ and $\nabla \dot{p} = \operatorname{grad}(\dot{p})$ is used. An additive split of the infinitesimal strain $\boldsymbol{\varepsilon}$ into a purely elastic part $\boldsymbol{\varepsilon}^{\mathrm{e}}$ and a part $\boldsymbol{\varepsilon}^{\mathrm{p}}(p)$ related to the additional DOF p is assumed, i.e., $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}}$ holds true. The free energy density is assumed to depend on $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\mathrm{p}}, p, \nabla p, \theta$, i.e.,

$$\psi = \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\mathrm{p}}, p, \nabla p, \theta) \tag{39}$$

holds true. It is assumed that the elastic properties are not affected by $\boldsymbol{\varepsilon}^{\mathrm{p}}$ during the deformation process, similar to Bertram and Krawietz [7, p. 2262]. This motivates that ψ only depends on the elastic strain $\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}$. Furthermore, for simplicity, it is assumed that the free energy density ψ can be additively decomposed into an elastic contribution ψ_{e}, a contribution ψ_{p} that depends on the additional DOF p, a gradient contribution ψ_{g} that accounts for the effects of the gradient of the additional DOF, and a thermal contribution ψ_θ, i.e.,

$$\psi(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}, p, \nabla p, \theta) = \psi_{\mathrm{e}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}) + \psi_{\mathrm{p}}(p) + \psi_{\mathrm{g}}(\nabla p) + \psi_\theta(\theta). \tag{40}$$

Naturally, this assumed split of the free energy density represents a special case [7]. Regarding rate-dependent material behavior, the Clausius–Duhem inequality, cf. Eq. (38), reads

$$\begin{aligned} \rho\delta = {} & \left(\boldsymbol{\sigma} - \rho\frac{\partial\psi_{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}} \right) \cdot \dot{\boldsymbol{\varepsilon}} - \rho\frac{\partial\psi_{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}} \cdot \frac{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}{\partial p}\dot{p} - \rho\left(\eta + \frac{\partial\psi_\theta}{\partial\theta} \right)\dot{\theta} - \boldsymbol{q} \cdot \boldsymbol{g}/\theta \\ & + \left(\operatorname{div}(\boldsymbol{\xi}) - \rho\frac{\partial\psi_{\mathrm{p}}}{\partial p} \right)\dot{p} + \left(\boldsymbol{\xi} - \rho\frac{\partial\psi_{\mathrm{g}}}{\partial\nabla p} \right) \cdot \nabla\dot{p} \geq 0. \end{aligned} \tag{41}$$

The standard procedure of Coleman and Noll is applied [12]. It is assumed that the micro-stress $\boldsymbol{\xi}$ is purely energetic. This yields the potential relations for the Cauchy stress, the entropy and the generalized stress

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi_{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}}, \quad \eta = -\frac{\partial\psi_\theta}{\partial\theta}, \quad \boldsymbol{\xi} = \rho\frac{\partial\psi_{\mathrm{g}}}{\partial\nabla p}. \tag{42}$$

Thus, the reduced dissipation inequality is given by

$$\left(\mathrm{div}\,(\boldsymbol{\xi}) - \rho\frac{\partial\psi_{\mathrm{p}}}{\partial p} - \rho\frac{\partial\psi_{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}\cdot\frac{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}{\partial p}\right)\dot{p} - \boldsymbol{q}\cdot\boldsymbol{g}/\theta \geq 0. \tag{43}$$

While the first term of Eq. (43) refers to the mechanical dissipation, the thermal dissipation is represented by the second expression.

Evolution equation Subsequently, no coupling is assumed between the mechanical and the thermal dissipation . Thus, Fourier's law [6] ensures the positivity of the second term in Eq. (43). Linear irreversible thermodynamics yields an admissible choice for $\dot{p}$, consistent with the reduced dissipation inequality, reading

$$\dot{p} = \dot{p}_0\left(\mathrm{div}\,(\boldsymbol{\xi}) - \rho\frac{\partial\psi_{\mathrm{p}}}{\partial p} - \rho\frac{\partial\psi_{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}\cdot\frac{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}{\partial p}\right), \quad \text{with } \dot{p}_0 \geq 0. \tag{44}$$

Here, $\dot{p}_0$ denotes a referential rate. Equation (44) constitutes a nonlocal evolution equation for the additionally considered DOF. Partitioning of Eq. (44) leads to a partial differential equation (PDE) and a local evolution equation, given by

$$\pi - \mathrm{div}\,(\boldsymbol{\xi}) = 0, \quad \dot{p} = \dot{p}_0\left(\pi - \rho\frac{\partial\psi_{\mathrm{p}}}{\partial p} - \rho\frac{\partial\psi_{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}\cdot\frac{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}{\partial p}\right). \tag{45}$$

For vanishing rates, i.e., $\dot{p} = 0$, the nonlocal evolution equation according to Eq. (44) reduces to a partial differential equation, reading

$$\mathrm{div}\,(\boldsymbol{\xi}) - \rho\frac{\partial\psi_{\mathrm{p}}}{\partial p} - \rho\frac{\partial\psi_{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}\cdot\frac{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}{\partial p} = 0. \tag{46}$$

Equation (46) characterizes the distribution of the additional DOF p in thermodynamical equilibrium. It is exploited in the context of a slip gradient crystal plasticity theory in the subsequent section.

3.5 Connection to an extended principle of virtual power

Weak forms Subsequently, the connection of the presented framework to an extended principle of virtual power is discussed. To this end, the weak forms of the PDE according to Eq. $(45)_1$ and the balance of linear momentum Eq. (25) are provided, first. Multiplication of Eq. $(45)_1$ with a test function f, integration over $\mathcal{V}_t$ and application of the divergence theorem yield the corresponding weak form

$$-\int_{\mathcal{V}_t} \pi f + \boldsymbol{\xi}\cdot\mathrm{grad}\,(f)\;\mathrm{d}v + \int_{\partial\mathcal{V}_t} \tilde{t} f\;\mathrm{d}a = 0, \tag{47}$$

with $\tilde{t} = \boldsymbol{\xi}\cdot\boldsymbol{n}$. Moreover, the weak form of the balance of linear momentum in Eq. (25) is given by,

$$\int_{\mathcal{V}_t} \rho\,(\boldsymbol{a} - \boldsymbol{b})\cdot\boldsymbol{f} + \boldsymbol{\sigma}\cdot\mathrm{grad}\,(\boldsymbol{f})\;\mathrm{d}v - \int_{\partial\mathcal{V}_t} \boldsymbol{t}\cdot\boldsymbol{f}\;\mathrm{d}a = 0, \tag{48}$$

where $\boldsymbol{f}$ is the vectorial test function.

Extended principle of virtual power Subsequently, the quasi-static case is considered and body forces are neglected, i.e., $\boldsymbol{a} = \boldsymbol{0}$ and $\boldsymbol{b} = \boldsymbol{0}$. Replacing the test functions f and $\boldsymbol{f}$ by the virtual rates $\delta\tilde{v}$ and $\delta\boldsymbol{v}$, respectively, the sum of Eq. (47) and Eq. (48) yields $\delta\mathcal{P}_{\mathrm{int}} = \delta\mathcal{P}_{\mathrm{ext}}$, with

$$\begin{aligned}\delta\mathcal{P}_{\mathrm{int}} &= \int_{\mathcal{V}_t} \pi\,\delta\tilde{v} + \boldsymbol{\xi}\cdot\mathrm{grad}\,(\delta\tilde{v}) + \boldsymbol{\sigma}\cdot\mathrm{grad}\,(\delta\boldsymbol{v})\;\mathrm{d}v,\\ \delta\mathcal{P}_{\mathrm{ext}} &= \int_{\partial\mathcal{V}_t} \tilde{t}\,\delta\tilde{v} + \boldsymbol{t}\cdot\delta\boldsymbol{v}\;\mathrm{d}a\end{aligned} \tag{49}$$

denoting the internal and external virtual power, respectively. This extended principle of virtual power is equivalent to the extended energy balance and the partitioned flow rule in Eq. (45). The exploitation of an extended principle of virtual power is widely used to derive additional field equations regarding extended continua [24]. The extended principle of virtual power using Eq. (49) is structurally equivalent to Wulfinghoff et al. [72, Eqs. (3) and (4)], Bayerschen and Böhlke [2, Eqs. (3) and (4)], similar to, e.g., Cermelli and Gurtin [10, Eq. (3.2)], Gurtin et al. [36, Eq. (3.2)]. In this context, Eq. $(45)_1$ is referred to as additional balance equation. It is denoted as *micro-force balance*; however, the notion of a balance is misleading according to the previous discussions. It is rather part of a partitioned, time-dependent partial differential equation.

4 Application to slip gradient crystal plasticity

4.1 Single slip

Plastic slip as additional degree of freedom In the context of a slip gradient crystal plasticity theory, the additional DOF p is considered as the plastic slip within a slip system. Consequently, the contribution $\boldsymbol{\varepsilon}^{\mathrm{p}}$ denotes the plastic part of the strain tensor. Regarding single slip, $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{p}\boldsymbol{M}$ is assumed with the corresponding Schmid tensor $\boldsymbol{M} = \mathrm{sym}(\boldsymbol{d} \otimes \boldsymbol{n})$. Here, $\boldsymbol{n}$ denotes the normal of the slip system and $\boldsymbol{d}$ the slip direction. While the defect energy density ψ_{g} is related to hardening on the basis of geometrically necessary dislocations, the hardening contribution ψ_{p} accounts for isotropic hardening due to statistically stored dislocations [3] in the context of monotonic loadings.

Free energy and material parameters The contributions to the free energy density are given by

$$\begin{aligned} \psi_{\mathrm{e}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}) &= \frac{1}{2\rho}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}\right) \cdot \left(\mathbb{C}\left[\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}\right]\right), \\ \psi_{\mathrm{g}}(\nabla_{\mathrm{n}} p) &= \frac{1}{2\rho} K_{\mathrm{g}} \nabla_{\mathrm{n}} p \cdot \nabla_{\mathrm{n}} p, \\ \psi_{\mathrm{p}}(p) &= \frac{1}{2\rho}\Theta_0 p^2. \end{aligned} \tag{50}$$

The planar gradient with respect to the considered slip system is denoted as ∇_{n}. Accounting for the lattice stretch and rotation, the elastic contribution ψ_{e} is assumed to be quadratic in the difference of the total strain $\boldsymbol{\varepsilon}$ and the plastic strain $\boldsymbol{\varepsilon}^{\mathrm{p}}$. Subsequently, an analytical solution of Eq. (46) is discussed. Thus, it is convenient to assume the defect contribution ψ_{g} as well as the hardening contribution ψ_{p} to be quadratic. Regarding a numerical implementation of the presented theory, more complex contributions are feasible such as Voce-hardening [72] or latent hardening [52] or a power-law defect energy [1,2]. Thermal effects are neglected, i.e., the thermal contribution $\psi_\theta(\theta)$ vanishes. While $\mathbb{C}$ denotes the elastic stiffness tensor of fourth order, K_{g} denotes the defect parameter, introducing an internal length scale to the model [72]. The initial hardening modulus is referred to as Θ_0 and the density of mass as ρ. Subsequently, it is assumed that the stiffness tensor, the defect parameter and the initial hardening modulus are constant parameters. Assuming the elastic material behavior to be isotropic, the elastic constants are chosen to be $G = 27$ GPa and $\nu = 0.347$, representing the elastic behavior of aluminum. Moreover, the defect parameter and the initial hardening modulus are chosen as $K_{\mathrm{g}} = 84$ µN and $\Theta_0 = 1075$ MPa, respectively. The material parameters are in line with [3]. Under consideration of the contributions to the free energy density, cf. Eq. (50), Eq. (46) reads

$$\triangle_{\mathrm{n}} p - p\frac{\Theta_0}{K_{\mathrm{g}}} = -\frac{\tau}{K_{\mathrm{g}}}, \tag{51}$$

where $\tau = \boldsymbol{\sigma} \cdot \boldsymbol{M}$ is the resolved shear stress. Here, $\triangle_{\mathrm{n}}$ denotes the Laplacian that uses the planar gradient.

4.2 Analytic solution for a laminate material

Simplifications An analytical, one-dimensional solution of Eq. (51) is discussed in the context of a laminate material consisting of two phases. The material behavior of one phase is assumed to be purely elastic. Contrarily, the material behavior of the second phase is considered to be elastoplastic. The plastic behavior is characterized by an individual slip system of a face-centered cubic (FCC) single crystal. A schematic illustration of the considered laminate is given in Fig. 1. The elastic phase is illustrated in dark gray, the elastoplastic phase in light gray. The normal and slip direction of the considered slip system are given by $\boldsymbol{n} = \boldsymbol{e}_2$ and $\boldsymbol{d} = \boldsymbol{e}_1$. The coordinate system is located within the middle of the elastoplastic phase. While the elastoplastic phase has a width of $2h$, the width of the elastic phase is s.

Kinematics Subsequently, the following ansatz for the displacement field $\boldsymbol{u}(\boldsymbol{x})$ is considered

$$\boldsymbol{u} = \bar{\gamma} x_2 \boldsymbol{e}_1 + \tilde{u}(x_1)\boldsymbol{e}_2, \tag{52}$$

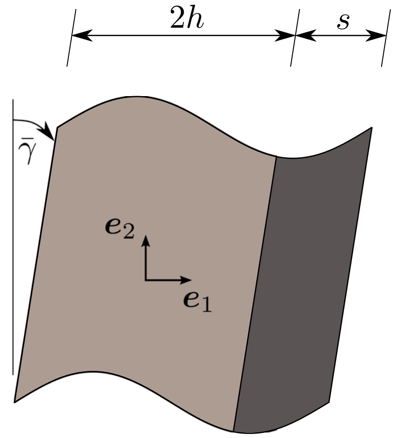

Fig. 1 Considered laminate material subjected to shear in $\boldsymbol{e}_1$- and periodic fluctuation in $\boldsymbol{e}_2$-direction. The elastoplastic phase is illustrated in light gray, the elastic phase in dark gray. The coordinate system is located in the center of the elastoplastic phase

cf. Forest [25], Forest and Guéninchault [26], Wulfinghoff et al. [73]. Here, $\bar{\gamma}$ denotes the constant macroscopic shear and $\tilde{u}(x_1)$ a periodic fluctuation. The corresponding infinitesimal strain tensor and the plastic strain tensor are given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\bar{\gamma} + \tilde{u}'(x_1)\right)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)\,, \quad \boldsymbol{\varepsilon}^{\mathrm{p}} = \frac{1}{2} p(x_1)\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)\,. \tag{53}$$

Based on the applied deformation, the plastic slip depends purely on x_1. The derivative of a quantity with respect to x_1 is denoted by $(\cdot)'$. The elastic shear $\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}$ is obtained by the assumed additive decomposition of the infinitesimal strain tensor in an elastic and a plastic contribution. For brevity, an isotropic elastic behavior is assumed in the following. Hooke's law for linear elasticity yields the corresponding Cauchy stress $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = G\left(\bar{\gamma} + \tilde{u}'(x_1) - p(x_1)\right)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)\,, \tag{54}$$

with the shear modulus G. The balance of linear momentum Eq. (25) yields the differential equation

$$\tilde{u}''(x_1) = p'(x_1). \tag{55}$$

Regarding Eq. (51), the Laplacian $\triangle_{\mathrm{n}} p$ can be replaced by $p''(x_1)$. Moreover, the resolved shear stress $\tau = G\left(\bar{\gamma} + \tilde{u}'(x_1) - p(x_1)\right)$ can be reformulated by means of Eq. (55). Thus, Eq. (51) reads

$$p'' - \frac{\Theta_0}{K_{\mathrm{g}}} p = -\frac{\sigma_0}{K_{\mathrm{g}}}, \qquad \sigma_0 = G\,(\bar{\gamma} + c)\,, \tag{56}$$

where c denotes the integration constant if Eq. (55) is integrated once with respect to x_1. At the boundaries between both phases, the plastic slip vanishes, i.e., $p\,(-h) = 0$ and $p\,(h) = 0$ hold true. The function $\tilde{u}(x_1)$ is considered to be a periodic fluctuation. Thus,

$$\int_{-h}^{h+s} \tilde{u}(x_1)\,\mathrm{d}x_1 = 0, \qquad \int_{-h}^{h+s} \tilde{u}'(x_1)\,\mathrm{d}x_1 = 0 \tag{57}$$

hold true, cf. Wulfinghoff et al. [73, Eq. (27)]. These conditions are used to determine the integration constants that arise in the context of Eq. (55). Finally, solving Eq. (56) closes the ansatz for the displacement field, given by Eq. (52).

Solution The ordinary, linear differential equation, cf. Eq. (56), can be solved analytically. The solution is given in dependency of σ_0 and reads

$$p(x_1) = -\frac{\sigma_0}{\Theta_0}\left(\mathrm{e}^{\sqrt{\frac{\Theta_0}{K_{\mathrm{g}}}}x_1} + \mathrm{e}^{-\sqrt{\frac{\Theta_0}{K_{\mathrm{g}}}}x_1}\right)\left(\mathrm{e}^{\sqrt{\frac{\Theta_0}{K_{\mathrm{g}}}}h} + \mathrm{e}^{-\sqrt{\frac{\Theta_0}{K_{\mathrm{g}}}}h}\right)^{-1} + \frac{\sigma_0}{\Theta_0}. \tag{58}$$

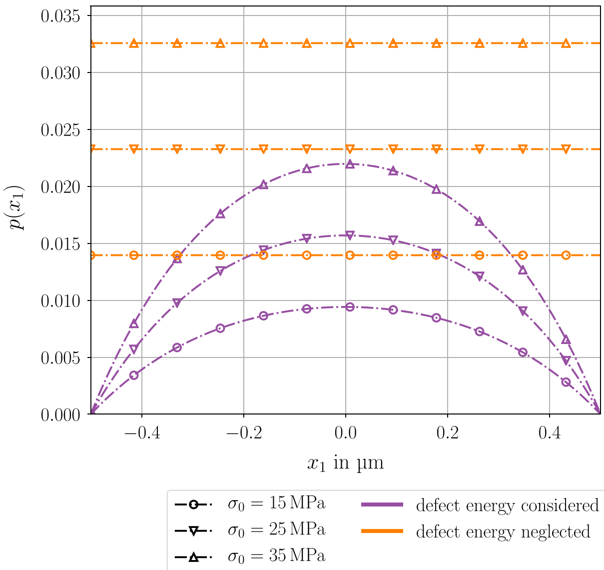

Fig. 2 Parabolic distribution of the plastic slip $p(x_1)$ as a consequence of the quadratic defect energy. Three different amplitudes σ_0 are considered. If the defect energy is neglected, the solution for the plastic slip is homogeneous

For brevity, the integration constants to determine $\tilde{u}(x_1)$ are not given explicitly, here. Regarding a sequence of equilibrium states, the distribution of the plastic slip $p(x_1)$ is depicted in Fig. 2. The amplitude σ_0 is chosen as 15 MPa, 25 MPa and 35 MPa, respectively. For the illustration, h is chosen as 0.5µm. The solution according to Eq. (58) is shown in purple. Due to the quadratic defect energy, a parabolic distribution of the plastic slip is obtained. The analytical solution is qualitatively in line with the numerical results presented in [3]. Neglecting the contribution due to the defect energy leads to a constant distribution of the plastic slip. This solution represents the classical distribution of the plastic slip without gradient effects and is illustrated in Fig. 2 in orange. The absolute value of the classical distribution of the plastic slip is significantly higher compared to the parabolic distribution.

5 Concluding remarks

The consideration of an additional, scalar-valued DOF does not provide an associated, additional force balance by means of the discussed invariance considerations. Moreover, the existence of a conservation law for micro-inertia cannot be obtained. However, the unambiguous existence of the Cauchy stress tensor, the micro-stress vector and the heat flux vector can be shown. This is in direct contrast to a corresponding extended principle of virtual power, which assumes the existence of stress quantities from the outset. The conservation of mass as well as the balance of linear momentum is obtained in the same form as for a classical continuum. In addition, the Cauchy stress tensor remains symmetric. The exploitation of the Clausius–Duhem inequality leads to a thermodynamically consistent, nonlocal flow rule for the scalar-valued DOF. Partitioning of the obtained flow rule yields the so-called micro-force balance as a constitutive equation. Moreover, the equivalence between the extended energy balance and an extended principle of virtual power is shown for a scalar-valued DOF neglecting micro-inertia and micro-body forces. In this context, it is outlined that the notion of balance is misleading with respect to the micro-force balance. A slip gradient crystal plasticity theory is considered as application of the approach presented. For brevity, considerations are limited to small deformations and single slip. Based on a quadratic defect energy, a parabolic distribution of the plastic slip is obtained by analytical means. The analytical results are in good agreement with numerical experiments.

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Appendix: Additional vector-valued DOF

Extended energy balance

Subsequently, the additionally considered DOF is referred to as $\boldsymbol{p}$ and defined on the tangent space of the Euclidean ambient space. It is given in terms of the microstructure function $\tilde{\boldsymbol{\varphi}}_t$. The spatial velocity $\tilde{\boldsymbol{v}}$ of $\boldsymbol{p}$ is calculated by means of the time derivative of $\tilde{\boldsymbol{\varphi}}_t$, i.e.,

$$\tilde{\boldsymbol{v}}(\boldsymbol{x}, t) = \tilde{\boldsymbol{V}}(\boldsymbol{X}, t) \circ \boldsymbol{\varphi}_t{}^{-1}, \quad \tilde{\boldsymbol{V}}(\boldsymbol{X}, t) = \left.\frac{\partial \tilde{\boldsymbol{\varphi}}_t(\boldsymbol{X})}{\partial t}\right|_{\boldsymbol{X}=\text{const.}}, \quad \boldsymbol{p} = \tilde{\boldsymbol{\varphi}}_t(\boldsymbol{X}). \tag{59}$$

Regarding Eq. (7), the considered extensions to the energy balance, based on the vectorial DOF, are given by

$$\kappa = \frac{1}{2}\tilde{A}\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{v}}, \quad \beta = \tilde{\boldsymbol{b}} \cdot \tilde{\boldsymbol{v}}\,, \quad s = \tilde{\boldsymbol{t}} \cdot \tilde{\boldsymbol{v}}\,. \tag{60}$$

Here, $\tilde{\boldsymbol{b}}$ denotes a generalized micro-body force and $\tilde{\boldsymbol{t}}$ a generalized micro-traction. The micro-inertia is referred to as $\tilde{A}$. Thus, the extended energy balance Eq. (7) can be written in the form

$$\begin{aligned}\epsilon = \int_{\mathcal{V}_t} &(\dot{\rho} + \rho \operatorname{div}(\boldsymbol{v}))\left(e + \frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v} + \frac{1}{2}\tilde{A}\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{v}}\right) + \rho\dot{e} + \rho\,(\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{v} \\ &+ \rho\left(\tilde{A}\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}}\right) \cdot \tilde{\boldsymbol{v}} + \frac{1}{2}\dot{\tilde{A}}\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{v}} - \rho r \,\mathrm{d}v - \int_{\partial\mathcal{V}_t} \boldsymbol{t} \cdot \boldsymbol{v} + \tilde{\boldsymbol{t}} \cdot \tilde{\boldsymbol{v}} + h \,\mathrm{d}a = 0.\end{aligned} \tag{61}$$

The additional vectorial DOF $\boldsymbol{p}$ is assumed to transform objectively concerning a change of observer, i.e.,

$$\boldsymbol{p}' = \boldsymbol{Q}(t)\boldsymbol{p} \tag{62}$$

holds true. Thus, $\boldsymbol{p}$ is unaffected if a pure translational transformation is considered. The implications for the spatial velocity fields are given in the first row of Table 1. The considered calculations are closely related to the discussion given in Marsden and Hughes [43, pp. 145-149]. As for the additional scalar-valued DOF, a pure translational transformation is considered first, and a pure rotational one subsequently.

Translational transformation

Since the same transformations are considered as discussed in Sect. (3.2), the additional DOF is not affected by the rigid-body translation. Thus, the existence of the Cauchy stress, cf. Eq. (19), as well as the conservation of mass, cf. Eq. (23), and the balance of linear momentum, cf. Eq. (25), are obtained, respectively.

Rotational transformation

Accounting for the results in Eqs. (19), (23) and (25), a pure rotational transformation is considered subsequently, according to the transformation laws in the third row of Table 1. In line with Eq. (27), the traction forces are assumed to transform objectively with respect to a change of observer, reading

$$\boldsymbol{t}' = \boldsymbol{Q}\boldsymbol{t}, \quad \tilde{\boldsymbol{t}}' = \boldsymbol{Q}\tilde{\boldsymbol{t}}. \tag{63}$$

Table 1 Transformations of the material points and the additionally considered vectorial DOF in the context of a change of observer

	$\boldsymbol{x}'$	$\boldsymbol{v}'$	$\boldsymbol{w}$	$\boldsymbol{p}'$	$\tilde{\boldsymbol{v}}'$	$\boldsymbol{z}$
Euclidean transformation	$\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}'$	$\boldsymbol{Q}\boldsymbol{v} + \boldsymbol{w}$	$\dot{\boldsymbol{Q}}\boldsymbol{x} + \dot{\boldsymbol{c}}'$	$\boldsymbol{Q}\boldsymbol{p}$	$\boldsymbol{Q}\tilde{\boldsymbol{v}} + \boldsymbol{z}$	$\dot{\boldsymbol{Q}}\boldsymbol{p}$
Pure translation	$\boldsymbol{x} + \boldsymbol{c}'$	$\boldsymbol{v} + \boldsymbol{w}$	$\dot{\boldsymbol{c}}'$	$\boldsymbol{p}$	$\tilde{\boldsymbol{v}}$	$\boldsymbol{0}$
Pure rotation	$\boldsymbol{Q}\boldsymbol{x}$	$\boldsymbol{Q}\boldsymbol{v} + \boldsymbol{w}$	$\dot{\boldsymbol{Q}}\boldsymbol{x}$	$\boldsymbol{Q}\boldsymbol{p}$	$\boldsymbol{Q}\tilde{\boldsymbol{v}} + \boldsymbol{z}$	$\dot{\boldsymbol{Q}}\boldsymbol{p}$

Similar to Eq. (28), the difference between body forces and accelerations is assumed to transform objectively

$$\left(\boldsymbol{a}' - \boldsymbol{b}'\right) = \boldsymbol{Q}\left(\boldsymbol{a} - \boldsymbol{b}\right), \quad \left(\tilde{\boldsymbol{a}}' - \tilde{\boldsymbol{b}}'\right) = \boldsymbol{Q}\left(\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}}\right), \tag{64}$$

cf. Marsden and Hughes [43]. As in Sect. (3.3), all scalar quantities are invariant with respect to the considered transformations. It is assumed that $\boldsymbol{Q}(t_0) = \boldsymbol{I}$ holds true.

Existence of micro-stress tensor Evaluation of Eq. (15) yields

$$\begin{aligned}\Delta\epsilon_{t_0} = &\int_{\mathcal{V}_t} -\boldsymbol{\sigma} \cdot \operatorname{grad}\left(\boldsymbol{w}\right) + \rho\left(\tilde{A}\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}}\right) \cdot \boldsymbol{z} + \rho\dot{\tilde{A}}\left(\tilde{\boldsymbol{v}} \cdot \boldsymbol{z} + \frac{1}{2}\boldsymbol{z} \cdot \boldsymbol{z}\right) \mathrm{d}v \\ &- \int_{\partial\mathcal{V}_t} \tilde{\boldsymbol{t}} \cdot \boldsymbol{z} \,\mathrm{d}a = 0.\end{aligned} \tag{65}$$

Application of Eq. (65) to an infinitesimal tetrahedron yields the existence of the micro-stress tensor $\tilde{\boldsymbol{\sigma}}$ given by

$$\tilde{\boldsymbol{\sigma}}\boldsymbol{n} = \tilde{\boldsymbol{t}}, \tag{66}$$

cf. Yavari and Marsden [74, p. 9].

Conservation of micro-inertia Accounting for Eq. (66) and applying the divergence theorem, Eq. (65) can be formulated as

$$\begin{aligned}\Delta\epsilon_{t_0} = &\int_{\mathcal{V}_t} -\boldsymbol{\sigma} \cdot \operatorname{grad}\left(\boldsymbol{w}\right) + \rho\left(\tilde{A}\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}}\right) \cdot \boldsymbol{z} + \rho\dot{\tilde{A}}\left(\tilde{\boldsymbol{v}} \cdot \boldsymbol{z} + \frac{1}{2}\boldsymbol{z} \cdot \boldsymbol{z}\right) \\ &- \operatorname{div}\left(\tilde{\boldsymbol{\sigma}}\right) \cdot \boldsymbol{z} - \tilde{\boldsymbol{\sigma}} \cdot \operatorname{grad}\left(\boldsymbol{z}\right) \,\mathrm{d}v = 0.\end{aligned} \tag{67}$$

Moreover, substituting $\boldsymbol{w}$ and $\boldsymbol{z}$ according to the third row of Table 1 yields

$$\begin{aligned}\Delta\epsilon_{t_0} = &\int_{\mathcal{V}_t} -\boldsymbol{\sigma} \cdot \operatorname{grad}\left(\dot{\boldsymbol{Q}}\boldsymbol{x}\right) + \rho\left(\tilde{A}\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}}\right) \cdot \left(\dot{\boldsymbol{Q}}\boldsymbol{p}\right) \\ &+ \rho\dot{\tilde{A}}\left(\tilde{\boldsymbol{v}} \cdot \left(\dot{\boldsymbol{Q}}\boldsymbol{p}\right) + \frac{1}{2}\left(\dot{\boldsymbol{Q}}\boldsymbol{p}\right) \cdot \left(\dot{\boldsymbol{Q}}\boldsymbol{p}\right)\right) \\ &- \operatorname{div}\left(\tilde{\boldsymbol{\sigma}}\right) \cdot \left(\dot{\boldsymbol{Q}}\boldsymbol{p}\right) - \tilde{\boldsymbol{\sigma}} \cdot \operatorname{grad}\left(\dot{\boldsymbol{Q}}\boldsymbol{p}\right) \,\mathrm{d}v = 0.\end{aligned} \tag{68}$$

Manipulations of Eq. (68) lead to

$$\begin{aligned}\Delta\epsilon_{t_0} = &\int_{\mathcal{V}_t} -\boldsymbol{\sigma} \cdot \dot{\boldsymbol{Q}} + \rho\left(\tilde{A}\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}}\right) \cdot \left(\dot{\boldsymbol{Q}}\boldsymbol{p}\right) \\ &+ \rho\dot{\tilde{A}}\left(\tilde{\boldsymbol{v}} \cdot \left(\dot{\boldsymbol{Q}}\boldsymbol{p}\right) + \frac{1}{2}\left(\dot{\boldsymbol{Q}}^{\mathsf{T}}\dot{\boldsymbol{Q}}\right) \cdot \left(\boldsymbol{p} \otimes \boldsymbol{p}\right)\right) \\ &- \operatorname{div}\left(\tilde{\boldsymbol{\sigma}}\right) \cdot \left(\dot{\boldsymbol{Q}}\boldsymbol{p}\right) - \tilde{\boldsymbol{\sigma}} \cdot \left(\dot{\boldsymbol{Q}}\operatorname{grad}\left(\boldsymbol{p}\right)\right) \,\mathrm{d}v = 0.\end{aligned} \tag{69}$$

In general, a rotation tensor can be expressed by means of its rotation axis $\boldsymbol{n}$ and its rotation angle θ, cf. [63], reading

$$\boldsymbol{Q} = \boldsymbol{n} \otimes \boldsymbol{n} + \cos\theta\left(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}\right) - \sin\theta\boldsymbol{\epsilon}[\boldsymbol{n}]. \tag{70}$$

If the rotation axis is considered as a constant unit vector, the time dependency of $\boldsymbol{Q}$ is due to the rotation angle $\theta = \theta(t)$. Thus, the time derivative of the rotation tensor is given by

$$\dot{\boldsymbol{Q}}(t) = -\left(\sin\theta\right)\dot{\theta}\boldsymbol{A} - \left(\cos\theta\right)\dot{\theta}\boldsymbol{B}, \tag{71}$$

with $\boldsymbol{A} = \left(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}\right)$ and $\boldsymbol{B} = \boldsymbol{\epsilon}[\boldsymbol{n}]$. Thus, the second derivative of Eq. (69) with respect to $\dot{\theta}$ reads

$$\begin{aligned}\frac{d^2\Delta\epsilon_{t_0}}{d\dot{\theta}^2} &= \int_{\mathcal{V}_t} \frac{1}{2}\rho\dot{\tilde{A}}\,\frac{d^2\left(\dot{\boldsymbol{Q}}^{\mathsf{T}}\dot{\boldsymbol{Q}}\right)}{d\dot{\theta}^2} \cdot \left(\boldsymbol{p} \otimes \boldsymbol{p}\right) \mathrm{d}v \\ &= \int_{\mathcal{V}_t} \rho\dot{\tilde{A}}\left(\left(\sin\theta\right)^2\boldsymbol{A}\boldsymbol{A} - \left(\cos\theta\right)^2\boldsymbol{B}\boldsymbol{B}\right) \cdot \left(\boldsymbol{p} \otimes \boldsymbol{p}\right) \,\mathrm{d}v = 0.\end{aligned} \tag{72}$$

Localization of Eq. (72) yields

$$\rho \dot{\tilde{A}} \left((\sin\theta)^2 \boldsymbol{A}\boldsymbol{A} - (\cos\theta)^2 \boldsymbol{B}\boldsymbol{B} \right) \cdot (\boldsymbol{p} \otimes \boldsymbol{p}) = 0. \tag{73}$$

Since $\rho \neq 0$, $(\sin\theta)^2 \boldsymbol{A}\boldsymbol{A} - (\cos\theta)^2 \boldsymbol{B}\boldsymbol{B} \neq \boldsymbol{0}$ and $\boldsymbol{p} \otimes \boldsymbol{p} \neq \boldsymbol{0}$, the conservation of micro-inertia is obtained, reading

$$\dot{\tilde{A}} = 0. \tag{74}$$

Balance of angular momentum Accounting for conservation of micro-inertia as stated in Eq. (74), Eq. (69) can be formulated as

$$\Delta\epsilon_{t_0} = \int_{\mathcal{V}_t} \left(-\boldsymbol{\sigma} + \left(\rho \left(\tilde{A}\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}} \right) - \operatorname{div}(\tilde{\boldsymbol{\sigma}}) \right) \otimes \boldsymbol{p} - \tilde{\boldsymbol{\sigma}} (\operatorname{grad}(\boldsymbol{p}))^{\mathsf{T}} \right) \cdot \dot{\boldsymbol{Q}} \, \mathrm{d}v = 0. \tag{75}$$

Since $\dot{\boldsymbol{Q}}(t_0) \in Skw$, the localization of Eq. (75) yields the modified balance of angular momentum

$$-\boldsymbol{\sigma} + \left(\rho \left(\tilde{A}\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}} \right) - \operatorname{div}(\tilde{\boldsymbol{\sigma}}) \right) \otimes \boldsymbol{p} - \tilde{\boldsymbol{\sigma}} (\operatorname{grad}(\boldsymbol{p}))^{\mathsf{T}} \in Sym. \tag{76}$$

Consequently, the Cauchy stress $\boldsymbol{\sigma}$ is not symmetric as in Eq. (4).

Existence of heat flux vector Accounting for the results in Eqs. (19), (23), (25), (66) and (74), Eq. (61) can be written as

$$\int_{\mathcal{V}_t} \rho\dot{e} - \boldsymbol{\sigma} \cdot \operatorname{grad}(\boldsymbol{v}) + \left(\rho \left(\tilde{A}\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}} \right) - \operatorname{div}(\tilde{\boldsymbol{\sigma}}) \right) \cdot \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{\sigma}} \cdot \operatorname{grad}(\tilde{\boldsymbol{v}}) - \rho r \, \mathrm{d}v - \int_{\partial\mathcal{V}_t} h \, \mathrm{d}a = 0. \tag{77}$$

Application of Eq. (77) to an infinitesimal tetrahedron yields the existence of the heat flux vector $\boldsymbol{q}$ given by

$$\boldsymbol{q} \cdot \boldsymbol{n} = -h, \tag{78}$$

cf. Marsden and Hughes [43, p. 148].

Balance of internal energy Under consideration of all previously discussed results, localization of Eq. (61) yields the local form of the balance of internal energy, reading

$$\rho\dot{e} - \boldsymbol{\sigma} \cdot \operatorname{grad}(\boldsymbol{v}) + \left(\rho \left(\tilde{A}\tilde{\boldsymbol{a}} - \tilde{\boldsymbol{b}} \right) - \operatorname{div}(\tilde{\boldsymbol{\sigma}}) \right) \cdot \tilde{\boldsymbol{v}} - \rho r - \tilde{\boldsymbol{\sigma}} \cdot \operatorname{grad}(\tilde{\boldsymbol{v}}) + \operatorname{div}(\boldsymbol{q}) = 0. \tag{79}$$

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